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A Note on the uncoupled Dirac-harmonic maps from Kähler spin manifolds to Kähler manifolds

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Abstract. In this manuscript, we consider uncoupled Dirac-harmonic maps from Kähler spin manifolds to Kähler manifolds and prove an existence theorem. Moreover, we construct some new uncoupled Dirac-harmonic maps from Riemann surfaces to Kähler manifolds.

1. Introduction

Motivated by the supersymmetric σ-model of quantum field theory, Dirac-harmonic maps were introduced by Chen et al. in [2,3]. They replaced the anticommuting spinor field of that model, which takes values in a Grassmannian algebra and makes the model supersymmetric, by a commuting field. Nevertheless, they preserve important symmetries, in particular conformal invariance. Mathematically, they can be seen as an extension of the harmonic map problem as they couple a harmonic map type field with a spinor field. Since all the fields are ordinary, commuting variables, we may apply the methods of the geometric calculus of variations. A technical difficulty, however, arises from the fact that the underlying action functional is not bounded from below, in contrast to standard harmonic maps where it is nonnegative.

First, we present the mathematical definitions. A spin manifold M is an oriented Riemannian manifold with a spin structure on its tangent bundle. Moreover, if M is spin, then the spin structures on M are 1-1 correspondence with elements of $H^1(M, \mathbf{Z}_2)$. It is well known that a complex manifold is spin if and only if its first Chern number is even. In this special case, the spin structures are 1-1 correspondence with the holomorphic square roots of the canonical bundle $K_M = \Lambda^{m,0}T^*M$ of M, where $m = \dim_{\mathbb{C}} M$. Fixed a holomorphic square root L of K_M, i.e., $K_M = L \otimes L$, the spin bundle ΣM of M which is determined by L can be identified with $\Lambda^{0,*}L$ ([5]), i.e.,

$$\Sigma M = \Lambda^{0,*}T^*M \otimes L.$$

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The classical connection ∇ on ΣM induced from the Levi–Civita connection on TM is compatible with the hermitian metric $\langle\cdot,\cdot\rangle$ on ΣM. The associated Dirac operator is

$$\partial\!\!\!/ := e_i \cdot \nabla_{e_i} = \sqrt{2}\left(\bar\partial_L + \bar\partial_L^*\right).$$

Here "·" stands for the Clifford multiplication on ΣM and $\{e_i\}$ is a local orthonormal frame of M. The Einstein's summation convention will be used if there is no confusion.

Let N be a closed Riemannian manifold (e.g., a Kähler manifold) and $\phi^{-1}TN$ be the pull-back bundle of TN by ϕ where $\phi : M \longrightarrow N$ is a smooth map. We also denote the metric induced from the metrics on ΣM and $\phi^{-1}TN$ on the twisted bundle $\Sigma M \otimes \phi^{-1}TN$ by $\langle\cdot,\cdot\rangle$. Likewise, we also denote the connection on $\Sigma M \otimes \phi^{-1}TN$ induced from those on ΣM and $\phi^{-1}TN$ by ∇. Therefore, $\Sigma M \otimes \phi^{-1}TN$ becomes a Dirac bundle, a bundle of left modules over the Clifford bundle $\mathrm{Cl}(M)$ together with a Riemannian metric and connection on $\Sigma M \otimes \phi^{-1}TN$ such that

$$\langle X\cdot\sigma_1, \sigma_2\rangle + \langle \sigma_1, X\cdot\sigma_2\rangle = 0,$$
$$\nabla(\varphi\cdot\sigma) = (\nabla\varphi)\cdot\sigma + \varphi\cdot(\nabla\sigma)$$

for all $X \in \Gamma(TM)$, $\varphi \in \Gamma(\mathrm{Cl}(M))$, σ_1, σ_2 and $\sigma \in \Gamma(\Sigma M)$.

A cross-section ψ of $\Sigma M \otimes \phi^{-1}TN$ can be locally written as $\psi = \sum_\alpha \psi^\alpha \otimes \theta_\alpha$, where $\{\psi^\alpha\}$ are local cross-sections of ΣM, and $\{\theta_\alpha\}$ are local cross-sections of $\phi^{-1}TN$. We always use the standard summation convention. The *Dirac operator along the map* ϕ is

$$\begin{aligned} D\!\!\!\!/^{\phi}\psi &:= e_i\cdot\nabla_{e_i}\psi \\ &= \partial\!\!\!/\psi^\alpha \otimes \theta_\alpha + e_i\cdot\psi^\alpha \otimes \nabla_{e_i}\theta_\alpha, \end{aligned}$$

where $\{e_i\}$ is a local orthonormal frame on M. We say that ψ *is harmonic along the map* ϕ if $D\!\!\!\!/^{\phi}\psi = 0$. We will identify $\Sigma M \otimes \phi^{-1}TN$ with $\Sigma M \otimes \phi^{-1}\mathbb{C}TN$ since ΣM is a complex vector bundle. Then every cross-sections $\psi \in \Gamma\left(\Sigma M \otimes \phi^{-1}TN\right)$ can be rewritten as follows:

$$\psi = \psi^\alpha \otimes \theta_\alpha,$$

where θ_α are local cross-sections of $\phi^{-1}\mathbb{C}TN$.

The action functional of the theory is

$$L(\phi,\psi) = \frac{1}{2}\int_M \left(\|\mathrm{d}\phi\|^2 + \left\langle \psi, D\!\!\!\!/^{\phi}\psi\right\rangle\right),$$

which couples the harmonic map type field ϕ with the spinor field ψ, because the Dirac operator $D\!\!\!\!/^{\phi}$ depends on ϕ. We see this coupling also from the Euler–Lagrange equations for $L(\phi,\psi)$ that critical points (ϕ,ψ) have to satisfy (c.f. [2]):

$$\begin{cases} \tau(\phi) = \frac{1}{2}\left\langle \psi^\alpha, e_i\cdot\psi^\beta\right\rangle R^N(\theta_\alpha, \overline{\theta_\beta})\phi_*(e_i), \\ D\!\!\!\!/^{\phi}\psi = 0, \end{cases} \tag{1.1}$$

where $R^N(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, $\forall X, Y \in \Gamma(TN)$ stands for the curvature operator of N, and $\tau(\phi)$ is the tension field of ϕ, i.e.,

$$\tau(\phi) := \left(\nabla_{e_i} \mathrm{d}\phi\right)(e_i).$$

Therefore, solutions of (1.1) are called *Dirac-harmonic maps from M to N*. We say that a Dirac-harmonic map (ϕ, ψ) is uncoupled if the underlying map ϕ is a harmonic map while is coupled otherwise.

There are two type of trivial solutions in the sense that the map part is trivial or the spinor part is trivial:

- ϕ is a harmonic map and $\psi = 0$.
- ϕ is a constant map and $\psi = \psi^\alpha \otimes \theta_\alpha$ with ψ^α is a harmonic spinor.

Ammann and Ginoux [1] studied uncoupled Dirac-harmonic maps using the index theoretical tools and obtained some existence theorems. Chen et al. [4] also proved some existence theorems of uncoupled Dirac-harmonic maps between closed Riemann surfaces using Riemann–Roch theorem.

Our first main result is as follows:

Theorem 1.1. *Let M^{2m} be a closed Kähler spin manifold with a fixed spin structure, and N be a closed Kähler manifold such that $\pi_2(N) = 0$. If $m > 1$, assume additionally there is a positive strictly convex function ρ on the universal covering $\tilde{N}$ of N with the following growth condition: for some constant C,*

$$\rho(y) \le C \exp\left(\frac{1}{4}\tilde{d}(y)^{1/m}\right),$$

where $\tilde{d}$ is the distance function of $\tilde{N}$ from some fixed point $y_0 \in \tilde{N}$. Then for every smooth map $\phi_0 : M \longrightarrow N$, there exists a harmonic map $\phi : M \longrightarrow N$ which is homotopic to ϕ_0 and a complex vector space V (consisting of some pairs (ϕ, ψ)) with complex dimension

$$2\left|\left\{\phi_0^* \mathrm{ch}(N) \cdot \hat{\mathbf{A}}(M)\right\}[M]\right|,$$

such that every $(\phi, \psi) \in V$ is an uncoupled Dirac-harmonic map.

The first nontrivial solution, which appears in [3] for $M = N = \mathrm{S}^2$ and then in [6] for general Riemann surface M and general Riemannian manifold N, couples a harmonic map ϕ with a harmonic spinor $\psi_{\phi,\eta}$ along the map ϕ defined by a twistor spinor η on M as follows:

$$\psi_{\phi,\eta} := e_i \cdot \eta \otimes \phi_*(e_i).$$

We say that η is a twistor spinor if it satisfies the following twistor equation (in the case that M is a Riemann surface)

$$\nabla_X \eta + \frac{1}{2} X \cdot \partial\!\!\!/ \eta = 0, \quad \forall X \in TM. \tag{1.2}$$

Yang [13] and Chen et al. [4] proved that for Riemann surfaces M, N, if a smooth map $\phi : M \longrightarrow N$ satisfies

$$g_N = 0, \quad g_M < |\deg(\phi)| + 1,$$

then every Dirac-harmonic map (ϕ, ψ) from M to N can be constructed in this way with η possibly having isolated singularities, i.e., η is smooth except some finite points and satisfies the twistor Eq. (1.2) except these points. Moreover, the map part is (anti-)holomorphic. This is so called the structure theorem of Dirac-harmonic maps. For more details, we would like to refer the reader to [13] for $M = N = \mathrm{S}^2$, $\deg(\phi) \neq 0$ and [4] for the remaining case.

Finally, we will construct new uncoupled Dirac-harmonic maps from Riemann surface M to Kähler manifold N. Let $\eta, \zeta \in \Gamma(\Sigma M)$, define a spinor $\tilde{\psi}_{\phi,\eta,\zeta}$ along the map ϕ as follows:

$$\tilde{\psi}_{\phi,\eta,\zeta} := e_i \cdot \eta \otimes \phi_*(e_i) + e_i \cdot \zeta \otimes J^N \phi_*(e_i),$$

where J^N is the complex structure of N.

The following theorem is our second main result.

Theorem 1.2. *Let (M, g) be a closed Riemann surface with a fixed spin structure, N be a Kähler manifold and $\phi : M \longrightarrow N$ be a harmonic map. Then for every two twistor spinors η, ζ satisfying* $\mathrm{Re}\,\langle X \cdot \eta, \zeta \rangle = 0$ *for all $X \in TM$, the pair $(\phi, \tilde{\psi}_{\phi,\eta,\zeta})$ constructed above is an uncoupled Dirac-harmonic map from M to N.*

As an immediate consequence, if ϕ is neither holomorphic nor anti-holomorphic, then the structure theorem of Dirac-harmonic maps between closed Riemann surfaces (c.f. [4,13]) is false in general.

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2. Hitchin representation

Recall the Kähler form ω

$$\omega(X, Y) = g(JX, Y) =: \langle JX, Y \rangle .$$

Let $\{e_A\}_{A=1}^{2m} = \{e_i, Je_i\}_{i=1}^{m}$ be a local orthonormal frame of M and denote $\omega_{AB} = \omega(e_A, e_B)$. As an endomorphism of the spinor bundle, the Kahler form ω acts on spinors as follows:

$$\begin{aligned}
\omega \cdot \psi &= \sum_{A<B} \omega_{AB} e_A \cdot e_B \cdot \psi \\
&= \frac{1}{2} \langle Je_A, e_B \rangle \, e_A \cdot e_B \cdot \psi \\
&= \frac{1}{2} e_A \cdot Je_A \cdot \psi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} e_i \cdot J e_i \cdot \psi - \frac{1}{2} J e_i \cdot e_i \cdot \psi \\
&= e_i \cdot J e_i \cdot \psi .
\end{aligned}
$$

In other words,

$$\omega = \frac{1}{2} \sum_{A=1}^{2m} e_A \cdot J e_A = \sum_{i=1}^{m} e_i \cdot J e_i .$$

Fixed i, j, define

$$
\begin{aligned}
&\varepsilon_j = \frac{1}{2} \left(e_j - \sqrt{-1} J e_j \right), \quad \bar{\varepsilon}_j = \frac{1}{2} \left(e_j + \sqrt{-1} J e_j \right), \\
&\omega_j = -\varepsilon_j \bar{\varepsilon}_j, \text{ and } \bar{\omega}_j = -\bar{\varepsilon}_j \varepsilon_j .
\end{aligned}
$$

We will omit the symbol "·" for convenience if there is no confusion. It is easy to check that

$$\varepsilon_i \varepsilon_j = -\varepsilon_j \varepsilon_i, \quad \bar{\varepsilon}_i \bar{\varepsilon}_j = -\bar{\varepsilon}_j \bar{\varepsilon}_i, \quad \varepsilon_i \bar{\varepsilon}_j + \bar{\varepsilon}_j \varepsilon_i = \bar{\varepsilon}_i \varepsilon_j + \varepsilon_j \bar{\varepsilon}_i = -\delta_{ij} .$$

Hence,

$$
\begin{aligned}
&\omega_i + \bar{\omega}_i = 1, \\
&\omega_i^* = \omega_i, \quad \bar{\omega}_i^* = \bar{\omega}_i, \quad \omega_i^2 = \omega_i, \quad \bar{\omega}_i^2 = \bar{\omega}_i, \\
&\omega_i \omega_j = \omega_j \omega_i, \quad \bar{\omega}_i \bar{\omega}_j = \bar{\omega}_j \bar{\omega}_i, \quad \omega_i \bar{\omega}_j = \bar{\omega}_j \omega_i, \quad \omega_i \bar{\omega}_i = 0.
\end{aligned}
$$

Rewrite the Kähler form ω as follows:

$$\omega = \sqrt{-1} \sum_{i=1}^{m} \left(\omega_i - \bar{\omega}_i \right) .$$

Notice that

$$1 = \Pi_{i=1}^{m} \left(\omega_i + \bar{\omega}_i \right) = \sum_{r=0}^{m} \frac{1}{r!(m-r)!} \sum_{|I|=r, |J|=m-r} \omega_I \bar{\omega}_J ,$$

where

$$
\begin{aligned}
&\omega_I = \omega_{i_1} \omega_{i_2} \cdots \omega_{i_p}, \quad \bar{\omega}_J = \bar{\omega}_{j_1} \bar{\omega}_{j_2} \cdots \bar{\omega}_{j_q}, \\
&I = (i_1, i_2, \ldots, i_p), \quad J = (j_1, j_2, \ldots, j_q), \quad p = |I|, q = |J| .
\end{aligned}
$$

Define

$$\pi_r = \frac{1}{r!(m-r)!} \sum_{|I|=r, |J|=m-r} \omega_I \bar{\omega}_J .$$

Then π_r's are globally defined and

$$\nabla \pi_r = 0, \quad \pi_r^* = \pi_r, \quad \pi_r \pi_s = \delta_{rs} \pi_r, \quad \varepsilon_j \pi_{r-1} = \pi_r \varepsilon_j, \quad \bar{\varepsilon}_j \pi_r = \pi_{r-1} \bar{\varepsilon}_j .$$

Moreover,

$$\omega\pi_r = \sqrt{-1}(2r - m)\pi_r,$$

i.e., $\pi_r \cdot \Sigma M$ is the eigenvector space associated to the eigenvalue $\sqrt{-1}(2r - m)$. The above calculations can be found in [5,7].

Under the action of the Kähler form ω, the spinor bundle splits into the orthogonal sum of holomorphic subbundles (c.f. Lemma 1 in[11], so-called *Hitchin representation*)

$$\Sigma M = \Lambda^{0,*}T^*M \otimes L \tag{2.1}$$

which is associated to the decomposition

$$\Lambda^{0,r}T^*M \otimes L = \pi_r \cdot \Sigma M, \quad \mathrm{rank}_{\mathbb{C}}\left(\pi_r \cdot \Sigma M\right) = \binom{m}{r}.$$

Denote

$$\Lambda^{0,odd}T^*M \otimes L = \bigoplus_{\text{k is odd}} \Lambda^{0,k}T^*M \otimes L = \bigoplus_{\text{k is odd}} \pi_k \cdot \Sigma M,$$
$$\Lambda^{0,even}T^*M \otimes L = \bigoplus_{\text{k is even}} \Lambda^{0,k}T^*M \otimes L = \bigoplus_{\text{k is even}} \pi_k \cdot \Sigma M.$$

Consider the volume form $G = \left(\sqrt{-1}\right)^m \Pi_{i=1}^m e_i \cdot Je_i \cdot$. It is easy to check that as an endomorphism of the spinor bundle,

$$G^2 = 1, \quad G^* = G, \quad \nabla G = 0, \text{ and } \quad X \cdot G = -G \cdot X, \quad \forall X \in TM.$$

As a consequence, G has the eigenvalues 1 and -1 with the corresponding eigenspaces

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M. \tag{2.2}$$

Rewrite G as follows:

$$G = \Pi_{i=1}^m \left(\bar{\omega}_i - \omega_i\right).$$

We know that $G\pi_r = (-1)^r$. In particular, comparing the decompositions (2.1) and (2.2), we have

$$\Sigma^+ M = \Lambda^{0,even}T^*M \otimes L, \quad \Sigma^- M = \Lambda^{0,odd}T^*M \otimes L.$$

3. Proof of the main results

In this section, we will give the proof of the main results. First, we have

Lemma 3.1. *Let M be a Kähler manifold with a fixed spin structure, N be a Kähler manifold. Suppose $\phi : M \longrightarrow N$ is a harmonic map and $\psi \in \Gamma\left(\Sigma^{\pm}M \otimes \phi^{-1}T_{1,0}N\right)$ or $\psi \in \Gamma\left(\Sigma^{\pm}M \otimes \phi^{-1}T_{0,1}N\right)$ is harmonic along the map ϕ, then (ϕ, ψ) is an uncompled Dirac-harmonic map from M to N.*

Proof. Notice that for every tangent vector X, we have

$$X : \Sigma^{\pm}M \longrightarrow \Sigma^{\mp}M, \quad \psi \mapsto X \cdot \psi.$$

Therefore, for each $\psi \in \Gamma\left(\Sigma^{\pm}M \otimes \phi^{-1}T_{1,0}N\right)$ or $\psi \in \Gamma\left(\Sigma^{\pm}M \otimes \phi^{-1}T_{0,1}N\right)$, we have

$$\frac{1}{2}\left\langle \psi^{\alpha}, e_i \cdot \psi^{\beta} \right\rangle R^N(\theta_\alpha, \overline{\theta_\beta})\phi_*(e_i) = 0.$$

Thus, (ϕ, ψ) is Dirac-harmonic if and only if

$$\begin{cases} \tau(\phi) = 0, \\ \not{D}^{\phi}\psi = 0, \end{cases}$$

i.e., ϕ is a harmonic map and ψ is harmonic along the map ϕ. □

Second, we need the following existence theorem of harmonic maps.

Theorem 3.2. *Suppose M^m $(m \geq 3)$ is a closed Riemannian manifold and N is a compact Riemannian manifold such that there is a positive strictly convex function ρ on the universal covering $\tilde{N}$ of N with the following growth condition: for some constant C,*

$$\rho(y) \leq C \exp\left(\frac{1}{4}\tilde{d}(y)^{2/m}\right),$$

where $\tilde{d}$ is the distance function of $\tilde{N}$ from some fixed point $y_0 \in \tilde{N}$. Then every smooth map $\phi_0 : M \longrightarrow N$ has a harmonic representative in its homotopy class, i.e., ϕ_0 is homotopic to a harmonic map.

Proof. According to the work of Lin and Wang (c.f. Theorem D [10]), we know that if there is no harmonic sphere S^l in N for $2 \leq l \leq m-1$ and no quasi-harmonic sphere from S^l in N for $3 \leq l \leq m$, then the harmonic map heat flow is smooth. Moreover, under the same assumption, Li and Zhu [9] claimed that the heat flow subconverges in C^2 norm to a smooth harmonic map at infinity.

Recently, Li and the author [8] proved that if the universal covering $\tilde{N}$ of N admits a positive strictly convex function ρ with the following exponential growth: for some constant C

$$\rho(y) \leq C \exp\left(\frac{1}{4}\tilde{d}(y)^{2/m}\right),$$

then there is no quasi-harmonic sphere S^m in N. Also, there is no quasi-harmonic sphere S^l in N for $3 \le l \le m-1$ and no harmonic sphere S^l in N for $2 \le l \le m-1$. Hence, there is a harmonic map $\phi : M \longrightarrow N$ in the homotopy class of $\phi_0 : M \longrightarrow N$. □

Remark 3.1. If $m = 2$ and $\pi_2(N) = 0$, the same result holds (c.f. [12]). In particular, the same result holds if $\tilde{N}$ admits a strictly convex function.

Theorem 3.3. *Let M be a closed Kähler manifold with a fixed spin structure, N be a closed Kähler manifold, and* $\phi : M \longrightarrow N$ *is smooth. Then*

$$\dim_{\mathbb{C}} \{\textit{harmonic spinors along the map } \phi\} \ge 2 \left| \left\{ \phi^* \mathrm{ch}(N) \cdot \hat{\mathbf{A}}(M) \right\} [M] \right| .$$

Proof. Consider the operator $\not{D}^+_{1,0} : \Gamma\left(\Sigma^+ M \otimes \phi^{-1} T_{1,0} N\right) \longrightarrow \Gamma\left(\Sigma^- M \otimes \phi^{-1} T_{1,0} N\right)$. The index theorem (c.f. Theorem 13.10 [7]) gives the following formula

$$\begin{aligned} \mathrm{ind}(\not{D}^+_{1,0}) &= \left\{ \mathrm{ch}(\phi^{-1} T_{1,0} N) \cdot \hat{\mathbf{A}}(M) \right\} [M] \\ &= \left\{ \phi^* \mathrm{ch}(N) \cdot \hat{\mathbf{A}}(M) \right\} [M]. \end{aligned}$$

Similarly, we get

$$\mathrm{ind}(\not{D}^+_{0,1}) = (-1)^m \left\{ \phi^* \mathrm{ch}(N) \cdot \hat{\mathbf{A}}(M) \right\} [M],$$

where $\not{D}^+_{0,1} : \Gamma\left(\Sigma^+ M \otimes \phi^{-1} T_{0,1} N\right) \longrightarrow \Gamma\left(\Sigma^- M \otimes \phi^{-1} T_{0,1} N\right)$. Now this theorem follows from

$$\ker \not{D}^\phi \ge \left| \mathrm{ind}\, \not{D}^+_{1,0} \right| + \left| \mathrm{ind}\, \not{D}^+_{0,1} \right| .$$

□

Before the proof of Theorem 1.1, we give some applications. The first application is to consider the Dirac-harmonic maps from a closed Riemann surface M. In this case, $c_1(M) = 2(1 - g_M)$ is of course an even number and hence M is a spin manifold. Now

$$\left\{ \phi^* \mathrm{ch}(N) \cdot \hat{\mathbf{A}}(M) \right\} [M] = \left\{ \phi^* c_1(N) \right\} [M],$$

since $\hat{\mathbf{A}}(M) = 1$. In particular, if N is a Riemann surface (this case was considered in [4]), then

$$\left\{ \phi^* \mathrm{ch}(N) \cdot \hat{\mathbf{A}}(M) \right\} [M] = \deg(\phi) \chi(N) = 2 \deg(\phi)(1 - g_N).$$

The second application is to consider the Dirac-harmonic maps from a complex surface. Suppose M^4 is a Kähler manifold whose the first Chern number $c_1(M)$ is even. Then $\hat{\mathbf{A}}(M) = 1 - p_1(M)/24 = 1 - (c_1^2(M) - 2c_2(M))/24$ where p_1 is the first Pontryagin class. Thus,

$$
\begin{aligned}
\left\{\phi^*\mathrm{ch}(N)\cdot\hat{\mathbf{A}}(M)\right\}[M] &= \left\{\phi^*\mathrm{ch}^2(N) - \frac{n}{24}p_1(M)\right\}[M] \\
&= \left\{\frac{1}{2}\phi^*c_1^2(N) - \phi^*c_2(N) - \frac{n}{24}c_1^2(M) + \frac{n}{12}c_2(M)\right\}[M] \\
&= \left\{\frac{1}{2}\phi^*c_1^2(N) - \phi^*c_2(N)\right\}[M] + n\hat{A}(M).
\end{aligned}
$$

Therefore, if N is a Riemann surface, then $c_2(N) = 0$, $c_1^2(N) = 0$, which implies that

$$
\left\{\phi^*\mathrm{ch}(N)\cdot\hat{\mathbf{A}}(M)\right\}[M] = \hat{A}(M).
$$

If N is a complex surface, i.e., $\dim_{\mathbb{C}} N = 2$, then

$$
\left\{\phi^*\mathrm{ch}(N)\cdot\hat{\mathbf{A}}(M)\right\}[M] = -12\deg(\phi)\hat{A}(N) + 2\hat{A}(M).
$$

Summarizing this analysis and applying Theorem 3.3, we obtain the following Corollary.

Corollary 3.4. *Suppose M^{2m} be a closed Kähler spin manifold with a fixed spin structure, N^{2n} be a closed Kähler manifold, and $\phi : M \longrightarrow N$ be a smooth map. Then*

(1) *If $m = 1$, then the harmonic spinors along the map ϕ form a complex vector space with complex dimension at least*

$$
2\left|\left\{\phi^*c_1(N)\right\}[M]\right|.
$$

In particular, if N is a Riemann surface, then the harmonic spinors along the map ϕ forms a complex vector space with complex dimension at least

$$
4\left|\deg(\phi)(1-g_N)\right|,
$$

where g_N is the genus of N.

(2) *If $m = 2$, then the complex dimension of the harmonic spinor along the map ϕ is at least*

$$
2\left|\left\{\frac{1}{2}\phi^*c_1^2(N) - \phi^*c_2(N)\right\}[M] + n\hat{A}(M)\right|.
$$

Therefore, if $n = 1$, then

$$
2\left|\left\{\frac{1}{2}\phi^*c_1^2(N) - \phi^*c_2(N)\right\}[M] + n\hat{A}(M)\right| = 2\left|\hat{A}(M)\right|.
$$

If $n = 2$, then

$$
2\left|\left\{\frac{1}{2}\phi^*c_1^2(N) - \phi^*c_2(N)\right\}[M] + n\hat{A}(M)\right| = 2\left|-12\deg(\phi)\hat{A}(N) + 2\hat{A}(M)\right|.
$$

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. According to Theorem 3.2 and Remark 3.1, we first choose a harmonic map $\phi : M \to N$ which is homotopic to $\phi_0 : M \to N$. Consider

$$\begin{aligned}
V^+_{even} &= \left\{\psi \in \Gamma\left(\Sigma^+ M \otimes \phi^{-1} T_{1,0} N\right) : \psi \text{ is harmonic}\right\} \\
&\quad \oplus \left\{\psi \in \Gamma\left(\Sigma^+ M \otimes \phi^{-1} T_{0,1} N\right) : \psi \text{ is harmonic}\right\}, \\
V^-_{even} &= \left\{\psi \in \Gamma\left(\Sigma^- M \otimes \phi^{-1} T_{1,0} N\right) : \psi \text{ is harmonic}\right\} \\
&\quad \oplus \left\{\psi \in \Gamma\left(\Sigma^- M \otimes \phi^{-1} T_{0,1} N\right) : \psi \text{ is harmonic}\right\}, \\
V^+_{odd} &= \left\{\psi \in \Gamma\left(\Sigma^+ M \otimes \phi^{-1} T_{1,0} N\right) : \psi \text{ is harmonic}\right\} \\
&\quad \oplus \left\{\psi \in \Gamma\left(\Sigma^- M \otimes \phi^{-1} T_{0,1} N\right) : \psi \text{ is harmonic}\right\}, \\
V^-_{odd} &= \left\{\psi \in \Gamma\left(\Sigma^- M \otimes \phi^{-1} T_{1,0} N\right) : \psi \text{ is harmonic}\right\} \\
&\quad \oplus \left\{\psi \in \Gamma\left(\Sigma^+ M \otimes \phi^{-1} T_{0,1} N\right) : \psi \text{ is harmonic}\right\}.
\end{aligned}$$

Thanks to Lemma 3.1, we know that (ϕ, ψ) is Dirac-harmonic if ψ belongs to either $V^\pm_{even}$ or $V^\pm_{odd}$. According to the proof of Theorem 3.3, we know that if m is even, then either

$$\begin{aligned}
\dim_{\mathbb{C}} V^+_{even} &\geq 2\left|\left\{\phi^* \mathrm{ch}(N) \cdot \hat{\mathbf{A}}(M)\right\}[M]\right|, \quad \text{or} \\
\dim_{\mathbb{C}} V^-_{even} &\geq 2\left|\left\{\phi^* \mathrm{ch}(N) \cdot \hat{\mathbf{A}}(M)\right\}[M]\right|
\end{aligned}$$

while if m is odd, then either

$$\begin{aligned}
\dim_{\mathbb{C}} V^+_{odd} &\geq 2\left|\left\{\phi^* \mathrm{ch}(N) \cdot \hat{\mathbf{A}}(M)\right\}[M]\right|, \quad \text{or} \\
\dim_{\mathbb{C}} V^-_{odd} &\geq 2\left|\left\{\phi^* \mathrm{ch}(N) \cdot \hat{\mathbf{A}}(M)\right\}[M]\right|.
\end{aligned}$$

Thus, we can find a complex linear space V, consisting of some uncoupled Dirac-harmonic maps, with complex dimension at least $2\left|\left\{\phi^* \mathrm{ch}(N) \cdot \hat{\mathbf{A}}(M)\right\}[M]\right|$. □

Finally, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Choose a local orthonormal frame $\{e_i\}$ of TM such that $\nabla e_i = 0$ at the point where the computation is done. A direct computation implies

$$\begin{aligned}
\not{D}^\phi \tilde{\psi}_{\phi,\eta,\zeta} &= e_j \cdot \nabla_{e_j} \tilde{\psi}_{\phi,\eta,\zeta} \\
&= e_j \cdot e_i \cdot \nabla_{e_j} \eta \otimes \phi_*(e_i) + e_j \cdot e_i \cdot \eta \otimes \left(\nabla_{e_j} \mathrm{d}\phi\right)(e_i) \\
&\quad + e_j \cdot e_i \cdot \nabla_{e_j} \zeta \otimes J^N \phi_*(e_i) + e_j \cdot e_i \cdot \zeta \otimes J^N \left(\nabla_{e_j} \mathrm{d}\phi\right)(e_i) \\
&= -e_i \cdot e_j \cdot \nabla_{e_j} \eta \otimes \phi_*(e_i) - 2\nabla_{e_i} \eta \otimes \phi_*(e_i) \\
&\quad + e_j \cdot e_i \cdot \eta \otimes \left(\nabla_{e_j} \mathrm{d}\phi\right)(e_i) \\
&\quad - e_i \cdot e_j \cdot \nabla_{e_j} \zeta \otimes J^N \phi_*(e_i) - 2\nabla_{e_i} \zeta \otimes J^N \phi_*(e_i)
\end{aligned}$$

$$
\begin{aligned}
&+ e_j \cdot e_i \cdot \zeta \otimes J^N \left(\nabla_{e_j} \mathrm{d}\phi\right)(e_i) \\
= &-2 \left(\nabla_{e_i} \eta + \frac{1}{2} e_i \cdot \partial\!\!\!/ \eta\right) \otimes \phi_*(e_i) - \eta \otimes \tau(\phi) \\
&- 2 \left(\nabla_{e_i} \zeta + \frac{1}{2} e_i \cdot \partial\!\!\!/ \zeta\right) \otimes J^N \phi_*(e_i) - \zeta \otimes J^N \tau(\phi).
\end{aligned}
$$

Here we used the fact that J^N is parallel. Since η, ζ are twistor spinors and ϕ is a harmonic map, we obtain

$$
D\!\!\!/^{\phi} \tilde{\psi}_{\phi,\eta,\zeta} = 0.
$$

To complete the proof of the theorem, we need only to check that the curvature term appearing in the first Eq. (1.1) equals to zero, i.e.,

$$
\begin{aligned}
&\langle e_i \cdot \eta, e_k \cdot e_j \cdot \eta \rangle R^N(\phi_*(e_i), \phi_*(e_j))\phi_*(e_k) \\
&\quad + \langle e_i \cdot \zeta, e_k \cdot e_j \cdot \zeta \rangle R^N(J^N \phi_*(e_i), J^N \phi_*(e_j))\phi_*(e_k) \\
&\quad + \langle e_i \cdot \eta, e_k \cdot e_j \cdot \zeta \rangle R^N(\phi_*(e_i), J^N \phi_*(e_j))\phi_*(e_k) \\
&\quad + \langle e_i \cdot \zeta, e_k \cdot e_j \cdot \eta \rangle R^N(J^N \phi_*(e_i), \phi_*(e_j))\phi_*(e_k) \\
&= 0.
\end{aligned}
$$

In fact, choose $\{e_1, e_2\}$ such that $\langle \phi_*(e_1), \phi_*(e_2) \rangle = 0$ at a considering point. Then

$$
\begin{aligned}
&\langle e_i \cdot \eta, e_k \cdot e_j \cdot \eta \rangle R^N(\phi_*(e_i), \phi_*(e_j))\phi_*(e_k) \\
&= (\langle e_1 \cdot \eta, e_k \cdot e_2 \cdot \eta \rangle - \langle e_2 \cdot \eta, e_k \cdot e_1 \cdot \eta \rangle) R^N(\phi_*(e_1), \phi_*(e_2))\phi_*(e_k) \\
&= (\langle e_1 \cdot \eta, e_1 \cdot e_2 \cdot \eta \rangle - \langle e_2 \cdot \eta, e_1 \cdot e_1 \cdot \eta \rangle) R^N(\phi_*(e_1), \phi_*(e_2))\phi_*(e_1) \\
&\quad + (\langle e_1 \cdot \eta, e_2 \cdot e_2 \cdot \eta \rangle - \langle e_2 \cdot \eta, e_2 \cdot e_1 \cdot \eta \rangle) R^N(\phi_*(e_1), \phi_*(e_2))\phi_*(e_2) \\
&= (\langle \eta, e_2 \cdot \eta \rangle + \langle e_2 \cdot \eta, \eta \rangle) R^N(\phi_*(e_1), \phi_*(e_2))\phi_*(e_1) \\
&\quad + (- \langle e_1 \cdot \eta, \eta \rangle - \langle \eta, e_1 \cdot \eta \rangle) R^N(\phi_*(e_1), \phi_*(e_2))\phi_*(e_2) \\
&= 0.
\end{aligned}
$$

Similarly,

$$
\langle e_i \cdot \zeta, e_k \cdot e_j \cdot \zeta \rangle R^N(J^N \phi_*(e_i), J^N \phi_*(e_j))\phi_*(e_k) = 0.
$$

Finally,

$$
\begin{aligned}
&\langle e_i \cdot \eta, e_k \cdot e_j \cdot \zeta \rangle R^N(\phi_*(e_i), J^N \phi_*(e_j))\phi_*(e_k) \\
&\quad + \langle e_i \cdot \zeta, e_k \cdot e_j \cdot \eta \rangle R^N(J^N \phi_*(e_i), \phi_*(e_j))\phi_*(e_k) \\
&= (\langle e_i \cdot \eta, e_k \cdot e_i \cdot \zeta \rangle - \langle e_i \cdot \zeta, e_k \cdot e_i \cdot \eta \rangle) R^N(\phi_*(e_i), J^N \phi_*(e_i))\phi_*(e_k) \\
&= 2 \operatorname{Re} (\langle e_i \cdot \eta, e_k \cdot e_i \cdot \zeta \rangle) R^N(\phi_*(e_i), J^N \phi_*(e_i))\phi_*(e_k) \\
&= 2 \operatorname{Re} (\langle \eta, (-e_k + 2\delta_{ik} e_i) \cdot \zeta \rangle) R^N(\phi_*(e_i), J^N \phi_*(e_i))\phi_*(e_k) \\
&= 0.
\end{aligned}
$$

The last identity follows from the assumption that $\operatorname{Re} \langle X \cdot \eta, \zeta \rangle = 0$ for all $X \in TM$. □

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