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Quiver GIT for varieties with tilting bundles

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Abstract. In the setting of a variety X admitting a tilting bundle T we consider the problem of constructing X as a quiver GIT quotient of the algebra $A := \mathrm{End}_X(T)^{\mathrm{op}}$. We prove that if the tilting equivalence restricts to a bijection between the skyscraper sheaves of X and the closed points of a quiver representation moduli functor for $A = \mathrm{End}_X(T)^{\mathrm{op}}$ then X is indeed a fine moduli space for this moduli functor, and we prove this result without any assumptions on the singularities of X. As an application we consider varieties which are projective over an affine base such that the fibres are of dimension 1, and the derived pushforward of the structure sheaf on X is the structure sheaf on the base. In this situation there is a particular tilting bundle on X constructed by Van den Bergh, and our result allows us to reconstruct X as a quiver GIT quotient for an easy to describe stability condition and dimension vector. This result applies to flips and flops in the minimal model program, and in the situation of flops shows that both a variety and its flop appear as moduli spaces for algebras produced from different tilting bundles on the variety. We also give an application to rational surface singularities, showing that their minimal resolutions can always be constructed as quiver GIT quotients for specific dimension vectors and stability conditions. This gives a construction of minimal resolutions as moduli spaces for all rational surface singularities, generalising the G-Hilbert scheme moduli space construction which exists only for quotient singularities.

1. Introduction

1.1. Overview

Any variety X equipped with a tilting bundle T induces a derived equivalence between the bounded derived category of coherent sheaves on X and the bounded derived category of finitely generated left modules for the algebra $A := \mathrm{End}_X(T)^{\mathrm{op}}$. This situation is similar to the case of an affine variety $\mathrm{Spec}(R)$ where we can construct the commutative algebra $R = \mathrm{End}_X(\mathcal{O}_X)^{\mathrm{op}}$ and there is an abelian equivalence between coherent sheaves on $\mathrm{Spec}(R)$ and finitely generated left R-modules. However, whereas in the affine case we can recover the variety $\mathrm{Spec}(R)$ from the algebra R, it is not so clear how to recover the variety X from the algebra A. One possibility is to present A as the path algebra of a quiver with relations, construct a moduli space of quiver representations for some dimension vector and stability condition, and attempt to relate this moduli space back to X.

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While this approach may not work in general there are many examples where this is known to be successful, such as del Pezzo surfaces [14,25], minimal resolutions of Kleinian singularities [8,15,27], and crepant resolutions of Gorenstein quotient singularities in dimension 3 [5,12], which lead us to hope it may work in some other interesting settings.

In this paper we will determine conditions for X to be a fine moduli space for the quiver representation moduli functor $\mathcal{F}_A$, (Sect. 2.6), and this will allow us to prove that X is a quiver GIT quotient for a specific stability condition and dimension vector in a large class of examples. These examples include applications to the minimal model program and to resolutions of rational surface singularities.

This problem was also considered by Bergman and Proudfoot [2], who study embeddings of closed points and tangent spaces to show that a smooth variety is a connected component of the quiver GIT quotient for 'great' stability condition and dimension vector. However, their approach cannot be extended to singular varieties and it can be difficult to identify which conditions are 'great'. The methods developed in this paper have the advantages of applying to singular varieties, such as those occurring in the minimal model program, and allowing us to identify a specific stability condition and dimension vector in applications.

1.2. Comparing moduli functors

In developing methods to understand quiver representation moduli functors we are inspired by the following result of Sekiya and Yamaura [34].

Theorem 1.2.1. ([34, Theorem 4.20]) *Let B be an algebra with tilting module T. Define $A = \mathrm{End}_B(T)^{\mathrm{op}}$, suppose that both A and B are presented as path algebras of quivers with relations, and let $\mathcal{F}_A$ and $\mathcal{F}_B$ denote quiver representation moduli functors on A and B for some choice of stability conditions and dimension vectors. Then if the tilting equivalences*

$$\begin{tikzcd} D^b(B\text{-mod}) \arrow[r, bend left, "\mathbb{R}\mathrm{Hom}_B(T,-)"] & D^b(A\text{-mod}) \arrow[l, bend left, "T \otimes^{\mathbb{L}}_A (-)"] \end{tikzcd}$$

restrict to a bijection between $\mathcal{F}_B(\mathbb{C})$ and $\mathcal{F}_A(\mathbb{C})$ then $\mathcal{F}_B$ is naturally isomorphic to $\mathcal{F}_A$.

This leads us to the idea of working with a moduli functor for which X is a fine moduli space instead of working with X itself, and we then prove the following variant of Sekiya and Yamaura's result.

Theorem. (Theorem 4.0.1) *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of varieties. Suppose X is equipped with a tilting bundle T, define $A = \mathrm{End}_X(T)^{\mathrm{op}}$, and suppose that A is presented as a quiver with relations. Let $\mathcal{F}_A$ be a quiver representation moduli functor on A for some dimension vector and stability condition. Then if the tilting equivalences*

$$\begin{tikzcd} D^b(\mathrm{Coh}\, X) \arrow[rr, bend left, "\mathbb{R}\mathrm{Hom}_X(T,-)"] & & D^b(A\text{-mod}) \arrow[ll, bend left, "T \otimes^{\mathbb{L}}_A (-)"] \end{tikzcd}$$

restrict to a bijection between $\mathcal{F}_X(\mathbb{C})$ and $\mathcal{F}_A(\mathbb{C})$ then $\mathcal{F}_X$ is naturally isomorphic to $\mathcal{F}_A$.

We recall the definitions of the moduli functors $\mathcal{F}_A$ and $\mathcal{F}_X$ in Sects. 2.6 and 2.7, and note in "Appendix 7" that [34, Theorem 4.20] should be stated for the functor defined in Sect. 2.6 rather than the functor originally defined in [34, Section 4.2]. The moduli functor $\mathcal{F}_X$ is similar to the Hilbert functor of one point on a variety, which is well-known to be represented by X, but for completeness we provide a proof in this setting.

Proposition. (Proposition 4.0.2) *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of varieties. Then there is an natural isomorphism between the functor of points $\mathrm{Hom}_{\mathfrak{Sch}}(-, X)$ and the moduli functor $\mathcal{F}_X$. In particular X is a fine moduli space for $\mathcal{F}_X$ with tautological object $\Delta_* \mathcal{O}_X$ on $X \times_{\mathrm{Spec}(\mathbb{C})} X$ where Δ is the diagonal inclusion.*

Combining these two results we have a method to show when a variety X with tilting bundle T can be recovered via quiver GIT as a fine moduli space for representations of the algebra $A = \mathrm{End}_X(T)^{\mathrm{op}}$.

Corollary 1.2.2. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of varieties. Suppose X is equipped with a tilting bundle, T, define $A = \mathrm{End}_X(T)^{\mathrm{op}}$, and suppose that A is presented as a quiver with relations. Let $\mathcal{F}_A$ be a quiver representation moduli functor on A for some indivisible dimension vector d and generic stability condition θ. Then if the tilting equivalences*

$$\begin{tikzcd} D^b(\mathrm{Coh}\, X) \arrow[rr, bend left, "\mathbb{R}\mathrm{Hom}_X(T,-)"] & & D^b(A\text{-mod}) \arrow[ll, bend left, "T \otimes^{\mathbb{L}}_A (-)"] \end{tikzcd}$$

restrict to a bijection between the skyscraper sheaves on X and the θ-stable A-modules with dimension vector d then X is a fine moduli space for $\mathcal{F}_A$ and the tautological bundle is the dual of the tilting bundle T.

1.3. Applications

To give an application of this theorem we need a class of varieties with tilting bundles and well-understood tilting equivalences. We consider the situation arising in following theorem of Van den Bergh.

Theorem 1.3.1. ([37, Theorem A]) *Let $\pi : X \to \operatorname{Spec}(R)$ be a projective morphism of Noetherian schemes such that $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$ and π has fibres of dimension ≤ 1. Then there are tilting bundles T_0 and $T_1 = T_0^\vee$ on X such that the derived equivalences $\mathbb{R}\mathrm{Hom}_X(T_i, -) : D^b(\mathrm{Coh}\, X) \to D^b(A_i\text{-mod})$ restrict to equivalences of abelian categories between ${}^{-i}\mathrm{Per}(X/R)$ and A_i-mod, where $A_i = \mathrm{End}_X(T_i)^{\mathrm{op}}$.*

This gives us a large class of varieties with well-understood tilting equivalences. We recall the definition of ${}^{-i}\mathrm{Per}(X/R)$ for $i = 0, 1$ in Definition 5.2.1. We then show that in this situation there is a particular choice of dimension vector d_{T_0} and stability condition θ_{T_0} such that X occurs as the quiver GIT quotient of A_0.

Corollary. (Corollary 5.2.4) *Suppose we are in the situation of Theorem 1.3.1 and that X and $\operatorname{Spec}(R)$ are both varieties. Then X is the fine moduli space for the quiver representation moduli functor of $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$ for dimension vector d_{T_0} and stability condition θ_{T_0}.*

See Sect. 5.1 for the definitions of θ_{T_0} and d_{T_0}. We note they are easy to define and depend only on a decomposition of T into indecomposable summands.

1.4. Applications to the minimal model program

The class of varieties in the above corollary includes flips and flops of dimension 3 in the minimal model program. In the setting of smooth, projective threefold flops were constructed as components of moduli spaces and shown to be derived equivalent in the work of Bridgeland [4], and this work was extended to include projective threefold with Gorenstein terminal singularities by Chen [9]. These results were reinterpreted more generally via tilting bundles by Van den Bergh [37]. We can now reinterpret these results once again by combining Corollary 5.2.4 with Van den Bergh's results.

It is immediate from Corollary 5.2.4 that if $\pi : X \to \operatorname{Spec}(R)$ is either a flipping or flopping contraction with fibres of dimension ≤ 1 then both X and its flip/flop can be reconstructed as fine moduli spaces with tilting tautological bundles. Further, in the case of flops, the following corollary shows that both X and its flop can be constructed as quiver representation moduli spaces arising from tilting bundles on X.

Corollary. (Corollary 5.3.2) *Suppose we are in the situation of Corollary 5.2.4 and that $\pi : X \to \operatorname{Spec}(R)$ is a flopping contraction with flop $\pi' : X' \to \operatorname{Spec}(R)$. Then X is the quiver GIT quotient of the algebra $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$ for dimension vector d_{T_0} and stability condition θ_{T_0} with tautological bundle $T_0^\vee$, and the flop X' is the quiver GIT quotient of the algebra $A_1 = \mathrm{End}_X(T_1)^{\mathrm{op}}$ for dimension vector d_{T_1} and stability condition θ_{T_1}.*

This fits into a general philosophy of having a preferred stability condition defined by a tilting bundle and realising all minimal models via quiver GIT by changing the tilting bundle rather than changing the stability condition.

1.5. Applications to resolutions of rational surface singularities

Minimal resolutions of affine rational surface singularities automatically satisfy the conditions of Corollary 5.2.4 hence provide another class of examples.

Corollary. (Example 5.4.2) *Suppose that X is a variety and that $\pi : X \to \operatorname{Spec}(R)$ is the minimal resolution of a rational surface singularity. Then there is a tilting bundle T_0 on X such that X is the fine moduli space of $A_0 = \operatorname{End}_X(T_0)^{\mathrm{op}}$ for dimension vector d_{T_0} and stability condition θ_{T_0} with tautological bundle $T_0^\vee$.*

For quotient surface singularities this result was already known when either $G < \mathrm{SL}_2(\mathbb{C})$ [15], or when G was a cyclic or dihedral subgroup of $\mathrm{GL}_2(\mathbb{C})$ [11,39, 41,42], but is new in other cases. In particular, for quotient surface singularities the minimal resolution is known to have moduli space interpretation as G-Hilb($\mathbb{C}^2$), see [21,22], but in general the tautological bundle is not tilting. This corollary extends a similar moduli space interpretation to minimal resolutions of all rational surface singularities such that the tautological bundle is tilting.

1.6. Outline

In Sect. 2 we recall a number of preliminary definitions and theorems relating to tilting bundles and quiver GIT which we will need in later sections. Section 3 consists of a collection of preliminary lemmas which form the bulk of the proofs of our main results. We then prove our main results in Sect. 4, and give an application to a class of examples motivated from the minimal model program, and also to resolutions of rational singularities, in Sect. 5. "Appendix 7" notes and corrects a small error in the results of [34].

2. Preliminaries

In this section we recall a number of definitions and theorems we will use later, in particular relating to tilting bundles and Quiver GIT.

2.1. Geometric and notational preliminaries

We begin by giving some geometric and notational preliminaries. Throughout this paper all schemes will be over $\mathbb{C}$ and a variety will be a scheme which is separated, reduced, irreducible and of finite type over $\mathbb{C}$. In the introduction we stated our results for varieties projective over an affine base, but in fact we will prove our results in the generality of schemes, X, arising from projective morphisms $\pi : X \to \operatorname{Spec}(R)$ of finite type schemes over $\mathbb{C}$. Such schemes are quasi-projective over $\mathbb{C}$, and hence separated, so are a slight generalisation of varieties projective over an affine base in that they may not be reduced or irreducible. For an affine scheme $\operatorname{Spec}(R)$ we will let $\mathcal{O}_R$ denote $\mathcal{O}_{\operatorname{Spec}(R)}$. We denote the category of coherent sheaves on a scheme X by $\operatorname{Coh} X$, we denote the skyscraper sheaf of a closed point

$x \in X$ by $\mathcal{O}_x$, and for a locally free sheaf $\mathcal{F} \in \operatorname{Coh} X$ we let $\mathcal{F}^\vee$ denote the dual $\mathcal{H}om_X(\mathcal{F}, \mathcal{O}_X)$. For an algebra A we let A^{op} denote the opposite algebra of A, and A-mod denote the category of finitely generated left A-modules.

2.2. Derived categories and tilting

We recall the definitions of tilting bundles on schemes and several notions related to derived categories that we will make use of later.

Consider a triangulated $\mathbb{C}$-linear category $\mathfrak{C}$ with small direct sums. A subcategory is *localising* if it is triangulated and also closed under all small direct sums. A localising subcategory is necessarily closed under direct summands [32, Proposition 1.6.8]. An object $T \in \mathfrak{C}$ *generates* if the smallest localising category containing T is $\mathfrak{C}$.

Definition 2.2.1. Let $\mathfrak{C}$ be a triangulated category closed under small direct sums. An object T in $\mathfrak{C}$ is *tilting* if:

(i) $\operatorname{Ext}^k_{\mathfrak{C}}(T, T) = 0$ for $k \neq 0$.
(ii) T generates $\mathfrak{C}$.
(iii) The functor $\operatorname{Hom}_{\mathfrak{C}}(T, -)$ commutes with small direct sums.

For X a quasi-projective scheme let $D(X)$ denote the derived category of quasi-coherent sheaves on X, and $D^b(X)$ denote the bounded derived category of coherent sheaves. For X a Noetherian quasi-projective scheme $D(X)$ is closed under small direct sums [33, Example 1.3], and $D(X)$ is compactly generated with compact objects the perfect complexes [33, Proposition 2.5]. We let $\operatorname{Perf}(X)$ denote the category of perfect complexes on X. When X is smooth the category of perfect complexes equals $D^b(X)$.

For an algebra A we let $D(A)$ be the derived category of left modules over A, and $D^b(A)$ the bounded derived category of finitely generated left A-modules. When $D(X)$ has tilting object a sheaf, T, then define $A := \operatorname{End}_X(T)^{\mathrm{op}}$. When T is a locally free coherent sheaf on X then T is a *tilting bundle* and this gives a derived equivalence between $D(X)$ and $D(A)$.

Theorem 2.2.2. ([20, Theorem 7.6], [6, Remark 1.9]) *Let X be a scheme that is projective over an affine scheme of finite type, $\pi : X \to \operatorname{Spec}(R)$, with tilting bundle T on X and define $A = \operatorname{End}_X(T)^{\mathrm{op}}$. Then:*

(i) *The functor $T_* := \mathbb{R}\operatorname{Hom}_X(T, -)$ is an equivalence between $D(X)$ and $D(A)$. An inverse equivalence is given by the left adjoint $T^* = T \otimes^{\mathbb{L}}_A (-)$.*
(ii) *The functors T_*, T^* remain equivalences when restricted to the bounded derived categories of finitely generated modules and coherent sheaves.*
(iii) *If X is smooth then A has finite global dimension.*

Moreover the equivalence T_* is R-linear, and A is a finite R-algebra.

2.3. Quivers and quiver GIT

We set our notation for quivers and then recall the definitions required for quiver geometric invariant theory, following the definitions of King [26].

A *quiver* is a directed multigraph. We will denote a quiver Q by $Q = (Q_0, Q_1)$, with Q_0 the set of vertices and Q_1 the set of arrows. The set of arrows is equipped with head and tail maps $h, t : Q_1 \to Q_0$ which take an arrow to the vertices that are its head and tail respectively. We compose arrows from right to left, that is

$$b.a = \begin{cases} b.a & \text{if } h(a) = t(b); \\ 0 & \text{otherwise}; \end{cases}$$

and we extend this definition to paths. We recall that there is a trivial path e_i for each vertex $i \in Q_0$ and that these form a set of orthogonal idempotents.

We denote the path algebra by $\mathbb{C}Q$, define S to be the subalgebra of $\mathbb{C}Q$ generated by the trivial paths, and define V to be the $\mathbb{C}$-vector subspace of $\mathbb{C}Q$ spanned by the arrows $a \in Q_1$. Then S is a semisimple $\mathbb{C}$-algebra, V is an $S^e := S \otimes_{\mathbb{C}} S^{\mathrm{op}}$-module, and $\mathbb{C}Q = T_S(V) := \bigoplus_{i \geq 0} V^{\otimes s i}$. Given $\Lambda \subset \mathbb{C}Q$ an S^e-module we define $I(\Lambda)$ to be the two sided ideal in $\mathbb{C}Q$ generated by Λ. We then define

$$\frac{\mathbb{C}Q}{\Lambda} := \frac{\mathbb{C}Q}{I(\Lambda)}$$

and refer to it as the path algebra with *relations* Λ.

We can now recall the definitions required for quiver GIT.

Definition 2.3.1. Let $Q = (Q_1, Q_0)$ be a quiver.

(i) A *dimension vector* for Q is defined to be an element $d \in \mathbb{N}^{Q_0}$ assigning a non-negative integer to each vertex.
(ii) A *dimension d representation* of Q is given by assigning to each vertex i the vector space $V_i = \mathbb{C}^{d(i)}$, to each arrow a a linear map $\phi_a : V_{t(a)} \to V_{h(a)}$, and to each trivial path e_i the linear map id_{V_i}.
(iii) A *morphism*, $\psi : (V_i, \rho_a) \to (W_i, \chi_a)$, between two finite dimensional representations is given by a linear map $\psi_i : V_i \to W_i$ for each vertex i such that for every arrow a we have $\chi_a \circ \psi_{t(a)} = \psi_{h(a)} \circ \rho_a$.
(iv) The *representation variety*, $\mathrm{Rep}_d(Q)$, is defined to be the set of all representations of Q of dimension d, and we note that this is an affine variety.

We then suppose that the quiver has relations Λ defining the algebra $A = \mathbb{C}Q/\Lambda$.

(v) A *representation of the quiver with relations*, (Q, Λ), is a representation of Q such that the linear maps assigned to the arrows satisfy the relations among the paths in the quiver. We recall that a representation of a quiver with relations corresponds to a left $\mathbb{C}Q/\Lambda$-module.
(vi) The *representation scheme* $\mathrm{Rep}_d(Q, \Lambda)$ is the closed subscheme of the affine variety $\mathrm{Rep}_d(Q)$ cut out by the ideal corresponding to the relations Λ. Closed points of $\mathrm{Rep}_d(Q, \Lambda)$ correspond to representations of (Q, Λ).

An action of a reductive group on the affine scheme $\mathrm{Rep}_d(Q, \Lambda)$ can now be defined. For $\{\phi_a : a \in Q_1\}$, a dimension d representation, there is an action of $\mathrm{GL}_{d(i)}(\mathbb{C})$ at vertex i by base change;

$$g.\phi_a = \begin{cases} g \circ \phi_a & \text{if } t(a) = i; \\ \phi_a \circ g^{-1} & \text{if } h(a) = i; \\ 0 & \text{otherwise.} \end{cases}$$

Then $G := \mathrm{GL}_d(\mathbb{C}) := \prod_{i \in Q_0} \mathrm{GL}_{d(i)}(\mathbb{C})$ acts on $\mathrm{Rep}_d(Q, \Lambda)$ with kernel $\mathbb{C}^* = \Delta$. We note that orbits of G correspond to isomorphism classes of representations.

Definition 2.3.2. The *affine quotient* with dimension vector d is defined to be

$$\mathrm{Rep}_d(Q, \Lambda)/\!/G := \mathrm{Spec}(\mathbb{C}[\mathrm{Rep}_d(Q, \Lambda)]^G).$$

We now recall the definition of stability conditions in order to consider more general GIT quotients of $\mathrm{Rep}_d(Q, \Lambda)$.

Definition 2.3.3. (i) For a dimension vector d a *stability condition* is defined to be a $\theta \in \mathbb{Z}^{Q_0}$ assigning an integer to each vertex of Q such that $\sum_{i \in Q_0} d(i)\theta(i) = 0$. For a finite dimensional representation M let d_M be the dimension vector of M, and define $\theta(M) = \sum_{i \in Q_0} \theta(i) d_M(i)$.

(ii) A representation M of dimension d is θ-*semistable* if any subrepresentation $N \subset M$ satisfies $\theta(N) \geq \theta(M)$.

(iii) A θ-semistable representation M of dimension d is θ-*stable* if there are no non-zero proper subrepresentations $N \subset M$ with $\theta(N) = \theta(M)$. For a dimension vector d, a stability condition θ is *generic* if all θ-semistable dimension d representations are stable.

(iv) For a stability condition θ and dimension vector d define $\mathrm{Rep}_d(Q, \Lambda)^s_\theta$ to be the set of θ-stable dimension d representations, and $\mathrm{Rep}_d(Q, \Lambda)^{ss}_\theta$ to be the set of θ-semistable dimension d representations.

Lemma 2.3.4. *Let d be a dimension vector and θ a stability condition on some quiver with relations. If M and N are dimension d θ-stable representations then:*

(i) *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence with M' a non-zero and proper submodule of M then $\theta(M') > \theta(M) = 0 > \theta(M'')$.*

(ii) *Any non-zero morphism of representations $f : M \to N$ is an isomorphism.*

(iii) *Any morphism of representations $f : M \to M$ is a multiple of the identity.*

Proof. Firstly, if $0 \to M' \to M \to M' \to 0$ is a short exact sequence and M' is non-zero and proper submodule of M then by the definition of stability $\theta(M') > 0 = \theta(M)$ and hence $\theta(M'') < 0$.

Secondly, suppose $f : M \to N$ is non-zero and so the kernel is a proper submodule of M. If the kernel is trivial than f is an injection and hence an isomorphism as M and N are finite dimensional with the same dimension vector. If the kernel is non-trivial then $\theta(\mathrm{Im} f) < 0$ by part (i). However, as $\mathrm{Im} f$ is a subrepresentation of N this is a contradiction to the stability of N, hence the kernel is trivial and f is an isomorphism.

Finally, if $f : M \to M$ is a morphism of representations then f defines a morphism of vector spaces $\mathbb{C}^d \to \mathbb{C}^d$. In particular this map has an eigenvalue λ and defines the map of representations $M \xrightarrow{f-\lambda\cdot\mathrm{id}} M$ which is not a surjection. As such it is not an isomorphism and so by part (ii) $f = \lambda \cdot \mathrm{id}$. □

Definition 2.3.5. Every finite dimensional θ-semistable representation M has a Jordan-Holder filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that each M_i is θ-semistable and each quotient is θ-stable. Two θ-semistable representations are defined to be *S-equivalent* if their Jordan-Holder filtrations have matching composition factors.

We note that θ-stable objects have length one filtrations hence are S-equivalent if and only if they are isomorphic.

Any character of G is given by powers of the determinant character and is of the form

$$\chi_\theta(g) := \prod_{i \in Q_0} \det(g_i)^{\theta_i}$$

for some collection of integers θ_i. For a given dimension vector d we will restrict our attention to characters which are trivial on the kernel of the action, Δ, which translates to the condition $\sum \theta(i)d(i) = 0$. Hence these characters are in correspondence with stabilities.

We recall that $\mathrm{Rep}_d(Q, \Lambda)$ is affine, and that $f \in \mathbb{C}[\mathrm{Rep}_d(Q, \Lambda)]$ is a *semi-invariant of weight* χ if $f(g.x) = \chi(g)f(x)$ for all $g \in G$ and all $x \in \mathrm{Rep}_d(Q, \Lambda)$. We denote the set of such f as $\mathbb{C}[\mathrm{Rep}_d(Q, \Lambda)]^{G,\chi}$.

Definition 2.3.6. ([26]) The *quiver GIT quotient*, for dimension vector d and stability condition θ, is defined to be the scheme

$$\mathcal{M}^{ss}_{d,\theta} := \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathbb{C}[\mathrm{Rep}_d(Q, \Lambda)]^{G,\chi^n_\theta}\right).$$

It is immediate from this definition that for any stability condition θ the quiver GIT quotient $\mathcal{M}^{ss}_{d,\theta}$ is projective over the affine quotient $\mathcal{M}^{ss}_{d,0} = \mathrm{Spec}(\mathbb{C}[\mathrm{Rep}_d(Q, \Lambda)]^G)$.

2.4. Quivers and tilting bundles

We recall the construction of a quiver with relations from a tilting bundle.

Let $X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Given a tilting bundle T' on X and a decomposition into indecomposable summands $T' = \bigoplus_{i=0}^n E_i^{\oplus\alpha_i}$, with E_i and E_j non-isomorphic for $i \neq j$, then $T = \bigoplus_{i=0}^n E_i$ is also a tilting bundle on X and $\mathrm{End}_X(T')^{\mathrm{op}}$ is Morita equivalent to $\mathrm{End}_X(T)^{\mathrm{op}}$.

Hence we will always assume, without loss of generality, that our tilting bundles have a given multiplicity free decomposition into indecomposables, $T = \bigoplus_{i=0}^{n} E_i$.

We then recall from Theorem 2.2.2 that $A = \mathrm{End}_X(T)^{\mathrm{op}}$ is a finite R-algebra for R a finite type commutative $\mathbb{C}$-algebra, and we wish to present A as the path algebra of a quiver with relations such that each indecomposable E_i corresponds to the unique idempotent $e_i = id_{E_i} \in \mathrm{Hom}_X(E_i, E_i) \subset A = \mathrm{End}_X(T)^{\mathrm{op}}$ that is the trivial path at vertex i. In particular $1 = \sum e_i$ and we have a diagonal inclusion $\bigoplus_{i=0}^{n} e_i R \subset A$.

Indeed, we can construct a quiver by creating a vertex i corresponding to each idempotent e_i. We then choose a finite set of generators of $e_i A e_j$ as an R-module, which is possible as A is finite R-module, and create corresponding arrows from vertex j to i for all $0 \leq i, j \leq n$. We then consider a presentation of R over $\mathbb{C}$ with finitely many generators, possible as it has finite type, and at each vertex add arrows corresponding to each generator of R. If we call this quiver Q then by this construction there is a surjection of R-algebras $\mathbb{C}Q \to A$ given by mapping each trivial path to the corresponding idempotent, and each arrow to the corresponding generator. We then take the kernel of this map, I, and $\mathbb{C}Q/I \cong A$ as an R-algebra.

We note that this presentation has many unpleasant properties, for example it may be the case that the ideal of relations I is not a subset of the paths of length greater than 1. In nice situations it is possible to simplify the presentation, see for example the situation considered in [2, Section 1].

We also note that there is a decomposition of A considered as a left A-module into projective modules $A = \bigoplus_{i=0}^{n} \mathrm{Hom}_X(T, E_i)$ where the module $\mathrm{Hom}_X(T, E_i)$ corresponds to paths in the quiver starting at vertex i.

2.5. Functor of points and moduli spaces

We recall the definition of the functor of points and the definition of a fine moduli space. Let $\mathfrak{Sch}$ denote the category of finite type schemes over $\mathbb{C}$, let $\mathfrak{Sets}$ denote the category of sets, and let $\mathfrak{R}$ denote the category of finite type commutative $\mathbb{C}$-algebras. Suppose $X \in \mathfrak{Sch}$, then the functor of points for X is defined to be the functor

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{Sch}}(-, X) : \mathfrak{R} &\to \mathfrak{Sets} \\ S &\mapsto \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(S), X) \end{aligned}$$

and by Yoneda's lemma this gives an embedding of $\mathfrak{Sch}$ into the category of functors from $\mathfrak{R}$ to $\mathfrak{Sets}$.

A functor $\mathcal{F} : \mathfrak{R} \to \mathfrak{Sets}$ is *representable* if there is some $Y \in \mathfrak{Sch}$ with a natural isomorphism $\nu : \mathcal{F} \to \mathrm{Hom}_{\mathfrak{Sch}}(-, Y)$. Then Y is said to be a *fine moduli space* for $\mathcal{F}$.

A functor $\mathcal{F}$ is said to be *corepresentable* if there is a natural transformation $\nu : \mathcal{F} \to \mathrm{Hom}_{\mathfrak{Sch}}(-, Y)$ such that for any scheme Y' with a natural transformation $\nu' : \mathcal{F} \to \mathrm{Hom}_{\mathfrak{Sch}}(-, Y')$ there is a unique morphism $Y \to Y'$ factoring ν' through ν.

The Yoneda embedding $X \mapsto \mathrm{Hom}_{\mathfrak{Sch}}(-, X)$ also defines a fully faithful functor from $\mathfrak{Sch}$ into the category of functors $\mathfrak{Sch}^{\mathrm{op}} \to \mathfrak{Sets}$, and so the above definitions could be given in terms of functors $\mathfrak{Sch}^{\mathrm{op}} \to \mathfrak{Sets}$ rather than functors $\mathfrak{R} \to \mathfrak{Sets}$. Such functors are presheaves on the categories $\mathfrak{Sch}$ and $\mathfrak{R}^{\mathrm{op}}$ respectively, and indeed the Yoneda embedding actually defines a fully faithful functor from $\mathfrak{Sch}$ into the category of sheaves on the respective big Zariski sites for $\mathfrak{R}^{\mathrm{op}}$ and $\mathfrak{Sch}$. As schemes in $\mathfrak{Sch}$ admit affine open covers by affine schemes in $\mathfrak{R}^{\mathrm{op}}$ the comparison lemma identifies sheaves on the big Zariski site of schemes of locally finite type over $\mathbb{C}$ with sheaves on the big affine site Zariski site of finite type affine schemes over $\mathbb{C}$ (see [23, Section C2.2, Theorem 2.2.3] or [35, Tag 020W]). As such it is equivalent to use functors of either type in the above definitions.

We consider functors $\mathfrak{R} \to \mathfrak{Sets}$ to automatically simplify later arguments and definitions to considering affine cases. One advantage of the alternative description is that it is clear to see that if X is a fine moduli space for a moduli functor $\mathcal{F}$ then there is a *tautological element* in $\mathcal{F}(X)$ corresponding to $id \in \mathrm{Hom}_{\mathfrak{Sch}}(X, X)$ under the natural isomorphism.

2.6. *Quiver representation moduli functors*

We recall the definition of a moduli functor for (semi)stable quiver representations. Let A be a $\mathbb{C}$-algebra of finite type. Suppose that A is presented as a quiver with relations and for $B \in \mathfrak{R}$ define $A^B := A \otimes_{\mathbb{C}} B$. We recall that left A-modules correspond to quiver representations. For a dimension vector d, stability condition θ, and $B \in \mathfrak{R}$ define the set

$$\mathcal{S}^{(s)s}_{A,d,\theta}(B) := \left\{ M \in A^B\text{-}\mathrm{mod} \;\middle|\; \begin{array}{l} \bullet\ M \text{ is a finitely generated and flat } B- \text{ module .} \\ \bullet\ \text{The } A- \text{ module } B/\mathfrak{m} \otimes_B M \text{ has dimension vector } d \\ \text{and is } \theta \text{ -(semi)stable for all maximal ideals } \mathfrak{m} \text{ of } B. \end{array} \right\}$$

and define the *quiver representation moduli functor* to be

$$\begin{aligned} \mathcal{F}^{(s)s}_{A,d,\theta} : \mathfrak{R} &\to \mathfrak{Sets} \\ B &\mapsto \mathcal{S}^{(s)s}_{A,d,\theta}(B)/\sim \end{aligned}$$

where the equivalence $\sim$ is defined by two modules being equivalent if they are isomorphic after tensoring by an invertible B-module: $M \sim N$ if there is a locally free rank one B-module L such that $M \otimes_B L \cong N$ as A^B modules. We note that two stable modules are equivalent if and only if they are locally isomorphic.

Lemma 2.6.1. *If* $M, N \in \mathcal{S}^s_{A,d,\theta}(B)$ *then* $M \sim N$ *if and only if* $M \otimes_B B_{\mathfrak{m}} \cong N \otimes_B B_{\mathfrak{m}}$ *for all* $\mathfrak{m} \in \mathrm{MaxSpec}(B)$.

Proof. If there exists a rank one locally free L such that $M \otimes_B L \cong N$ then it is clear that M and N are locally isomorphic.

If M and N are locally isomorphic then consider the B-module $L := \mathrm{Hom}_{A^B}(M, N)$. This is a submodule of $\mathrm{Hom}_B(M, N)$ hence is a finitely generated B-module as M and N are finitely generated as B-modules. For any $\mathfrak{m} \in \mathrm{MaxSpec}(B)$ $M_\mathfrak{m} \cong N_\mathfrak{m}$ as $A^{B_\mathfrak{m}}$-modules, and as M and N are locally free B-modules of the same rank $M_\mathfrak{m}$ and $N_\mathfrak{m}$ are free $B_\mathfrak{m}$-modules of the same rank, n say. In particular

$$\mathrm{Hom}_{B_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) \cong \mathrm{End}_{B_\mathfrak{m}}(M_\mathfrak{m}) \cong \mathrm{Mat}_n(B_\mathfrak{m}) \cong B_\mathfrak{m}^{\oplus n^2}.$$

As $M_\mathfrak{m}$ is an $A^{B_\mathfrak{m}}$-module there is a map $\theta : A \to \mathrm{End}_{B_\mathfrak{m}}(M_\mathfrak{m})$ defining the A-module structure and

$$L_\mathfrak{m} \cong \mathrm{End}_{A^{B_\mathfrak{m}}}(M_\mathfrak{m}) \cong \left\{ f \in \mathrm{End}_{B_\mathfrak{m}}(M_\mathfrak{m}) : \theta(a)f - f\theta(a) = 0 \quad \text{for all } a \in A \right\}.$$

Moreover, the $A^{B_\mathfrak{m}}$-module structure on M defines an A-module structure on $M/\mathfrak{m}$ via the map $\bar{\boldsymbol{\theta}} : A \to \mathrm{End}_A(M/\mathfrak{m})$ defined as θ followed by reduction modulo $\mathfrak{m}$, and it follows that

$$\mathrm{End}_A(M/\mathfrak{m}) \cong \left\{ \bar{\boldsymbol{f}} \in \mathrm{End}_{B/\mathfrak{m}}(M/\mathfrak{m}) : \bar{\boldsymbol{\theta}}(a)\bar{\boldsymbol{f}} - \bar{\boldsymbol{f}}\bar{\boldsymbol{\theta}}(a) = 0 \quad \text{for all } a \in A \right\}.$$

For each $a \in A$ the map $f \mapsto \theta(a)f - f\theta(a)$ defines a $B_\mathfrak{m}$-linear map $\phi_a : B_\mathfrak{m}^{\oplus n^2} \to B_\mathfrak{m}^{\oplus n^2}$ whose reduction modulo $\mathfrak{m}$ is a $B/\mathfrak{m}$-linear map $\bar{\boldsymbol{\phi}}_a : (B/\mathfrak{m})^{\oplus n^2} \to (B/\mathfrak{m})^{\oplus n^2}$ defined by $\bar{\boldsymbol{f}} \mapsto \bar{\boldsymbol{\theta}}(a)\bar{\boldsymbol{f}} - \bar{\boldsymbol{f}}\bar{\boldsymbol{\theta}}(a)$. Choosing a generating set $I \subset A$ for A as a finitely generated $\mathbb{C}$-algebra there is then an associated $B_\mathfrak{m}$-linear map $\Phi = \bigoplus_{a \in I} \phi_a : B_\mathfrak{m}^{\oplus n^2} \to \left(B_\mathfrak{m}^{\oplus n^2}\right)^{\oplus I}$ such that $L_\mathfrak{m}$ is the kernel of Φ

$$0 \to L_\mathfrak{m} \to B_\mathfrak{m}^{\oplus n^2} \xrightarrow{\Phi} \left(B_\mathfrak{m}^{\oplus n^2}\right)^{\oplus I}.$$

Similarly, $\mathrm{End}_A(M/\mathfrak{m})$ is the kernel of $\bar{\boldsymbol{\Phi}} = \bigoplus_{a \in I} \bar{\boldsymbol{\phi}}_a$

$$0 \to \mathrm{End}_A(M/\mathfrak{m}) \to (B/\mathfrak{m})^{\oplus n^2} \xrightarrow{\bar{\boldsymbol{\Phi}}} \left((B/\mathfrak{m})^{\oplus n^2}\right)^{\oplus I}.$$

As $M \in \mathcal{S}^s_{A,d,\theta}(B)$ the A-module $M/\mathfrak{m}$ is θ-stable and by Lemma 2.3.4 (iii) there is an isomorphism $\mathrm{End}_A(M/\mathfrak{m}) \cong B/\mathfrak{m}$ given by the inclusion of the identity endomorphism. In particular this shows that $\bar{\boldsymbol{\Phi}}$ is a $B/\mathfrak{m}$-linear map of rank $n^2 - 1$.

The identity inclusion $B_\mathfrak{m} \to L_\mathfrak{m} \cong \mathrm{End}_{A^{B_\mathfrak{m}}}(M_\mathfrak{m})$ is defined by $b \mapsto b \cdot id_M$. This extends to the identity inclusion $B_\mathfrak{m} \hookrightarrow L_\mathfrak{m} \hookrightarrow \mathrm{End}_{B_\mathfrak{m}}(M_\mathfrak{m})$ which splits, and hence $B_\mathfrak{m} \hookrightarrow L_\mathfrak{m}$ splits as $L_\mathfrak{m} = B_\mathfrak{m} \oplus K$ for some $B_\mathfrak{m}$-module K. Factoring out the split inclusion $B_\mathfrak{m} \hookrightarrow L_\mathfrak{m}$ produces an exact sequence

$$0 \to K \to B_\mathfrak{m}^{\oplus n^2 - 1} \xrightarrow{\Phi'} \left(B_\mathfrak{m}^{\oplus n^2}\right)^{\oplus I}$$

such that the image of Φ equals the image of Φ'. In particular, as $\bar{\boldsymbol{\Phi}}$ has rank $n^2 - 1$ the reduction $\bar{\boldsymbol{\Phi}}'$ of Φ' modulo $\mathfrak{m}$ also has rank $n^2 - 1$ so is an injection

$$0 \to (B/\mathfrak{m})^{\oplus n^2 - 1} \xrightarrow{\bar{\boldsymbol{\Phi}}'} \left((B/\mathfrak{m})^{\oplus n^2}\right)^{\oplus I}.$$

A morphism between finitely generated free modules over a local ring whose reduction modulo $\mathfrak{m}$ is injective is itself injective [31, Theorem 22.5] (or see [35, Tag 00ME]). Hence Φ' is an injection of $B_{\mathfrak{m}}$-modules and $K \cong 0$.

Hence $L_{\mathfrak{m}} \cong B_{\mathfrak{m}}$ and we can deduce that L is a locally free B-module of rank 1. Then the natural map $M \otimes_B \operatorname{Hom}_{A^B}(M, N) \to N$ can be seen to be an isomorphism by considering localisations at all maximal ideals $\mathfrak{m} \in \operatorname{MaxSpec}(B)$ where it reduces to the composition of isomorphisms $L_{\mathfrak{m}} \cong B_{\mathfrak{m}}$, $M_{\mathfrak{m}} \otimes_{B_{\mathfrak{m}}} B_{\mathfrak{m}} \to M_{\mathfrak{m}}$, and $M_{\mathfrak{m}} \cong N_{\mathfrak{m}}$. □

By the results of King in [26] the quiver representation moduli functor is corepresentable.

Theorem 2.6.2. ([26, Proposition 5.2]) *The scheme $\mathcal{M}^{ss}_{d,\theta}$ corepresents the functor $\mathcal{F}^{ss}_{A,d,\theta}$. In particular, closed points of $\mathcal{M}^{ss}_{d,\theta}$ correspond to S-equivalence classes of dimension d, θ-semistable A-modules.*

If we restrict to stable representations then the functor is representable and has a fine moduli space.

Theorem 2.6.3. ([26, Proposition 5.3]) *Suppose d is indivisible and let $\mathcal{M}^{s}_{d,\theta}$ be the open subscheme of $\mathcal{M}^{ss}_{d,\theta}$ corresponding to the stable points. Then $\mathcal{M}^{s}_{d,\theta}$ is a fine moduli space for $\mathcal{F}^{s}_{A,d,\theta}$.*

We note that when d is indivisible and θ is generic all semistable points are stable and $\mathcal{M}^{ss}_{d,\theta} = \mathcal{M}^{s}_{d,\theta}$ is a fine moduli space. We will later restrict ourselves to considering such cases.

We will often just refer to the functor as $\mathcal{F}_A$, recalling the choices of semistablity or stability, d, and θ only when necessary. We also note that the tautological element for $\mathcal{F}^{s}_{A,d,\theta}$ is a vector bundle on $\mathcal{M}^{s}_{d,\theta}$ with each fibre corresponding to a θ-stable representation of A with dimension vector d which we refer to as the *tautological bundle*.

Remark 2.6.4. The functor here differs from the functor considered in Sekiya Yamaura, [34, Definition 4.1], but their results also hold for this functor. See "Appendix 7" for more details.

We also note that the assumption that A is presented as a quiver with relations is not necessary; for any algebra which is finitely generated over a commutative Noetherian ring Van den Bergh defines a functor analogous to $\mathcal{F}^{s}_{A,d,\theta}$ and proves that such a functor is representable when d is indivisible and θ is generic [36, Proposition 6.2.1]. We note that local equivalence is used in this setting.

2.7. Geometric moduli functors

We define a similar functor for a scheme, X, arising in a projective morphism, $\pi : X \to \operatorname{Spec}(R)$, of finite type schemes over $\mathbb{C}$.

We first introduce several pieces of notation which we will frequently use. Let $\rho : X \to \mathrm{Spec}(\mathbb{C})$ denote the structure morphism. For $B \in \mathfrak{R}$ we define $X^B := X \times_{\mathrm{Spec}(\mathbb{C})} \mathrm{Spec}(B)$ and consider the following pullback diagram

$$\begin{tikzcd} X^B \arrow[r, "\rho^X"] \arrow[d, "\rho^B"'] & X \arrow[d, "\rho"] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

which defines the morphisms ρ^B and ρ^X from the structure morphism $\rho : X \to \mathrm{Spec}(\mathbb{C})$. We note that X^B is also of finite type over $\mathbb{C}$, and has a projective morphism $\pi^B : X^B \to \mathrm{Spec}(R \otimes_{\mathbb{C}} B)$, see [6, Remark 1.7]. Also if X has a tilting bundle T the following result, which is a particular case of the result [6, Proposition 2.9] of Buchweitz and Hille, defines a tilting bundle T^B on X^B.

Proposition 2.7.1. ([6, Proposition 2.9]) *If T is a tilting bundle on X and $A = \mathrm{End}_X(T)^{\mathrm{op}}$ then $T^B := \mathbb{L}\rho^{X*}T$ is a tilting bundle on X^B, and $A^B := \mathrm{End}_{X^B}(T^B)^{\mathrm{op}} = A \otimes_{\mathbb{C}} B$.*

We introduce a further piece of notation. For any $B \in \mathfrak{R}$ we let $\mathrm{MaxSpec}(B)$ denote the closed points of $\mathrm{Spec}(B)$, and any $p \in \mathrm{MaxSpec}(B)$ there is a closed immersion $i_p : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(B)$ and a pullback diagram

$$\begin{tikzcd} X \arrow[r, hook, "j_p"] \arrow[d, "\rho"'] & X^B \arrow[d, "\rho^B"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, hook, "i_p"] & \mathrm{Spec}(B) \end{tikzcd} \qquad (i_p/j_p)$$

which we later refer to as the diagram (i_p/j_p).

We can now define the geometric moduli functor. We define $\mathcal{F}_X(\mathbb{C})$ to be the set of skyscraper sheaves of X considered up to isomorphism, and for $B \in \mathfrak{R}$ define the sets

$$\mathcal{S}_X(B) := \left\{ \mathcal{E} \in D^b(X^B) \mid \mathbb{L}j_p^*\mathcal{E} \in \mathcal{F}_X(\mathbb{C}) \quad \text{for all } p \in \mathrm{MaxSpec}(B). \right\}$$

and the moduli functor

$$\begin{aligned} \mathcal{F}_X : \mathfrak{R} &\to \mathfrak{Sets} \\ B &\mapsto \mathcal{S}_X(B)/\sim \end{aligned}$$

where the equivalence $\sim$ is defined by $\mathcal{E}_1$ being equivalent to $\mathcal{E}_2$ if there is a line bundle L on $\mathrm{Spec}(B)$ such that $\mathcal{E}_1 \otimes_{X^B} \rho^{B*}L \cong \mathcal{E}_2$. We later recall in Proposition 4.0.2 that X is a fine moduli space for this functor.

Remark 2.7.2. It follows immediately from Lemma 2.7.3, which we state below, that the set $\mathcal{S}_X(B)$ is equivalent to the set

$$\left\{ \mathcal{E} \in \mathrm{Coh}(X^B) \;\middle|\; \begin{array}{l} \bullet\ \mathcal{E} \text{ is flat as a } B\text{-module.} \\ \bullet\ j_p^*\mathcal{E} \in \mathcal{F}_X(\mathbb{C}) \text{ for all } p \in \mathrm{MaxSpec}(B). \end{array} \right\}.$$

Lemma 2.7.3. ([3, Lemma 4.3]) *Let $f : X \to Y$ be a morphism of finite type schemes over $\mathbb{C}$, and for each closed point $y \in Y$ let j_y denote the inclusion of the fibre $f^{-1}(y)$. Suppose $\mathcal{E} \in D^b(X)$ is such that $\mathbb{L}j_y^*\mathcal{E}$ is a sheaf for all y. Then $\mathcal{E}$ is a coherent sheaf on X which is flat over Y.*

Remark 2.7.4. In the definition of the moduli functor $\mathcal{F}_X$ we could change the set $\mathcal{F}_X(\mathbb{C})$ of skyscraper sheaves up to isomorphism to, for example, the set of perverse point sheaves as defined by Bridgeland [4, Section 3], to obtain a functor mirroring Bridgeland's perverse point sheaf moduli functor. Indeed, the results of Sect. 3 and Theorem 4.0.1 do not rely on the fact that $\mathcal{F}_X(\mathbb{C})$ consists of skyscraper sheaves up to isomorphism, but Proposition 4.0.2 and our applications in Sect. 5 do.

We now note some properties of coherent sheaves $\mathcal{E} \in \mathcal{F}_X(B)$ that will be used later.

Lemma 2.7.5. *Suppose that $\mathcal{E} \in \mathcal{F}_X(B)$ and $\mathcal{E}$ has scheme theoretic support $\iota : Z \to X^B$. Then*

(i) *The coherent sheaf $\mathcal{E}$ is a line bundle over its support $Z \subset X^B$,*
(ii) *The morphism $\rho^B \circ \iota : Z \to \mathrm{Spec}\, B$ is flat,*
(iii) *The morphism $\rho^B \circ \iota : Z \to \mathrm{Spec}\, B$ is an isomorphism, and*
(iv) *There exists a line bundle $\mathcal{L}$ on $\mathrm{Spec}\, B$ such that $\mathcal{E} \cong \mathcal{O}_Z \otimes_{X^B} \rho^{B*}\mathcal{L}$.*

Proof. By Remark 2.7.2 we can assume that $\mathcal{E}$ is a coherent sheaf on X^B that is flat over $\mathrm{Spec}\, B$ such that for any closed point $p \in \mathrm{Spec}\, B$ there is an isomorphism $j_p^*\mathcal{E} \cong \mathcal{O}_x$ for some closed point $x \in X$. As $\mathcal{E}$ is a coherent sheaf ι is a closed immersion.

We first show that $\mathcal{E}$ is a line bundle over its support. Suppose that $z \in X^B$ is a closed point that is in the support of $\mathcal{E}$ and consider the point $p = \rho^B(z) \in \mathrm{Spec}\, B$. As the schemes are of finite type over $\mathbb{C}$ the point p is a closed point in $\mathrm{Spec}\, B$. As $\mathcal{E} \in \mathcal{F}_X(B)$ it follows that $j_p^*\mathcal{E} \cong \mathcal{O}_x$ for some closed point $x \in X$ and there is a surjection

$$\mathcal{O}_X \xrightarrow{f|_p} j_p^*\mathcal{E} \to 0.$$

Further, as $j_p^*\mathcal{E} \cong \mathcal{O}_x$ the point $z = (x, p) \in X^B$ is the unique closed point in the support of $\mathcal{E}$ that maps to p. As there is a surjection $\mathrm{Hom}(\mathcal{O}_{X^B}, \mathcal{E}) \to \mathrm{Hom}(\mathcal{O}_X, j^*\mathcal{E})$ there exists some $f : \mathcal{O}_{X^B} \to \mathcal{E}$ lifting $f|_p$. Hence Nakayama's lemma implies the stalk of f at z is surjective

$$\mathcal{O}_{X^B,z} \xrightarrow{f_z} \mathcal{E}_z \to 0.$$

As $\mathcal{E}$ locally has a surjection from $\mathcal{O}_{X^B}$ at all closed points of Z it follows that $\mathcal{E}$ is locally isomorphic to the closed subscheme of $Z \subset X^B$, and hence $\mathcal{E}$ is a line bundle over its support Z. This proves part (i).

As the sheaf $\mathcal{E}$ is flat over Spec B and $\mathcal{E}$ is a line bundle over Z it follows that $\mathcal{O}_Z$ is also flat over Spec B, proving part (ii).

In particular for all closed points $p \in \operatorname{Spec} B$ the fibre $Z_p \to p$ is an isomorphism. This implies that the morphism is unramified and universally injective (see [35, Tag 05VH]). The morphism is flat, unramified, and universally injective so it is an open immersion, in particular a flat monomorphism (see [35, Tag 025G]). Then the morphism is surjective on closed points hence surjective, and a surjective open immersion is an isomorphism (see [35, Tag 06NG]). Hence part (iii) is proved.

Part (iv) is a consequence of (i) and (iii). □

In particular the following result is used later.

Lemma 2.7.6. *If T is a locally free coherent sheaf T on X^B and $\mathcal{E} \in \mathcal{F}_X(B)$, then $\mathbb{R}\rho^B_* \mathcal{RH}\mathrm{om}_{X^B}(T, \mathcal{E})$ is a locally free coherent sheaf on* Spec B.

Proof. By Remark 2.7.2 $\mathcal{E}$ is a coherent sheaf on X^B. As T is locally free $\mathcal{RH}\mathrm{om}_{X^B}(T, \mathcal{E}) \cong T^\vee \otimes_{X^B} \mathcal{E}$ and the scheme theoretic support of $T^\vee \otimes_{X^B} \mathcal{E}$ equals the scheme theoretic support $\iota : Z \to X^B$ of $\mathcal{E}$. Hence there exists a locally free coherent sheaf $\mathcal{G} \in \operatorname{Coh} Z$ such that $\iota(\mathcal{G}) \cong T^\vee \otimes_{X^B} \mathcal{E}$. In particular $\mathbb{R}\rho^B_* T^\vee \otimes_{X^B} \mathcal{E}$ is equivalent to $(\mathbb{R}\rho^B \circ \iota)_* \mathcal{G}$. As $\rho^B \circ \iota$ is an isomorphism by Lemma 2.7.5 (iii) and $\mathcal{G}$ is a locally free coherent sheaf on Z it follows that $\mathbb{R}\rho^B_* T^\vee \otimes_{X^B} \mathcal{E} \cong \mathbb{R}\rho^B_* \mathcal{RH}\mathrm{om}_{X^B}(T, \mathcal{E})$ is a locally free coherent sheaf on Spec B. □

3. Preliminary lemmas

In this section we give a series of lemmas required to prove the main results in the next section.

3.1. *Derived base change*

We first recall the following property, which we will make use of several times.

Lemma 3.1.1. *Let $f : X \to Y$ be a quasi-compact, separated morphism of Noetherian schemes over $\mathbb{C}$. Then if $T \in \operatorname{Perf}(Y)$*

$$\mathbb{L}f^* \mathcal{RH}\mathrm{om}_Y(T, \mathcal{E}) \cong \mathcal{RH}\mathrm{om}_X(\mathbb{L}f^* T, \mathbb{L}f^* \mathcal{E})$$

for any $\mathcal{E} \in D^b(Y)$.

Proof. We consider the two functors

$$\operatorname{Hom}_{D^b(X)}(\mathbb{L}f^* \mathcal{RH}\mathrm{om}_Y(T, \mathcal{E}), -) : D^b(X) \to \mathfrak{Sets}, \quad \text{and}$$
$$\operatorname{Hom}_{D^b(X)}(\mathcal{RH}\mathrm{om}_X(\mathbb{L}f^* T, \mathbb{L}f^* \mathcal{E}), -) : D^b(X) \to \mathfrak{Sets}.$$

We will show these are naturally isomorphic, and it then follows that $\mathbb{L}f^* \mathcal{RH}\mathrm{om}_Y(T, \mathcal{E}) \cong \mathcal{RH}\mathrm{om}_X(\mathbb{L}f^*T, \mathbb{L}f^*\mathcal{E})$ as they represent the same functor under the Yoneda embedding. This follows from the chain of natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{D(X)}(\mathbb{L}f^*\mathcal{RH}\mathrm{om}_Y(T, \mathcal{E}), -) &\cong \mathrm{Hom}_{D(Y)}(\mathcal{RH}\mathrm{om}_Y(T, \mathcal{E}), \mathbb{R}f_*(-)) \\
&\hspace{18em}\text{(adjunction)}\\
&\cong \mathrm{Hom}_{D(Y)}(\mathcal{E}, T \otimes^{\mathbb{L}}_Y \mathbb{R}f_*(-)) \quad (T \text{ perfect})\\
&\cong \mathrm{Hom}_{D(Y)}(\mathcal{E}, \mathbb{R}f_*(\mathbb{L}f^*T \otimes^{\mathbb{L}}_X (-)))\\
&\hspace{18em}\text{(projection)}\\
&\cong \mathrm{Hom}_{D(X)}(\mathbb{L}f^*\mathcal{E}, \mathbb{L}f^*T \otimes^{\mathbb{L}}_Y (-))\\
&\hspace{18em}\text{(adjunction)}\\
&\cong \mathrm{Hom}_{D(X)}(\mathcal{RH}\mathrm{om}_X(\mathbb{L}f^*T, \mathbb{L}f^*\mathcal{E}), -).\\
&\hspace{16em}(\mathbb{L}f^*T \text{ perfect})
\end{aligned}$$

□

We then recall the following derived base change results.

Lemma 3.1.2. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$, and let $B, C \in \mathfrak{R}$. Consider the following pullback diagram for a morphism $u : \mathrm{Spec}(B) \to \mathrm{Spec}(C)$, where we use the notation of Sect. 2.7.*

$$\begin{tikzcd}
X^B \arrow[r, "v"] \arrow[d, "\rho^B"'] & X^C \arrow[d, "\rho^C"] \\
\mathrm{Spec}(B) \arrow[r, "u"] & \mathrm{Spec}(C)
\end{tikzcd}$$

Suppose $\mathcal{E} \in D^b(X^C)$. Then

$$\mathbb{L}u^*\mathbb{R}\rho^C_*\mathcal{E} \cong \mathbb{R}\rho^B_*\mathbb{L}v^*\mathcal{E}.$$

Suppose further that X has a tilting bundle T and define $A = \mathrm{End}_X(T)^{\mathrm{op}}$. If the A^C-module $\mathbb{R}\mathrm{Hom}_{X^C}(T^C, \mathcal{E})$ is flat as a C-module then

$$B \otimes_C \mathbb{R}\mathrm{Hom}_{X^C}(T^C, \mathcal{E}) \cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathbb{L}v^*\mathcal{E})$$

as A^B-modules.

Also, if L is a line bundle on $\mathrm{Spec}(B)$ *then*

$$\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E} \otimes_{X^B} \rho^{B*}L) \cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \otimes_B L$$

as A^B-modules.

Proof. As X^C is flat over $\mathrm{Spec}(C)$, for any $x \in X^C$ and any $b \in \mathrm{Spec}(B)$ such that $\rho^C(x) = u(b) = c$ we have that $\mathrm{Tor}_i^{\mathcal{O}_{C,c}}(\mathcal{O}_{B,b}, \mathcal{O}_{X^C,x}) = 0$ for all $i \neq 0$. Hence X^C and $\mathrm{Spec}(B)$ are Tor independent over $\mathrm{Spec}(C)$, and so the first result follows from Tor independent base change [30, Theorem 3.10.3] (or see [35, Tag 08IR]).

The second result follows by applying the first result and the previous lemma:

$$\begin{aligned} B \otimes_C \mathbb{R}\mathrm{Hom}_{X^C}(T^C, \mathcal{E}) &\cong \mathbb{L}u^* \mathbb{R}\rho^C_* \mathcal{RH}om_{X^C}(T^C, \mathcal{E}) \\ &\cong \mathbb{R}\rho^B_* \mathbb{L}v^* \mathcal{RH}om_{X^C}(T^C, \mathcal{E}) \quad (\mathbb{L}u^* \mathbb{R}\rho^C_* \cong \mathbb{R}\rho^B_* \mathbb{L}v^*) \\ &\cong \mathbb{R}\rho^B_* \mathcal{RH}om_{X^B}(\mathbb{L}v^* T^C, \mathbb{L}v^* \mathcal{E}) \qquad (\text{Lemma 3.1.1}) \\ &\cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathbb{L}v^* \mathcal{E}). \end{aligned}$$

The final assertion follows by the projection formula [33, Proposition 5.3] (or see [35, Tag 0B54]):

$$\begin{aligned} \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E} \otimes_{X^B} \rho^{B*} L) &:= \mathbb{R}\rho^B_* \mathcal{H}om_{X^B}(T^B, \mathcal{E} \otimes_{X^B} \rho^{B*} L) \\ &= \mathbb{R}\rho^B_*((T^B)^\vee \otimes_{X^B} \mathcal{E} \otimes_{X^B} \rho^{B*} L) \quad (T^B \text{ perfect}) \\ &= \mathbb{R}\rho^B_*((T^B)^\vee \otimes_{X^B} \mathcal{E}) \otimes_B L \\ &\qquad\qquad (\text{projection formula}) \\ &= \mathbb{R}\rho^B_* \mathcal{H}om_{X^B}(T^B, \mathcal{E}) \otimes_B L \qquad (T^B \text{ perfect}) \\ &= \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \otimes_B L. \end{aligned}$$

□

3.2. Natural transformations

In this section let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Suppose that X has a tilting bundle T and that $A = \mathrm{End}_X(T)^{\mathrm{op}}$ is presented as a quiver with relations. Choose some dimension vector d and stability condition θ in order to define $\mathcal{F}_A := \mathcal{F}^{ss}_{A,d,\theta}$. We aim to define a natural transformation, η, between the moduli functors $\mathcal{F}_X$ and $\mathcal{F}_A$ defined in Sects. 2.7 and 2.6. We define $\eta : \mathcal{F}_X \to \mathcal{F}_A$ by

$$\begin{aligned} \eta_B : \mathcal{F}_X(B) &\to \mathcal{F}_A(B) \\ \mathcal{E} &\mapsto \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \end{aligned}$$

for any $B \in \mathfrak{R}$, and we must check when this is well defined.

Lemma 3.2.1. *Suppose $\eta_{\mathbb{C}}$ is well defined. Then η is a well defined natural transformation and η_B is injective for all $B \in \mathfrak{R}$.*

Proof. To prove that η is well defined we must check the following for any $B \in \mathfrak{R}$ and any $\mathcal{E} \in \mathcal{F}_X(B)$:

(i) $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E})$ is a B-module which is flat and finitely generated.
(ii) For all maximal ideals $\mathfrak{m}$ of B the A-module $B/\mathfrak{m} \otimes_B \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E})$ is in $\mathcal{F}_A(\mathbb{C})$.
(iii) If $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent in $\mathcal{F}_X(B)$ then $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{E}_1)$ and $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{E}_2)$ are equivalent in $\mathcal{F}_A(B)$.

Firstly, part (i) is an immediate consequence of Lemma 2.7.6.

Secondly, to prove (ii), we note for any $\mathfrak{m} \in \operatorname{MaxSpec} B$ corresponds to a closed point p and diagram (i_p/j_p). As $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E})$ is a flat B-module $B/\mathfrak{m} \otimes_B \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \cong \mathbb{R}\mathrm{Hom}_X(T, \mathbb{L}j_p^*\mathcal{E})$ for each maximal ideal $\mathfrak{m}$ by Lemma 3.1.2. As $\eta_{\mathbb{C}}$ is well defined and $\mathbb{L}j_p^*\mathcal{E} \in \mathcal{F}_X(\mathbb{C})$ it follows that $B/\mathfrak{m} \otimes_B \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \in \mathcal{F}_A(\mathbb{C})$.

To prove part (iii) let $\mathcal{E}_1$ and $\mathcal{E}_2$ be equivalent elements of $\mathcal{F}_X(B)$. Then there exists some line bundle L on $\operatorname{Spec}(B)$ such that $\mathcal{E}_1 \otimes_{X^B} \rho^{B*}L \cong \mathcal{E}_2$ and so by Lemma 3.1.2

$$\begin{aligned}\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_2) &\cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_1 \otimes_{X^B} \rho^{B*}L)\\ &\cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_1) \otimes_B L.\end{aligned}$$

This shows that $\eta_B(\mathcal{E}_1)$ and $\eta_B(\mathcal{E}_2)$ are equivalent in $\mathcal{F}_X(B)$ and proves part (iii).

We now show that η is a natural transformation. Suppose that $B, C \in \mathfrak{R}$ and $u : \operatorname{Spec}(B) \to \operatorname{Spec}(C)$, then we have the base change diagram

$$\begin{tikzcd} X^B \arrow[r, "v"] \arrow[d, "\rho^B"'] & X^C \arrow[d, "\rho^C"] \\ \operatorname{Spec}(B) \arrow[r, "u"] & \operatorname{Spec}(C) \end{tikzcd}$$

and we consider the diagram

$$\begin{tikzcd} \mathcal{F}_X(C) \arrow[r, "\mathbb{R}\mathrm{Hom}_{X^C}(T^C, -)"] \arrow[d, "\mathbb{L}v^*"'] & \mathcal{F}_A(C) \arrow[d, "B \otimes_C (-)"] \\ \mathcal{F}_X(B) \arrow[r, "\mathbb{R}\mathrm{Hom}_{X^B}(T^B, -)"] & \mathcal{F}_A(B) \end{tikzcd}$$

and to show that η is natural we must check that this commutes. For $\mathcal{E} \in \mathcal{F}_X(C)$ as $\mathbb{R}\mathrm{Hom}_{X^C}(T^C, \mathcal{E})$ is a flat C-module

$$B \otimes_C \mathbb{R}\mathrm{Hom}_{X^C}(T^C, \mathcal{E}) \cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathbb{L}v^*\mathcal{E})$$

as A^B-modules by Lemma 3.1.2. Hence η is natural.

We now show that η_B is injective. Suppose that $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{F}_X(B)$ and $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_1)$ is equivalent to $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_2)$, hence there exists an invertible B-module L such that $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_1) \otimes_B L \cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_2)$. By Lemma 3.1.2

$$\begin{aligned}\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_2) &\cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_1) \otimes_B L\\ &\cong \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}_1 \otimes_{X^B} \rho^{B*}L)\end{aligned}$$

and hence $\mathcal{E}_1 \otimes_{X^B} \rho^{B*}L \cong \mathcal{E}_2$ as $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, -)$ is an equivalence of derived categories. Hence η_B is injective. □

Lemma 3.2.2. *With the assumptions as in Lemma 3.2.1, if $\eta_{\mathbb{C}}$ is also bijective with inverse $T \otimes^{\mathbb{L}}_A (-)$ then η_B is bijective for all $B \in \mathfrak{R}$.*

Proof. We suppose that $\eta_{\mathbb{C}}$ is bijective with inverse $T \otimes^{\mathbb{L}}_A (-)$ and we show that η_B is surjective. We consider $M \in \mathcal{F}_A(B)$ and note that as T^B is a tilting bundle there exists some $\mathcal{E} \in D^b(X^B)$ such that $\mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E}) \cong M$. If we can show that $\mathcal{E} \in \mathcal{F}_X(B)$ then we have proved that η_B is surjective. We check that $\mathbb{L}j_p^*\mathcal{E} \in \mathcal{F}_X(\mathbb{C})$ for any maximal ideal $\mathfrak{m}$ of B with corresponding closed point $p \in \mathrm{MaxSpec}(B)$ and diagram (i_p/j_p). By Lemma 3.1.2

$$B/\mathfrak{m} \otimes_B M \cong \mathbb{R}\mathrm{Hom}_X\left(T, \mathbb{L}j_p^*\mathcal{E}\right)$$

as M is flat over B. As $B/\mathfrak{m} \otimes_B M \cong \mathbb{R}\mathrm{Hom}_X(T, \mathbb{L}j_p^*\mathcal{E}) \in \mathcal{F}_A(\mathbb{C})$ and $\eta_{\mathbb{C}}$ is bijective with inverse $T \otimes^{\mathbb{L}}_A (-)$ it follows that $\mathbb{L}j_p^*\mathcal{E} \cong T \otimes^{\mathbb{L}}_A \mathbb{R}\mathrm{Hom}_X(T, \mathbb{L}j_p^*\mathcal{E}) \in \mathcal{F}_X(\mathbb{C})$. Hence $\mathcal{E} \in \mathcal{F}_X(B)$ and η_B is surjective. □

4. Results

In this section we state our main result, which follows from the previous lemmas, and we also show that the moduli functor $\mathcal{F}_X$ is represented by X. We will find several applications of these results in the next section.

Theorem 4.0.1. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Suppose X is equipped with a tilting bundle T, define $A = \mathrm{End}_X(T)^{\mathrm{op}}$, and suppose that A is presented as a quiver with relations. If there exists an dimension vector d and stability condition θ defining the moduli functor $\mathcal{F}_A := \mathcal{F}^{ss}_{A,\theta,d}$ such that the tilting equivalence*

$$\begin{tikzcd} D^b(X) \arrow[r, bend left, "\mathbb{R}\mathrm{Hom}_X(T,-)"] & D^b(A) \arrow[l, bend left, "T \otimes^{\mathbb{L}}_A (-)"] \end{tikzcd}$$

restricts to a bijection between $\mathcal{F}_X(\mathbb{C})$ and $\mathcal{F}_A(\mathbb{C})$ then the map $\eta : \mathcal{F}_X \to \mathcal{F}_A$ defined by $\eta_B : \mathcal{E} \mapsto \mathbb{R}\mathrm{Hom}_{X^B}(T^B, \mathcal{E})$ is a natural isomorphism.

Proof. This follows from Lemmas 3.2.1 and 3.2.2. □

We now prove that the moduli functor $\mathcal{F}_X$ has X as a fine moduli space; this is just a restatement of the well known result that a finite type scheme over $\mathbb{C}$ can be reconstructed as the Hilbert scheme of closed points.

Proposition 4.0.2. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Then there is a natural isomorphism between the functor of points $\mathrm{Hom}_{\mathfrak{Sch}}(-, X)$ and the moduli functor $\mathcal{F}_X$. In particular X is a fine moduli space for $\mathcal{F}_X$ with tautological object $\Delta_*\mathcal{O}_X$ on $X \times_{\mathrm{Spec}(\mathbb{C})} X$ where $\Delta : X \to X \times_{\mathrm{Spec}(\mathbb{C})} X$ is the diagonal map.*

Remark 4.0.3. Combining Theorem 4.0.1 and Proposition 4.0.2 we can deduce that if there exists a dimension vector d and stability condition θ such that $\mathbb{R}\mathrm{Hom}_X(T, -)$ and $T \otimes^{\mathbb{L}}_A (-)$ restrict to bijections between $\mathcal{F}_X(\mathbb{C})$ and $\mathcal{F}_A(\mathbb{C})$ then X is a fine moduli space for the functor $\mathcal{F}_A$. In particular, for a bijection between $\mathcal{F}_X(\mathbb{C})$ and $\mathcal{F}_A(\mathbb{C})$ to exist d must be indivisible and θ must be generic. Further, in this situation the tautological bundle on X is in fact $T^\vee$, as can be seen by translating the tautological element $\Delta_* \mathcal{O}_X$ across the natural isomorphism between $\mathcal{F}_X$ and $\mathcal{F}_A$, so $\mathrm{End}_X(T^\vee) \cong \mathrm{End}_X(T)^{\mathrm{op}} = A$ and the dual of the tautological bundle is the tilting bundle T.

Proof of Proposition 4.0.2. Consider

$$\mu : \mathrm{Hom}_{\mathfrak{Sch}}(-, X) \to \mathcal{F}_X$$

defined by

$$\mu_C : (g : \mathrm{Spec}(C) \to X) \mapsto ((\Gamma_g)_* \mathcal{O}_C)$$

for $C \in \mathfrak{R}$, where $\Gamma_g : \mathrm{Spec}(C) \to X^C$ is the graph of g. The graph is a closed immersion as X is separated, and hence Γ_g is affine and $(\Gamma_g)_*$ is exact.

We now show this is a well defined natural transformation. To show that it is well defined we consider a morphism $g : \mathrm{Spec}(C) \to X$ and check that $(\Gamma_g)_* \mathcal{O}_C \in \mathcal{F}_X(C)$. Firstly, as Γ_g is a closed immersion it is proper, hence $(\Gamma_g)_* \mathcal{O}_C$ is a coherent sheaf [18, Theorémè 3.2.1](or see [35, Tag 02O5]). Further, as Γ_g is a closed immersion and $\mathcal{O}_C$ is flat over $\mathrm{Spec}(C)$ it follows by considering stalks that $(\Gamma_g)_* \mathcal{O}_C$ is also flat over $\mathrm{Spec}(C)$. Then as Γ_g is affine $j_p^* (\Gamma_g)_* \mathcal{O}_C \cong (\Gamma_{g \circ i_p})_* i_p^* \mathcal{O}_C$ for all $p \in \mathrm{MaxSpec}(C)$ with diagrams (i_p/j_p) by affine base change [17, Corollaire 1.5.2] (or see [35, Tag 02KE]), hence

$$\mathbb{L} j_p^* (\Gamma_g)_* \mathcal{O}_C \cong j_p^* (\Gamma_g)_* \mathcal{O}_C \cong (\Gamma_{g \circ i_p})_* i_p^* \mathcal{O}_C \cong \mathcal{O}_{g(p)}.$$

Hence μ_C is well defined as $(\Gamma_g)_* \mathcal{O}_{\mathrm{Spec}(C)} \in \mathcal{F}_A(C)$ for any $g \in \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(C), X)$. It is natural as if $B, C \in \mathfrak{R}$ with a morphism $u : \mathrm{Spec}(B) \to \mathrm{Spec}(C)$ and $g : \mathrm{Spec}(C) \to X \in \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(C), X)$ we have the diagram

$$\begin{tikzcd}
X^B \arrow[r, "v"] \arrow[d, "\rho^B"] & X^C \arrow[r, "\rho^X"] \arrow[d, "\rho^C"] & X \arrow[d, "\rho"] \\
\mathrm{Spec}(B) \arrow[r, "u"] \arrow[u, bend left, "\Gamma_{g \circ u}"] & \mathrm{Spec}(C) \arrow[r] \arrow[u, bend left, "\Gamma_g"] \arrow[ur, "g"] & \mathbb{C}
\end{tikzcd}$$

where $g = \rho^X \circ \Gamma_g$, $g \circ u = \rho^X \circ v \circ \Gamma_{g \circ u}$ and the squares can be seen to be pullback squares using the universal property of pullback squares and the fact that $\rho^B \circ \Gamma_{g \circ u}$ is the identity. As above, as Γ_g and $\Gamma_{g \circ u}$ are closed immersions

$$(\Gamma_{g \circ u})_* u^* \mathcal{E} \cong v^* (\Gamma_g)_* \mathcal{E}$$

for any $\mathcal{E} \in \mathrm{Coh}(\mathrm{Spec}(C))$ by affine base change [17, Corollaire 1.5.2] (or see [35, Tag 02KE]). Hence

$$\mu_B(g \circ u) \cong \Gamma_{(g \circ u)*}\mathcal{O}_B \cong \Gamma_{(g \circ u)*}u^*\mathcal{O}_C \cong v^*(\Gamma_g)_*\mathcal{O}_C \cong v^*\mu_B(g).$$

To show it is a natural isomorphism we need to check that μ_B is bijective for all $B \in \mathfrak{R}$. We do this now by constructing an inverse ν_B. For $B \in \mathfrak{R}$, given $\mathcal{E} \in \mathcal{F}_X(B)$ we consider its schematic support $\iota : Z \hookrightarrow X^B$ and we then have the diagram

$$\begin{tikzcd}
Z \arrow[r, hook, "\iota"] \arrow[dr, "\psi"'] & X^B \arrow[r, "\rho^X"] \arrow[d, "\rho^B"] & X \arrow[d, "\rho"] \\
 & \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(\mathbb{C})
\end{tikzcd}$$

where we define $\psi = \rho^B \circ \iota$. We recall that ψ is an isomorphism from Lemma 2.7.5 (iii), and we then consider the map $\rho^X \circ \iota \circ \psi^{-1} : \mathrm{Spec}(B) \to X \in \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(B), X)$, and our inverse is defined by sending $\mathcal{E} \in \mathcal{F}_X(B)$ to this element of $\mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(B), X)$:

$$\nu_B : \mathcal{F}_X(B) \to \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}(B), X)$$
$$\mathcal{E} \mapsto \left(\rho_X \circ \iota \circ \psi^{-1} : \mathrm{Spec}(B) \to X\right).$$

Finally we note that this is an inverse, as

$$\nu_B \circ \mu_B(g) = \nu_B(\Gamma_{g_*}\mathcal{O}_B) = g$$

and

$$\mu_B \circ \nu_B(\mathcal{E}) = \mu_B\left(\left(\rho_X \circ \iota \circ \psi^{-1}\right) : \mathrm{Spec}(B) \to X^B\right) = \Gamma_{(\rho_X \circ \iota \circ \psi^{-1})_*}(\mathcal{O}_B)$$

where we note that $\Gamma_{(\rho_X \circ \iota \circ \psi^{-1})_*}(\mathcal{O}_B)$ is isomorphic to the structure sheaf of $\mathcal{Z}$ hence is equivalent to $\mathcal{E}$ in $\mathcal{F}_X(B)$ by Lemma 2.7.5 (iv). Hence $\mathrm{Hom}_{\mathfrak{Sch}}(-, X)$ is naturally isomorphic $\mathcal{F}_X$.

Finally, under this identification the identity morphism $id \in \mathrm{Hom}_{\mathfrak{Sch}}(X, X)$ is mapped to the bundle $\Gamma_{id*}\mathcal{O}_X = \Delta_*\mathcal{O}_X$, so this is the tautological element. □

Remark 4.0.4. A scheme X that is not locally finite type over $\mathbb{C}$ is not necessarily a fine moduli space for the functor $\mathcal{F}_X$. For example, there may exist exist coherent sheaves $\mathcal{E}$ on X^B that are flat over $\mathrm{Spec}(B)$ but not supported at any closed point of X^B that maps to a closed point of Spec B. Consider $R = \mathbb{C}[x]$, $X = \mathrm{Spec}\, R$, $B = \mathrm{Spec}\,\mathbb{C}[t]_{(t)}$ the localisation of $\mathbb{C}[t]$ at the maximal ideal (t), and $\mathcal{E} := R^B/(xt-1)$. Then $\mathcal{E} \in \mathrm{Coh}\, X^B$, $\mathcal{E}$ is flat as a B-module, and the fibre of $\mathcal{E}$ over the closed point of B is $R^B/(xt-1, t) \cong 0$. In particular $\mathcal{E}$ is not finitely generated as a B-module and it is an example of a flat module over a local ring that is not free.

In this generality stronger conditions must be imposed; such as requiring objects of $\mathcal{F}_X(B)$ to have proper support over Spec B. For a more general result reconstructing a scheme from the data of its point objects in the derived category see [7, Theorem 2.10] of Calabrese and Groechenig.

5. Applications

Let $\pi : X \to \operatorname{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$, suppose X has a tilting bundle T, and suppose that $A = \operatorname{End}_X(T)^{\text{op}}$ is presented as a quiver with relations. In this section we will introduce an indivisible dimension vector d_T and generic stability condition θ_T defined by a decomposition of the tilting bundle and give general conditions for the map $\eta : \mathcal{F}_X \to \mathcal{F}_A$ introduced in the previous sections to be a natural isomorphism for this stability condition and dimension vector. We will then use these general conditions to produce the applications outlined in the introduction.

5.1. Dimension vectors and stability

We introduce a certain dimension vector and stability condition defined from a decomposition of a tilting bundle and then, using Theorem 4.0.1, we give criterion for η to be a natural isomorphism with respect to this stability condition and dimension vector. In order to do this we make the following assumption on T, a tilting bundle on a scheme X.

Assumption 5.1.1. The tilting bundle T has a decomposition into non-isomorphic indecomposables $T = \bigoplus_{i=0}^{n} E_i$ such that there is a unique indecomposable, E_0, isomorphic to $\mathcal{O}_X$.

We then consider a presentation of $A = \operatorname{End}_X(T)^{\text{op}}$ as the path algebra of a quiver with relations such that each indecomposable E_i corresponds to a vertex i of the quiver, as in Sect. 2.4. In particular the 0 vertex in the quiver corresponds to the summand $\mathcal{O}_X$.

Definition 5.1.2. Suppose T is a tilting bundle T with decomposition $T = \bigoplus_{i=0}^{n} E_i$.

(i) The dimension vector d_T is defined by

$$d_T(i) = \operatorname{rk} E_i.$$

In particular $d_T(0) = 1$ as E_0 is assumed to be isomorphic to $\mathcal{O}_X$ so d_T is indivisible.

(ii) The stability condition θ_T is defined by

$$\theta_T(i) = \begin{cases} -\sum_{i \neq 0} \operatorname{rk} E_i & \text{if } i = 0; \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 5.1.3. *The stability condition θ_T has the following properties:*

(i) *Let $P_0 := \mathbb{R}\operatorname{Hom}_X(T, \mathcal{O}_X)$ and M be an A-module with dimension vector d_T. Then* $\operatorname{Hom}_A(P_0, M)$ *is one dimensional, and M is θ_T-stable if and only if there is a surjection $P_0 \to M \to 0$.*

(ii) *The stability θ_T is generic for A-modules of dimension d_T.*

Proof. The A-module P_0 is the projective module consisting of paths in the quiver starting at the vertex 0. For any representation M with dimension vector d_T a homomorphism from P_0 to M is determined by the image of the trivial path $e_0 \in P_0$ in the vector space $\mathbb{C} \subset M$ at vertex 0, which we denote by 1_0. This is as any path p starting at 0 must be sent to the evaluation in M of the linear map corresponding to p on the element 1_0. Hence $\mathrm{Hom}_A(P_0, M) = \mathbb{C}$, and any nonzero element of $\mathrm{Hom}_A(P_0, M)$ is surjective precisely when the linear maps in M corresponding to paths starting at 0 form a surjection from the vector space at the zero vertex onto M. By the definition of θ_T the module M is θ_T-semistable if and only if there are no proper submodules $N \subset M$ such that $d_N(0) = 1$, and this property is equivalent to the linear maps in M corresponding to paths starting at 0 forming a surjection. This proves part (i).

We now prove (ii). It is clear by the definitions of θ_T and d_T that any θ_T-stable dimension d_T module M can have no proper submodules $N \subset M$ such that $\theta_T(N) = 0$ as if N is a nontrivial submodule, either $d_N(0) = 0$ and $\theta_T(N) > 0$, or $d_N(0) = 1$ and $N = M$. □

We now give conditions for $\eta : \mathcal{F}_X \to \mathcal{F}_A$ to be a natural isomorphism for this stability condition and dimension vector. We note that there is an abelian category $\mathcal{A}$ corresponding to the abelian category A-mod under the tilting equivalence between $D^b(X)$ and $D^b(A)$ such that T is a projective generator of $\mathcal{A}$. Then $\mathbb{R}\mathrm{Hom}_X(T, -)$ and $T \otimes^{\mathbb{L}}_A (-)$ define an equivalence of abelian categories between $\mathcal{A}$ and A-mod. Our conditions are defined on this category $\mathcal{A}$.

Lemma 5.1.4. *Take the dimension vector d_T and stability condition θ_T as above. Then the structure sheaf $\mathcal{O}_X$ is in $\mathcal{A}$ and for all closed points $x \in X$ the skyscraper sheaf $\mathcal{O}_x$ is in $\mathcal{A}$. Suppose the following condition holds:*

(i) *For all closed points $x \in X$ there are surjections $\mathcal{O}_X \to \mathcal{O}_x \to 0$ in $\mathcal{A}$.*

Then η is a well defined natural transformation and η_B is injective for all $B \in \mathfrak{R}$. Suppose further that the following condition also holds:

(ii) *The set*

$$S := \left\{ \mathcal{E} \in \mathcal{A} \;\middle|\; \begin{array}{l} \bullet \mathbb{R}\mathrm{Hom}_X(T, \mathcal{E}) \text{ has dimension vector } d_T. \\ \bullet \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{O}_x) = 0 \text{ for all closed points } x \in X. \\ \bullet \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_x, \mathcal{E}) = 0 \text{ for all closed points } x \in X. \end{array} \right\}$$

is empty.

Then η is a natural isomorphism.

Proof. We first prove that $\mathcal{O}_X$ and all skyscraper sheaves $\mathcal{O}_x$ are in $\mathcal{A}$. An object $\mathcal{E} \in D^b(X)$ is in $\mathcal{A}$ if $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{E})$ is an A-module. It follows from assumption 5.1.1 that $\mathcal{O}_X$ is a summand of T and hence $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_X) = \mathrm{Hom}_X(T, \mathcal{O}_X)$ is an A-module. Then any closed point $x \in X$ is defined by a closed embedding $\iota_x : \mathrm{Spec}\,\mathbb{C} \to X$ and by definition $\mathcal{O}_x = \iota_x \mathbb{C}$, hence as T is locally free $\mathbb{R}\mathrm{Hom}_X(T, \iota_x \mathbb{C}) \cong \mathbb{R}\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\mathrm{rk}\, T}, \mathbb{C}) \cong \mathbb{C}^{\mathrm{rk}\, T}$ by adjunction and this an A-module so $\mathcal{O}_x \in \mathcal{A}$.

To prove the remaining claims we first assume that condition (i) holds and prove that $\eta_{\mathbb{C}}$ is well defined. Then it follows from Lemmas 3.2.1 and 3.2.2 that η is a natural transformation and η_B is injective for all $B \in \mathfrak{R}$.

Any element of $\mathcal{F}_X(\mathbb{C})$ is a skyscraper sheaf on X up to isomorphism. For any closed point $x \in X$ the A-module $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x)$ has dimension vector d_T, hence the map $\eta_{\mathbb{C}}$ is well defined if and only if all $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x)$ are θ_T-semistable A-modules. These are A-modules as $\mathcal{O}_x \in \mathcal{A}$. By considering the surjections of condition (i), $\mathcal{O}_X \to \mathcal{O}_x \to 0$ in $\mathcal{A}$, and applying the abelian equivalence $\mathbb{R}\mathrm{Hom}_X(T, -)$ we see that all $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x)$ are θ_T-stable by Lemma 5.1.3 (i). Hence $\eta_{\mathbb{C}}$ is well defined.

We now also assume that condition (ii) holds and prove that $\eta_{\mathbb{C}}$ is also surjective with inverse $T \otimes^{\mathbb{L}}_A (-)$. It then follows from Theorem 4.0.1 that η is a natural isomorphism. Take an A-module, M, with dimension vector d_T and which is θ_T-stable. As M is θ_T-stable by Lemma 5.1.3 (ii) there is a surjection

$$P_0 \to M \to 0$$

which under the abelian equivalence gives an exact sequence in $\mathcal{A}$

$$\mathcal{O}_X \to \mathcal{E} \to 0$$

where $\mathcal{E} \cong M \otimes^{\mathbb{L}}_A T \in D^b(X)$. Then by condition (ii) there must be some closed point $x \in X$ such that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{O}_x) \neq 0$ or $\mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_x, \mathcal{E}) \neq 0$.

If $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{O}_x) \neq 0$ then we apply $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{O}_x)$ to the surjection $\mathcal{O}_X \to \mathcal{E} \to 0$ to obtain an injection

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{O}_x) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_X, \mathcal{O}_x) = \mathbb{C}$$

and hence the surjection $\mathcal{O}_X \to \mathcal{O}_x \to 0$ factors through $\mathcal{E}$, and there is a surjection $\mathcal{E} \to \mathcal{O}_x \to 0$. We then apply the abelian equivalence functor $\mathbb{R}\mathrm{Hom}_X(T, -)$ to obtain a surjection of finite dimensional A-modules

$$M \to \mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x) \to 0$$

and by comparing dimension vectors we see that the map is an isomorphism, hence that $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x) \cong M$ and $\mathcal{E} \cong T \otimes^{\mathbb{L}}_A M \cong \mathcal{O}_x$.

If $\mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_x, \mathcal{E})$ is nonzero then by applying $\mathrm{Hom}(-, \mathcal{E})$ to the short exact sequence $\mathcal{O}_X \to \mathcal{O}_x \to 0$ an analogous argument deduces that $\mathcal{E} \cong \mathcal{O}_x$, where we note that $\dim \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_X, \mathcal{E}) = 1$ as M has dimension vector d_T. □

Corollary 5.1.5. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$. Let T be a tilting bundle on X which defines an equivalence of an abelian category $\mathcal{A}$ with A-mod, where $A = \mathrm{End}_X(T)^{\mathrm{op}}$. Choose the stability condition θ_T and dimension vector d_T as above, define $\mathcal{F}_A = \mathcal{F}^{ss}_{A,d_T,\theta_T}$, and assume that condition (i) of Lemma 5.1.4 holds for $\mathcal{A}$. Then:*

(i) *The map $\eta : \mathcal{F}_X \to \mathcal{F}_A$ defined in Sect. 3.2 is a natural transformation and induces a morphism $f : X \to \mathcal{M}^{ss}_{d_T,\theta_T}$ between X and the quiver GIT quotient of A for stability condition θ_T and dimension vector d_T. This morphism is a monomorphism in the sense of* [35, Tag 01L2].

(ii) *If condition (ii) of Lemma 5.1.4 also holds for $\mathcal{A}$ then the morphism f is an isomorphism.*

Proof. We note that $\mathcal{M}^{ss}_{d_T,\theta_T} = \mathcal{M}^{s}_{d_T,\theta_T}$ as θ_T is generic by Lemma 5.1.3 (ii) and that $\mathcal{M}^{s}_{d_T,\theta_T}$ is a fine moduli space for $\mathcal{F}_A$ by Theorem 2.6.3 as the dimension vector d_T is indivisible. The map $\eta : \mathcal{F}_X \to \mathcal{F}_A$ is a natural transformation as conditions (i) of Lemma 5.1.4 hold for $\mathcal{A}$. It then follows that there is a corresponding morphism $f : X \to \mathcal{M}^{ss}_{d_T,\theta_T}$ as $\mathcal{F}_A$ is represented by $\mathcal{M}^{ss}_{d_T,\theta_T}$ and $\mathcal{F}_X$ is represented by X by Proposition 4.0.2. For all $B \in \mathfrak{R}$ the map η_B is injective by Lemma 5.1.4, hence the corresponding morphism, f, is a monomorphism.

If condition (ii) of Lemma 5.1.4 also holds for $\mathcal{A}$ then η is actually a natural isomorphism by Lemma 5.1.4. Hence f is an isomorphism, proving (ii). □

Remark 5.1.6. While we make no further use of the monomorphism property we note that it can be a useful notion as proper monomorphisms are exactly closed immersions, [35, Tag 04XV], and étale monomorphisms are exactly open immersions, [35, Tag 025G].

While the remainder of this paper is focused on the application of these results in the case of projective morphisms $\pi : X \to Y$ with one-dimensional fibres, here we first give an example of an application in a situation with higher dimensional fibres. In the example below we verify that the conditions of Lemma 5.1.4 hold in the well known case of $\mathbb{P}^n$ with the Beilinson tilting bundle.

Example 5.1.7. Consider projective space $X := \mathbb{P}^n$ equipped with the Beilinson tilting bundle $T := \bigoplus_{i=0}^{n} \mathcal{O}_X(-i)$ [1]. In this situation $\langle \mathcal{O}_X(-n), \dots, \mathcal{O}_X(-1), \mathcal{O}_X \rangle$ is actually a full exceptional sequence and the the algebra $A := \mathrm{End}_X(T)^{\mathrm{op}}$ is finite dimensional with a presentation as the Beilinson quiver with relations. We set the notation that the algebra A has idempotents e_i such that the projective Ae_i corresponds to the tilting summand $\mathcal{O}_X(-i)$, we denote the corresponding simple by s_i, and we define $\varepsilon_i = \sum_{j=i}^{n} e_j$. We note that if $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{E}) = M$ is an A-module then as $\langle \mathcal{O}_X(-n), \dots, \mathcal{O}_X(-1), \mathcal{O}_X \rangle$ is an exceptional sequence it follows that $\mathbb{R}\mathrm{Hom}_X\left(\bigoplus_{j=0}^{i} \mathcal{O}_X(-j), \mathcal{E}\right) \cong M/M\varepsilon_{i+1}$ as an A-module.

It follows from the construction of $\mathbb{P}^n$ as the moduli space of lines in $\mathbb{C}^{n+1}$ with tautological bundle $\mathcal{O}_X \oplus \mathcal{O}_X(-1)$ that $\mathrm{Hom}_X(\mathcal{O}_X \oplus \mathcal{O}_X(-1), -)$ restricts to a surjection from skyscraper sheaves on $\mathbb{P}^n$ to $(-1, 1)$ stable $(1, 1)$ dimension representations of $\bar{A} := \mathrm{End}(\mathcal{O}_X \oplus \mathcal{O}_X(-1))^{\mathrm{op}}$, see [10,13,14] for more general examples of varieties with tilting bundles that extend tautological bundles on a particular realisation of the variety as a moduli space. Making use of this fact we can verify that condition (i) and (ii) of Lemma 5.1.4 hold in this example.

Firstly, any skyscraper sheaf on $\mathbb{P}^n$ has a length n resolution by locally free coherent sheaves constructed from the Koszul resolution

$$0 \to \mathcal{O}_X(-n) \to \mathcal{O}_X(-n+1)^{\oplus \binom{n}{n-1}} \to \cdots \to \mathcal{O}_X(-1)^{\oplus \binom{n}{1}} \to \mathcal{O}_X \to \mathcal{O}_x \to 0.$$

As each individual line bundle occurring in the sequence is a summand of T this long exact sequence in $\mathrm{Coh}\,\mathbb{P}^n$ is also a projective resolution of $\mathcal{O}_x$ in the abelian category $\mathcal{A}$. In particular the surjection $\mathcal{O}_X \to \mathcal{O}_x \to 0$ is a surjection in $\mathcal{A}$. This verifies condition (i). It is also straightforward to deduce from this resolution that $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{O}_x, s_i) = 0$ for $i > 1$, and we will use this fact below.

Secondly, suppose for contradiction that S is non-empty. Then there exists some $\mathcal{E} \in D^b(\mathbb{P}^n)$ such that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{O}_x) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_x, \mathcal{E}) = 0$ for all closed points $x \in \mathbb{P}^n$ and the dimension vector of the A-module $\mathbb{R}\mathrm{Hom}_X(T, \mathcal{E})$ is $d_T = (1, 1, \ldots, 1)$. We define the A-modules $T_x := \mathbb{R}\mathrm{Hom}_X(T, \mathcal{O}_x)$ for any closed point $x \in \mathbb{P}^n$, $M := \mathbb{R}\mathrm{Hom}_X(T, \mathcal{E})$, and $M_i := \mathbb{R}\mathrm{Hom}_X(\bigoplus_{j=0}^i \mathcal{O}_X(-j), \mathcal{E})$. In particular we note that $M_n = M$ and $M_i \cong \frac{M_{i+1}}{M_{i+1}e_{i+1}} \cong \frac{M}{M\varepsilon_{i+1}}$ for $0 < i < n$. We now prove that $\mathrm{Hom}_A(T_x, M_i) = 0$ for $i = n, \ldots, 1$. This is true for $i = n$ by the assumption that $\mathcal{E} \in S$. Then we consider the short exact sequences

$$0 \to M_i e_i \to M_i \to M_{i-1} \to 0$$

in A-mod, and by considering dimension vectors we can deduce that $M_i e_i \cong s_i$, the simple module at vertex i. It follows that there are long exact sequences

$$0 \to \mathrm{Hom}_A(T_x, s_i) \to \mathrm{Hom}_A(T_x, M_i) \to \mathrm{Hom}_A(T_x, M_{i-1}) \to \mathrm{Ext}^1_A(T_x, s_i).$$

By induction we can assume $\mathrm{Hom}_A(T_x, M_i) = 0$, and using the fact $\mathrm{Ext}^1_A(T_x, s_j) \cong \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{O}_x, s_j) = 0$ for $j > 1$ that is noted above we can deduce that $\mathrm{Hom}_A(T_x, M_{i-1}) = 0$ when $i > 1$. It follows by induction that $\mathrm{Hom}_A(T_x, M_1) = 0$ for all closed points $x \in \mathbb{P}^n$.

This is a contradiction. By construction the A-module M_1 is a $(-1, 1)$ stable $\bar{A}$-module with dimension vector $(1, 1)$ and as stated above all such $\bar{A}$-modules are the image of a skyscraper sheaf on $\mathbb{P}^n$ under $\mathrm{Hom}_{\mathbb{P}^n}(\mathcal{O}_X \oplus \mathcal{O}_X(-1), -)$. In particular $M_1 \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathcal{O}_X \oplus \mathcal{O}_X(-1), \mathcal{O}_y) \cong \frac{T_y}{T_y\varepsilon_2}$ for some closed point $y \in \mathbb{P}^n$ as an A-module and so there must be a nonzero A-module morphism $T_y \to M_1$ corresponding to the nonzero A-module morphism $T_y \to \frac{T_y}{T_y\varepsilon_2}$. Hence the conclusion that $\mathrm{Hom}(T_x, M_1) = 0$ for all closed points $x \in \mathbb{P}^n$ is absurd, no such $\mathcal{E}$ can exist, and $S = \emptyset$. This verifies condition (ii).

5.2. *One dimensional fibres*

To apply Lemma 5.1.4 and Corollary 5.1.5 we need a class of varieties with tilting bundles such that we understand the abelian categories $\mathcal{A}$. Such a class was introduced in Theorem 1.3.1; if $\pi : X \to \mathrm{Spec}(R)$ is a projective morphism of Noetherian schemes such that $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$ and the fibres of π have dimension ≤ 1 then there exist tilting bundles T_i on X such that the abelian category $\mathcal{A}$ is ${}^{-i}\mathrm{Per}(X/R)$, defined as follows.

Definition 5.2.1. ([37, Section 3]) Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of Noetherian schemes such that $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$ and π has fibres of dimension ≤ 1. Define $\mathfrak{C}$ to be the abelian subcategory of Coh X consisting of $\mathcal{F} \in$ Coh X such that $\mathbb{R}\pi_*\mathcal{F} \cong 0$. For $i = 0, 1$ the abelian category ${}^{-i}\mathrm{Per}(X/R)$ is defined to contain $\mathcal{E} \in D^b(X)$ which satisfy the following conditions:

(i) The only non-vanishing cohomology of $\mathcal{E}$ lies in degrees -1 and 0.
(ii) $\pi_*\mathcal{H}^{-1}(\mathcal{E}) = 0$ and $\mathbb{R}^1\pi_*\mathcal{H}^0(\mathcal{E}) = 0$, where $\mathcal{H}^j$ denotes taking the jth cohomology sheaf.
(iii) For $i = 0$, $\mathrm{Hom}_X(C, \mathcal{H}^{-1}(\mathcal{E})) = 0$ for all $C \in \mathfrak{C}$.
(iv) For $i = 1$, $\mathrm{Hom}_X(\mathcal{H}^0(\mathcal{E}), C) = 0$ for all $C \in \mathfrak{C}$.

We note that the abelian categories ${}^{-i}\mathrm{Per}(X/R)$ are hearts of t-structures on $D^b(X)$ so short exact sequences in ${}^{-i}\mathrm{Per}(X/R)$ correspond to triangles in $D^b(X)$ whose vertices are in ${}^{-i}\mathrm{Per}(X/R)$.

Any projective generator of the abelian category ${}^{-i}\mathrm{Per}(X/R)$ gives a tilting bundle T_i with the properties defined in Theorem 1.3.1, and we can assume that such a tilting bundle contains $\mathcal{O}_X$ as a summand by the following proposition.

Proposition 5.2.2. ([37, Proposition 3.2.7]) *Define $\mathfrak{V}_X$ to be the category of vector bundles $\mathcal{M}$ on X which are generated by global sections and such that* $\mathrm{H}^1(X, \mathcal{M}^\vee) = 0$, *and define $\mathfrak{V}_X^\vee := \{\mathcal{M}^\vee : \mathcal{M} \in \mathfrak{V}_X\}$. The projective generators of* ${}^{-1}\mathrm{Per}(X/R)$ *are the $\mathcal{M} \in \mathfrak{V}_X$ such that $\wedge^{\mathrm{rk}\,\mathcal{M}}\mathcal{M}$ is ample and $\mathcal{O}_X$ is a summand of $\mathcal{M}^{\oplus a}$ for some $a \in \mathbb{N}$. The projective generators of* ${}^{0}\mathrm{Per}(X/R)$ *are the elements of $\mathfrak{V}_X^\vee$ which are dual to projective generators of* ${}^{-1}\mathrm{Per}(X/R)$.

Hence we let T_i be a projective generator of ${}^{i}\mathrm{Per}(X/R)$ with a decomposition as required in Assumption 5.1.1. Then the algebra $A_i = \mathrm{End}_X(T_i)^{\mathrm{op}}$ can be presented as a quiver with relations with vertex 0 corresponding to $\mathcal{O}_X$ and the stability condition θ_{T_i} and dimension vector d_{T_i} are well defined.

We now check that the conditions of Lemma 5.1.4 hold for ${}^{0}\mathrm{Per}(X/R)$.

Theorem 5.2.3. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$ such that π has fibres of dimension ≤ 1 and $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$. Then the abelian category* ${}^{0}\mathrm{Per}(X/R)$ *satisfies conditions (i) and (ii) of Lemma 5.1.4.*

Proof. We begin by checking $\mathcal{A}$ satisfies condition (i) of Lemma 5.1.4. All skyscraper sheaves $\mathcal{O}_x$ and the structure sheaf $\mathcal{O}_X$ are in $\mathcal{A}$ as they satisfy the conditions of Definition 5.2.1. Then, for any $x \in X$, the short exact sequence of sheaves $0 \to I \to \mathcal{O}_X \to \mathcal{O}_x \to 0$ corresponds to a triangle in $D^b(X)$, and the ideal sheaf I is also in $\mathcal{A}$ as $\mathbb{R}^1\pi_*I = 0$ due to the exact sequence $0 \to \pi_*I \to \pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_x \to \mathbb{R}^1\pi_*I \to 0$ where $\pi_*\mathcal{O}_X \cong \mathcal{O}_R$ and the third arrow is a surjection. Hence the map $\mathcal{O}_X \to \mathcal{O}_x \to 0$ is in fact a surjection in $\mathcal{A}$. We then note, for all $x \in X$, that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_X, \mathcal{O}_x) \cong \mathrm{Hom}_{D^b(X)}(\mathcal{O}_X, \mathcal{O}_x) \cong \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_x) \cong \mathbb{C}$, hence $\mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_X, \mathcal{O}_x) \cong \mathbb{C}$ corresponding to the map of sheaves $\mathcal{O}_X \to \mathcal{O}_x \to 0$ which is surjective in $\mathcal{A}$.

To check condition (ii) we suppose S is not empty and we will deduce a contradiction. The argument below should be thought of as a translation to our setting of the proof of Nakamura's conjecture for the G-Hilbert scheme in [5, Section 8] which derives a contradiction between the facts that the Euler pairing of a coherent sheaf shifted by [1] with a very ample line bundle must be negative whereas the Euler pairing of a G-cluster with any locally free sheaf must be positive.

As S is assumed nonempty there exists $\mathcal{E} \in S$. In particular, $M \cong \mathrm{Hom}_X(T_0, \mathcal{E})$ has dimension vector d_{T_0} so $\mathbb{R}\pi_*\mathcal{E} \cong \mathcal{O}_p$ for some closed point $p \in \mathrm{Spec}(R)$. As $\mathcal{E} \in \mathcal{A}$ there is a short exact sequence in $\mathcal{A}$

$$0 \to \mathcal{H}^{-1}(\mathcal{E})[1] \to \mathcal{E} \to \mathcal{H}^0(\mathcal{E}) \to 0$$

where [1] is the shift in $D^b(X)$. Hence, for all closed points $x \in X$, there is an injection

$$0 \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}^0(\mathcal{E}), \mathcal{O}_x) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{O}_x).$$

Then it follows that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}^0(\mathcal{E}), \mathcal{O}_x) \cong \mathrm{Hom}_{D^b(X)}(\mathcal{H}^0(\mathcal{E}), \mathcal{O}_x) = 0$ for all $x \in X$ as by assumption $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{O}_x) = 0$, and hence $\mathcal{H}^0(\mathcal{E}) = 0$ as a nonzero coherent sheaf must be supported at some closed point. So $\mathcal{E} = \mathcal{H}^{-1}(\mathcal{E})[1]$, and we now seek to reach a contradiction to the existence of such an $\mathcal{E}$.

If the indecomposable summands E_i of T_0 formed a basis for the Grothendieck group $K(\mathrm{Perf}(X))$ then $\mathcal{E}$ having dimension vector d_{T_0} would imply that it has the same class as a skyscraper sheaf in $K(\mathrm{Perf}(X))$ and we could then use the Euler form to deduce a contradiction. However, as the Krull-Schmidt property may not hold in general the E_i may not be a basis and hence we must first restrict to the complete local case. We do this now. As $\mathbb{R}\pi_*\mathcal{E} \cong \mathcal{O}_p$ we calculate $\pi_*\mathcal{H}^{-1}(\mathcal{E}) = 0$ and $\mathbb{R}^1\pi_*\mathcal{H}^{-1}(\mathcal{E}) = \mathcal{O}_p$. By [37, Lemma 3.1.3] there is an injection of sheaves

$$0 \to \mathcal{H}^{-1}(\mathcal{E}) \to \mathcal{H}^{-1}(\pi^!\mathcal{O}_p)$$

and hence $\mathcal{H}^{-1}(\mathcal{E})$ is set-theoretically supported on $\pi^{-1}(p)$. In particular p corresponds to a maximal ideal $\mathfrak{m}$ of R and we consider the completion $R \to \hat{R} = \varprojlim(R/\mathfrak{m}^n)$. This produces the following pullback diagram

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "\hat{\pi}"'] & X \arrow[d, "\pi"] \\ \mathrm{Spec}(\hat{R}) \arrow[r, "i"] & \mathrm{Spec}(R) \end{tikzcd}$$

where $Y := \hat{R} \times_{\mathrm{Spec}(R)} X$, the morphisms i and j are both flat and affine, and the morphism $\hat{\pi}$ is projective. Then we have the following isomorphism, where we recall that the morphisms i and j are both flat and affine so we need not derive them,

$$\begin{aligned} \mathbb{R}\mathrm{Hom}_X(T_0, j_*j^*\mathcal{E}) &\cong i_*\mathbb{R}\mathrm{Hom}_Y(j^*T_0, j^*\mathcal{E}) && (j_*, j^* \text{ adjoint pair}) \\ &\cong i_*\mathbb{R}\hat{\pi}_*j^*\mathcal{RH}om_X(T_0, \mathcal{E}) && (\text{Lemma } 3.1.1) \\ &\cong i_*i^*\mathbb{R}\mathrm{Hom}_X(T_0, \mathcal{E}). && (\text{Flat base change}) \end{aligned}$$

As $M \cong \mathbb{R}\mathrm{Hom}_X(T_0, \mathcal{E})$ is finite dimensional and supported above $\mathfrak{m}$ it follows that completion in $\mathfrak{m}$ followed by restriction of scalars acts as the identity, see [16, Theorem 2.13] and [28, Lemma 2.5], hence $i_*i^*M := \hat{R} \otimes_R M \cong M$. We deduce that $\mathbb{R}\mathrm{Hom}_X(T_0, j_*j^*\mathcal{E}) \cong \mathbb{R}\mathrm{Hom}_X(T_0, \mathcal{E})$ and so $\mathcal{E} \cong j_*j^*\mathcal{E}$ as T_0 is a tilting bundle. We define $\hat{\mathcal{E}} := j^*\mathcal{E}$ and note that as j is flat and $\mathcal{E} \cong \mathcal{H}^{-1}(\mathcal{E})$ it is also true that $\mathcal{H}^{-1}(\hat{\mathcal{E}}) \cong \hat{\mathcal{E}}$.

There is an Euler form defined between $\mathrm{Perf}(Y) \subset D^b(Y)$ and the full triangulated subcategory $D^b_c(Y) \subset D^b(Y)$ consisting of objects with compact support

$$\chi : K(\mathrm{Perf}(Y)) \times K(D^b_c(Y)) \to \mathbb{Z}$$
$$\chi(\mathcal{F}, \mathcal{G}) = \sum_{i\in\mathbb{Z}} (-1)^n \mathrm{Hom}_{D(Y)}(\mathcal{F}, \mathcal{G}[i]).$$

As T_0 is a projective generator of ${}^0\mathrm{Per}(X/R)$ by [37, Proposition 3.1.4] $P := j^*T_0 \in \mathfrak{V}_Y^{\vee}$ is a projective generator of ${}^0\mathrm{Per}(Y/\hat{R})$. As $P \in {}^{-0}\mathrm{Per}(Y/\hat{R})$ and $\hat{R}$ is a complete local ring by [37, Lemma 3.5.1] there exists a short exact sequence

$$0 \to \mathcal{O}_Y^{\oplus d-1} \to P^\vee \to \wedge^d(P^\vee) \to 0$$

where $d = \mathrm{rk}\, P = \mathrm{rk}\, T_0$, and as P is a projective generator the line bundle $\mathcal{L} := \wedge^d(P^\vee)$ is ample by Proposition 5.2.2. By [37, Lemma 3.5.1] there are also short exact sequences

$$0 \to \mathcal{L}^{\otimes -n} \to P^{\oplus n} \to \mathcal{O}_Y^{\oplus(nd-1)} \to 0 \qquad \dagger_n$$

for any $n \in \mathbb{N}$. We then calculate

$$\begin{aligned} \chi(P, \hat{\mathcal{E}}) &= \dim \mathrm{Hom}_Y(P, \hat{\mathcal{E}}) && (P \in {}^0\mathrm{Per}(Y/\hat{R}) \text{ projective}) \\ &= \dim \mathrm{Hom}_X(T_0, \mathcal{E}) && (j^*, j_* \text{ adjunction}) \\ &= d && (\mathcal{E} \text{ has dimension vector } d_{T_0}) \end{aligned}$$

and

$$\begin{aligned} \chi(\mathcal{O}_Y, \hat{\mathcal{E}}) &= \dim \mathrm{Hom}_Y(\mathcal{O}_Y, \hat{\mathcal{E}}) && (\mathcal{O}_Y \in {}^0\mathrm{Per}(Y/\hat{R}) \text{ projective}) \\ &= \dim \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{E}) && (j^*, j_* \text{ adjunction}) \\ &= 1 && (\mathcal{E} \text{ has dimension vector } d_{T_0}(0) = 1) \end{aligned}$$

hence additivity of the Euler character applied to a short exact sequence ($\dagger_n$) implies

$$\begin{aligned} \chi(\mathcal{L}^{\otimes -n}, \hat{\mathcal{E}}) &= n \cdot \chi(P, \hat{\mathcal{E}}) - (nd-1) \cdot \chi(\mathcal{O}_Y, \hat{\mathcal{E}}) \\ &= nd - (nd-1) \\ &= 1 \end{aligned}$$

for any $n \in \mathbb{N}$.

Alternatively, by Serre vanishing there exists some $n \in \mathbb{N}$ such that

$$\mathrm{Ext}^i_Y(\mathcal{L}^{\otimes -n}, \mathcal{H}^{-1}(\hat{\mathcal{E}})) \cong \mathrm{Ext}^i_Y(\mathcal{O}_Y, \mathcal{L}^{\otimes n} \otimes_Y \mathcal{H}^{-1}(\hat{\mathcal{E}})) = 0$$

for $i > 0$, see [19, III Theorem 5.2]. As $\hat{\mathcal{E}} \cong \mathcal{H}^{-1}(\hat{\mathcal{E}})[1]$ this implies

$$\chi(\mathcal{L}^{\otimes -n}, \hat{\mathcal{E}}) = \chi(\mathcal{L}^{\otimes -n}, \mathcal{H}^{-1}(\hat{\mathcal{E}})[1]) = -\dim \mathrm{Hom}(\mathcal{L}^{\otimes -n}, \mathcal{H}^{-1}(\hat{\mathcal{E}})) \leq 0.$$

This is a contradiction to the previous calculation that $\chi(\mathcal{L}^{\otimes -n}, \hat{\mathcal{E}}) = 1 > 0$, and so we conclude that S is empty. $\square$

Combining this theorem with Corollary 5.1.5 gives us the following result, showing that in this situation schemes can be reconstructed as fine moduli spaces by quiver GIT.

Corollary 5.2.4. *Let $\pi : X \to \mathrm{Spec}(R)$ be a projective morphism of finite type schemes over $\mathbb{C}$ such that π has fibres of dimension ≤ 1 and $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$. Let T_0 be a tilting bundle which is a projective generator of ${}^0\mathrm{Per}(X/R)$ as defined by Theorem 1.3.1, define $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$, and choose the stability condition θ_{T_0} and dimension vector d_{T_0} as above. Then X is the fine moduli space of the quiver representation moduli functor for $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$ with dimension vector d_{T_0} and stability condition θ_{T_0} and the tautological bundle is the tilting bundle $T_0^{\vee}$.*

5.3. Example: flops

The class of varieties considered in Sect. 5.2 were originally motivated by flops in the minimal model program. In the paper [4] Bridgeland proves that smooth varieties in dimension three which are related by a flop are derived equivalent, and in the process constructs the flop of such a variety as a moduli space of perverse point sheaves. In this section we show that this moduli space construction can in fact be done using quiver GIT. Recall the following theorem.

Theorem 5.3.1. ([37, Theorems 4.4.1, 4.4.2]) *Suppose $\pi : X \to \mathrm{Spec}(R)$ is a projective birational map of quasiprojective Gorenstein varieties of dimension ≥ 3, with π having fibres of dimension ≤ 1, the exceptional locus of π having codimension ≥ 2, and Y having canonical hypersurface singularities of multiplicity ≤ 2. Then the flop $\pi' : X' \to \mathrm{Spec}(R)$ exists and is unique. Further X and X' are derived equivalent such that ${}^{-1}\mathrm{Per}(X/R)$ corresponds to ${}^0\mathrm{Per}(X'/R)$. In particular, for a tilting bundle T_1 on X which is a projective generator of ${}^{-1}\mathrm{Per}(X/R)$ there is a tilting bundle T_0' on X' which is a projective generator of ${}^0\mathrm{Per}(X'/R)$ such that $A_1 = \mathrm{End}_X(T_1)^{\mathrm{op}} \cong \mathrm{End}_{X'}(T_0')^{\mathrm{op}} = A_0'$ and $\pi_*T_1 \cong \pi'_*T_0'$.*

We refer the reader to [37, Theorem 4.4.1] for the definition of a flop in this setting. The results from the previous sections now imply the following corollary, showing that the variety X and its flop X' can both be constructed as quiver GIT quotients from tilting bundles on X.

Corollary 5.3.2. *Suppose we are in the situation of Theorem 5.3.1. Then X is the quiver GIT quotient of $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$ for stability condition θ_{T_0} and dimension vector d_{T_0}, and X' is the quiver GIT quotient of $A_1 = \mathrm{End}_X(T_1)^{\mathrm{op}}$ for stability condition θ_{T_1} and dimension vector d_{T_1}.*

Proof. Corollary 5.2.4 tells us both that X is the quiver GIT quotient of A_0 for stability condition θ_{T_0} and dimension vector d_{T_0}, and that X' is the quiver GIT quotient of $A_0' = \mathrm{End}_{X'}(T_0')^{\mathrm{op}}$ for stability condition $\theta_{T_0'}$ and dimension vector $d_{T_0'}$. We now relate A_0', $\theta_{T_0'}$ and $d_{T_0'}$ to A_1, θ_{T_1} and d_{T_1}.

We note that by Theorem 5.3.1 $A_0' \cong A_1$, and we choose a presentation of A_1 as a quiver with relations matching that of A_0' in order to identify the stability condition and dimension vector matching $\theta_{T_0'}$ and $d_{T_0'}$. In particular there is a decomposition of $T_1 = \bigoplus_{i=0}^n E_i$ and $T_0' = \bigoplus_{i=0}^n E_i'$ such that $\pi_* E_i \cong \pi'_* E_i'$. We note that under this correspondence the vertices corresponding to $\mathcal{O}_X$ and $\mathcal{O}_{X'}$ correspond by [37, Lemma 4.2.1] as $\pi_*\mathcal{O}_X \cong \pi'_*\mathcal{O}_{X'} \cong \mathcal{O}_R$, and since π and π' are birational $\mathrm{rk}_X\, E_i = \mathrm{rk}_R\, \pi_* E_i = \mathrm{rk}_R\, \pi'_* E_i' = \mathrm{rk}_{X'}\, E_i'$. Hence $A_0' \cong A_1$, $d_{T_0'} = d_{T_1}$ and $\theta_{T_0'} = \theta_{T_1}$ so X' is the quiver GIT quotient of $A_1 = \mathrm{End}_X(T_1)^{\mathrm{op}}$ for stability condition θ_{T_1} and dimension vector d_{T_1}. □

5.4. Example: resolutions of rational singularities

We give a further application of Theorem 5.2.4 to the case of rational singularities, extending and recapturing several well-known examples.

Definition 5.4.1. Let Y be a (possibly singular) variety. A smooth variety X with a projective birational map $\pi : X \to Y$ that is bijective over the smooth locus of Y is called a *resolution* of Y. A resolution, X, is a *minimal resolution* of Y if any other resolution factors through it. In general minimal resolutions do not exist, but they always exist for surfaces, [29, Corollary 27.3]. A resolution, X, is a *crepant resolution* of Y if $\pi^*\omega_Y = \omega_X$, where ω_X and ω_Y are the canonical classes of X and Y which we assume are normal. In general crepant resolutions do not exist. A singularity, Y, is *rational* if for any resolution $\pi : X \to Y$

$$\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_Y.$$

If this holds for one resolution it holds for all resolutions, [38, Lemma 1].

Minimal resolutions of rational affine singularities $\pi : X \to \mathrm{Spec}(R)$ satisfy the condition $\mathbb{R}\pi_*\mathcal{O}_X \cong \mathcal{O}_R$ by definition, and in the case of surface singularities it is immediate that the dimensions of the fibres of π are ≤ 1. Hence the following corollary is immediate from Corollary 5.2.4 (ii).

Corollary 5.4.2. *Suppose that $\pi : X \to \mathrm{Spec}(R)$ is the minimal resolution of a rational surface singularity. Then there is a tilting bundle T_0 on X as in Theorem 1.3.1, and by Corollary 5.2.4 (ii) X is the fine moduli space of quiver representations of $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$ for dimension vector d_{T_0} and stability condition θ_{T_0} with tautological bundle $T_0^\vee$.*

This gives a moduli interpretation of minimal resolutions for all rational surface singularities. In certain examples the tilting bundles and algebras are well-understood and this corollary recovers previously known examples.

Example 5.4.3. (Kleinian Singularities) Kleinian singularities are quotient singularities $\mathbb{C}^2/G$ for G a non-trivial finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. These have crepant resolutions, and in particular $\mathrm{Hilb}^G(\mathbb{C}^2) = X \to \mathbb{C}^2/G$ is a crepant resolution, [22]. There is a tilting bundle T on X constructed by Kapranov and Vasserot [24], which, if we take the multiplicity free version, matches the T_0 of Theorem 1.3.1. Then $A = \mathrm{End}_X(T)^{\mathrm{op}}$ is presentable as the McKay quiver with relations, the preprojective algebra, and G-$\mathrm{Hilb}(\mathbb{C}^2)$ is the quiver GIT quotient of the preprojective algebra for stability condition θ_T and dimension vector d_T. The crepant resolutions were previously constructed as hyper-Kähler quotients by Kronheimer [27], this approach was interpreted as a GIT quotient construction by Cassens and Slodowy [8], and as a quiver GIT quotient by Crawley-Boevey [15].

Example 5.4.4. (Surface Quotient Singularities) As an expansion of the previous example we consider G a non-trivial, pseudo-reflection-free, finite subgroup of $\mathrm{GL}_2(\mathbb{C})$. Then $\mathbb{C}^2/G$ is a rational singularity with a minimal resolution $\pi : G\text{-}\mathrm{Hilb}(\mathbb{C}) = X \to \mathbb{C}^2/G$ by [21]. The variety X has the tilting bundle T_0, and the algebras $A = \mathrm{End}_X(T_0)^{\mathrm{op}}$ can be presented as the path algebras of quivers with relations, the reconstruction algebras, which are defined and explicitly calculated in [39–42]. If $G < \mathrm{SL}_2(\mathbb{C})$ then this example falls into the case of Kleinian singularities above, otherwise these fall into a classification in types $\mathbb{A}$, $\mathbb{D}$, $\mathbb{T}$, $\mathbb{I}$, and $\mathbb{O}$, [40, Section 5]. It was shown by explicit calculation in [39,41,42] that in types $\mathbb{A}$ and $\mathbb{D}$ the minimal resolutions X are quiver GIT quotients of A with stability condition θ_{T_0} and dimension vector d_{T_0}. Corollary 5.4.2 recovers these cases without needing to perform explicit calculations, and also includes the same result for the remaining cases $\mathbb{T}$, $\mathbb{I}$, and $\mathbb{O}$.

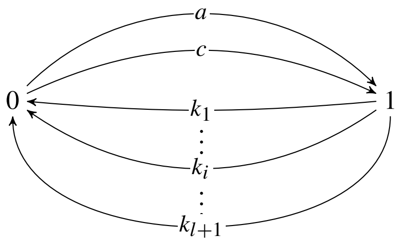
Corollary 5.4.5. *Suppose $G < \mathrm{GL}_2(\mathbb{C})$ is a finite, non-trivial, pseudo-reflection-free group. Then the minimal resolution of the quotient singularity $\mathbb{C}^2/G$ can be constructed as the fine moduli space of the quiver representation moduli functor of the corresponding reconstruction algebra for stability condition θ_{T_0} and dimension vector d_{T_0}, and the tautological bundle is the tilting bundle $T_0^\vee$.*

Proof. We note that in Theorem 1.3.1 $T_1 = T_0^\vee$ and that $\mathrm{End}_X(T_0^\vee) \cong \mathrm{End}_X(T_0)^{\mathrm{op}}$. Hence our definition of $A = \mathrm{End}_X(T_0)^{\mathrm{op}}$ as the reconstruction algebra matches that given in [39–42] as $A = \mathrm{End}_X(T_1)$. Then the result is an immediate corollary of Corollary 5.4.2. □

Example 5.4.6. (Determinantal Singularities) We give one higher dimensional example. Let R be the $\mathbb{C}$-algebra $\mathbb{C}[X_0, \dots X_l, Y_1, \dots Y_{l+1}]$ subject to the relations generated by all two by two minors of the matrix

$$\begin{pmatrix} X_0 & X_1 & \dots & X_i & \dots & X_l \\ Y_1 & Y_2 & \dots & Y_{i+1} & \dots & Y_{l+1} \end{pmatrix}.$$

Then $\mathrm{Spec}(R)$ is a $l+2$ dimensional rational singularity and has an isolated singularity at the origin. This has a resolution given by $\pi : X = \mathrm{Tot}\left(\bigoplus_{i=1}^l \mathcal{O}_{\mathbb{P}^1}(-1)\right) \to \mathrm{Spec}(R)$, the total space of the locally free sheaf $\bigoplus_{i=1}^l \mathcal{O}_{\mathbb{P}^1}(-1)$ mapping onto its affinisation. The variety X has a tilting bundle T_0 by Theorem 1.3.1, which, considering the bundle map $f : X \to \mathbb{P}^1$, we can identify as $T_0 = \mathcal{O}_X \oplus f^*\mathcal{O}_{\mathbb{P}^1}(-1)$. We can then present $A_0 = \mathrm{End}_X(T_0)^{\mathrm{op}}$ as the following quiver with relations, (Q, Λ).



$$\begin{gathered} k_i a k_j = k_j a k_i \\ k_i c k_j = k_j c k_i \\ a k_j c = c k_j a \\ \text{for } 1 \leq i, j \leq l+1 \end{gathered}$$

By Theorem 5.2.4 we know that X can be reconstructed as the quiver GIT quotient of A_0 with dimension vector $d_{T_0} = (1, 1)$ and stability condition $\theta_{T_0} = (-1, 1)$. In this example we will explicitly verify this. A dimension d_{T_0} representation is defined by assigning a value $\lambda_i \in \mathbb{C}$ to each k_i and $(\alpha, \gamma) \in \mathbb{C}^2$ to (a, c). The relations are all automatically satisfied so $\mathrm{Rep}_{d_{T_0}}(Q, \Lambda) = \mathbb{C}^{l+1} \times \mathbb{C}^2$. Then a representation is θ_{T_0} stable if it has no dimension $(1, 0)$ submodules, so these correspond to the subvariety with $(\alpha, \gamma) \in \mathbb{C}^2/(0, 0)$, hence $\mathrm{Rep}_d(Q, \Lambda)^{ss} = \mathbb{C}^{l+1} \times \mathbb{C}^2/(0, 0)$. We then find that the corresponding quiver GIT quotient is given by the action of $\mathbb{C}^*$ on $\mathbb{C}^{l+1} \times \mathbb{C}^2/(0, 0)$ with weights -1 on $\mathbb{C}^2/(0, 0)$ and 1 on $\mathbb{C}^{l+1}$. This produces the total bundle X.

When $l = 2$ this is the motivating example of the Atiyah flop given as the opening example of [37] and A_0 is the conifold quiver. In this case, by Theorem 5.3.2, we can calculate the flop as the quiver GIT quotient of $A_1 \cong A_0^{\mathrm{op}}$.

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7. Appendix: comparing quiver moduli functors

As we noted in the introduction, our results are inspired by a theorem of Sekiya and Yamaura that compares quiver GIT quotients for algebras related by tilting modules. This is done by

constructing natural transformations between moduli functors which should have the quiver GIT quotients as moduli spaces, however the quiver representation moduli functor considered in [34] is different to the one defined in Sect. 2.6 and does not always have the quiver GIT quotient as a moduli space. In this appendix we outline the minimal changes required to reinterpret the results of [34] for a correct moduli functor.
The moduli functor for quiver representations defined in [34, Section 4.1] is

$$\begin{aligned} \mathcal{F}^{SY}_{A,d,\theta} : \mathfrak{R} &\to \mathfrak{Sets} \\ R &\mapsto \mathcal{S}^{ss}_{A,d,\theta}(R)\big/ \sim_{SY} \end{aligned}$$

with the set $\mathcal{S}^{ss}_{A,d,\theta}(R)$ defined as in Sect. 2.6 and the equivalence condition $M_1 \sim_{SY} M_2$ if $M_1 \otimes_R R/\mathfrak{m}$ is S-equivalent to $M_2 \otimes_R R/\mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{MaxSpec}(R)$. This differs from the functor $\mathcal{F}^{ss}_{A,d,\theta}$ defined in Sect. 2.6 by using the equivalence $\sim_{SY}$ rather than the equivalence $\sim$. However, as the following example shows, the equivalence $\sim_{SY}$ is too restrictive.

Example. Let $A = \mathbb{C}[x]$, $d = 1$, and $\theta = 0$. Then A can be presented as the path algebra of a quiver with a single vertex and single loop, and the quiver GIT quotient is $\mathrm{Spec}(A)$. In particular, if this were a fine moduli space for $\mathcal{F}^{SY}_{A,d,\theta}$ then $\mathcal{F}^{SY}_{A,d,\theta}(\mathbb{C}[\epsilon]/\epsilon^2) \cong \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}\,\mathbb{C}[\epsilon]/\epsilon^2, \mathrm{Spec}\,A) \cong \mathbb{C}^2$. However

$$\{M_{a,b} := \mathbb{C}[x,\epsilon]/(x - a - b\epsilon) \mid a, b \in \mathbb{C}\} = \mathcal{S}^{ss}_{A,d,\theta}(\mathbb{C}[\epsilon]/\epsilon^2)$$

and $M_{a,b} \sim_{SY} M_{\alpha,\beta} \Leftrightarrow a = \alpha$ so $\mathcal{F}^{SY}_{A,d,\theta}(\mathbb{C}[\epsilon]/\epsilon^2) \cong \mathbb{C}$. Hence the quiver GIT quotient is not a fine moduli space for the functor $\mathcal{F}^{SY}_{A,d,\theta}$.

This indicates that $\sim_{SY}$ is not the correct equivalence to use to define a quiver representation moduli functor. Below we note a brief amendment that adapts the results of [34] to work with the functor used in this paper instead.

Firstly, the moduli functor defined in [34, Section 4.1] should be replaced by the moduli functor $\mathcal{F}^{ss}_{A,d,\theta}$ defined in Sect. 2.6 and the statement that $\mathcal{M}^{ss}_{d,\theta}$ is a coarse moduli space can then be replaced by the statement that $\mathcal{F}^{ss}_{A,d,\theta}$ is corepresented by $\mathcal{M}^{ss}_{d,\theta}$ and when d is indivisible and θ generic this is a fine moduli space.

There are then minimal changes to make; the majority of the work in [34] concerns only the sets $\mathcal{S}^{ss}_{A,d,\theta}(R)$ so needs no alteration. The moduli functor enters the results via [34, Proposition 4.5], which gives conditions for a family of functors $F^R : \mathcal{S}^{ss}_{B,d',\theta'}(R) \to \mathcal{S}^{ss}_{A,d,\theta}(R)$ to define a natural transformation between quiver representation moduli functors and shows that such a natural transformation induces a morphism of schemes between the quiver GIT quotients. A natural transformation of functors induces a morphism between corepresenting schemes by the universal property, and to adapt the conditions for a family to induce a natural transformation for the moduli functor with equivalence $\sim$ rather than $\sim_{SY}$ we need only add an additional condition to ensure that the natural transformation is well defined under the equivalence $\sim$:

$$\begin{aligned} &F^R(M \otimes_R L) \cong F^R(M) \otimes_R L \text{ for any invertible } R\text{-module } L \text{ and} \\ &M \in \mathcal{S}^{ss}_{A,d,\theta}(R). \end{aligned}$$

The only other results which involve the moduli functor are [34, Theorems 4.6 and 4.11] which check that the conditions of [34, Proposition 4.5] are satisfied by the specific functors $\mathrm{Hom}_{A^R}(T^R, -)$ and $T^R \otimes_{A^R} (-)$ when T has a finite length resolution by projective A-modules, and [34, Theorem 4.20] which combines these two results in the case where

T is a tilting module. It is easy to see that the functors $\mathrm{Hom}_{A^R}(T^R, -)$ and $T^R \otimes_{A^R} (-)$ also satisfy the additional condition: this follows from [34, Lemmas 4.7 and 4.14] in the case of an invertible R-module. As such the main result [34, Theorem 4.20] holds when the moduli functor is taken to be $\mathcal{F}^{ss}_{A,d,\theta}$ rather than $\mathcal{F}^{SY}_{A,d,\theta}$.

Proposition 7.1. ([34, Theorem 4.20]) *Let B be an algebra with tilting module T. Define $A = \mathrm{End}_B(T)^{\mathrm{op}}$, suppose that both A and B are presented as path algebras of quivers with relations, and let $\mathcal{F}^{ss}_{A,d,\theta}$ and $\mathcal{F}^{ss}_{B,d',\theta'}$ denote quiver representation moduli functors on A and B for some choice of dimension vectors d, d' and stability conditions θ, θ'. Then if the tilting equivalences*

$$\begin{tikzcd} D^b(B\text{-mod}) \arrow[r, bend left, "\mathbb{R}\mathrm{Hom}_B(T,-)"] & D^b(A\text{-mod}) \arrow[l, bend left, "T \otimes^{\mathbb{L}}_A (-)"] \end{tikzcd}$$

restrict to a bijection between $\mathcal{F}^{ss}_{B,d',\theta'}(\mathbb{C})$ and $\mathcal{F}^{ss}_{A,d,\theta}(\mathbb{C})$ then $\mathcal{F}^{ss}_{B,d',\theta'}$ is naturally isomorphic to $\mathcal{F}^{ss}_{A,d,\theta}$. Hence by the universal property of corepresenting schemes the corresponding quiver GIT quotients are isomorphic as schemes.

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