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On the space of projective curves of maximal regularity

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Abstract. Let $\Gamma_{r,d}$ be the space of smooth rational curves of degree d in $\mathbb{P}^r$ of maximal regularity. Then the automorphism group $\mathrm{Aut}(\mathbb{P}^r) = \mathrm{PGL}(r+1)$ acts naturally on $\Gamma_{r,d}$ and thus the quotient $\Gamma_{r,d}/\mathrm{PGL}(r+1)$ classifies those rational curves up to projective motions. In this paper, we show that $\Gamma_{r,d}$ is an irreducible variety of dimension $3d + r^2 - r - 1$. The main idea of the proof is to use the canonical form of rational curves of maximal regularity which is given by the $(d - r + 2)$-secant line. Also, through the geometric invariant theory, we discuss how to give a scheme structure on the $\mathrm{PGL}(r+1)$-orbits of rational curves.

1. Motivation and results

Rational curves in projective varieties have played useful roles in algebraic geometry. Their moduli spaces have been studied in the view point of birational geometry ([3–6]). The main purpose of this paper is to study the space which parameterizes projective curves with a fixed regularity condition. Due to Mumford ([19]), a nondegenerate irreducible projective curve $C \subset \mathbb{P}^r$ is said to be m-regular if its sheaf of ideal $\mathcal{I}_C$ satisfies the vanishing

$$H^i(\mathbb{P}^r, \mathcal{I}_C(m-i)) = 0 \quad \text{for all } i \geq 1.$$

The Castelnuovo–Mumford regularity (or simply the regularity) of C, denoted by $\mathrm{reg}(C)$, is defined as the least integer m such that C is m-regular. The regularity of curves (or more general, one dimensional subschemes) contained in a projective space gives an essential role for the construction of Hilbert scheme ([15, Theorem 1.5]). Another interest of this notion stems partly from the fact that C is m-regular if and only if for every $j \geq 0$ the minimal generators of the j-th syzygy module of the homogeneous ideal $I(C)$ of C occur in degree $\leq m + j$ ([7]). In

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particular, $I(C)$ is generated by forms of degree $\leq m$. In their fundamental paper ([10]) Gruson, Lazarsfeld and Peskine have shown that

$$\mathrm{reg}(C) \leq d - r + 2$$

where d denotes the degree of C. Nowadays, C is called a *curve of maximal regularity* if $\mathrm{reg}(C)$ takes the maximally possible value $d - r + 2$. In the same paper, it is also shown that if $d \geq r + 2$, then C is a curve of maximal regularity if and only if it is a smooth rational curve with a $(d - r + 2)$-secant line ([10, Theorem 3.1]), which we will call an *extremal secant line to* C. Later M. Brodmann and P. Schenzel investigate various algebraic properties of curves of maximal regularity by using their extremal secant lines ([1,2]). Recently, their results are partially extended to the next case by the second named author of the present paper ([16]).

In this paper, we study the set $R_{r,d}^{d-r+2}$ of all maximal regularity curves in $\mathbb{P}^r$ of degree d. There is a natural group action of the automorphism group $\mathrm{PGL}(r+1)$ of $\mathbb{P}^r$ on $R_{r,d}^{d-r+2}$. Moreover the set

$$\Lambda_{r,d}^{d-r+2} := R_{r,d}^{d-r+2}/\mathrm{PGL}(r+1)$$

classifies all maximal regularity curves of degree d in $\mathbb{P}^r$, up to projective equivalence. Recall that two projective subvarieties X and Y of $\mathbb{P}^r$ are said to be *projectively equivalent* if there exists an automorphism of $\mathbb{P}^r$ which maps X onto Y. To state our results precisely, we require some notation. Let $R_{r,d}$ be the set of all nondegenerate smooth rational curves of degree d in $\mathbb{P}^r$. A natural scheme structure on $R_{r,d}$ is obtained by regarding it as a subscheme of the Hilbert scheme $Hilb^{dn+1}(\mathbb{P}^r)$ of all subschemes with Hilbert polynomial $dn + 1$. It is easy to see that $R_{r,d}$ is a smooth quasi-projective variety of dimension $(r+1)(d+1) - 4$ (for details, see Sect. 3). For each $m \geq 3$, consider the subset

$$R_{r,d}^{m} := \{C \in R_{r,d} \mid \mathrm{reg}(C) \geq m\}$$

of $R_{r,d}$. Note that $R_{r,d}^m$ is locally closed since it is the locus of all members in $R_{r,d}$ satisfying the non-vanishing condition $H^1(\mathbb{P}^r, \mathcal{I}_C(m-2)) \neq 0$. Thus we obtain a stratification of the quasi-projective variety $R_{r,d}$ by its locally closed subsets:

$$Hilb^{dn+1}(\mathbb{P}^r) \supseteq R_{r,d} \supseteq R_{r,d}^2 \supseteq R_{r,d}^3 \supseteq R_{r,d}^4 \supseteq \cdots \supseteq R_{r,d}^{d-r+1} \supseteq R_{r,d}^{d-r+2}.$$

Furthermore, $\mathrm{PGL}(r+1)$ acts naturally on $R_{r,d}^m$ and the corresponding set

$$\Lambda_{r,d}^{m} := R_{r,d}^{m}/\mathrm{PGL}(r+1)$$

of orbits classifies all $(m-1)$-irregular nondegenerate smooth rational curves of degree d, up to projective equivalence.

The main result of this paper is the following:

Theorem 1.1. *Under the above assumption and notations,*

(1) *the variety* $R_{r,d}^{d-r+2}$ *is irreducible of dimension* $3d + r^2 - r - 1$ *and*

(2) *the stabilizer group* $\mathrm{Aut}(C, \mathbb{P}^r)$ *of a rational curve* $[C] \in R^{d-r+2}_{r,d}$ *is finite where the rational curve C meets the extremal secant line with at least four distinct points.*

Remark 1.2. (1) The first part of Theorem 1.1 tells us that the deepest strata $R^{d-r+2}_{r,d}$ achieves the explict minimal bound in [12, Lemma 2.4].
(2) The finiteness of the group $\mathrm{Aut}(C, \mathbb{P}^r)$ in Corollary 2.3 provides us that the quotient space $[R^{d-r+2}_{r,d}/\mathrm{PGL}(r+1)]$ exists as an algebraic stack and its dimension is $3(d-r)-1$ ([17, §11]).

For the proof of Theorem 1.1, see Theorem 3.5 and Corollary 2.3. Indeed, we provide a canonical form of curves of maximal regularity by using the $(d-r+2)$-secant line (see Sect. 2.1). This enables us to consider the geometric properties of the moduli space of such curves of maximal regularity.

To give a scheme structure on the space $\Lambda^{d-r+2}_{r,d}$, one can consider the geometric invariant theoretic (GIT) quotient of the Hilbert scheme $Hilb^{dn+1}(\mathbb{P}^r)$ (or the Chow variety) by the reductive group $\mathrm{PGL}(r+1)$ (For detail, see [18,20] and compare the Kapranov's definition of Chow quotient [13, Definition 0.1.7]). The main obstacle to do this is to check the stability of the rational curves with given regularity. As a clue, we show that each point in $R^{d-r+2}_{r,d}$ is the Chow linearly unstable (cf. [20, Theorem 4.12] and [18, Corollary 3.5]). For details, see Proposition 4.1. In Sect. 4, we discuss how to give a scheme structure on the set $\Lambda^{d-r+2}_{r,d}$ by using the principal of "reduction in stages" about the good quotients (cf. [13, §2.2] and [8, §1]). The canonical form of rational curves studied in Sect. 2 leads us to consider the parameter space $M = \mathbb{P}(\mathbb{C}^{r+1} \otimes Sym^d(\mathbb{C}^2))$ which parameterizes all of the rational curves in $\mathbb{P}^r$. For details, see Sect. 3. One can give natural commutative group actions on M by $\mathrm{PGL}(r+1)$ and $\mathrm{PGL}(2)$. So we take the quotient of M by $\mathrm{PGL}(r+1)$ firstly and $\mathrm{PGL}(2)$ secondly. Then, one can easily check that $M//\mathrm{PGL}(r+1) \cong Gr(r+1, d+1)$. By taking further the quotient on $Gr(r+1, d+1)$ by the group $\mathrm{PGL}(2)$, we obtain the GIT-quotient space $Gr(r+1, d+1)//\mathrm{PGL}(2)$. By using the numerical criterion ([21]), one can easily see that the general points in the Grassimanian are stable with respect to the action $\mathrm{PGL}(2)$. Specifically, our rational curves of maximal regularity are stable with some open condition. In this way, one can give a scheme structure on the set of the $\mathrm{PGL}(r+1)$ orbits in $\Lambda^m_{r,d}$ (Proposition 4.5). Throughout this paper we work over the complex number field $\mathbb{C}$.

2. Some results on curves of maximal regularity

For $r \geq 3$ and $d \geq r+2$, let $C \subset \mathbb{P}^r$ be a non-degenerate integral curve of degree d whose regularity $\mathrm{reg}(C)$ takes the maximal possible value $d-r+2$. In their fundamental paper (cf. [10]) Gruson, Lazarsfeld and Peskine have shown that

(i) C is a smooth rational curve and
(ii) C admits a $(d-r+2)$-secant line.

The aim of this section is to prove a few interesting geometric properties of C which are caused by the above two properties of C.

2.1. Uniqueness of extremal secant line

Let $C \subset \mathbb{P}^r$ be as above. We say that a line $\mathbb{L}$ in $\mathbb{P}^r$ is an *extremal secant line* to C if $\text{length}(C \cap \mathbb{L}) = d - r + 2$. It is known that C has only one such extremal secant line with one exceptional case. For lack of suitable references we give a brief proof here.

Proposition 2.1. *Either C has a unique extremal secant line or else $r = 3$ and C is contained in a smooth quadric surface whose divisor class is $(1, d-1)$.*

Proof. This is an immediate consequence of degree considerations. Suppose $r \geq 4$ and there are two such secant lines. They either span a $\mathbb{P}^3$ or they intersect. In the former case, the $\mathbb{P}^3$ they span intersects the curve with length at least $2d - 2r + 4$. We can take $r - 4$ additional points on the curve to span a $\mathbb{P}^{r-1}$. Since the curve is non-degenerate, we get the inequality $2d - r \leq d$. In other words, $d \leq r$, which contradicts $d \geq r + 2$. In the latter case, the two secant lines may intersect at a point of the curve, so the $\mathbb{P}^2$ intersects the curve with length at least $2d - 2r + 3$. Now we can choose $(r - 3)$ points to get a $\mathbb{P}^{r-1}$ and the same argument applies. If $r = 3$, and there are two $(d - 1)$ secant lines, the lines have to be skew since $d > 4$. Choose three additional points on the curve. There exists a quadric surface containing the two lines and the three points. By Bezóut, the quadric must contain the curve since it contains $2d + 1$ points of the curve. If the quadric surface is singular, then C must be arithmetically Cohen-Macaulay. But our C is non-linearly normal. This shows that the quadric surface is smooth. Furthermore, one can see that C has a class $(1, d-1)$ on the quadric, and hence there is a 1-parameter family of such secant lines. □

2.2. A canonical parametrization

Let $T := \mathbb{C}[s, t]$ be the homogeneous coordinate ring of $\mathbb{P}^1$. For each $k \geq 1$, we denote by T_k the k-th graded component of T. Since C is a rational curve, there exists a subset $\{f_0, f_1, \ldots, f_r\} \subset T_d$ of $\mathbb{C}$-linearly independent forms of degree d such that

$$C = \{[f_0(P) : f_1(P) : \cdots : f_r(P)] \mid P \in \mathbb{P}^1\}. \tag{2.1}$$

By using (ii) at the beginning of this section and an appropriate projective coordinate change of $\mathbb{P}^r$, we can simplify the parametrization in (2.1).

Proposition 2.2. *Let $C \subset \mathbb{P}^r$ and T be as above. Then there are forms $f_0, f_1 \in T_d$ and $f \in T_{d-r+2}$ such that C is projectively equivalent to the curve*

$$C' := \{[f_0(P) : f_1(P) : fs^{r-2}(P) : fs^{r-3}t(P) : \ldots : ft^{r-2}(P)] \mid P \in \mathbb{P}^1\} \subset \mathbb{P}^r. \tag{2.2}$$

Proof. The parametrization in the above (2.1) comes from the embedding

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^r, \;\; P \mapsto [f_0(P) : f_1(P) : \cdots : f_r(P)]$$

of $\mathbb{P}^1$ into $\mathbb{P}^r$. Now, let $S = \mathbb{C}[x_0, x_1, \dots, x_r]$ be the homogeneous coordinate ring of $\mathbb{P}^r$ and assume that $\mathbb{L} = \mathbb{V}(x_2, \dots, x_r)$ is an extremal secant line to C. Thus we have $\varphi^{-1}(C \cap \mathbb{L}) = \mathbb{V}(f_2, \dots, f_r)$. Since $\mathbb{L}$ is a $(d - r + 2)$-secant line to C, we can write

$$f_2 = fg_2, \ \dots, f_r = fg_r$$

for some $\mathbb{C}$-linearly independent forms $g_2, \dots, g_r \in T_{r-2}$. Obviously $\{g_2, \dots, g_r\}$ is a basis of T_{r-2}. Therefore, after an appropriate coordinate change of the subspace $\mathbb{V}(x_0, x_1) = \mathbb{P}^{r-2}$ of $\mathbb{P}^r$, we can choose $g_i = s^{r-i}t^{i-2}$ for all $2 \le i \le r$. □

Corollary 2.3. *Let $C \subset \mathbb{P}^r$ be as above which has a unique $(d - r + 2)$-secant line, say $\mathbb{L}$, and let $\Gamma = C \cap \mathbb{L}$. If the number $|\Gamma|$ of distinct points of Γ satisfies the condition $|\Gamma| \ge 4$, then* $\mathrm{Aut}(C, \mathbb{P}^r)$ *is a finite group.*

Proof. Let $\mathrm{Aut}(\Gamma, \mathbb{L})$ be the group of automorphisms of $\mathbb{L}$ which maps Γ onto itself. Our assumption $|\Gamma| \ge 4$ implies that the order of this group is finite. Now, observe that any element ψ in $\mathrm{Aut}(C, \mathbb{P}^r)$ maps $\mathbb{L}$ and Γ respectively onto themselves. This follows from the uniqueness of the extremal secant line $\mathbb{L}$ of C. Thus we have a natural map $\pi : \mathrm{Aut}(C, \mathbb{P}^r) \to \mathrm{Aut}(\Gamma, \mathbb{L})$. Finally, this map π is injective since $\mathrm{Aut}(C, \mathbb{P}^r)$ is a subgroup of $\mathrm{Aut}(C) = \mathrm{PGL}(2)$ (because of the non-degeneracy of C) and any element in $\mathrm{PGL}(2)$ is uniquely determined by its action on three points of C. □

2.3. Construction of the maximal regularity curve

Recall that $T = \mathbb{C}[s, t]$ is the homogeneous coordinate ring of $\mathbb{P}^1$ and T_k denotes its k-th component of T.

Definition 2.4. Let $r \ge 3$ and $d \ge r + 2$. We define a nonempty open subset $\mathcal{U}_{r,d}$ of $T_d \times T_d \times T_{d-r+2}$ as

$$\mathcal{U}_{r,d} := \{(f_0, f_1, f) \mid f_0, f_1 \text{ and } f \text{ are all nonzero}\}.$$

Also for each $(f_0, f_1, f) \in \mathcal{U}_{r,d}$, we define $C(f_0, f_1, f) \subset \mathbb{P}^r$ as the parameterized curve

$$C(f_0, f_1, f) := \overline{\{[f_0(P) : f_1(P) : f_2(P) : \cdots : f_r(P)] \mid P \in \mathbb{P}^1 \setminus \mathbb{V}(f_0, f_1, f)\}}$$

where $f_i(s, t) = s^{r-i}t^{i-2}f(s, t)$ for all $2 \le i \le r$.

Proposition 2.2 says that any curve of maximal regularity is projectively equivalent to $C(f_0, f_1, f)$ for some $(f_0, f_1, f) \in \mathcal{U}_{r,d}$. Conversely, it is natural to ask for a criterion determining whether the curve $C(f_0, f_1, f)$ coming from an element $(f_0, f_1, f) \in \mathcal{U}_{r,d}$ is a curve of maximal regularity or not. In this vein, we have

Proposition 2.5. *For an element $(f_0, f_1, f) \in \mathcal{U}_{r,d}$, let $\Gamma := \{P_1, \dots, P_\ell\}$ be the set of all distinct zeros of f on $\mathbb{P}^1$. Then the following three conditions are equivalent:*

(α) $\{f_0, f_1, fs^{r-2}, fs^{r-3}t, \ldots, ft^{r-2}\}$ *spans an* r*-dimensional very ample linear system on* $\mathbb{P}^1$*;*

(β) $C(f_0, f_1, f) \subset \mathbb{P}^r$ *is a nondegenerate curve with* $\text{reg}(C(f_0, f_1, f)) = d - r + 2$*;*

(γ) the following two conditions hold:

(a) *The determinant of every* (2×2)*-minor of the matrix*

$$A := \begin{bmatrix} f_0(P_1) & f_0(P_2) \cdots & f_0(P_l) \\ f_1(P_1) & f_1(P_2) \cdots & f_1(P_l) \end{bmatrix}$$

is nonzero;

(b) $\text{Ord}_P(f_1(P)f_0 - f_0(P)f_1) = 1$ *for all* $P \in \Gamma$ *with* $\text{Ord}_P(f) \geq 2$.

Proof. Let $\varphi : \mathbb{P}^1 \dashrightarrow C(f_0, f_1, f)$ be the rational map defined by sending P to the point $[f_0(P) : f_1(P) : f_2(P) : \cdots : f_r(P)]$ for all $P \in \mathbb{P}^1 \setminus \mathbb{V}(f_0, f_1, f)$. Also, let V be the linear system on $\mathbb{P}^1$ spanned by $\{f_0, f_1, \ldots, f_r\}$ where $f_i = fs^{r-i}t^{i-2}$ for $2 \leq i \leq r$.

$(\alpha) \iff (\beta)$: Condition (α) means that φ is an isomorphism and hence $C(f_0, f_1, f)$ is a nondegenerate curve of degree d in $\mathbb{P}^r$. Furthermore, for the line $\mathbb{L} = \mathbb{V}(x_2, \ldots, x_r)$ we have $\varphi^{-1}(C(f_0, f_1, f) \cap \mathbb{L}) = \text{Proj}(T/\langle f \rangle)$ which is a finite scheme of length $d - r + 2$. Therefore $\mathbb{L}$ is a $(d - r + 2)$-secant line to $C(f_0, f_1, f)$. This completes the proof that $\text{reg}(C(f_0, f_1, f)) = d - r + 2$ by Theorem 3.1 in [10]. Conversely, condition (β) implies that $C(f_0, f_1, f)$ is a smooth curve of degree d and hence (α) is true by Proposition 2.2.

$(\alpha) \iff (\gamma)$: First we see that if condition (γ) holds then $\{f_0, f_1, \ldots, f_r\}$ is linearly independent. Indeed, suppose that $u_0 f_0 + u_1 f_1 + u_2 f_2 + \cdots + u_r f_r = 0$ for some $u_0, \ldots, u_r \in \mathbb{C}$. If $|\Gamma| \geq 2$ and, u_0 or u_1 is non-zero then one can easily see that the rank of A is strictly less than 2. If $|\Gamma| = 1$ and, u_0 or u_1 is non-zero then

$$\text{Ord}_P(f_1(P)f_0 - f_0(P)f_1) = \text{Ord}_P(f) \geq 2 \quad \text{for } P \in \Gamma.$$

Thus $u_0 = u_1 = 0$ and hence $u_2 = \ldots = u_r = 0$. Now we show that the conditions (a) and (b) are respectively equivalent to the two conditions in [11, Remark 7.8.2,II], in turn. First suppose that $|\Gamma| \geq 2$ and take two distinct points $P, Q \in \mathbb{P}^1$. If at least one of them is outside of $\mathbb{V}(f)$, then $\langle f_0, f_1, \ldots, f_r \rangle$ separates P and Q since it includes $\{fs^{r-2}, fs^{r-3}t, \ldots, ft^{r-2}\}$. On the other hand, if $P, Q \in \mathbb{V}(f)$ then the two points $\varphi(P) = [f_0(P) : f_1(P) : 0 : \ldots : 0]$ and $\varphi(Q) = [f_0(Q) : f_1(Q) : 0 : \ldots : 0]$ are different if and only if the determinant of the submatrix $\begin{bmatrix} f_0(P) & f_0(Q) \\ f_1(P) & f_1(Q) \end{bmatrix}$ of A is nonzero. Thus we conclude that the condition (1) in [11, Remark 7.8.2] is equivalent to the (a). If $|\Gamma| = 1$, then the condition (1) in [11, Remark 7.8.2] holds automatically. Now it remains to show that (b) is equivalent to the condition (2) in [11, Remark 7.8.2]. Let $P \in \mathbb{P}^1$. If $P \notin \mathbb{V}(f)$ (resp. $P \in \mathbb{V}(f)$ and $\text{Ord}_P(f) = 1$), then there exists an element $h \in T_{r-2}$ such that $\text{Ord}_P(h) = 1$ (resp. $\text{Ord}_P(h) = 0$) and hence for each case we have $\text{Ord}_p(fh) = 1$. Finally, if $P \in \mathbb{V}(f)$ and $\text{Ord}_P(f) \geq 2$ then $\langle f_0, f_1, \ldots, f_r \rangle$ separates tangent vectors at P if and only if

$$\text{Ord}_P(f_1(P)f_0 - f_0(P)f_1) = 1.$$

Thus (b) is exactly the condition that $\langle f_0, f_1, \ldots, f_r \rangle$ separates tangent vectors. □

Example 2.6. Let $\{[0:1], [1:1], [2:1], [1:2], [3:1]\}$ be the set of the secant points which are determined by the form

$$f = (s)(s-t)(s-2t)(2s-t)(s-3t).$$

Also, let us choose f_0, f_1 such that all of the determinants of (2×2)-minors of the matrix A in Proposition 2.5.(γ) are not zero. For example,

$$f_0 = s^7 + st^6, \ \ f_1 = s^6 t + t^7.$$

Then, by the computer algebra system “Singular”[9], one can easily check that the canonical curve $C = C(f_0, f_1, f) \subset \mathbb{P}^4$ has the maximal regularity $\mathrm{reg}(C) = 5$ as follows:

i	$\beta_{1,i}$	$\beta_{2,i}$	$\beta_{3,i}$	$\beta_{4,i}$
1	3	2	0	0
2	1	0	0	0
3	1	6	5	1
4	1	3	3	1

3. Parameter space of smooth rational curves

Suppose that $r \geq 3$ and $d \geq r+2$. Let us define

$$M = \mathbb{P}(\mathbb{C}^{r+1} \otimes Sym^d(\mathbb{C}^2))$$

be the space of $(d+1)\times(r+1)$ matrices up to the scalar multiplication. An element $\alpha \in M$ is represented by the $(r+1)$-tuples of homogenuous bivariate polynomials of degree d. Also the reductive groups $\mathrm{PGL}(r+1)$ and $\mathrm{PGL}(2)$ naturally act on M in the obvious fashion. From the canonical evaluation map

$$Sym^d(\mathbb{C}^2)^* \otimes (\mathbb{C}^{r+1} \otimes Sym^d(\mathbb{C}^2)) \longrightarrow \mathbb{C}^{r+1}$$

we obtain a family of rational maps parameterized by M into the projective space $\mathbb{P}^r$

$$\begin{array}{c} \phi : M \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^r \\ \downarrow \\ M. \end{array} \tag{3.1}$$

The rational map ϕ is well-defined morphism from $\mathbb{P}^1$ whenever the $(r+1)$-tuples of homogenuous polynomials is base point free. Let M^{ns} be the locus of $(r+1)$-tuples α such that the rational map $\phi_\alpha : \mathbb{P}^1 \to \mathbb{P}^r$ is a non-degenerate closed embedding into $\mathbb{P}^r$. Since all of the conditions are open, one can easily see that the locus M^{ns} is a smooth quasi-projective variety of dimension $(r+1)(d+1)-1$.

On the other hand, the PGL(2)-quotient

$$R_{r,d} := M^{ns}//\mathrm{PGL}(2)$$

parameterizes all nondegenrate smooth rational curves of degree d in $\mathbb{P}^r$. Note that every point in M^{ns} is stable with respect to the PGL(2)-action ([14, §5]). The space $R_{r,d}$ is naturally embedded into the Hilbert scheme

$$i : R_{r,d} \hookrightarrow \mathrm{Hilb}^{dn+1}(\mathbb{P}^r),$$

defined by $[\phi_\alpha : \mathbb{P}^1 \to \mathbb{P}^r] \mapsto C = \phi_\alpha(\mathbb{P}^1) \subset \mathbb{P}^r$. Obviously the map i is an injective map. Furthermore, $T_C R_{r,d} = H^0(T_{\mathbb{P}^r}|_C)/H^0(T_C) \cong H^0(C, N_{C/\mathbb{P}^r}) \cong \mathrm{Hom}(I_C, \mathcal{O}_C) = T_C \mathrm{Hilb}^{dn+1}(\mathbb{P}^r)$. Hence the map i is an embedding. For each m, $3 \le m \le d - r + 2$, let $R^m_{r,d} := \{C \in R_{r,d} \mid \mathrm{reg}(C) \ge m\}$. Then, by the upper semi-continuity theorem ([11]), there exist a locally closed stratification of $R_{r,d}$ as follow:

$$R_{r,d} \supseteq R^2_{r,d} \supseteq R^3_{r,d} \supseteq R^4_{r,d} \supseteq \cdots \supseteq R^{d-r+1}_{r,d} \supseteq R^{d-r+2}_{r,d}.$$

The automorphism group $\mathrm{Aut}(\mathbb{P}^r) \cong \mathrm{PGL}(r+1)$ acts naturally on M and thus on each $R^m_{r,d}$. Let

$$\Lambda^m_{r,d} := R^m_{d,r}/\mathrm{Aut}(\mathbb{P}^r)$$

be the set of orbits of this action. Then $\Lambda^m_{d,r}$ is a set which classifies all $(m-1)$-irregular nondegenerate smooth rational curves, up to projective equivalence. In this paper, we focus on the deepest orbit space $\Lambda^{d-r+2}_{r,d}$. In this section, we will show that the orbit space $R^{d-r+2}_{r,d}$ is irreducible and pure dimensional. In Sect. 4, we will discuss how to give the scheme structure by using the *reduction in a stage* with the group actions. To do this, let us start by describing the locus of matrices in M for which curve represented by the matrix is of maximal regularity. Let us recall that the space $\mathcal{U}_{r,d}$ is the set of triples $(f_0, f_1, f) \in T_d \times T_d \times T_{d-r+2}$ such that f_0, f_1 and f are all nonzero. From Definition 2.4, we obtain a map

$$\varphi : \mathcal{U}_{r,d} \longrightarrow M, \quad (f_0, f_1, f) \mapsto (f_0(s,t), f_1(s,t), f_2(s,t), \ldots, f_r(s,t))$$

where $f_i(s,t) = f(s,t)s^{r-i}t^{i-2}$ for all $2 \le i \le r$. Recall that every curve in $\mathbb{P}^r$ of maximal regularity is projectively equivalent to a canonical curve $C(f_0, f_1, f)$ for some $(f_0, f_1, f) \in \mathcal{U}_{r,d}$ (Proposition 2.2).

Lemma 3.1. *Let $\mathcal{W}_{r,d}$ be the sub-locus of the triples $(f_0, f_1, f) \in \mathcal{U}_{r,d}$ such that the canonical curve $C(f_0, f_1, f)$ is of maximal regularity. The variety $\mathcal{W}_{r,d}$ is a non-empty open subset of the affine space $T_d \times T_d \times T_{d-r+2} \cong \mathbb{C}^{3d-r+5}$.*

Proof. Let us consider the map

$$\varphi : \mathcal{U}_{r,d} \longrightarrow M$$

and the image $(f_0(s,t), f_1(s,t), f_2(s,t), \ldots, f_r(s,t))$ of (f_0, f_1, f). Then the condition that the canonical curve $C(f_0, f_1, f)$ corresponds to (f_0, f_1, f) is of maximal regularity is equivalent to the condition that $\{f_0(s,t), f_1(s,t), f_2(s,t), \ldots, f_r(s,t)\}$ spans an r-dimensional very ample linear system on $\mathbb{P}^1$ (Proposition 2.5) and hence the image $(f_0(s,t), f_1(s,t), f_2(s,t), \ldots, f_r(s,t))$ is an element of M^{ns}. This guarantees that the inverse image $\varphi^{-1}(M^{ns})$ of the open subset $M^{ns} \subset M$ is exactly $\mathcal{W}_{r,d}$. □

3.1. Irreducibility of $R^{d-r+2}_{r,d}$

In this subsection, we prove the irreducibility of the space $\Gamma_{r,d} := R^{d-r+2}_{r,d}$. The key idea is to use the uniqueness of $(d-r+2)$-secant line. To do this we closely investigate the fiber of the natural morphism $\Gamma_{r,d} \longrightarrow Gr(2, r+1)$, so called *the secant map*, which associates curve to its secant line.

Proposition 3.2. *Let $\Gamma_{r,d} \subset Hilb^{dn+1}(\mathbb{P}^r)$ be the locus of the smooth rational curves of maximal regularity. Then there exists a surjective morphism*

$$\Gamma_{r,d} \longrightarrow Gr(2, r+1)$$

associating the curve C to its secant line L.

Proof. Let $W = \pi^{-1}(\Gamma_{r,d})$ be the inverse image of $\Gamma_{r,d}$ where $\pi : M^{ns} \longrightarrow M^{ns}//\mathrm{PGL}(2) \subset Hilb^{dn+1}(\mathbb{P}^r)$ is the quotient map. From the diagram (3.1), there exists a flat family of degree d closed embedding map

$$\begin{array}{c} \phi : W \times \mathbb{P}^1 \longrightarrow \mathbb{P}^r \\ \downarrow \\ W \end{array}$$

parameterized by W. The construction of the secant line in [10, Theorem 3.1] can be relativized on W. Note that, since our curves are smooth, we do not need to normalize the rational curve. Let M' be the kernel of the sheaf homomorphism $\mathcal{O}^{\oplus r+1}_{W\times\mathbb{P}^1} \twoheadrightarrow \mathcal{O}_{W\times\mathbb{P}^1}(1, d)$ defined by the map ϕ. Let $A = q^*\mathcal{O}_{\mathbb{P}^1}(d-r)$ be the pull-back where $q : W \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection. Then the sheaf homomorphism v in the proof of [10, Theorem 3.1] is defined on $(\mathbb{P}^1 \times W) \times \mathbb{P}^r$. Let us denote the cokernel of v by $\mathcal{F}_1$. By taking the push-forward of the sheaf homomorphism v into the factor $W \times \mathbb{P}^r$, one defines the higher direct image sheaf $\mathcal{G} := R^1(p_{23})_*\mathcal{F}_1$ on $W \times \mathbb{P}^r$. Here $p_{23} : (\mathbb{P}^1 \times W) \times \mathbb{P}^r \to W \times \mathbb{P}^r$ is the projection map. Since the support of the sheaf $\mathcal{G}_w$ for each $w \in W$ is a line in $\mathbb{P}^r$ ([10, Theorem 3.1]), there exists a natural morphism

$$W \longrightarrow Gr(2, r+1).$$

Clearly, this map is PGL(2)-invariant and thus it descents to the quotient

$$W//\mathrm{PGL}(2) = \Gamma_{r,d} \longrightarrow Gr(2, r+1).$$

□

Remark 3.3. In the proof of the above proposition, the locus W consists of the $(r+1)$-tuples of homogenous polynomials of degree d in the matrix space $M = \mathbb{P}(\mathbb{C}^{r+1} \otimes Sym^d(\mathbb{C}^2))$ where the representing curves have the maximal regularity.

Lemma 3.4. *The locus $W \subset M$ is an irreducible variety of dimension $3d + r^2 - r + 2$.*

Proof. Because of the technical reason, we do the work on the affine setting. Let $V = \mathbb{C}^{r+1} \otimes Sym^d(C^2) \setminus \{0\}$. Let $j : V \longrightarrow V/\mathbb{C}^* := M$ be the canonical $\mathbb{C}^*$-quotient map. Now let $\widetilde{W} = j^{-1}(W)$ be the inverse image of W. Let us denote the composition map by

$$\kappa = \pi \circ j : \widetilde{W} \longrightarrow Gr(2, r+1),$$

where $\pi : W \to Gr(2, r+1)$ is the secant map. Obviously the map κ is surjective. Let $L \in Gr(2, r+1)$ be a line. We show that the fiber is isomorphic to

$$\kappa^{-1}(L) \cong (\mathcal{W}_{r,d} \times GL(r-1))/\mathbb{C}^*,$$

where the group $\mathbb{C}^*$ diagonally acts on $\mathcal{W}_{r,d} \times GL(r-1)$ by $(f_0, f_1, f) \times (g) \mapsto (f_0, f_1, \alpha f) \times (\alpha^{-1} g)$. Here $\mathcal{W}_{r,d}$ is the set of the triples of (f_0, f_1, f) such that the canonical curve $C(f_0, f_1, f)$ is of maximal regularity (see Lemma 3.1). In detail, let us define the map $\phi : \mathcal{W}_{r,d} \times GL(r-1) \to \kappa^{-1}(L) \subset W$ by $(f_0, f_1, f) \times (g) \mapsto (f_0, f_1, f \cdot g)$. If $\phi((f_0, f_1, f) \times (g)) = \phi((f_0', f_1', f') \times (g'))$, then $f_0 = f_0'$, $f_1 = f_1'$ and $f \cdot g = f' \cdot g'$. But if $f \neq kf'$ for some $k \in \mathbb{C}^*$, then all g_i' has the non-constant common factor $f/gcd(f, f')$. This contradicts the choice of $g' \in GL(r-1)$. So the map $\phi : \mathcal{W}_{r,d} \times GL(r-1) \to \kappa^{-1}(L)$ factors through the quotient $(\mathcal{W}_{r,d} \times GL(r-1))/\mathbb{C}^*$. Obviously the desecent map $\overline{\phi}$ is injective. Furthermore the map $\overline{\phi}$ is an embedding. To see this, we construct an local inverse map of $\overline{\phi}$. For an element $(f_0, f_1, \alpha_1, \alpha_2, \cdots, \alpha_{r-1}) \in W$, we know that $\{\alpha_i\}$ has the common factor f for all i. Without loss of generality, one can assume that the coefficient of s^{r-2} in α_1/f is 1. For this one we have the inverse map $(f_0, f_1, \alpha_1, \alpha_2, \cdots, \alpha_{r-1}) \mapsto (f_0, f_1, f) \times (\alpha_i/f)$. Hence, W is irreducible and

$$\begin{aligned} \dim(W) &= \dim(Gr(2, r+1)) + \dim(\mathcal{W}_{r,d} \times GL(r-1)/\mathbb{C}^*) - \dim(\mathbb{C}^*) \\ &= 2(r-1) + (3d - r + 5) + (r-1)^2 - 1 - 1 = 3d + r^2 - r + 2. \end{aligned}$$

□

Theorem 3.5. *The space* $\Gamma_{r,d}$ *is an irreducible variety of dimension* $3d + r^2 - r - 1$.

Proof. Since $\Gamma_{r,d} = W//\mathrm{PGL}(2)$ and the stabilizer of any point in W is trivial with respect to the action $\mathrm{PGL}(2)$ ([14]), we obtain the result by Lemma 3.4. □

3.2. Space curves of maximal regularity

When $r = 3$, the space $\Gamma_{3,d}$ consists of two types of rational curves as follows.

(1) **Type 1:** $\dim H^0(I_C(2)) = 0$, there is a unique line $(d-1)$-secant $\mathbb{L}_{d-1}$. Or else,
(2) **Type 2:** $\dim H^0(I_C(2)) = 1$, rational curves in the rational normal surface scroll $S(1, 1)$ with numerical type $H + (d-2)F$, where H and F are respectively the hyperplane divisor and a ruling of $S(1, 1)$. In this case, all ruling lines are the $(d-1)$-secant lines to C.

Proposition 3.6. *The space $\Gamma_{3,d}$ is an irreducible variety.*

Proof. Using the same argument in the proof of Proposition 3.2, one can show that the locus of rational curves consisting of Type 1 is irreducible. Also, the locus of curves in Type 2 is a boundary of Type 1. Without loss of generality, one can assume that the rational curve defined by $[g(s,t)s : g(s,t)t : f(s,t)s : f(s,t)t] \subset \mathbb{P}^3$ be a general point of Type 2 where $g = g(s,t)$, $f = f(s,t) \in T_{d-1}$. Let us consider a flat family of rational curves

$$\mathcal{C}_\alpha := \{[gs + \alpha ft, gt, fs, ft] \,|\} \subset \mathbb{C} \times \mathbb{P}^3 \to \mathbb{C}$$

parameterized by $\alpha \in \mathbb{C}$. For each $\alpha \in \mathbb{C}^*$, one can easily check that the curve $\mathcal{C}_\alpha$ is of Type 1 by using Proposition 2.5. So the claim is proved. □

4. Toward GIT-quotient of the space of maximal regularity curves

To find the underlying parameter space of the PGL$(r+1)$-orbits in $\Gamma_{r,d}$, one can consider the Chow (or Hilbert) stability in the sense of [20]. But it seems too hard to check the stability of the general points in $\Gamma_{r,d}$. In fact,

Proposition 4.1. *Any curve parameterized by the closed points in $\Gamma_{r,d}$ is linearly unstable.*

Proof. Let $C \subset \mathbb{P}^r$ be a rational curve of maximal regularity $d - r + 2$ with $(d-r+2)$-secant line $\mathbb{L}$. Now consider the projection $\pi_{\mathbb{L}} : \mathbb{P}^r \setminus \mathbb{L} \to \mathbb{P}^{r-2}$ and let $C' := \overline{\pi_{\mathbb{L}}(C \setminus (C \cap \mathbb{L}))}$. Then $C' \subset \mathbb{P}^{r-2}$ is a rational normal curve of degree $r-2$ and hence $\pi_{\mathbb{L}}$ extends to a unique isomorphism $\pi' : C \to C'$. Thus one can see that

$$\frac{d}{r} = \mathrm{reddeg}(C) > \mathrm{reddeg}(\pi_{\mathbb{L}}(C)) = \mathrm{reddeg}(C') = 1.$$

Hence, the curve C is linearly unstable. For details, we refer to the reader to [20, Definition 2.16]. □

This implies that one can not directly apply Theorem 4.12 in [20]. So, in this section, we will discuss a method to give a scheme structure on the set of the orbits by using the principle of “reduction in stages". Let us denote X^{ss} (resp. X^s) by the semistable (resp. stable) locus of the reductive group action G on a projective variety X. The group PGL$(r+1)$ and PGL(2) commutatively acts on the space $M = \mathbb{P}(\mathbb{C}^{r+1} \otimes Sym^d(\mathbb{C}^2))$ as a obvious fashion.

Lemma 4.2. *Under the above assumption, there is a natural isomorphism*

$$M^{ss}//\big(\mathrm{PGL}(r+1) \times \mathrm{PGL}(2)\big) \cong (M//\mathrm{PGL}(r+1))^{ss}//\mathrm{PGL}(2).$$

Proof. It suffice to show that, if $\alpha \in M$ is semi-stable with respect to the product group PGL$(r+1) \times$ PGL(2), then α is an injective map. Let us apply the Hilbert-Mumford criterion ([21]). If α is not injective, then one can assume that the first row is zero, then let us choose the 1-parameter family $\lambda : t \mapsto (\mathrm{diag}(1, t, t, \ldots, t), \mathrm{Id}) \in \mathrm{PGL}(r+1) \times \mathrm{PGL}(2)$. Then $\lim \lambda(t) \cdot \alpha = 0$ and so M is not semistable. □

From this lemma, to study the semistable points in M with respect to the reductive group $\mathrm{PGL}(r+1) \times \mathrm{PGL}(2)$ it is enough to study the semistable points in the Grassmannian variety

$$\mathbb{P}(\mathbb{C}^{r+1} \otimes Sym^d(\mathbb{C}^2))//\mathrm{PGL}(r+1) \cong Gr(r+1, Sym^d(\mathbb{C}^2)).$$

By Proposition 2.5, each point in W is stable with respect to the group action $\mathrm{PGL}(r+1)$ and W is invariant under the $\mathrm{PGL}(r+1)$-action. Hence the GIT-quotient $\bar{W} := W//\mathrm{PGL}(r+1)$ exists as a variety. Now, by applying the Hilbert-Mumford criterion ([21]), we will prove that some open subvariety of $\bar{W}$ is stable by the action $\mathrm{PGL}(2)$. In Proposition 3.1, for a canonical form $(f_0, f_1, f) \in \mathcal{W}_{r,d}$, let us define

$$\alpha := (f_0(s,t), f_1(s,t), f_2(s,t), \ldots, f_r(s,t))$$

where $f_i(s,t) = f(s,t)s^{r-i}t^{i-2}$ for all $2 \le i \le r$. Note $f_0, f_1, \ldots, f_r$ are $\mathbb{C}$-linearly independent, $\alpha \in M^{ss}$. Let us denote $[\alpha] \in \bar{W}$ by its image of the quotient map $W \to W//\mathrm{PGL}(r+1)$.

Lemma 4.3. *Under the above notation, for the general* $[\alpha] \in \bar{W} \subset Gr(r+1, Sym^d(\mathbb{C}^2))$ $[\alpha]$ *is stable with respect to the group action* $\mathrm{PGL}(2)$.

Proof. Let us apply the Hilbert-Mumford criterion ([21]). Under the Plücker embedding $Gr(r+1, Sym^d(C^2)) \subset \mathbb{P}^N$ into the projective space, let $\{\begin{bmatrix} u^a & 0 \\ 0 & u^{-a} \end{bmatrix} | u \in \mathbb{C}^*\}$ be a maximal torus in $\mathrm{PGL}(2) \cong SL(2)$. Then the weights are given by the determinants of $(r+1) \times (r+1)$-minors of

$$[\alpha] = \begin{bmatrix} a_0 & a_1 & \cdots & a_{d-r+2} & a_{d-r+3} & \cdots & a_d \\ b_0 & b_1 & \cdots & b_{d-r+2} & b_{d-r+3} & \cdots & b_d \\ c_0 & c_1 & \cdots & c_{d-r+2} & 0 & \cdots & 0 \\ 0 & c_0 & \cdots & c_{d-r+1} & c_{d-r+2} & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & c_0 & c_1 & \cdots & c_{d-r+2} \end{bmatrix},$$

where $f_0(s,t) = \sum_{i=0}^{d} a_i s^{d-i} t^i$, $f_1(s,t) = \sum_{i=0}^{d} b_i s^{d-i} t^i$, and $f(s,t) = \sum_{i=0}^{d-r+2} c_i s^{d-r+2-i} t^i$. Whenever we choose the points $[\alpha]$ such that the first and last maximal minor of the matrix are not zero, then the weights has the positive/negative one and thus the point $[\alpha]$ is stable. □

Example 4.4. Let us consider the curve in Example 2.6. As a point in $\bar{W}$,

$$[\alpha] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & -13 & 28 & -23 & 6 & 0 & 0 & 0 \\ 0 & 2 & -13 & 28 & -23 & 6 & 0 & 0 \\ 0 & 0 & 2 & -13 & 28 & -23 & 6 & 0 \end{bmatrix}.$$

Since both of the first and the last maximal minors of the matrix are not zero, $[\alpha]$ is a stable point in $\bar{W}$.

Let $W^{gen} \subset \bar{W}$ be the sub-locus satisfying the conditions in the proof of Lemma 4.3.

Proposition 4.5. *The GIT-quotient* $\bar{W}//\mathrm{PGL}(2)$ *is an irreducible,* $3(d-r)-1$*-dimensional variety.*

Proof. Irreducibility of the quotient space $\bar{W}//\mathrm{PGL}(2)$ directly comes from that of W (see Proposition 3.4). Also, the locus W^{gen} is an open subset of the stable locus $\bar{W}^s$. The dimension is given by

$$\dim(\bar{W}//\mathrm{PGL}(2)) = \dim(W) - \dim(\mathrm{PGL}(r+1)) - \dim(\mathrm{PGL}(2)) = 3(d-r) - 1.$$

□

Remark 4.6. Like other geometric invariant theoretic quotient, it seems to be a very interesting problem to determine for which points in $\bar{W}$ are (semi)stable.

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