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Gradient estimates and the fundamental solution for higher-order elliptic systems with rough coefficients

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Abstract. This paper considers the theory of higher-order divergence-form elliptic differential equations. In particular, we provide new generalizations of several well-known tools from the theory of second-order equations. These tools are the Caccioppoli inequality, Meyers's reverse Hölder inequality for gradients, and the fundamental solution. Our construction of the fundamental solution may also be of interest in the theory of second-order operators, as we impose no regularity assumptions on our elliptic operator beyond ellipticity and boundedness of coefficients.

1. Introduction

In this paper we will study divergence-form elliptic operators L of order $2m$, given formally by

$$(L\mathbf{u})_j = (-1)^m \sum_{k=1}^{N} \sum_{|\alpha|=m} \sum_{|\beta|=m} \partial^\alpha \left(A^{jk}_{\alpha\beta} \partial^\beta u_k \right)$$

and in particular systems of equations of the form

$$(L\mathbf{u})_j = (-1)^m \sum_{|\alpha|=m} \partial^\alpha F_{j,\alpha}.$$

(We will write this system of equations as $L\mathbf{u} = \operatorname{div}_m \dot{\boldsymbol{F}}$.)

The theory of second-order operators, that is, operators with $m = 1$, has a long and celebrated history. Important tools in the theory of second-order elliptic systems include the Caccioppoli inequality, Meyers's reverse Hölder inequality for derivatives, and the fundamental solution.

The boundary Caccioppoli inequality states that, if $L\mathbf{u} = \operatorname{div} \dot{\boldsymbol{F}}$ in some domain Ω for some second-order elliptic operator L, and if either $\mathbf{u} = 0$ or $\nu \cdot \boldsymbol{A}\nabla\mathbf{u} = 0$ on $\partial\Omega \cap B(x_0, r)$, where ν is the unit outward normal vector, then the gradient of $\mathbf{u}$ may be controlled by $\mathbf{u}$ and the inhomogeneous term $\dot{\boldsymbol{F}}$, as

$$\int_{B(x_0,r)\cap\Omega} |\nabla\mathbf{u}|^2 \leq \frac{C}{r^2} \int_{B(x_0,2r)\cap\Omega} |\mathbf{u}|^2 + C \int_{B(x_0,2r)\cap\Omega} |\dot{\boldsymbol{F}}|^2. \tag{1}$$

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Meyers's reverse Hölder estimate (see [34]) states that, if $L\mathbf{u} = \operatorname{div} \dot{\boldsymbol{F}}$ in some ball $B(x_0, 2r)$, then $\nabla\mathbf{u}$ satisfies the reverse Hölder estimate

$$\left(\fint_{B(x_0,r)} |\nabla\mathbf{u}|^p\right)^{1/p} \leq \frac{C}{r^{d/2-d/p}}\left(\fint_{B(x_0,2r)} |\nabla\mathbf{u}|^2\right)^{1/2} + C\left(\fint_{B(x_0,2r)} |\dot{\boldsymbol{F}}|^p\right)^{1/p} \tag{2}$$

for some $p > 2$ depending only on the operator L. With some care, Meyers's estimate may also be extended to the boundary case, at least in relatively nice domains. Both of these inequalities have been used extensively in the literature.

Much less is known in the case of higher-order elliptic systems in the rough setting. In the case of continuous coefficients and C^m domains, some regularity results are available; see [1]. In the interior case the Caccioppoli inequality

$$\int_{B(x_0,r)} |\nabla^m\mathbf{u}|^2 \leq \sum_{j=0}^{m-1} \frac{C}{r^{2m-2j}} \int_{B(x_0,2r)} |\nabla^j\mathbf{u}|^2 + C\int_{B(x_0,2r)} |\dot{\boldsymbol{F}}|^2 \tag{3}$$

was established in [9] for general bounded and strongly elliptic coefficients. It would of course be preferable to establish this bound with only a norm of $\mathbf{u}$, and not of $\nabla^j\mathbf{u}$, on the right-hand side. In [5], the bound

$$\int_{B(x_0,r)} |\nabla^m\mathbf{u}|^2 \leq \frac{C(\varepsilon)}{r^{2m}} \int_{B(x_0,2r)} |\mathbf{u}|^2 + \varepsilon\int_{B(x_0,2r)} |\nabla^m\mathbf{u}|^2 \tag{4}$$

was established for solutions $\mathbf{u}$ to the equation $L\mathbf{u} = 0$ in $B(x_0, 2r)$, where ε is an arbitrary positive number and $C(\varepsilon)$ a constant depending on ε. Either of the bounds (3) or (4) suffices to generalize Meyers's estimate (2) to the higher-order case, and in fact this was done in both [5,9].

The boundary Caccioppoli inequality in the case of rough domains has not been established; we mention that some pointwise estimates were established in [30,31] in the case where $L = \Delta^2$ is the biharmonic operator.

In Sect. 3, we will establish the higher-order Caccioppoli inequality with no terms involving derivatives of $\mathbf{u}$ on the right-hand side; we will also establish this inequality in the Dirichlet and Neumann boundary cases. The main results of this section are Lemma 16 and Corollaries 22 and 23. In Sect. 4, we will provide boundary versions and some refinements to the generalization of Meyers's inequality (2), and in particular will carefully state the consequences for the lower-order derivatives of the solution $\mathbf{u}$. The main results of this section are Theorems 24 and 36.

Another important tool in the second-order case is the fundamental solution $\boldsymbol{E}^L(x, y)$. This solution is a (matrix-valued) distribution defined on $\mathbb{R}^d \times \mathbb{R}^d$ such that, formally, $L\boldsymbol{E}^L(\,\cdot\,, y) = \boldsymbol{I}\delta_y$, where δ_y denotes the Dirac mass and $\boldsymbol{I}$ denotes the identity matrix. In Sect. 5 we will construct the fundamental solution for higher-order elliptic systems.

The fundamental solution was constructed for second-order equations with real coefficients (that is, if $N = m = 1$, $A_{\alpha\beta}$ real) in [29] (in the case of symmetric coefficients $A_{\alpha\beta} = A_{\beta\alpha}$), in [23] (in dimension $d \geq 3$) and in [28] (in dimension $d = 2$). In dimension $d = 2$ these results were extended to the case of complex coefficients in [4]; as observed in [15] their strategy carries over to the case of systems with $d = 2$, $m = 1$ and $N \geq 1$.

In the case of second-order systems (that is, $m = 1$ and $N \geq 1$), the fundamental solution was constructed in the papers [14,21,24,40] under progressively weaker conditions on the operator L.

Specifically, the paper [40] constructs the fundamental solution for the operator L under the assumption that, if $L\mathbf{u} = 0$ in some ball $B(x, r)$, then $\mathbf{u}$ is continuous in $B(x, r)$ and satisfies the local boundedness estimate

$$|\mathbf{u}(x)| \leq C\left(\frac{1}{r^d}\int_{B(x,r)} |\mathbf{u}|^2\right)^{1/2} \tag{5}$$

for some constant C depending only on L and not on u, X or r. This assumption is not true for all elliptic operators; see [19].

All of the above papers made the same or stronger assumptions. Specifically, [14,21] constructed the fundamental solution in the case of systems with continuous coefficients, for which the bound (5) is always valid; see [36, Theorem 6.4.8] or [14, Section 3]. [24] constructed the fundamental solution using the stronger assumption of local Hölder continuity of solutions. The papers [23,28,29] considered only the case $N = m = 1$ with real coefficients; in this case the bound (5) was established by Moser in [37]. The paper [4] constructed the fundamental solution in dimension $d = 2$. In this case Meyers's estimate (2) implies that solutions $\mathbf{u}$ locally satisfy $\nabla\mathbf{u} \in L^p$ for some $p > d$; Morrey's inequality then implies that solutions are necessarily locally Hölder continuous. The papers [10,15,27] investigate the related topic of Green's functions in domains; they too require local boundedness of solutions (either as an explicit assumption or by virtue of working in dimension $d = 2$).

Fewer results are available in the case of higher-order equations. In the case of the polyharmonic operator $L = (-\Delta)^m$ we have an explicit formula for the fundamental solution, and this solution has been used extensively in the theory of biharmonic and polyharmonic functions. The fundamental solution in the case of general constant coefficients has also been studied and used; see, for example, [12,13,20,32,33,39,41], or the survey paper [38] and the references therein. In the case of variable analytic coefficients the fundamental solution was constructed in [25], and in the case of smooth coefficients the Green's function in domains was constructed in [16].

Our initial construction of the fundamental solution will require solutions to be continuous and satisfy the local bound (5). Again by Morrey's inequality and the higher-order generalizations of the Caccioppoli inequality (1), this is true whenever the elliptic operator L is of order $2m > d$. Thus, we will begin by constructing the fundamental solution in the case of low dimension or high order. Then, given an operator L of order $2m \leq d$, we will construct an appropriate auxiliary operator $\widetilde{L}$ of order $2\widetilde{m} > d$ and construct the fundamental solution $\boldsymbol{E}^L$ for L from the fundamental solution $\boldsymbol{E}^{\widetilde{L}}$ for $\widetilde{L}$. This technique was used in [3] in the proof of the Kato conjecture for higher-order operators. Our main results concerning the fundamental solution are summarized as Theorem 62 and the following remarks.

This paper may be of some interest to the reader interested only in second-order operators (in the case $d \geq 3$ and in the case of complex coefficients or systems) as our construction extends to the case of operators whose solutions do not satisfy local bounds.

2. Definitions

Throughout we work with a divergence-form elliptic system of N partial differential equations of order $2m$ in dimension d.

We will often use multiindices in $\mathbb{N}^d$. If $\gamma = (\gamma_1, \dots, \gamma_d)$ is a multiindex, then $|\gamma| = \gamma_1 + \gamma_2 + \dots + \gamma_d$. If $\delta = (\delta_1, \dots, \delta_d)$ is another multiindex, then we say that $\delta \le \gamma$ if $\delta_i \le \gamma_i$ for all $1 \le i \le d$, and we say that $\delta < \gamma$ if in addition the strict inequality $\delta_i < \gamma_i$ holds for at least one such i.

We will routinely consider arrays $\dot{\boldsymbol{F}} = \big(F_{j,\gamma}\big)$ indexed by integers j with $1 \le j \le N$ and by multiindices γ with $|\gamma| = k$ for some k. In particular, if $\boldsymbol{\varphi}$ is a vector-valued function with weak derivatives of order up to k, then we view $\nabla^k \boldsymbol{\varphi}$ as such an array, with

$$(\nabla^k \boldsymbol{\varphi})_{j,\gamma} = \partial^\gamma \varphi_j.$$

The L^2 inner product of two such arrays of numbers $\dot{\boldsymbol{F}}$ and $\dot{\boldsymbol{G}}$ is given by

$$\big\langle \dot{\boldsymbol{F}}, \dot{\boldsymbol{G}} \big\rangle = \sum_{j=1}^N \sum_{|\gamma|=k} \overline{F_{j,\gamma}}\, G_{j,\gamma}.$$

If $\dot{\boldsymbol{F}}$ and $\dot{\boldsymbol{G}}$ are two arrays of L^2 functions defined in a measurable set $\Omega \subseteq \mathbb{R}^d$, then the inner product of $\dot{\boldsymbol{F}}$ and $\dot{\boldsymbol{G}}$ is given by

$$\big\langle \dot{\boldsymbol{F}}, \dot{\boldsymbol{G}} \big\rangle_\Omega = \sum_{j=1}^N \sum_{|\gamma|=k} \int_\Omega \overline{F_{j,\gamma}}\, G_{j,\gamma}.$$

If $E \subset \mathbb{R}^d$ is a set of finite measure, we let $\fint_E f = \frac{1}{|E|} \int_E f$, where $|E|$ denotes Lebesgue measure. We let $\mathbf{e}_k$ be the unit vector in $\mathbb{R}^d$ in the kth direction. We let $\dot{\boldsymbol{e}}_{j,\gamma}$ be the "unit array" corresponding to the multiindex γ and the number j; thus, $\big\langle \dot{\boldsymbol{e}}_{j,\gamma}, \dot{\boldsymbol{F}} \big\rangle = F_{j,\gamma}$. We let $L^p(U)$ and $L^\infty(U)$ denote the standard Lebesgue spaces with respect to Lebesgue measure.

The inhomogeneous and homogeneous Sobolev spaces are denoted as

$$W_k^p(U) = \Big\{ u : \|u\|_{W_k^p(U)} = \sum_{j=0}^k \|\nabla^j u\|_{L^p(U)} < \infty \Big\},$$
$$\dot{W}_k^p(U) = \Big\{ u : \|u\|_{\dot{W}_k^p(U)} = \|\nabla^k u\|_{L^p(U)} < \infty \Big\}.$$

(Elements of $\dot{W}_k^p(U)$ are then defined only up to adding polynomials of order $k-1$.) In Sects. 3 and 4, we will use only the inhomogeneous Sobolev spaces W_k^p, while in Sect. 5, we will use only the homogeneous spaces $\dot{W}_k^p$.

We say that $u \in L^p_{loc}(U)$ or $u \in \dot{W}^p_{k,loc}(U)$ if $u \in L^p(V)$ or $u \in \dot{W}_k^p(V)$ for every bounded set V with $\overline{V} \subset U$.

2.1. Elliptic operators

Let $\boldsymbol{A} = \big(A^{jk}_{\alpha\beta}\big)$ be an array of measurable coefficients defined on $\mathbb{R}^d$, indexed by integers $1 \leq j \leq N$, $1 \leq k \leq N$ and by multtiindices α, β with $|\alpha| = |\beta| = m$. If $\dot{\boldsymbol{F}} = \big(F_{j,\alpha}\big)$ is an array, then $\boldsymbol{A}\dot{\boldsymbol{F}}$ is the array given by

$$(\boldsymbol{A}\dot{\boldsymbol{F}})_{j,\alpha} = \sum_{k=1}^{N} \sum_{|\beta|=m} A^{jk}_{\alpha\beta} F_{k,\beta}.$$

Throughout we consider coefficients that satisfy the bound

$$\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^d)} \leq \Lambda \tag{6}$$

for some $\Lambda > 0$. In our construction of the fundamental solution in Sect. 5, we will consider only operators that satisfy the strict Gårding inequality

$$\operatorname{Re}\big\langle \nabla^m \boldsymbol{\varphi}, \boldsymbol{A}\nabla^m \boldsymbol{\varphi}\big\rangle_{\mathbb{R}^d} \geq \lambda \|\nabla^m \boldsymbol{\varphi}\|^2_{L^2(\mathbb{R}^d)} \tag{7}$$

for all $\boldsymbol{\varphi}$ with $\nabla^m \boldsymbol{\varphi} \in L^2(\mathbb{R}^d)$ and for some $\lambda > 0$ independent of $\boldsymbol{\varphi}$. In Sect. 3 we will consider weaker and stronger versions of the Gårding inequality.

We let L be the $2m$th-order divergence-form operator associated with $\boldsymbol{A}$. That is, we say that $L\mathbf{u} = \operatorname{div}_m \dot{\boldsymbol{F}}$ in Ω in the weak sense if, for every $\boldsymbol{\varphi}$ smooth and compactly supported in Ω, we have that

$$\big\langle \nabla^m \boldsymbol{\varphi}, \boldsymbol{A}\nabla^m \mathbf{u}\big\rangle_\Omega = \big\langle \nabla^m \boldsymbol{\varphi}, \dot{\boldsymbol{F}}\big\rangle_\Omega, \tag{8}$$

that is, we have that

$$\sum_{j=1}^{N}\sum_{k=1}^{N} \sum_{|\alpha|=|\beta|=m} \int_\Omega \partial^\alpha \bar{\varphi}_j \, A^{jk}_{\alpha\beta} \, \partial^\beta u_k = \sum_{j=1}^{N} \sum_{|\alpha|=m} \int_\Omega \partial^\alpha \bar{\varphi}_j \, F_{j,\alpha}.$$

In particular, if the left-hand side is zero for all such $\boldsymbol{\varphi}$ then we say that $L\mathbf{u} = 0$.

If $\boldsymbol{A}$ is such an array of coefficients, we let the adjoint array $\boldsymbol{A}^*$ be given by $(A^*)^{jk}_{\alpha\beta} = \overline{A^{kj}_{\beta\alpha}}$; we then let L^* be the operator associated with $\boldsymbol{A}^*$.

Throughout the paper we will let C denote a constant whose value may change from line to line, but which depends only on the dimension d, the ellipticity constants λ and Λ in the bounds (6) and (7) (or variants thereof), and the order $2m$ of the operator L. Any other dependencies will be indicated explicitly.

3. The Caccioppoli inequality

In this section we will generalize the Caccioppoli inequality (1) to the case of higher-order elliptic systems. Because the Caccioppoli inequality involves norms both of the solution $\mathbf{u}$ and its gradient $\nabla^m \mathbf{u}$, in this section we will use the inhomogeneous Sobolev spaces

$$W^p_k(U) = \Big\{\mathbf{u} : \sum_{j=0}^{k} \|\nabla^j \mathbf{u}\|_{L^p(U)} < \infty\Big\}.$$

The first step in our generalization of the Caccioppoli inequality is the following lemma.

Lemma 9. *Let L be the operator of order* $2m$ *associated to the coefficients* $\boldsymbol{A}$*, where* $\boldsymbol{A}$ *satisfies the bound* (6) *and the weak Gårding inequality*

$$\operatorname{Re}\langle \nabla^m \boldsymbol{\varphi}, \boldsymbol{A}\nabla^m \boldsymbol{\varphi}\rangle_{\mathbb{R}^d} \geq \lambda \|\nabla^m \boldsymbol{\varphi}\|^2_{L^2(\mathbb{R}^d)} - \delta \|\boldsymbol{\varphi}\|^2_{L^2(\mathbb{R}^d)} \tag{10}$$

for some $\lambda > 0$ *and some* $\delta \geq 0$*, and for all smooth, compactly supported functions* $\boldsymbol{\varphi}$*.*

Let $x_0 \in \mathbb{R}^d$ *and let* $R > 0$*. Suppose that* $\mathbf{u} \in W^2_m(B(x_0, R))$*, that* $\dot{\boldsymbol{F}} \in L^2(B(x_0, R))$*, and that one of the following two conditions holds.*

$$L\mathbf{u} = \operatorname{div}_m \dot{\boldsymbol{F}} \text{ in } \Omega = B(x_0, R), \text{ or} \tag{11}$$

$$L\mathbf{u} = \operatorname{div}_m \dot{\boldsymbol{F}} \text{ in some domain } \Omega \subsetneq B(x_0, R), \text{ and } \mathbf{u} \text{ lies in the closure in } W^2_m(B(x_0, R)) \text{ of } \{\boldsymbol{\varphi} \in C^\infty(\mathbb{R}^d) : \boldsymbol{\varphi} \equiv 0 \text{ in } B(x_0, R)\setminus\Omega\}. \tag{12}$$

Then, for any $0 < r < R$*, we have that*

$$\int_{\Omega\cap B(x_0,r)} |\nabla^m \mathbf{u}|^2 \leq \sum_{i=0}^{m-1} \frac{C}{(R-r)^{2m-2i}} \int_{\Omega\setminus B(x_0,r)} |\nabla^i \mathbf{u}|^2 + C\int_\Omega |\dot{\boldsymbol{F}}|^2 + C\delta \int_\Omega |\mathbf{u}|^2 \tag{13}$$

where C is a constant depending only on the dimension d, the order $2m$ *of the elliptic operator L and the numbers* λ *and* Λ *in the bounds* (6) *and* (10)*.*

In Theorem 18 we will strengthen this lemma by replacing the sum on the right-hand side by the $i = 0$ term alone. Our Theorem 18 will thus be stronger than the bound (4) of [5]; we have chosen to follow the example of [5] and establish the Caccioppoli inequality for operators that satisfy the weak Gårding inequality (10), as well as operators that satisfy the strong Gårding inequality (7).

Lemma 9 was proven in [9] in the interior case (11) for coefficients $\boldsymbol{A}$ that satisfy the strong pointwise Gårding inequality

$$\operatorname{Re}\langle \dot{\boldsymbol{\eta}}, \boldsymbol{A}(x)\dot{\boldsymbol{\eta}}\rangle \geq \lambda\langle \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{\eta}}\rangle \quad \text{for almost every } x \in \mathbb{R}^d \text{ and any array } \dot{\boldsymbol{\eta}}. \tag{14}$$

Thus the main new result of Lemma 9 is the case (12), which corresponds to zero Dirichlet boundary values.

In the higher-order case, the condition that $\mathbf{u}$ have zero Neumann boundary values along $\partial\Omega \cap B(x_0, R)$ may best be expressed by the following condition.

$\mathbf{u} \in W^2_m(B(x_0, R))$, and the equation

$$\langle \nabla^m \boldsymbol{\varphi}, \dot{\boldsymbol{F}}\rangle_\Omega = \langle \nabla^m \boldsymbol{\varphi}, \boldsymbol{A}\nabla^m \mathbf{u}\rangle_\Omega \tag{15}$$

is true for all $\boldsymbol{\varphi}$ smooth and supported in $B(x_0, R)$, not only all $\boldsymbol{\varphi}$ supported in Ω.

We refer the reader to the author's survey paper with Svitlana Mayboroda [8] for a discussion of the meaning of Neumann boundary data for higher order operators. See also the papers [2,7,11,35,42,43], which treat various special cases of the Neumann problem.

Lemma 16. *If* $L\mathbf{u} = \mathrm{div}_m \dot{\boldsymbol{F}}$ *in* $\Omega \subset B(x_0, R)$ *and* $\mathbf{u}$ *satisfies the Neumann boundary condition* (15)*, then the conclusion* (13) *of Lemma 9 is still true provided that the coefficients* $\boldsymbol{A}$ *associated with the operator* L *satisfy the bound* (6) *and the local Gårding inequality*

$$\mathrm{Re}\big\langle \nabla^m \boldsymbol{\varphi}, \boldsymbol{A}\nabla^m \boldsymbol{\varphi}\big\rangle_\Omega \geq \lambda \|\nabla^m \boldsymbol{\varphi}\|^2_{L^2(\Omega)} - \delta \|\boldsymbol{\varphi}\|^2_{L^2(\Omega)} \tag{17}$$

for all $\boldsymbol{\varphi} \in W^2_m(B(x_0, R))$.

Notice that the pointwise ellipticity condition (14) implies the local Gårding inequality (17) with $\delta = 0$.

In all cases we assume that $\mathbf{u}$ is defined in the ball $B(x_0, R)$; equivalently, we assume that we may extend $\mathbf{u}$ from Ω to the ball. This extension is very natural in the interior or Dirichlet cases but must be explicitly assumed in the Neumann case. If Ω is a Lipschitz domain and $\nabla^m \mathbf{u} \in L^2(\Omega)$, then by a well-known result of Calderón and Stein, an extension of $\mathbf{u}$ to $B(x_0, R)$ (indeed, to $\mathbb{R}^d$) exists. Such extensions are also guaranteed to exist under weaker conditions on Ω; see, for example, [26]. Notice further that in the interior and Neumann cases (11) and (15) the conclusion (13) remains valid if we modify $\mathbf{u}$ by adding a polynomial of order $m-1$; however, this is not true in the Dirichlet case (12), as in this case we must maintain the condition $\mathbf{u} \equiv 0$ in $B(x_0, R)\setminus\Omega$.

Proof (Proof of Lemmas 9 and 16). Let φ be a smooth, real-valued test function with $0 \leq \varphi \leq 1$, supported in $B(x_0, R)$ and identically equal to 1 in $B(x_0, r)$. We require $|\nabla^k \varphi| \leq C_k (R-r)^{-k}$.

Observe that $\boldsymbol{\psi} = \varphi^{4m}\mathbf{u}$ is a function supported in $B(x_0, R)$ with $\nabla^m \boldsymbol{\psi} \in L^2(B(x_0, R))$. By definition of $L\mathbf{u}$ or condition (15), and by density of smooth functions, we have that

$$\big\langle \nabla^m(\varphi^{4m}\mathbf{u}), \dot{\boldsymbol{F}}\big\rangle_\Omega = \big\langle \nabla^m(\varphi^{4m}\mathbf{u}), \boldsymbol{A}\nabla^m \mathbf{u}\big\rangle_\Omega.$$

Observe that for all suitably differentiable functions v and w,

$$\partial^\alpha(w\,v) = \sum_{\gamma\leq\alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!}\partial^\gamma w\,\partial^{\alpha-\gamma} v$$

where $\gamma! = \gamma_1!\gamma_2!\ldots\gamma_d!$. Let $a_{\alpha,\gamma} = \alpha!/\gamma!(\alpha-\gamma)!$. Notice that $a_{\alpha,0} = a_{\alpha,\alpha} = 1$.

By definition of the inner product, we have that

$$\big|\big\langle \nabla^m(\varphi^{4m}\mathbf{u}), \dot{\boldsymbol{F}}\big\rangle_\Omega\big| = \Big|\sum_{j=1}^N \sum_{|\alpha|=m} \int_\Omega \partial^\alpha(\varphi^{4m}\bar{u}_j)\dot{\boldsymbol{F}}_{j,\alpha}\Big|.$$

Then

$$\left|\langle\nabla^m(\varphi^{4m}\mathbf{u}),\dot{\boldsymbol{F}}\rangle_\Omega\right| \le \left|\sum_{j=1}^N\sum_{|\alpha|=m}\int_\Omega \partial^\alpha(\varphi^{2m}\bar{u}_j)\,\varphi^{2m}F_{j,\alpha}\right| + \left|\sum_{j=1}^N\sum_{|\alpha|=m}\sum_{\gamma<\alpha} a_{\alpha,\gamma}\int_\Omega \partial^{\alpha-\gamma}(\varphi^{2m})\,\partial^\gamma(\varphi^{2m}\bar{u}_j)\dot{\boldsymbol{F}}_{j,\alpha}\right|.$$

Thus

$$\left|\langle\nabla^m(\varphi^{4m}\mathbf{u}),\dot{\boldsymbol{F}}\rangle_\Omega\right| \le \|\nabla^m(\varphi^{2m}\mathbf{u})\|_{L^2(\Omega)}\|\dot{\boldsymbol{F}}\|_{L^2(\Omega)} + C\sum_{i=0}^{m-1}\frac{\|\nabla^i u\|_{L^2(\Omega\setminus B(x_0,r))}}{(R-r)^{m-i}}\|\dot{\boldsymbol{F}}\|_{L^2(\Omega)}.$$

We now consider the right-hand side. We have that

$$\langle\nabla^m\left(\varphi^{4m}\mathbf{u}\right),\boldsymbol{A}\nabla^m\mathbf{u}\rangle_\Omega = \sum_{j,k,\alpha,\beta}\int_\Omega\sum_{\gamma<\alpha}a_{\alpha,\gamma}\partial^{\alpha-\gamma}(\varphi^{2m})\partial^\gamma\left(\varphi^{2m}\bar{u}_j\right)A^{j,k}_{\alpha\beta}\,\partial^\beta u_k + \sum_{j,k,\alpha,\beta}\int_\Omega\varphi^{2m}\partial^\alpha\left(\varphi^{2m}\bar{u}_j\right)A^{j,k}_{\alpha\beta}\,\partial^\beta u_k$$

where the sums are taken over all j, k, α, β with $1 \le j \le N$, $1 \le k \le N$ and $|\alpha| = |\beta| = m$. Now, we may write

$$\sum_{\gamma<\alpha}a_{\alpha,\gamma}\partial^{\alpha-\gamma}(\varphi^{2m})\partial^\gamma\left(\varphi^{2m}\bar{u}_j\right) = \sum_{\zeta<\alpha}\varphi^{2m}\Phi_{\alpha,\zeta}\partial^\zeta\bar{u}_j$$

for some functions $\Phi_{\alpha,\zeta}$ supported in $B(x_0,R)\setminus B(x_0,r)$ with $|\Phi_{\alpha,\zeta}| \le C(R-r)^{|\zeta|-|\alpha|}$. Therefore

$$\langle\nabla^m\left(\varphi^{4m}\mathbf{u}\right),\boldsymbol{A}\nabla^m\mathbf{u}\rangle_\Omega = \sum_{j,k,\alpha,\beta}\int_\Omega\partial^\alpha\left(\varphi^{2m}\bar{u}_j\right)A^{j,k}_{\alpha\beta}\left(\varphi^{2m}\partial^\beta u_k\right) + \sum_{j,k,\alpha,\beta}\int_{\Omega\setminus B(x_0,r)}\sum_{\zeta<\alpha}\Phi_{\alpha,\zeta}\partial^\zeta\bar{u}_j\,A^{j,k}_{\alpha\beta}\,(\varphi^{2m}\partial^\beta u_k).$$

We rewrite the two terms $\varphi^{2m}\partial^\beta u_k$ to see that

$$\begin{aligned}\langle\nabla^m(\varphi^{4m}\mathbf{u}),\boldsymbol{A}\nabla^m\mathbf{u}\rangle_\Omega &= \langle\nabla^m(\varphi^{2m}\mathbf{u}),\boldsymbol{A}\nabla^m(\varphi^{2m}\mathbf{u})\rangle_\Omega \\ &\quad+\sum_{j,k,\alpha,\beta}\int_\Omega\sum_{\zeta<\alpha}\Phi_{\alpha,\zeta}\partial^\zeta\bar{u}_j\,A^{j,k}_{\alpha\beta}\,\partial^\beta\left(\varphi^{2m}u_k\right) \\ &\quad-\sum_{j,k,\alpha,\beta}\int_\Omega\sum_{\gamma<\beta}a_{\beta,\gamma}\partial^\alpha\left(\varphi^{2m}\bar{u}_j\right)A^{j,k}_{\alpha\beta}\,\partial^{\beta-\gamma}\left(\varphi^{2m}\right)\partial^\gamma u_k \\ &\quad-\sum_{j,k,\alpha,\beta}\int_\Omega\sum_{\gamma<\beta}a_{\beta,\gamma}\sum_{\zeta<\alpha}\Phi_{\alpha,\zeta}\partial^\zeta\bar{u}_j\,A^{j,k}_{\alpha\beta}\,\partial^{\beta-\gamma}\left(\varphi^{2m}\right)\partial^\gamma u_k.\end{aligned}$$

Observe that the integrands in the second and third terms are zero in $B(x_0, r)$.

By the Gårding inequality (10) or (17),

$$\lambda \int_\Omega |\nabla^m(\varphi^{2m}\mathbf{u})|^2 \leq \operatorname{Re}\langle\nabla^m(\varphi^{2m}\mathbf{u}), \boldsymbol{A}\nabla^m(\varphi^{2m}\mathbf{u})\rangle_\Omega + \delta\|\varphi^{2m}\mathbf{u}\|^2_{L^2(\Omega)}.$$

Thus

$$\begin{aligned}\lambda \int_\Omega |\nabla^m(\varphi^{2m}\mathbf{u})|^2 &\leq |\langle\nabla^m(\varphi^{4m}\mathbf{u}), \boldsymbol{A}\nabla^m\mathbf{u}\rangle_\Omega| + \delta\|\mathbf{u}\|^2_{L^2(\Omega)} \\ &\quad + C\|\nabla^m(\varphi^{2m}\mathbf{u})\|_{L^2(\Omega)} \sum_{i=0}^{m-1} \frac{\|\nabla^i u\|_{L^2(\Omega\setminus B(x_0,r))}}{(R-r)^{m-i}} \\ &\quad + \sum_{i=0}^{m-1} \frac{C}{(R-r)^{2m-2i}} \|\nabla^i u\|^2_{L^2(\Omega\setminus B(x_0,r))}.\end{aligned}$$

Recalling that

$$\begin{aligned}|\langle\nabla^m(\varphi^{4m}\mathbf{u}), \boldsymbol{A}\nabla^m\mathbf{u}\rangle_\Omega| &= |\langle\nabla^m(\varphi^{4m}\mathbf{u}), \dot{\boldsymbol{F}}\rangle_\Omega| \\ &\leq \|\nabla^m(\varphi^{2m}\mathbf{u})\|_{L^2(\Omega)}\|\dot{\boldsymbol{F}}\|_{L^2(\Omega)} \\ &\quad + C\sum_{i=0}^{m-1} \frac{\|\nabla^i u\|_{L^2(\Omega\setminus B(x_0,r))}}{(R-r)^{m-i}} \|\dot{\boldsymbol{F}}\|_{L^2(\Omega)}\end{aligned}$$

we may derive the desired bound on $\|\nabla^m\mathbf{u}\|_{L^2(\Omega\cap B(x_0,r))}$. □

We now wish to improve this inequality to a bound in terms of $\|u\|_{L^2}$ rather than in terms of all of the lower-order derivatives. This will be done by the following theorem and its corollaries.

Theorem 18. *Let $x_0 \in \mathbb{R}^d$ and let $R > 0$. Let $\mathbf{u} \in W^2_m(B(x_0, R))$ be a function that satisfies the inequality*

$$\int_{B(x_0,\rho)} |\nabla^m\mathbf{u}|^2 \leq \sum_{i=0}^{m-1} \frac{C_0}{(r-\rho)^{2m-2i}} \int_{B(x_0,r)\setminus B(x_0,\rho)} |\nabla^i\mathbf{u}|^2 + F \tag{19}$$

whenever $0 < \rho < r < R$, for some number $F > 0$.

Then $\vec{u}$ satisfies the stronger inequality

$$\int_{B(x_0,r)} |\nabla^m\mathbf{u}|^2 \leq \frac{C}{(R-r)^{2m}} \int_{B(x_0,R)\setminus B(x_0,r)} |\mathbf{u}|^2 + CF \tag{20}$$

for some constant C depending only on m, the dimension d and the constant C_0.

Furthermore, if $0 \leq j \leq m$, then u satisfies

$$\int_{B(x_0,r)} |\nabla^j\mathbf{u}|^2 \leq \frac{C}{(R-r)^{2j}} \int_{B(x_0,R)} |\mathbf{u}|^2 + CR^{2m-2j}F. \tag{21}$$

Notice that in the bound (20), the right-hand side involves the quantity $|\mathbf{u}|^2$ integrated over an annulus $B(x_0, R)\setminus B(x_0, r)$, while in the bound (21) $|\mathbf{u}|^2$ is integrated over the full ball $B(x_0, R)$. It is possible to use the Poincaré inequality and the bound (20) to improve the bound (21) to an estimate involving the integral of $|\mathbf{u}|^2$ over an annulus, but this comes at a cost of introducing powers of $(R-r)/r$, and so we have chosen to state the bound (21) as above.

Combined with Lemma 9, we immediately have the following corollaries.

Corollary 22. *Let $x_0 \in \mathbb{R}^d$ and let $R > 0$. Suppose that $L\mathbf{u} = \operatorname{div}_m \dot{\boldsymbol{F}}$ in $B(x_0, R)$, for some operator L of order $2m$ that satisfies the bounds (6) and (10), some $\mathbf{u} \in W_m^2(B(x_0, R))$, and some $\dot{\boldsymbol{F}} \in L^2(B(x_0, R))$. If $0 < r < R$ and $0 \le j \le m$, then*

$$\int_{B(x_0,r)} |\nabla^j \mathbf{u}|^2 \le \frac{C}{(R-r)^{2j}} \int_{B(x_0,R)} |\mathbf{u}|^2 + C R^{2m-2j} \int_{B(x_0,R)} \big(|\dot{\boldsymbol{F}}|^2 + \delta|\mathbf{u}|^2\big),$$

$$\int_{B(x_0,r)} |\nabla^m \mathbf{u}|^2 \le \frac{C}{(R-r)^{2m}} \int_{B(x_0,R)\setminus B(x_0,r)} |\mathbf{u}|^2 + C \int_{B(x_0,R)} \big(|\dot{\boldsymbol{F}}|^2 + \delta|\mathbf{u}|^2\big).$$

Recall that if we allow a term of the form $\varepsilon \|\nabla^m \mathbf{u}\|^2_{L^2(B(x_0,R))}$ on the right-hand side, then this corollary was proven in [5] in the homogeneous case $L\mathbf{u} = 0$.

Corollary 23. *Let $x_0 \in \mathbb{R}^d$ and let $R > 0$, and let $\Omega \subset B(x_0, R)$. Suppose that $L\mathbf{u} = \operatorname{div}_m \dot{\boldsymbol{F}}$ in Ω, for some operator L of order $2m$ that satisfies the bounds (6) and (10), some $\mathbf{u} \in W_m^2(\Omega)$, and some $\dot{\boldsymbol{F}} \in L^2(\Omega)$. Suppose in addition that $\mathbf{u}$ may be extended by zero to all of $B(x_0, R)$, in the sense of condition (12) of Lemma 9.*

If $0 < r < R$ and $0 \le j \le m$, then

$$\int_{B(x_0,r)\cap\Omega} |\nabla^j \mathbf{u}|^2 \le \frac{C}{(R-r)^{2j}} \int_{\Omega\cap B(x_0,R)} |\mathbf{u}|^2 + C R^{2m-2j} \int_{\Omega} \big(|\dot{\boldsymbol{F}}|^2 + \delta|\mathbf{u}|^2\big),$$

$$\int_{B(x_0,r)\cap\Omega} |\nabla^m \mathbf{u}|^2 \le \frac{C}{(R-r)^{2m}} \int_{\Omega\cap B(x_0,R)\setminus B(x_0,r)} |\mathbf{u}|^2 + C \int_{\Omega} \big(|\dot{\boldsymbol{F}}|^2 + \delta|\mathbf{u}|^2\big).$$

Our methods will not allow us to improve upon Lemma 16 in the case of Neumann boundary data.

Proof (Proof of Theorem 18). Let $A(r,\zeta)$ denote either the annulus $B(x_0, r+\zeta)\setminus B(x_0, r-\zeta)$, or simply the ball $B(x_0, r+\zeta)$, depending on whether we are establishing the bound (20) on $\nabla^m \mathbf{u}$ or the bound (21) on $\nabla^k \mathbf{u}$.

Consider the following claim.

Claim. *If $1 \le k \le m$, and if $R/2 < r < R$ and $0 < \zeta < \min(R-r, r)$, then*

$$\int_{A(r,\zeta)} |\nabla^k \mathbf{u}|^2 \le \sum_{i=0}^{k-1} \frac{C_k}{(\xi-\zeta)^{2k-2i}} \int_{A(r,\xi)} |\nabla^i \mathbf{u}|^2 + R^{2m-2k} F.$$

If this claim is true for all such k, then clearly the bound (21) is valid. To establish the bound (20), we combine the above claim with the assumed bound (19); it is this that allows us to bound $\nabla^m \mathbf{u}$ by the integral of $|\mathbf{u}|^2$ over an annulus rather than a ball.

Thus we need only prove the claim. That the claim is true for $k = m$ follows by our assumption (19). We work by induction. Suppose that the claim is true for some $k + 1 \leq m$; we will show that it is valid for k as well.

Let $A_j = A(r, \rho_j)$, where $\zeta = \rho_0 < \rho_1 < \cdots < \xi$ for some sequence $\{\rho_j\}_{j=0}^{\infty}$ to be chosen momentarily. Let $\delta_j = \rho_{j+1} - \rho_j$, and let $\widetilde{A}_j = A(r, \rho_j + \delta_j/2)$, so $A_j \subset \widetilde{A}_j \subset A_{j+1}$. Let φ_j be smooth, supported in $\widetilde{A}_j$, and identically equal to 1 in A_j; we may require that $\|\nabla\varphi_k\| \leq C/\delta_j$ and $\|\nabla^2\varphi_k\| \leq C/\delta_j^2$ for some absolute constant C.

Now, for any $j \geq 0$,

$$\int_{A_j} |\nabla^k \mathbf{u}|^2 \leq \int_{\widetilde{A}_j} |\nabla(\varphi_j \nabla^{k-1}\mathbf{u})|^2.$$

By Plancherel's theorem, if $f \in W_2^2(\mathbb{R}^d)$ then

$$\|\nabla f\|^2_{L^2(\mathbb{R}^d)} \leq C\|\nabla^2 f\|_{L^2(\mathbb{R}^d)}\|f\|_{L^2(\mathbb{R}^d)}.$$

We will apply this inequality to $f = (\varphi_j \nabla^{k-1}\mathbf{u})$; it is this step that fails in the case of Neumann boundary data. We have that

$$\begin{aligned}\int_{A_j} |\nabla^k \mathbf{u}|^2 &\leq C\Big(\int_{\widetilde{A}_j} |\nabla^2(\varphi_j \nabla^{k-1}\mathbf{u})|^2\Big)^{1/2}\Big(\int_{\widetilde{A}_j} |\varphi_j \nabla^{k-1}\mathbf{u}|^2\Big)^{1/2}\\ &\leq C\Big(\int_{\widetilde{A}_j} |\nabla^{k+1}\mathbf{u}|^2 + \frac{|\nabla^k\mathbf{u}|^2}{\delta_j^2} + \frac{|\nabla^{k-1}\mathbf{u}|^2}{\delta_j^4}\Big)^{1/2}\Big(\int_{\widetilde{A}_j} |\nabla^{k-1}\mathbf{u}|^2\Big)^{1/2}.\end{aligned}$$

Applying the claim to bound $|\nabla^{k+1}\mathbf{u}|^2$, we see that

$$\begin{aligned}\int_{A_j} |\nabla^k \mathbf{u}|^2 \leq &\Big(\sum_{i=0}^{k} \frac{C_k}{\delta_j^{2k+2-2i}} \int_{A_{j+1}} |\nabla^i \mathbf{u}|^2 + CR^{2m-2k-2}F\Big)^{1/2}\\ &\times\Big(\int_{\widetilde{A}_j} |\nabla^{k-1}\mathbf{u}|^2\Big)^{1/2}.\end{aligned}$$

We move a factor of C_k/δ_j^2 from the first term to the second, and then use the inequality $\sqrt{a}\sqrt{b} \leq (1/2)a + (1/2)b$ to see that

$$\int_{A_j} |\nabla^k \mathbf{u}|^2 \leq \frac{1}{2}\sum_{i=0}^{k} \frac{1}{\delta_j^{2k-2i}} \int_{A_{j+1}} |\nabla^i \mathbf{u}|^2 + \frac{1}{2}R^{2m-2k}F + \frac{C_k}{\delta_j^2}\int_{\widetilde{A}_j} |\nabla^{k-1}\mathbf{u}|^2.$$

Separating out the term $i = k$, we see that

$$\int_{A_j} |\nabla^k \mathbf{u}|^2 \leq C_k\sum_{i=0}^{k-1} \frac{1}{\delta_j^{2k-2i}} \int_{A_{j+1}} |\nabla^i \mathbf{u}|^2 + \frac{1}{2}R^{2m-2k}F + \frac{1}{2}\int_{A_{j+1}} |\nabla^k \mathbf{u}|^2.$$

This bound is valid for all $j > 0$. We may iterate to see that

$$\int_{A_0} |\nabla^k \mathbf{u}|^2 \leq \sum_{j=0}^{\infty} 2^{-j} \left(C_k \sum_{i=0}^{k-1} \frac{1}{\delta_j^{2k-2i}} \int_{A_{j+1}} |\nabla^i \mathbf{u}|^2 + \frac{1}{2} R^{2m-2k} F \right)$$
$$\leq C_k \sum_{i=0}^{k-1} \left(\sum_{j=0}^{\infty} 2^{-j} \frac{1}{\delta_j^{2k-2i}} \right) \int_{A_\infty} |\nabla^i \mathbf{u}|^2 + R^{2m-2k} F.$$

Now, choose $\rho_j = \zeta + (\xi - \zeta)(1-\tau) \sum_{i=1}^{j} \tau^i$ for some $0 < \tau < 1$. Then $\rho_0 = \zeta$ and $\lim_{j\to\infty} \rho_j = \xi$. So

$$\int_{A_0} |\nabla^k \mathbf{u}|^2 \leq C_{k,\tau} \sum_{i=0}^{k-1} \left(\sum_{j=1}^{\infty} \frac{1}{(2\tau^{2k-2i})^j} \frac{1}{(\xi - \zeta)^{2k-2i}} \right) \int_{A_\infty} |\nabla^i \mathbf{u}|^2 + R^{2m-2k} F.$$

Choosing τ so that $2\tau^{2k} > 1$ and $\tau < 1$, we see that the sum in j converges and the proof is complete. □

4. Meyers's reverse Hölder inequality for gradients

In this section we will generalize Meyers's reverse Hölder inequality (2) to the higher-order case. We will use many of the techniques of the second-order case. As in Sect. 3, we will use the inhomogeneous Sobolev spaces

$$W_k^p(U) = \left\{ \mathbf{u} : \sum_{j=0}^{k} \|\nabla^j \mathbf{u}\|_{L^p(U)} < \infty \right\}.$$

The main result of this section in the interior and Dirichlet boundary case is the following theorem; the Neumann boundary version is Theorem 36 below.

Theorem 24. *Let L be an operator of order $2m$ that satisfies the bounds (6) and (10). Let $c_\Omega > 0$. Then there is some number $p^+ = p_L^+ > 2$ depending only on the standard constants and the number c_Ω such that the following statement is true.*

Let $x_0 \in \mathbb{R}^d$ and let $R > 0$. Suppose that $\mathbf{u} \in W_m^2(B(x_0, R))$, that $\dot{\boldsymbol{F}} \in L^2(B(x_0, R))$, and that either

(25) *$L\mathbf{u} = \operatorname{div}_m \dot{\boldsymbol{F}}$ in $\Omega = B(x_0, R)$, or*
(26) *$L\mathbf{u} = \operatorname{div}_m \dot{\boldsymbol{F}}$ in some domain $\Omega \subsetneq B(x_0, R)$, and $\mathbf{u}$ lies in the closure in $W_m^2(B(x_0, R))$ of $\{\boldsymbol{\varphi} \in C^\infty(\mathbb{R}^d) : \boldsymbol{\varphi} \equiv 0$ in $B(x_0, R)\setminus\Omega\}$. Furthermore, if $x \in \partial\Omega$ and $\rho > 0$, then $|B(x_0, \rho)\setminus\Omega| \geq c_\Omega \rho^d$, where $|E|$ denotes the Lebesgue measure of E.*

Suppose that $0 < p \le 2 < q < p^+$. *Then*

$$\left(\int_{B(x_0,r)\cap\Omega} |\nabla^m \mathbf{u}|^q\right)^{1/q} \le \frac{C(c_\Omega, p, q)}{(R-r)^{d/p-d/q}}\left(\int_\Omega |\nabla^m \mathbf{u}|^p\right)^{1/p} + C(c_\Omega, p, q)\left(\int_\Omega |\dot{\boldsymbol{F}}|^q + \delta^{q/2}|\mathbf{u}|^q\right)^{1/q} \tag{27}$$

for some constant $C(c_\Omega, p, q)$ *depending only on* p, q, c_Ω *and the standard parameters.*

We may also bound the lower-order derivatives. Suppose that $m - d/2 < m - k < m$ *and that* $0 \le m - k$. *Let* $0 < p \le 2 \le q < \min(p_L^+, d/k)$. *Then*

$$\left(\int_{B(x_0,r)\cap\Omega} |\nabla^{m-k} \mathbf{u}|^{q_k}\right)^{1/q_k} \le \frac{C(c_\Omega, p, q)}{(R-r)^{d/p_k-d/q_k}}\left(\int_\Omega |\nabla^{m-k} \mathbf{u}|^{p_k}\right)^{1/p_k} + C(c_\Omega, p, q)R^k\left(\int_\Omega |\dot{\boldsymbol{F}}|^q + \delta^{q/2}|\mathbf{u}|^q\right)^{1/q} \tag{28}$$

where $q_k = q\,d/(d-k\,q)$ *and* $p_k = p\,d/(d-k\,p)$. (*Notice that the condition* $0 < p \le 2 \le q < \min(p_L^+, d/k)$ *is equivalent to the condition* $0 < p_k \le 2_k \le q_k < p_k^+$, *where* $2_k = 2\,d/(d-2k)$ *and* $p_k^+ = p_L^+\,d/(d-k\,p_L^+)$ *if* $d > k\,p_L^+$ *and* $p_k^+ = \infty$ *if* $d \le k\,p_L^+$.)

Finally, if $0 \le m - k \le m - d/2$ *and* $0 < p < \infty$, *then* $\nabla^{m-k}\mathbf{u}$ *is Hölder continuous and satisfies the bound*

$$\sup_{B(x_0,r)\cap\Omega} |\nabla^{m-k}\mathbf{u}| \le \frac{C(p,q)}{(R-r)^{d/p}}\left(\int_\Omega |\nabla^{m-k}\mathbf{u}|^p\right)^{1/p} + C(p,q)R^{k-d/q}\left(\int_\Omega |\dot{\boldsymbol{F}}|^q + \delta^{q/2}|\mathbf{u}|^q\right)^{1/q} \tag{29}$$

provided that $0 < p \le \infty$ *and that either* $q \ge 2$ *and* $k > d/2$ *or* $q > 2$ *and* $k \ge d/2$.

Of course if $p > q$, then we may use Hölder's inequality to bound $\|\nabla^{m-k}\mathbf{u}\|_{L^q}$ by $\|\nabla^{m-k}\mathbf{u}\|_{L^p}$; however, we then no longer have the coefficient $(R-r)^{d/q-d/p}$. In the interior case $\Omega = B(x_0, R)$, the bound (27) with $p = 2$ was proven in [5] in the homogeneous case $L\mathbf{u} = 0$, and in [9] under the strong pointwise Gårding inequality; the lower-order bounds (28) and (29) are relatively straightforward consequences of the bound (27) but it will be convenient later to have them stated explicitly.

We will prove Theorem 24 as in the second-order case; we will need the following lemmas. The first two given lemmas are standard in the theory of Sobolev spaces; see, for example, [17, Section 5.6.3].

Lemma 30. (The Gagliardo–Nirenberg–Sobolev inequality in balls). *Let* $x_0 \in \mathbb{R}^d$ *and let* $\rho > 0$. *Suppose that* $1 \le q < d$, *that* $1 \le k < d/q$, *and that* $\nabla^k v \in L^q(B(x_0, \rho))$. *Let* $q_k = q\,d/(d-k\,q)$.

Then $v \in L^{q_k}(B(x_0, \rho))$. *More precisely,*

$$\|v\|_{L^{q_k}(B(x_0,\rho))} \leq C(q,k) \sum_{i=0}^{k} \rho^{i-k} \|\nabla^i v\|_{L^q(B(x_0,\rho))}.$$

Lemma 31. (Morrey's inequality). *Suppose that* $1 \leq q \leq \infty$, *that* $k > d/q$, *and that* $\nabla^k v \in L^q(B(x_0, \rho))$ *for some ball* $B(x_0, \rho) \subset \mathbb{R}^d$.

Then v *is Hölder continuous in* $B(x_0, \rho)$. *Furthermore,* v *satisfies the local bound*

$$\|v\|_{L^\infty(B(x_0,\rho))} \leq C(q,k) \sum_{i=0}^{k} \rho^{i-d/q} \|\nabla^i v\|_{L^q(B(x_0,\rho))}.$$

The next lemma comes from the book [22], where it was used for a relatively straightforward proof of Theorem 24 in the second-order case.

Lemma 32. ([22, Chapter V, Theorem 1.2]). *Let* $Q \subset \mathbb{R}^d$ *be a cube and let* g *and* f *be two nonnegative, locally integrable functions defined on* Q. *Suppose that, for any* $x \in Q$, *we have that*

$$\sup_{0<r<\mathrm{dist}(x,\partial Q)/2} \fint_{B(x,r)} g^p \leq b\biggl(\sup_{0<r} \fint_{B(x,r)} g\biggr)^p + \sup_{0<r} \fint_{B(x,r)} f^p$$

for some constant $b > 0$ *and some* $p > 1$. *Then there is some* $\varepsilon > 0$ *depending only on* b, p *and the dimension* d, *such that if* $p < q < p + \varepsilon$ *and* $f \in L^p(B(x_0, R))$, *then*

$$\biggl(\fint_{(1/2)Q} g^q\biggr)^{1/q} \leq C(b,p,q)\biggl(\fint_{Q} g^p\biggr)^{1/p} + C(b,p,q)\biggl(\fint_{Q} f^q\biggr)^{1/q}$$

where $(1/2)Q$ *is the cube concentric to* Q *with side-length half that of* Q.

The following lemma was established in [18, Section 9, Lemma 2] in the case of harmonic functions. We must now generalize it.

Lemma 33. *Let* $0 < p_0 < q \leq \infty$. *Let* $x_0 \in \mathbb{R}^d$ *and let* $R > 0$. *Suppose that* $u \in L^q(B(x_0, R))$ *is a function with the property that, whenever* $0 < \rho < r < R$, *we have the bound*

$$\biggl(\int_{B(x_0,\rho)} |u|^q\biggr)^{1/q} \leq \frac{C_0}{(r-\rho)^{d/p_0-d/q}} \biggl(\int_{B(x_0,r)} |u|^{p_0}\biggr)^{1/p_0} + F \tag{34}$$

for some constants C_0 *and* F *depending only on* u.

Then for every p *with* $0 < p \leq p_0$, *there is some constant* $C(p,q)$, *depending only on* p, p_0, q *and* C_0, *such that for any such* ρ *and* r,

$$\biggl(\int_{B(x_0,\rho)} |u|^q\biggr)^{1/q} \leq \frac{C(p,q)}{(r-\rho)^{d/p-d/q}} \biggl(\int_{B(x_0,r)} |u|^p\biggr)^{1/p} + C(p,q)\, F.$$

Proof. Let $\rho = \rho_0 < \rho_1 < \rho_2 < \cdots < r$ for some ρ_k to be chosen momentarily, and let $B_k = B(x_0, \rho_k)$. If $0 < \tau < 1$, then

$$\|u\|_{L^{p_0}(B_k)} = \left(\int_{B_k} |u|^{p_0}\right)^{1/p_0} = \left(\int_{B_k} |u|^{\tau p_0} |u|^{(1-\tau)p_0}\right)^{1/p_0}.$$

If $0 < \tau \leq p/p_0$, then $p/\tau p_0 \geq 1$ and so we may apply Hölder's inequality to see that

$$\|u\|_{L^{p_0}(B_k)} \leq \|u\|^{\tau}_{L^p(B_k)} \|u\|^{1-\tau}_{L^\gamma(B_k)}$$

where γ satisfies $1/p_0 = \tau/p + (1-\tau)/\gamma$. Choose τ so that $\gamma = q$; observe that this means that $\tau = (p/p_0)(q - p_0)/(q - p)$, and thus if $0 < p < p_0 < q$ then τ does satisfy the condition $0 < \tau < p/p_0$.

In order for our estimates to scale correctly, we rewrite this estimate as

$$\frac{\|u\|_{L^{p_0}(B_k)}}{(r-\rho)^{d/p_0}} \leq \left(\frac{\|u\|_{L^p(B_k)}}{(r-\rho)^{d/p}}\right)^{\tau} \left(\frac{\|u\|_{L^q(B_k)}}{(r-\rho)^{d/q}}\right)^{1-\tau}. \tag{35}$$

By the bound (34),

$$\frac{\|u\|_{L^q(B_k)}}{(r-\rho)^{d/q}} \leq \frac{C(p_0,q)\|u\|_{L^{p_0}(B_{k+1})}}{(\rho_{k+1}-\rho_k)^{\beta}(r-\rho)^{d/q}} + \frac{C(p_0)\, F}{(r-\rho)^{d/q}}$$

where we have set $\beta = d/p_0 - d/q$. Notice $\beta > 0$.

Recall that $\rho_0 = \rho$. Let $\rho_{k+1} = \rho_k + (r-\rho)(1-\sigma)\sigma^k$ for some constant $0 < \sigma < 1$ to be chosen momentarily. Notice that $\lim_{k\to\infty} \rho_k = r$. Because $\sigma^{-k\beta} > 1 > (1-\sigma)^{\beta}$, we have that

$$\begin{aligned}\frac{\|u\|_{L^q(B_k)}}{(r-\rho)^{d/q}} &\leq \frac{C(p_0,q)\, F}{(r-\rho)^{d/q}} + \sigma^{-k\beta}\frac{C(p_0,q)\|u\|_{L^{p_0}(B_{k+1})}}{(1-\sigma)^{\beta}(r-\rho)^{d/p_0}} \\ &\leq C(p_0,q,\sigma)\sigma^{-k\beta}\left(\frac{F}{(r-\rho)^{d/q}} + \frac{\|u\|_{L^{p_0}(B_{k+1})}}{(r-\rho)^{d/p_0}}\right).\end{aligned}$$

By the bound (35) and Young's inequality, we have that

$$\begin{aligned}\frac{\|u\|_{L^{p_0}(B_k)}}{(r-\rho)^{d/p_0}} &\leq \tau C(p_0,q,\sigma)\sigma^{-k\beta(1-\tau)/\tau}\frac{\|u\|_{L^p(B_k)}}{(r-\rho)^{d/p}} + (1-\tau)\frac{F}{(r-\rho)^{d/q}} \\ &\quad + (1-\tau)\frac{\|u\|_{L^{p_0}(B_{k+1})}}{(r-\rho)^{d/p_0}}.\end{aligned}$$

Applying this bound to $k = 0$ and iterating, we have that for any integer $K \geq 1$,

$$\begin{aligned}\frac{\|u\|_{L^{p_0}(B_0)}}{(r-\rho)^{d/p}} &\leq \sum_{k=0}^{K}(1-\tau)^k\left(\tau C(p_0,q,\sigma)\sigma^{-k\beta(1-\tau)/\tau}\frac{\|u\|_{L^p(B_k)}}{(r-\rho)^{d/p}}\right) \\ &\quad + \sum_{k=0}^{K}(1-\tau)^k\left((1-\tau)\frac{F}{(r-\rho)^{d/q}}\right) \\ &\quad + (1-\tau)^{K+1}\frac{\|u\|_{L^{p_0}(B_{K+1})}}{(r-\rho)^{d/p_0}}.\end{aligned}$$

We want to take the limit as $K \to \infty$. Choose σ so that $(1-\tau) < \sigma^{\beta(1-\tau)/\tau} < 1$; then the sums converge and we have that

$$\frac{\|u\|_{L^{p_0}(B(x_0,r))}}{(r-\rho)^{d/p}} \leq C(p_0, p, q)\frac{\|u\|_{L^p(B(x_0,r))}}{(r-\rho)^{d/p}} + C(p_0, p, q)\frac{F}{(r-\rho)^{d/q}}.$$

This completes the proof. □

Proof (Proof of Theorem 24). We begin with the bound (27).

Let $x_1 \in \mathbb{R}^d$ and let $\rho > 0$ be such that $B(x_1, 2\rho) \subset B(x_0, R)$. By Lemma 9,

$$\fint_{B(x_1,\rho)} |\nabla^m \mathbf{u}|^2 \leq \sum_{j=1}^{m} \frac{C}{\rho^{2j}} \fint_{B(x_1,(3/2)\rho)} |\nabla^{m-j}\mathbf{u}|^2 + C\fint_{B(x_1,(3/2)\rho)} h^2$$

where $h(x) = |\dot{\boldsymbol{F}}(x)| + \delta^{1/2}|\mathbf{u}(x)|$. (Recall that $\mathbf{u} = 0$ in $B(x_0, R)\setminus\Omega$; we may also take $\dot{\boldsymbol{F}} = 0$ in $B(x_0, R)\setminus\Omega$.)

If $B(x_1, (3/2)\rho) \subset \Omega$, then we normalize $\mathbf{u}$ by adding polynomials, so that $\fint_{B(x_1,(3/2)\rho)} \nabla^i \mathbf{u} = 0$ for all $0 \leq i \leq m-1$; if $L\mathbf{u} = \operatorname{div}_m \dot{\boldsymbol{F}}$ in all of $B(x_1, (3/2)\rho)$ then the above bound is still valid. We may then apply the Poincaré inequality to control the integral of $\nabla^{m-j}\mathbf{u}$ by the integral of $\nabla^{m-1}\mathbf{u}$. Thus,

$$\fint_{B(x_1,\rho)} |\nabla^m \mathbf{u}|^2 \leq \frac{C}{\rho^2} \fint_{B(x_1,(3/2)\rho)} |\nabla^{m-1}\mathbf{u}|^2 + C\fint_{B(x_1,(3/2)\rho)} h^2.$$

Now, let $2'_1 = 2d/(d+2)$. By Lemma 30,

$$\left(\fint_{B(x_1,(3/2)\rho)} |\nabla^{m-1}\mathbf{u}|^2\right)^{1/2} \leq C\rho\left(\fint_{B(x_1,(3/2)\rho)} |\nabla^m \mathbf{u}|^{2'_1}\right)^{1/2'_1} + C\left(\fint_{B(x_1,(3/2)\rho)} |\nabla^{m-1}\mathbf{u}|^{2'_1}\right)^{1/2'_1}.$$

Using the Poincaré inequality and the assumption that $\fint_{B(x_1,2\rho)} \nabla^{m-1}\mathbf{u} = 0$, we may control the second term on the right-hand side by the first; we thus have the bound

$$\left(\fint_{B(x_1,\rho)} |\nabla^m \mathbf{u}|^2\right)^{1/2} \leq C\left(\fint_{B(x_1,(3/2)\rho)} |\nabla^m \mathbf{u}|^{2'_1}\right)^{1/2'_1} + C\left(\fint_{B(x_1,(3/2)\rho)} h^2\right)^{1/2}.$$

If $B(x_1, (3/2)\rho) \not\subset \Omega$, then there is some $x_2 \in \partial\Omega \cap B(x_1, (3/2)\rho)$. By our assumption on Ω,

$$2^{-d} c_\Omega \rho^d \leq |B(x_2, \rho/2)\setminus\Omega| \leq |B(x_1, 2\rho)\setminus\Omega|.$$

Then $\nabla^{m-j}\mathbf{u} = 0$ in the substantial set $B(x_1, 2\rho)\setminus\Omega$ for all j. Thus, we may use the Poincaré inequality in $B(x_1, 2\rho)$ without renormalizing $\mathbf{u}$. Arguing as before we have the bound

$$\left(\fint_{B(x_1,\rho)} \left|\nabla^m \mathbf{u}\right|^2\right)^{1/2} \leq C\left(\fint_{B(x_1,2\rho)} \left|\nabla^m \mathbf{u}\right|^{2_1'}\right)^{1/2_1'} + C\left(\fint_{B(x_1,2\rho)} h^2\right)^{1/2}.$$

Observe that $2_1' < 2$. Thus we have established a reverse Hölder inequality. In particular, the bound (27) is valid for $R = 2r = 2\rho$, for $q = 2$ and for $p = 2_1'$.

We now use Lemma 32 to improve to $q > 2$. Observe that we may cover $B(x_0, r)$ by a grid of cubes Q_j, $1 \leq j \leq J$, with side-length $\ell(Q_j) = (R-r)/2c_0$, with pairwise-disjoint interiors. If we choose c_0 large enough (depending on the dimension), then $2Q_j \subset B(x_0, R)$ for all j. We then have that, for any p,

$$\int_{B(x_0,r)} |\nabla^m \mathbf{u}|^p \leq \sum_{j=1}^{J} \int_{Q_j} |\nabla^m \mathbf{u}|^p.$$

Fix some j. Let $g(x) = |\nabla^m \mathbf{u}(x)|^{2_1'}$, and let $f(x) = h(x)^{2_1'}$. Let $p = 2/2_1'$; notice $p > 1$.

If $x_1 \in Q_j$, and if $0 < \rho < \operatorname{dist}(x_1, \partial Q_j)/2$, then

$$\begin{aligned}
\fint_{B(x_1,\rho)} g^p &= \fint_{B(x_1,\rho)} |\nabla^m \mathbf{u}(x)|^2 \\
&\leq C\left(\fint_{B(x_1,2\rho)} \left|\nabla^m \mathbf{u}\right|^{2_1'}\right)^{2/2_1'} + C \fint_{B(x_1,2\rho)} h^2 \\
&= C\left(\fint_{B(x_1,2\rho)} g\right)^p + C \fint_{B(x_1,2\rho)} f^p.
\end{aligned}$$

Thus Lemma 32 applies, and so there is some $q^+ > 2$ such that

$$\left(\fint_{Q_j} |\nabla^m \mathbf{u}|^q\right)^{1/q} \leq C(q)\left(\fint_{2Q_j} |\nabla^m \mathbf{u}|^2\right)^{1/2} + C(q)\left(\fint_{2Q_j} h^q\right)^{1/q}$$

for all q with $2 < q < q^+$. Thus,

$$\begin{aligned}
\int_{B(x_0,r)} |\nabla^m \mathbf{u}|^q &\leq \sum_{j=1}^{J} \int_{Q_j} |\nabla^m \mathbf{u}|^q \\
&\leq \sum_{j=1}^{J} \frac{C(q)}{\ell(Q_j)^{dq/2-d}} \left(\int_{2Q_j} |\nabla^m \mathbf{u}|^2\right)^{q/2} + C(q) \sum_{j=1}^{J} \int_{2Q_j} h^q.
\end{aligned}$$

Recall that $\ell(Q_j) = (R-r)/2c_0$. Observe that almost every $x \in B(x_0, R)$ is in at most 2^d of the cubes $2Q_j$; thus,

$$\int_{B(x_0,r)} |\nabla^m \mathbf{u}|^q \leq \frac{C(q)}{(R-r)^{dq/2-d}} \left(\int_{B(x_0,R)} |\nabla^m \mathbf{u}|^2\right)^{q/2} + C(q) \int_{B(x_0,R)} h^q$$

as desired.

Applying Lemma 33, we see that we may replace the exponent 2 by any exponent p with $0 < p < 2$; this completes the proof of the bound (27).

Now, suppose that $0 < k < d/2$. We wish to prove the bound (28). We apply Lemma 30 to v a component of $\nabla^{m-k}\mathbf{u}$. This gives us the bound

$$\left(\int_{B(x_1,\rho)} |\nabla^{m-k}\mathbf{u}|^{q_k}\right)^{1/q_k} \leq C\sum_{i=0}^{k} \rho^{-i}\left(\int_{B(x_1,\rho)} |\nabla^{m-i}\mathbf{u}|^q\right)^{1/q}.$$

We have that

$$\left(\int_{B(x_1,\rho)} |\nabla^{m-i}\mathbf{u}|^q\right)^{1/q} \leq \left(\int_{B(x_1,\rho)} |\nabla^{m-i}\mathbf{u} - \fint_{B(x_1,\rho)} \nabla^{m-i}\mathbf{u}|^q\right)^{1/q} + C\rho^{d/q}|\fint_{B(x_1,\rho)} \nabla^{m-i}\mathbf{u}|$$

and so by the Poincaré inequality

$$\left(\int_{B(x_1,\rho)} |\nabla^{m-i}\mathbf{u}|^q\right)^{1/q} \leq C\rho\left(\int_{B(x_1,\rho)} |\nabla^{m-i+1}\mathbf{u}|^q\right)^{1/q} + C\rho^{d/q-d}\int_{B(x_1,\rho)} |\nabla^{m-i}\mathbf{u}|.$$

Iterating, we see that

$$\left(\int_{B(x_1,\rho)} |\nabla^{m-k}\mathbf{u}|^{q_k}\right)^{1/q_k} \leq C\left(\int_{B(x_1,\rho)} |\nabla^{m}\mathbf{u}|^q\right)^{1/q} + C\sum_{i=0}^{k} \rho^{-i+d/q-d}\int_{B(x_1,\rho)} |\nabla^{m-i}\mathbf{u}|.$$

Applying the known results for $\nabla^m\mathbf{u}$ and Corollary 22 or 23, we see that

$$\left(\int_{B(x_1,\rho)} |\nabla^{m-k}\mathbf{u}|^{q_k}\right)^{1/q_k} \leq \frac{C(q)}{\rho^{d/2-d/q+m}}\left(\int_{B(x_1,(3/2)\rho)} |\mathbf{u}|^2\right)^{1/2} + C(q)\left(\int_{B(x_1,(3/2)\rho)} h^q\right)^{1/q}.$$

As before, we either normalize $\mathbf{u}$ in $B(x_1, (3/2)\rho)$ by adding polynomials of degree $m - k - 1$ or observe that $\mathbf{u}$ and all its derivatives are zero on a substantial subset of $B(x_1, 2\rho)$; in either case we may use the Poincaré inequality to control $\mathbf{u}$ by $\nabla^{m-k}\mathbf{u}$. This yields the bound

$$\left(\int_{B(x_1,\rho)} |\nabla^{m-k}\mathbf{u}|^{q_k}\right)^{1/q_k} \leq \frac{C(q)}{\rho^{d/2-d/q+k}}\left(\int_{B(x_1,2\rho)} |\nabla^{m-k}\mathbf{u}|^2\right)^{1/2} + C(q)\left(\int_{B(x_1,2\rho)} h^q\right)^{1/q}.$$

By Hölder's inequality we may replace the exponent 2 by the exponent p_k provided $p_k \geq 2$. Using standard covering lemmas, if $q_k \geq \max(p_k, q)$ then we may improve to the estimate

$$\left(\fint_{B(x_1,\rho)} |\nabla^{m-k}\mathbf{u}|^{q_k}\right)^{1/q_k} \leq \frac{C(q)}{(r-\rho)^{d/p_k - d/q + k}} \left(\fint_{B(x_0,r)} |\nabla^{m-k}\mathbf{u}|^{p_k}\right)^{1/p_k} + C(q)\left(\fint_{B(x_0,r)} h^q\right)^{1/q}.$$

By Lemma 33 this inequality is still valid for $0 < p_k < 2$.

Identical arguments, using Lemma 31 in place of Lemma 30, establish the bound (29) on $\sup|\nabla^{m-k}\mathbf{u}|$ in the case $k > d/q$. □

In some domains we may also prove a boundary reverse Hölder estimate in the Neumann case.

Theorem 36. *Let Ω be a Lipschitz graph domain, that is, a domain of the form*

$$\Omega = \{(x', t) : x' \in \mathbb{R}^{d-1},\ t > \varphi(x')\}$$

for some function $\varphi : \mathbb{R}^{d-1} \mapsto \mathbb{R}$ with $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^{d-1})} = M < \infty$.

Let L be an operator of order $2m$ that satisfies the bound (6) and the bound (17) in Ω.

Then there is some number $p^+ = p^+_L > 2$ depending only on the standard constants and the number $M = \|\nabla\varphi\|_{L^\infty(\mathbb{R}^{d-1})}$ such that the following statement is true.

Let $x_0 \in \partial\Omega$ and let $R > 0$. Suppose that $\mathbf{u} \in W^2_m(B(x_0, R))$, that $\dot{\boldsymbol{F}} \in L^2(B(x_0, R))$, and that

$$\langle \nabla^m \boldsymbol{\varphi}, \boldsymbol{A}\nabla^m \mathbf{u}\rangle_\Omega = \langle \nabla^m \boldsymbol{\varphi}, \dot{\boldsymbol{F}}\rangle_\Omega$$

for all smooth functions $\boldsymbol{\varphi}$ supported in $B(x_0, R)$.

Then

$$\left(\fint_{B(x_0,r)\cap\Omega} |\nabla^m\mathbf{u}|^q\right)^{1/q} \leq \frac{C(M,p,q)}{(R-r)^{d/p-d/q}} \left(\int_{B(x_0,R)\cap\Omega} |\nabla^m\mathbf{u}|^p\right)^{1/p} + C(M,p,q)\left(\int_{B(x_0,R)\cap\Omega} |\dot{\boldsymbol{F}}|^q + \delta^{q/2}|\mathbf{u}|^q\right)^{1/q} \quad (37)$$

for some constant $C(M, p, q)$ depending only on p, q, M and the standard parameters.

Proof. If $x_1 = (x_1', t_1) \in \mathbb{R}^d$ and $\rho > 0$, then let $Q(x_1, \rho)$ be the Lipschitz cylinder

$$Q(x_1, \rho) = \{(x', t) : |x' - x_1'| < \rho,\ \varphi(x') + t_1 - \rho < t < \varphi(x') + t_1 + \rho\}.$$

Using either covering lemmas or a bilipschitz change of variables, we see that many results stated in terms of balls are valid in Lipschitz cylinders. In particular, Lemma 16, the Poincaré inequality, and the first-order Gagliardo–Nirenberg–Sobolev inequality

$$\|v\|_{L^{q_1}(Q(x_0,\rho))} \leq C\|\nabla v\|_{L^q(Q(x_0,\rho))} + C\rho\|v\|_{L^q(Q(x_0,\rho))},$$

Lemma 32, and Lemma 33 are valid in Lipschitz cylinders.

We now proceed much as in the proof of the estimate (27) of Theorem 24. Let $x_1 \in \mathbb{R}^d$ and let $\rho > 0$ be such that $Q(x_1, 2\rho) \subset B(x_0, R)$. By Lemma 16,

$$\left(\fint_{Q(x_1,\rho)} \mathbf{1}_\Omega |\nabla^m \mathbf{u}|^2\right)^{1/2} \leq \sum_{j=1}^m \frac{C}{\rho^{2j}} \left(\fint_{Q(x_1,(3/2)\rho)} \mathbf{1}_\Omega |\nabla^{m-j} \mathbf{u}|^2\right)^{1/2} + C\left(\fint_{Q(x_1,(3/2)\rho)} h^2\right)^{1/2}$$

where $h(x) = |\dot{\boldsymbol{F}}(x)| + \delta^{1/2}|\mathbf{u}(x)|$ in Ω and is zero outside Ω.

Notice that we may normalize $\mathbf{u}$ by adding polynomials, regardless of whether $Q(x_1, (3/2)\rho)$ is contained in Ω. If $Q(x_1, (3/2)\rho) \subset \Omega$, then may establish the reverse Hölder inequality

$$\left(\fint_{Q(x_1,\rho)} \mathbf{1}_\Omega |\nabla^m \mathbf{u}|^2\right)^{1/2} \leq C\left(\fint_{Q(x_1,(3/2)\rho)} \mathbf{1}_\Omega |\nabla^m \mathbf{u}|^{2'_1}\right)^{1/2'_1} + C\left(\fint_{Q(x_1,(3/2)\rho)} h^2\right)^{1/2}$$

as in the proof of Theorem 24. If $Q(x_1, (3/2)\rho) \not\subset \Omega$, either $Q(x_1, (3/2)\rho) \cap \Omega = \emptyset$ and so this reverse Hölder inequality is trivially true, or $Q(x_1, 2\rho) \cap \Omega$ is substantial. Specifically, in this final case there exists some c with $4/3 < c < 8$ such that the map $(x, t) \mapsto (x, ct)$ sends $Q(x_1, 2\rho) \cap \Omega$ to a Lipschitz cylinder. Thus, Lemma 30 and the Poincaré inequality are valid in $Q(x_1, 2\rho) \cap \Omega$ with constants independent of x_1 and ρ, and so we see that

$$\left(\fint_{Q(x_1,\rho)} \mathbf{1}_\Omega |\nabla^m \mathbf{u}|^2\right)^{1/2} \leq C\left(\fint_{Q(x_1,2\rho)} \mathbf{1}_\Omega |\nabla^m \mathbf{u}|^{2'_1}\right)^{1/2'_1} + C\left(\fint_{Q(x_1,2\rho)} h^2\right)^{1/2}.$$

This establishes a reverse Hölder inequality with $q = 2$ and $p = 2'_1$; as in the proof of Theorem 24, we may use Lemmas 32 and 33 and covering lemmas to improve to arbitrary p, q and to return to balls of radii r and R. □

5. The fundamental solution

In this section we will construct the fundamental solution for elliptic systems of arbitrary order $2m \geq 2$ in dimension $d \geq 2$. As in [23,24], we will construct the fundamental solution as the kernel of the solution operator to the equation $L\mathbf{u} = \operatorname{div}_m \dot{\boldsymbol{F}}$.

Specifically, in Sect. 5.1 we will construct this solution operator using the Lax-Milgram lemma and will discuss its adjoint. In Sect. 5.2 we will construct a preliminary version of the fundamental solution in the case of operators of high order.

In Sect. 5.3 we will refine our construction to produce some desirable additional properties, and finally in Sect. 5.4 we will extend these results to operators of arbitrary even order. A summary of the principal results concerning the fundamental solution is collected at the beginning of Sect. 5.4.

An important estimate in this section will be the norm estimate

$$\lambda\|\mathbf{u}\|^2_{\dot W^2_m(\mathbb{R}^d)} \leq \operatorname{Re}\big\langle \nabla^m \mathbf{u}, \boldsymbol{A}\nabla^m \mathbf{u}\big\rangle_{\mathbb{R}^d}.$$

This estimate is valid if the coefficients $\boldsymbol{A}$ are elliptic in the sense of the bound (7) [*not* the weaker sense of the bound (10)] and if we take the norm of $\mathbf{u}$ in a homogeneous space. Thus, in this section, we will work with strongly elliptic coefficients and with the homogeneous Sobolev spaces

$$\dot W^p_k(U) = \Big\{\mathbf{u} : \|\nabla^k \mathbf{u}\|_{L^p(U)} < \infty\Big\}.$$

5.1. The Newton potential

In this section we will construct the Newton potential, that is, the operator whose kernel is the fundamental solution. The Newton potential $\mathbf{u} = \boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$ is defined as the solution to $L\mathbf{u} = \operatorname{div}_m \dot{\boldsymbol{F}}$ in $\mathbb{R}^d$. If $\dot{\boldsymbol{F}} \in L^2(\mathbb{R}^d)$, then we may construct $\boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$ as follows.

Recall the (complex) Lax-Milgram lemma:

Theorem 38. ([6, Theorem 2.1]). *Let H_1 and H_2 be two Hilbert spaces, and let B be a bounded bilinear form on $H_1 \times H_2$ that is coercive in the sense that*

$$\sup_{w\in H_1\setminus\{0\}} \frac{|B(w,v)|}{\|w\|_{H_1}} \geq \lambda\|v\|_{H_2}, \qquad \sup_{w\in H_2\setminus\{0\}} \frac{|B(u,w)|}{\|w\|_{H_2}} \geq \lambda\|u\|_{H_1}$$

for every $u \in H_1$, $v \in H_2$, for some fixed $\lambda > 0$. Then for every linear functional T defined on H_1 there is a unique $u_T \in H_2$ such that $B(v, u_T) = \overline{T(v)}$. Furthermore, $\|u_T\|_{H_2} \leq \frac{1}{\lambda}\|T\|_{H_1'}$.

Let L be an operator of order $2m$ that is elliptic in the sense that the coefficients satisfy the conditions (6) and (7). Suppose that $\dot{\boldsymbol{F}} = \{F_{j,\alpha} : 1 \leq j \leq N, |\alpha| = m\}$ is an array of functions all lying in $L^2(\mathbb{R}^d)$. Then $T_{\dot{\boldsymbol{F}}}(\mathbf{v}) = \big\langle \dot{\boldsymbol{F}}, \nabla^m \mathbf{v}\big\rangle_{\mathbb{R}^d}$ is a bounded linear operator on the Hilbert space $\dot W^2_m(\mathbb{R}^d)$. We choose $B(\mathbf{w}, \mathbf{v}) = \big\langle \nabla^m \mathbf{w}, \boldsymbol{A}\nabla^m \mathbf{v}\big\rangle_{\mathbb{R}^d}$; by our ellipticity conditions (6) and (7), B is bounded and coercive on $\dot W^2_m(\mathbb{R}^d)$. Let $\boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$ be the element u_T of $\dot W^2_m(\mathbb{R}^d)$ given by the Lax-Milgram lemma. Then

$$\big\langle \nabla^m \boldsymbol{\varphi}, \boldsymbol{A}\nabla^m (\boldsymbol{\Pi}^L \dot{\boldsymbol{F}})\big\rangle_{\mathbb{R}^d} = \big\langle \nabla^m \boldsymbol{\varphi}, \dot{\boldsymbol{F}}\big\rangle_{\mathbb{R}^d} \tag{39}$$

for all $\boldsymbol{\varphi} \in \dot W^2_m(\mathbb{R}^d \mapsto \mathbb{C}^N)$.

We will need some properties of the Newton potential $\boldsymbol{\Pi}^L$. First, by the uniqueness of solutions provided by the Lax-Milgram lemma, $\boldsymbol{\Pi}^L$ is a well-defined operator; furthermore, $\boldsymbol{\Pi}^L$ is linear and bounded $L^2(\mathbb{R}^d) \mapsto \dot W^2_m(\mathbb{R}^d)$.

Next, observe that if $\boldsymbol{\Phi} \in \dot{W}^2_m(\mathbb{R}^d \mapsto \mathbb{C}^N)$, then by uniqueness of solutions to $L\mathbf{u} = \operatorname{div}_m \dot{\boldsymbol{F}}$,

$$\boldsymbol{\Pi}^L(\boldsymbol{A}\nabla^m\boldsymbol{\Phi}) = \boldsymbol{\Phi} \tag{40}$$

as $\dot{W}^2_m(\mathbb{R}^d \mapsto \mathbb{C}^N)$-functions, that is, up to adding polynomials of order $m-1$.

Next, we wish to show that the adjoint $(\nabla^m\boldsymbol{\Pi}^L)^*$ to the operator $\nabla^m\boldsymbol{\Pi}^L$ is $\nabla^m\boldsymbol{\Pi}^{L^*}$. To prove this we will need the following elementary result; this will let us identify vector fields that arise as mth-order gradients.

Lemma 41. *Let $\big(f_\alpha\big)_{|\alpha|=m}$ be a set of functions in $L^1_{loc}(\Omega)$, where Ω is a simply connected domain. Suppose that whenever $\alpha + \mathbf{e}_k = \beta + \mathbf{e}_j$, we have that*

$$\big\langle \partial_j\varphi, f_\beta\big\rangle_\Omega = \big\langle \partial_k\varphi, f_\alpha\big\rangle_\Omega$$

for all φ smooth and compactly supported in Ω.

Then there is some function $f \in \dot{W}^1_{m,loc}(\Omega)$ such that $f_\alpha = \partial^\alpha f$ for all α.

Proof. If $m = 1$ and the functions f_α are C^1, then this lemma is merely the classical result that irrotational vector fields may be written as gradients. We begin by generalizing to the case $m = 1$ and the case $f_\alpha \in L^1_{loc}(\Omega)$. We let $f_j = f_{\mathbf{e}_j}$.

Let η be a smooth, nonnegative function supported in $B(0,1)$ with $\int \eta = 1$, and let $\eta_\varepsilon(x) = \varepsilon^{-d}\eta(x/\varepsilon)$. Let $f^\varepsilon_j = f_j * \eta_\varepsilon$, so that f^ε_j is smooth. By assumption, $\partial_k f^\varepsilon_j(x) = \partial^j f^\varepsilon_k(x)$ provided $\varepsilon < \operatorname{dist}(x, \Omega^C)$. Let B be a ball with $\overline{B} \subset \Omega$, and assume that $\varepsilon < \operatorname{dist}(B, \Omega^C)/2$. Then there is some function f^ε such that $\partial^j f^\varepsilon = f^\varepsilon_j$ in B.

Now renormalize f^ε so that $\int_B f^\varepsilon = 0$. By Lemma 30, because $\nabla f^\varepsilon \in L^1(B)$, we have that $f^\varepsilon \in L^p(B)$, uniformly in ε, for some $p > 1$. Since $L^p(B)$ is weakly sequentially compact, we have that some subsequence f^{ε_i} has a weak limit f.

If φ is smooth and supported in B, then

$$\big\langle \partial^j\varphi, f\big\rangle_B = \lim_{i\to\infty}\big\langle \partial^j\varphi, f^{\varepsilon_i}\big\rangle_B = -\lim_{i\to\infty}\big\langle \varphi, f^{\varepsilon_i}_j\big\rangle_B = -\big\langle \varphi, f_j\big\rangle_B$$

and so f_j is the weak derivative of f in the jth direction for all $1 \leq j \leq d$.

We may cover any compact subset $\overline{V} \subset \Omega$ by such balls B; renormalizing f again, so as to be defined compatibly on different balls, we see that we may extend f to a function in $L^1_{loc}(\Omega)$.

Now we work by induction. Suppose that the theorem is true for $m = 1$ and for $m = M-1$. We wish to show that the theorem is true for $m = M$ as well.

Fix some γ with $|\gamma| = M-1$, and let $f_j = f_{\gamma+\mathbf{e}_j}$. By assumption

$$\big\langle \partial_k\varphi, f_j\big\rangle_\Omega = \big\langle \partial_k\varphi, f_{\gamma+\mathbf{e}_j}\big\rangle_\Omega = \big\langle \partial_j\varphi, f_{\gamma+\mathbf{e}_k}\big\rangle_\Omega = \big\langle \partial_j\varphi, f_k\big\rangle_\Omega$$

for all appropriate test functions φ.

Because the theorem is valid for $m = 1$, there is some $f = f_\gamma \in \dot{W}^1_{1,loc}(\Omega)$ such that $\partial_j f_\gamma = f_{\gamma+\mathbf{e}_j}$ in the weak sense.

If $|\gamma| = |\delta| = M-1$, and $\gamma + \mathbf{e}_j = \delta + \mathbf{e}_k$, then

$$\big\langle \partial_j\varphi, f_\gamma\big\rangle_\Omega = -\big\langle \varphi, f_{\gamma+\mathbf{e}_j}\big\rangle_\Omega = -\big\langle \varphi, f_{\delta+\mathbf{e}_k}\big\rangle_\Omega = \big\langle \partial_k\varphi, f_\delta\big\rangle_\Omega$$

and so the array $\left(f_\gamma\right)_{|\gamma|=M-1}$ satisfies the conditions of the theorem with $m = M-1$. Because the theorem is true for $m = M-1$, we have that there is some $f \in \dot{W}^1_{M-1,loc}(\Omega)$ such that $f_\gamma = \partial^\gamma f$ for all $|\gamma| = M-1$; because $\partial_k f_\gamma = f_{\gamma+\mathbf{e}_k}$ we have that $f_\alpha = \partial^\alpha f$ for all $|\alpha| = m$, and so the theorem is true for $m = M$ as well. This completes the proof. □

We now consider the adjoint operator to the Newton potential.

Lemma 42. *The adjoint* $(\nabla^m \mathbf{\Pi}^L)^*$ *to the operator* $\nabla^m \mathbf{\Pi}^L$ *is* $\nabla^m \mathbf{\Pi}^{L^*}$.

Proof. Observe that $\nabla^m \mathbf{\Pi}^L$ is bounded on $L^2(\mathbb{R}^d)$ and so $(\nabla^m \mathbf{\Pi}^L)^*$ is as well; that is, $(\nabla^m \mathbf{\Pi}^L)^* \dot{\boldsymbol{F}}$ is an element of $L^2(\mathbb{R}^d)$. We first show that it is an element of the subspace of gradients of $\dot{W}^2_m(\mathbb{R}^d)$-functions, that is, that there is some function $\mathbf{u} \in \dot{W}^2_m(\mathbb{R}^d)$ such that $(\nabla^m \mathbf{\Pi}^L)^* \dot{\boldsymbol{F}} = \nabla^m \mathbf{u}$.

By Lemma 41, it suffices to show that if $1 \leq i \leq N$, if φ is smooth and compactly supported in Ω, and if $\alpha + \mathbf{e}_k = \beta + \mathbf{e}_j$, then

$$\left\langle \partial_j \varphi \, \dot{\boldsymbol{e}}_{i,\beta}, (\nabla^m \mathbf{\Pi}^L)^* \dot{\boldsymbol{F}} \right\rangle_\Omega = \left\langle \partial_k \varphi \, \dot{\boldsymbol{e}}_{i,\alpha}, (\nabla^m \mathbf{\Pi}^L)^* \dot{\boldsymbol{F}} \right\rangle_\Omega.$$

That is, we seek to show that

$$\left\langle \nabla^m \mathbf{\Pi}^L (\partial_j \varphi \, \dot{\boldsymbol{e}}_{i,\beta} - \partial_k \varphi \, \dot{\boldsymbol{e}}_{i,\alpha}), \dot{\boldsymbol{F}} \right\rangle_\Omega = 0.$$

But $\left\langle \nabla^m \boldsymbol{\eta}, \partial_j \varphi \, \dot{\boldsymbol{e}}_{i,\beta} - \partial_k \varphi \, \dot{\boldsymbol{e}}_{i,\alpha} \right\rangle_{\mathbb{R}^d} = 0$ for all $\boldsymbol{\eta}$ smooth and compactly supported, and so $\mathbf{\Pi}^L(\partial_j \varphi \, \dot{\boldsymbol{e}}_{i,\beta} - \partial_k \varphi \, \dot{\boldsymbol{e}}_{i,\alpha}) = 0$.

Let $\mathbf{u}$ satisfy $\nabla^m \mathbf{u} = (\nabla^m \mathbf{\Pi}^L)^* \dot{\boldsymbol{F}}$. We now show that $\mathbf{u} = \mathbf{\Pi}^{L^*} \dot{\boldsymbol{F}}$. Choose some $\boldsymbol{\varphi}$ smooth and compactly supported in $\mathbb{R}^d$. Then

$$\begin{aligned} \left\langle \nabla^m \boldsymbol{\varphi}, \boldsymbol{A}^* \nabla^m \mathbf{u} \right\rangle_{\mathbb{R}^d} &= \left\langle \boldsymbol{A} \nabla^m \boldsymbol{\varphi}, (\nabla^m \mathbf{\Pi}^L)^* \dot{\boldsymbol{F}} \right\rangle_{\mathbb{R}^d} \\ &= \left\langle \nabla^m \mathbf{\Pi}^L (\boldsymbol{A} \nabla^m \boldsymbol{\varphi}), \dot{\boldsymbol{F}} \right\rangle_{\mathbb{R}^d}. \end{aligned}$$

By formula (40), we have that $\nabla^m \mathbf{\Pi}^L(\boldsymbol{A} \nabla^m \boldsymbol{\varphi}) = \nabla^m \boldsymbol{\varphi}$. Thus

$$\left\langle \nabla^m \boldsymbol{\varphi}, \boldsymbol{A}^* \nabla^m \mathbf{u} \right\rangle_{\mathbb{R}^d} = \left\langle \nabla^m \boldsymbol{\varphi}, \dot{\boldsymbol{F}} \right\rangle_{\mathbb{R}^d}$$

for all $\boldsymbol{\varphi}$ smooth and compactly supported. Because $\mathbf{\Pi}^{L^*} \dot{\boldsymbol{F}}$ is the unique element of $\dot{W}^2_m(\mathbb{R}^d)$ with this property, we must have that $\mathbf{u} = \mathbf{\Pi}^{L^*} \dot{\boldsymbol{F}}$ and the proof is complete. □

We conclude this section by showing that the Newton potential is bounded on a range of L^p spaces.

Lemma 43. *Let* L *be an operator of order* $2m$ *that satisfies the bounds* (6) *and* (7), *and let* p_L^+ *be as in Theorem* 24. *Let* $1/p_L^+ + 1/p_L^- = 1$. *If* $p_{L^*}^- < p < p_L^+$, *then* $\mathbf{\Pi}^L$ *extends to an operator that is bounded* $L^p(\mathbb{R}^d) \mapsto \dot{W}^p_m(\mathbb{R}^d)$.

Proof. Suppose first that $2 < p < p_L^+$. Let $\dot{\boldsymbol{F}} \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and let $\mathbf{u} = \boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$. By Theorem 24,

$$\left(\int_{B(x_0,r)} |\nabla^m \mathbf{u}|^p\right)^{1/p} \leq \frac{C(p)}{r^{d/2-d/p}} \left(\int_{B(x_0,2r)} |\nabla^m \mathbf{u}|^2\right)^{1/2} + C(p)\left(\int_{B(x_0,2r)} |\dot{\boldsymbol{F}}|^p\right)^{1/p}.$$

By taking the limit as $r \to \infty$, we see that $\|\nabla^m \boldsymbol{\Pi}^L \dot{\boldsymbol{F}}\|_{L^p(\mathbb{R}^d)} \leq C(p)\|\dot{\boldsymbol{F}}\|_{L^p(\mathbb{R}^d)}$, and so $\boldsymbol{\Pi}^L$ extends to an operator that is bounded $L^p(\mathbb{R}^d) \mapsto \dot{W}^p_m(\mathbb{R}^d)$.

By a similar argument $\nabla^m \boldsymbol{\Pi}^{L^*}$ is bounded $L^{p'}(\mathbb{R}^d) \mapsto L^{p'}(\mathbb{R}^d)$ for all $2 < p' < p_{L^*}^+$; thus by duality $\nabla^m \boldsymbol{\Pi}^L$ is bounded $L^p(\mathbb{R}^d) \mapsto L^p(\mathbb{R}^d)$ for all $p_{L^*}^- < p < 2$, as desired. □

5.2. The fundamental solution for operators of high order

This section will be devoted to the proof of the following theorem.

Theorem 44. *Let L be an operator of order $2m > d$ whose coefficients satisfy the bounds (6) and (7). For each $z_0 \in \mathbb{R}^d$ and each $r > 0$, there exist functions $E^L_{j,k,z_0,r}(x, y)$ with the following properties.*

First, if $x \in \mathbb{R}^d$ and $|\beta| = m$, then $f(y) = \partial_y^\beta E^L_{j,k,z_0,r}(x, y)$ lies in $L^2(\mathbb{R}^d)$, and if $\dot{\boldsymbol{F}} \in L^2(\mathbb{R}^d)$, then for all $1 \leq j \leq N$, we have that

$$\Pi_j^L \dot{\boldsymbol{F}}(x) = \sum_{k=1}^{N} \sum_{|\beta|=m} \int_{\mathbb{R}^d} \partial_y^\beta E^L_{j,k,z_0,r}(x, y)\, F_{k,\beta}(y)\, dy \tag{45}$$

as $\dot{W}_2^m(\mathbb{R}^d)$-functions, that is, up to adding polynomials of order $m-1$.

Next, for any x_0 and y_0, we have the bounds

$$\int_{B(x_0,r)} \int_{B(y_0,r)} |\nabla_x^m \nabla_y^m E^L_{j,k,z_0,r}(x, y)|^2\, dy\, dx \leq C, \quad r = |x_0 - y_0|/3. \tag{46}$$

If $1 \leq j \leq N$, $1 \leq k \leq N$, and if α, β are multiindices with $|\alpha| = |\beta| = m$, then

$$\partial_x^\alpha \partial_y^\beta E^{L^*}_{j,k,z_0,r}(x, y) = \overline{\partial_y^\beta \partial_x^\alpha E^L_{k,j,z_0,r}(y, x)}. \tag{47}$$

Finally, if $|x_0 - z_0| = |y_0 - z_0| = |x_0 - y_0| = 3r$, then we have the bounds

$$\int_{B(x_0,r)} \int_{B(y_0,r)} |\nabla_x^{m-q} \nabla_y^{m-s} E^L_{j,k,z_0,r}(x, y)|^2\, dy\, dx \leq C r^{2q+2s} \tag{48}$$

whenever $0 \leq q \leq m$ and $0 \leq s \leq m$.

By uniqueness of the Newton potential $\boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$ in $\dot{W}^2_m(\mathbb{R}^d)$, the array of highest-order derivatives $\nabla^m_x \nabla^m_y E^L_{j,k}(x, y)$ is unique; however, there are many possible normalizations of the lower-order derivatives $\nabla^{m-q}_x \nabla^{m-s}_y E^L_{j,k}(x, y)$. In Sect. 5.3 we will discuss some natural normalization conditions. In Sect. 5.4 we will extend this theorem to operators of order $2m \le d$.

We will now prove Theorem 44. We begin by constructing a fundamental solution $\boldsymbol{E}^L(x, y)$. For our preliminary argument, we will need $\boldsymbol{\Pi}^L \dot{\boldsymbol{F}}(x)$ to be well-defined for any specified x; that is, we will need to assume that $\boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$ is always continuous. By Lemma 31 if $\nabla^m \boldsymbol{\Pi}^L \dot{\boldsymbol{F}} \in L^2(\mathbb{R}^d)$ and $m > d/2$ then $\boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$ is continuous. It is for this reason that we begin with operators of order $2m > d$.

Recall that even if $\boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$ is continuous, it is still defined only up to adding polynomials of order $m - 1$. We will fix a normalization of $\boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$ as follows. Choose some points $h_1, h_2, \ldots h_q \in \mathbb{R}^d$ with $|h_i| = 1$, where q is the number of multiindices γ with $|\gamma| \le m - 1$. If the h_is are chosen appropriately, then for any numbers a_i, there is a unique polynomial $P(x) = \sum_{|\gamma| \le m-1} p_\gamma\, x^\gamma$, of order at most $m - 1$, such that $P(h_i) = a_i$ for all $1 \le i \le p$. Furthermore, there is some constant H depending only on our choice of h_i such that the bound $|p_\gamma| \le H \sup_i |a_i|$ is valid.

Now, choose some $z_0 \in \mathbb{R}^d$ and some $r > 0$. We fix an additive normalization of $\boldsymbol{\Pi}^L = \boldsymbol{\Pi}^L_{z_0,r}$ by requiring $\boldsymbol{\Pi}^L_{z_0,r} \dot{\boldsymbol{F}}(z_0 + r\, h_i) = 0$ for all $1 \le i \le q$.

Let $x \in \mathbb{R}^d$. Define $\mathbf{S}_x \dot{\boldsymbol{F}} = \boldsymbol{\Pi}^L_{z_0,r} \dot{\boldsymbol{F}}(x)$. Then $\mathbf{S}_x$ is a linear operator. We will use the Riesz representation theorem to construct the fundamental solution as the kernel of $\mathbf{S}_x$; to do this, we will need to establish boundedness of $\mathbf{S}_x$.

We will use the following lemma with $u(x)$ a component of $\boldsymbol{\Pi}^L \dot{\boldsymbol{F}}(x) = \mathbf{S}_x \dot{\boldsymbol{F}}$.

Lemma 49. *Let u be a function such that $\nabla^m u \in L^2(\mathbb{R}^d)$ and such that $u(z_0 + r\, h_i) = 0$ for all $1 \le i \le q$.*

Then

$$|u(x)| \le C \left(\frac{R}{r}\right)^{m-1} R^{m-d/2} \|\nabla^m u\|_{L^2(\mathbb{R}^d)}, \quad \textit{where } R = |x - z_0| + r.$$

Proof. By Lemma 31,

$$|u(x)| \le C \left(\sum_{k=0}^{m} R^{2k} \fint_{B(z_0,2R)} |\nabla^k u|^2 \right)^{1/2}.$$

Let $P(x)$ be the polynomial of degree at most $m - 1$ such that

$$\fint_{B(z_0,2R)} \partial^\gamma P(x)\, dx = \fint_{B(z_0,2R)} \partial^\gamma u(x)\, dx$$

for all $|\gamma| \le m - 1$. Then

$$|u(x)| \le C \left(\sum_{k=0}^{m} R^{2k} \fint_{B(z_0,2R)} |\nabla^k u - \nabla^k P|^2 + \sum_{k=0}^{m} R^{2k} \fint_{B(z_0,2R)} |\nabla^k P|^2 \right)^{1/2}.$$

If $k \le m-1$, then by the Poincaré inequality

$$R^{2k} \fint_{B(z_0,2R)} |\nabla^k u - \nabla^k P|^2 \le R^{2m} \fint_{B(z_0,2R)} |\nabla^m u|^2.$$

Therefore,

$$|u(x)| \le C R^{m-d/2} \biggl(\int_{B(z_0,2R)} |\nabla^m u|^2\biggr)^{1/2} + C\biggl(\sum_{k=0}^{m} R^{2k} \fint_{B(z_0,2R)} |\nabla^k P|^2\biggr)^{1/2}.$$

By Lemma 31 and the above bounds on $\nabla^k u - \nabla^k P$, if $1 \le i \le q$ then

$$|P(z_0 + r\, h_i)| = |P(z_0 + r\, h_i) - u(z_0 + r\, h_i)| \le C R^{m-d/2} \|\nabla^m u\|_{L^2(B(z_0,2R))}.$$

Let $P(x) = Q((x-z_0)/r)$, so that $Q(h_i) = P(z_0 + r\, h_i)$. By construction of Q and h_i, we have that

$$Q(x) = \sum_{|\gamma| \le m-1} q_\gamma\, x^\gamma \text{ for some } q_\gamma \text{ with } |q_\gamma| \le C R^{m-d/2} \|\nabla^m u\|_{L^2(B(z_0,2R))}.$$

Then

$$\partial^\delta P(x) = \sum_{\gamma \ge \delta} r^{-|\gamma|} q_\gamma \frac{\gamma!}{(\gamma-\delta)!} (x-z_0)^{\gamma-\delta}$$

where $\gamma! = \gamma_1!\gamma_2!\ldots\gamma_d!$. Thus, if $x \in B(z_0, 2R)$, then

$$|\nabla^k P| \le C\biggl(\frac{R}{r}\biggr)^{m-1} R^{-k} \sup_\gamma |q_\gamma| \le C\biggl(\frac{R}{r}\biggr)^{m-1} R^{m-k-d/2} \|\nabla^m u\|_{L^2(B(z_0,2R))}.$$

Combining these estimates, we have that

$$|u(x)| \le C\biggl(\frac{R}{r}\biggr)^{m-1} R^{m-d/2} \|\nabla^m u\|_{L^2(B(z_0,2R))}$$

as desired. □

We apply the lemma to the function $u = \mathbf{\Pi}^L_{z_0,r} \dot{\boldsymbol{F}}$. Recall that $\nabla^m \mathbf{\Pi}^L$ is bounded on $L^2(\mathbb{R}^d)$, and so

$$|\mathbf{S}_x \dot{\boldsymbol{F}}| \le C R^{m-d/2} \biggl(\frac{R}{r}\biggr)^{m-1} \|\dot{\boldsymbol{F}}\|_{L^2(\mathbb{R}^d)}, \quad R = |x - z_0| + r.$$

By the Riesz representation theorem, there is some array $\boldsymbol{E}^L$ such that

$$\Bigl(\mathbf{\Pi}^L_{z_0,r} \dot{\boldsymbol{F}}\Bigr)_j (x) = (\mathbf{S}_x \dot{\boldsymbol{F}})_j = \sum_{k=1}^{N} \sum_{|\beta|=m} \int_{\mathbb{R}^d} E^L_{j,k,\beta,z_0,r}(x,y)\, F_{k,\beta}(y)\, dy.$$

Furthermore, $\boldsymbol{E}^L$ satisfies the bound

$$\|E^L_{j,k,\beta,z_0,r}(x,\,\cdot\,)\|_{L^2(\mathbb{R}^d)} \leq C\,R^{m-d/2}\left(\frac{R}{r}\right)^{m-1}, \quad R = r + |x-z_0|. \tag{50}$$

As in the proof of Lemma 42, we may use Lemma 41 to see that there is some function $E^L_{j,k,z_0,r}$ such that $E^L_{j,k,\beta,z_0,r}(x,y) = \partial_y^\beta E^L_{j,k,z_0,r}(x,y)$. Again $E^L_{j,k,z_0,r}(x,y)$ is not unique; we may fix a normalization by requiring that

$$\boldsymbol{E}^L_{z_0,r}(x, z_0 + r\,h_i) = 0 \quad \text{for all } x \in \mathbb{R}^d \text{ and all } 1 \leq i \leq q.$$

Notice that by construction of $\boldsymbol{E}^L_{z_0,r}$,

$$\partial_y^\beta \boldsymbol{E}^L_{z_0,r}(z_0 + r\,h_i, y) = 0 \quad \text{for all } 1 \leq i \leq p$$

as an $L^2(\mathbb{R}^d)$-function; thus $P(y) = \boldsymbol{E}^L_{z_0,r}(z_0 + r\,h_i, y)$ is a polynomial in y of order $m-1$, and because it is equal to zero at the points $y = z_0 + r\,h_i$ we have that

$$\boldsymbol{E}^L_{z_0,r}(z_0 + r\,h_i, y) = 0 \quad \text{for all } y \in \mathbb{R}^d \text{ and all } 1 \leq i \leq p.$$

We also observe that by Lemma 49 and the bound (50), we have that

$$|\boldsymbol{E}^L_{z_0,r}(x,y)| \leq C r^{2m-d}\left(1 + \frac{|y-z_0|}{r}\right)^{2m-d/2-1}\left(1 + \frac{|x-z_0|}{r}\right)^{2m-d/2-1} \tag{51}$$

We have established the existence of $\boldsymbol{E}^L$ and the relation (45). To complete the proof of Theorem 44, we must show that the derivatives $\partial_x^\zeta \partial_y^\xi \boldsymbol{E}^L_{j,k,z_0,r}(x,y)$ exist in the weak sense and satisfy the bounds (46) and (48), and must establish the symmetry property (47).

Let η be a smooth cutoff function, that is, $\int_{\mathbb{R}^d} \eta = 1$, $\eta \geq 0$ and $\eta \equiv 0$ outside of the unit ball $B(0,1)$. Let $\eta_\varepsilon(x) = \varepsilon^{-d}\eta(x/\varepsilon)$. We will let $*_x$ denote convolution in the x variable, that is,

$$\eta_\varepsilon *_x E^L_{j,k,z_0,r}(x,y) = \int_{\mathbb{R}^d} \eta_\varepsilon(\tilde{x})\, E^L_{j,k,z_0,r}(x - \tilde{x}, y)\, d\tilde{x}.$$

For the sake of symmetry we will consider the function $\eta_\delta *_x E^L_{j,k,z_0,r}(x,y) *_y \eta_\varepsilon$ for some $\varepsilon, \delta > 0$.

For any multiindices ζ and ξ, let

$$E^L_{j,k,\zeta,\xi,\delta,\varepsilon}(x,y) = \partial_x^\zeta \partial_y^\xi (\eta_\delta *_x E^L_{j,k,z_0,r}(x,y) *_y \eta_\varepsilon).$$

We will then construct $\partial_x^\zeta \partial_y^\xi E^L_{j,k}(x,y)$ as the weak limit of $E^L_{j,k,\zeta,\xi,\delta,\varepsilon}(x,y)$ as $\varepsilon \to 0$, $\delta \to 0$.

We begin with the derivatives of highest order. Let $|\alpha| = |\beta| = m$. Observe that

$$E^L_{j,k,\alpha,\beta,\delta,\varepsilon}(x,y) = (\partial^\alpha \eta_\delta) *_x E^L_{j,k,\beta,z_0,r}(x,y) *_y \eta_\varepsilon.$$

Now, we have that

$$\int_{\mathbb{R}^d} E^L_{j,k,\alpha,\beta,\delta,\varepsilon}(x,y)\,F(y)\,dy = (\partial^\alpha \eta_\delta) *_x \int_{\mathbb{R}^d} E^L_{j,k,\beta,z_0,r}(x,y)\,(\eta_\varepsilon * F)(y)\,dy$$
$$= \eta_\delta * \partial^\alpha \Pi^L_j(\eta_\varepsilon * F\,\dot{\boldsymbol{e}}_{k,\beta})(x).$$

The operator $F \mapsto \eta_\delta * \partial^\alpha \Pi^L_j(\eta_\varepsilon * F\,\dot{\boldsymbol{e}}_{k,\beta})(x)$ is bounded $L^2(\mathbb{R}^d) \mapsto \mathbb{C}$, albeit with a bound depending on δ. Thus by the Riesz representation theorem, $K(y) = E^L_{j,k,\alpha,\beta,\delta,\varepsilon}(x,y)$ is the kernel of this operator, and so does not depend on z_0 and r. Furthermore, by Lemma 42,

$$E^L_{j,k,\alpha,\beta,\delta,\varepsilon}(x,y) = \overline{E^{L^*}_{k,j,\beta,\alpha,\varepsilon,\delta}(y,x)}.$$

In order to establish the bounds (46) and (48), we would like to use the Caccioppoli inequality in both x and y; it will be helpful to have a similar symmetry relation for $\boldsymbol{E}^L_{z_0,r}(x,y)$ as well as its highest derivatives.

Lemma 52. *We have that* $E^L_{j,k,z_0,r}(x,y) = \overline{E^{L^*}_{k,j,z_0,r}(y,x)}$.

Proof. Because $E^L_{j,k,\alpha,\beta,\delta,\varepsilon}(x,y) = \overline{E^{L^*}_{k,j,\beta,\alpha,\varepsilon,\delta}(y,x)}$, we have that

$$\nabla^m_x E^L_{j,k,0,\beta,\delta,\varepsilon}(x,y) = \nabla^m_x \overline{E^{L^*}_{k,j,\beta,0,\varepsilon,\delta}(y,x)}.$$

Thus $E^L_{j,k,0,\beta,\delta,\varepsilon}(x,y)$ and $\overline{E^{L^*}_{k,j,\beta,0,\varepsilon,\delta}(y,x)}$ differ by a polynomial in x of order $m-1$. But observe that

$$E^L_{j,k,0,\beta,\delta,\varepsilon}(z_0 + r\,h_i, y) = 0 = \overline{E^{L^*}_{k,j,\beta,0,\varepsilon,\delta}(y, z_0 + r\,h_i)}$$

for all $1 \le i \le q$; by construction of the points h_i, this implies that

$$E^L_{j,k,0,\beta,\delta,\varepsilon}(x,y) = \overline{E^{L^*}_{k,j,\beta,0,\varepsilon,\delta}(y,x)}.$$

By a similar argument,

$$E^L_{j,k,0,0,\delta,\varepsilon}(x,y) = \overline{E^{L^*}_{k,j,0,0,\varepsilon,\delta}(y,x)}.$$

By Morrey's inequality $\boldsymbol{E}^L$ is continuous. Taking the limits as $\varepsilon \to 0$ and $\delta \to 0$ completes the proof. □

We now wish to establish an L^2 bound on $E^L_{j,k,\zeta,\xi,\delta,\varepsilon}$, independent of δ and ε; this will allow us to prove the bounds (46) and (48), and also to construct the derivatives by taking the limits as $\delta, \varepsilon \to 0$. We will use the Caccioppoli inequality.

The first step is to show that $\boldsymbol{E}^L_{z_0,r}$ is a solution in some sense. Recall that if $\boldsymbol{\varphi} \in \dot{W}^2_m(\mathbb{R}^d)$, then by formula (40) $\varphi_j(x) = \Pi^L_j(\boldsymbol{A}\nabla^m\boldsymbol{\varphi})(x)$, and so by our construction of $\boldsymbol{E}^L$,

$$\varphi_j(x) = \sum_{k=1}^N \sum_{|\beta|=m} \int_{\mathbb{R}^d} \partial^\beta_y E^L_{j,k,z_0,r}(x,y) \sum_{\ell=1}^N \sum_{|\gamma|=m} A^{k\ell}_{\beta\gamma}\,\partial^\gamma \varphi_\ell(y)\,dy$$

as $\dot W^2_m$ functions; if $\boldsymbol{\varphi}(z_0 + r\,h_i) = 0$ for all $1 \le i \le q$, then this equation is true pointwise for all x. Thus, we have that for any x, j, z_0, r, the function $\mathbf{v}(y)$ given by $v_k(y) = E^L_{j,k,z_0,r}(x, y)$ is a solution to $L^*\mathbf{v} = 0$ in $\mathbb{R}^d\setminus\{x\}\setminus\overline{B(z_0, r)}$.

Fix some x_0, y_0. We wish to bound $E^L_{j,k,\zeta,\xi,\delta,\varepsilon}$. Choose z_0 and r so that $|x_0 - y_0| = |x_0 - z_0| = |y_0 - z_0| = 8r$.

For any $x \in B(x_0, r)$, we have by Corollary 22, if ε is small compared to r then

$$\begin{aligned}\int_{B(y_0,r)} |\boldsymbol{E}^L_{\zeta,\xi,\delta,\varepsilon}(x, y)|^2\,dy &= \int_{B(y_0,r)} |\eta_\varepsilon *_y (\partial^\xi_y(\partial^\zeta \eta_\varepsilon *_x \boldsymbol{E}^L_{z_0,r}(x, y))|^2\,dy\\ &\le \int_{B(y_0,2r)} |(\partial^\xi_y(\partial^\zeta \eta_\varepsilon *_x \boldsymbol{E}^L_{z_0,r}(x, y))|^2\,dy\\ &\le \frac{C}{r^{2|\xi|}} \int_{B(y_0,4r)} |(\partial^\zeta \eta_\varepsilon *_x \boldsymbol{E}^L_{z_0,r}(x, y))|^2\,dy.\end{aligned}$$

Again by Corollary 22 and by the bound (51),

$$\begin{aligned}\int_{B(x_0,r)} |(\partial^\zeta \eta_\varepsilon *_x \boldsymbol{E}^L_{z_0,r}(x, y))|^2\,dx &= \int_{B(x_0,r)} |(\eta_\varepsilon *_x \partial^\zeta_x \boldsymbol{E}^{L^*}_{z_0,r}(y, x))|^2\,dx\\ &\le \int_{B(x_0,2r)} |\partial^\zeta_x \boldsymbol{E}^{L^*}_{z_0,r}(y, x)|^2\,dx\\ &\le C r^{4m-d-2|\zeta|}.\end{aligned}$$

Thus

$$\begin{aligned}&\int_{B(x_0,r)}\int_{B(y_0,r)} |\boldsymbol{E}^L_{\zeta,\xi,\delta,\varepsilon}(x, y)|^2\,dy\,dx\\ &\quad\le \frac{C}{r^{2|\xi|}} \int_{B(y_0,4r)}\int_{B(x_0,r)} |(\partial^\zeta \eta_\varepsilon *_x \boldsymbol{E}^L_{z_0,r}(x, y))|^2\,dx\,dy \le C r^{4m-2|\zeta|-2|\xi|}.\end{aligned}$$

So $\boldsymbol{E}^L_{\zeta,\xi,\delta,\varepsilon}$ is in $L^2(B(x_0, r) \times B(y_0, r))$, uniformly in δ, ε; thus there is a weakly convergent subsequence as δ, $\varepsilon \to 0$. Observe that the weak limit must be the partial derivative $\partial^\zeta_x \partial^\xi_y \boldsymbol{E}^L_{z_0,r}(x, y)$, as desired.

5.3. Natural normalization conditions for the fundamental solution

Recall that our normalization of $\boldsymbol{E}^L$, in the construction given in Sect. 5.2, is highly artificial and depends on our choice of the normalization points $z_0 + r\,h_i$. In this section we will construct a somewhat more natural normalization of at least the higher derivatives of $\boldsymbol{E}^L$.

Our normalization will, loosely speaking, be a requirement that the higher-order derivatives of $\boldsymbol{E}^L$ decay at infinity. Thus, we begin with a decay result.

Lemma 53. *Let $A(x_0, R)$ denote the annulus $B(x_0, 2R)\setminus B(x_0, R)$. Let $p^+ = \min(p^+_L, p^+_{L^*})$, where p^+_L is as in Theorem 24. If $0 < \varepsilon < d(1 - 2/p^+)$, then there is some constant $C = C(\varepsilon)$ such that if $x_0 \in \mathbb{R}^d$ and $R > 4r > 0$, then*

$$\int_{A(x_0,R)}\int_{B(x_0,r)} |\nabla^m_x \nabla^m_y \boldsymbol{E}^L(x, y)|^2\,dy\,dx \le C(\varepsilon)\left(\frac{r}{R}\right)^\varepsilon.$$

Proof. Let η_δ be a smooth approximate identity, as in Sect. 5.2; we will establish a bound on $\eta_\delta *_x \nabla_x^m \nabla_y^m \boldsymbol{E}^L(x,y)$, uniform in δ, and then let $\delta \to 0$.

Fix some $\delta > 0$, $x \in \mathbb{R}^d$, and some j and α with $1 \leq j \leq N$ and $|\alpha| = m$. Let

$$v_k^\delta(y) = \eta_\delta *_x \partial_x^\alpha E_{j,k}^L(x,y).$$

As in Sect. 5.2, we begin by showing that $\mathbf{v}^\delta$ is a solution to an elliptic equation. By the bound (46), we have that $\mathbf{v}^\delta \in \dot{W}^2_{m,loc}$. Suppose that $\boldsymbol{\varphi}$ is smooth and compactly supported. If $\operatorname{dist}(x, \operatorname{supp} \boldsymbol{\varphi}) > \delta$, then by formula (40) and formula (45),

$$\begin{aligned} 0 &= \eta_\delta * \partial^\alpha \varphi_j(x) \\ &= \sum_{k=1}^N \sum_{\ell=1}^N \sum_{|\beta|=|\gamma|=m} \int_{\mathbb{R}^d} \eta_\delta *_x \partial_x^\alpha \partial_y^\beta E_{j,k}^L(x,y)\, A_{k\ell}^{\beta\gamma}(y) \partial^\gamma \varphi_\ell(y)\, dy. \end{aligned}$$

So $L^* \mathbf{v}^\delta = 0$ in $\mathbb{R}^d \setminus B(x,\delta)$, and so Theorem 24 applies.

Let p be such that $\varepsilon = d(1 - 2/p)$; notice that $2 < p < p^+$. By Hölder's inequality, we have that

$$\int_{B(x_0,r)} |\nabla_y^m \mathbf{v}^\delta(y)|^2\, dy \leq C r^\varepsilon \left(\int_{B(x_0,r)} |\nabla_y^m \mathbf{v}^\delta(y)|^p\, dy \right)^{2/p}.$$

Because $R > 4r$, we may replace the second integral by an integral over the ball $B(x_0, R/4)$. We then apply Theorem 24. This yields the bound

$$\int_{B(x_0,r)} |\nabla_y^m \mathbf{v}^\delta(y)|^2\, dy \leq C \frac{r^\varepsilon}{R^\varepsilon} \int_{B(x_0,R/2)} |\nabla_y^m \mathbf{v}^\delta(y)|^2\, dy$$

uniformly in δ. Taking the limit as $\delta \to 0$ and applying the bound (46), we see that

$$\begin{aligned} &\int_{A(x_0,R)} \int_{B(x_0,r)} |\nabla_x^m \nabla_y^m \boldsymbol{E}^L(x,y)|^2\, dy\, dx \\ &\quad \leq \frac{C r^\varepsilon}{R^\varepsilon} \int_{A(x_0,R)} \int_{B(x_0,R/2)} |\nabla_x^m \nabla_y^m \boldsymbol{E}^L(x,y)|^2\, dy\, dx \leq \frac{C r^\varepsilon}{R^\varepsilon} \end{aligned}$$

as desired. □

Because L^* is also elliptic, a similar bound is valid for $\boldsymbol{E}^{L^*}$. Notice that by formula (47), we have that $\nabla_x^m \nabla_y^m E_{j,k}^L(x,y) = \overline{\nabla_x^m \nabla_y^m E_{k,j}^{L^*}(y,x)}$. Thus, a similar bound on $\boldsymbol{E}^L$ is valid with the roles of x and y reversed.

Next, we use this bound to produce natural normalizations of certain higher-order derivatives.

Lemma 54. *Suppose that E is a function such that, for some $v \geq 0$, $c > 0$, $\varepsilon > 0$ and $t < d + \varepsilon$, the decay estimate*

$$\int_{y \in B(x_0,r)} \int_{x \in A(x_0,R)} |\nabla_x^m \nabla_y^v E(x,y)|^2\, dx\, dy \leq c R^t \left(\frac{r}{R}\right)^\varepsilon$$

is true for all $x_0 \in \mathbb{R}^d$ and all $R > 4r > 0$.

Then there is an array of functions p_γ *such that, if*

$$\widetilde{E}(x,y) = E(x,y) + \sum_{m+t/2-d/2-\varepsilon/2<|\gamma|\leq m-1} p_\gamma(y)\, x^\gamma$$

then there is a constant $C = C(\varepsilon)$ *depending only on* ε *such that, for all integers* q *with* $0 \leq q \leq m$ *and* $q < d/2 + \varepsilon/2 - t/2$, *we have that*

$$\int_{y\in B(x_0,r)} \int_{x\in A(x_0,R)} |\nabla_x^{m-q} \nabla_y^{v} \widetilde{E}(x,y)|^2 \, dx\, dy \leq C(\varepsilon)\, c\, R^{t+2q} \left(\frac{r}{R}\right)^{\varepsilon} \qquad (55)$$

for all $x_0 \in \mathbb{R}^d$ *and all* $R > 4r > 0$.

Furthermore, $p_\gamma(y)$ *is unique up to adding polynomials of order* $v - 1$.

By Lemma 53, if $E = E^L_{j,k}$ is a component of the fundamental solution for some elliptic operator L, then E satisfies the conditions of Lemma 53 for $v = m$ and $t = 0$; we will shortly need the lemma for $v < m$ as well.

Proof (Proof of Lemma 54). We begin with uniqueness. Suppose that there were two such arrays p and $\tilde{p}$. Let $P_\gamma(y) = p_\gamma(y) - \tilde{p}_\gamma(y)$. If $m + t/2 - d/2 - \varepsilon/2 < |\gamma|$ and $|\gamma| \leq m - 1$, then the difference $P_\gamma(y)\, x^\gamma$ must satisfy the bound (55) for $q = m - |\gamma|$. Thus, for any $x_0 \in \mathbb{R}^d$ and any $R > 4r > 0$, we have that

$$\int_{y\in B(x_0,r)} \int_{x\in A(x_0,R)} |\nabla_x^{|\gamma|} \nabla_y^{v} (P_\gamma(y)\, x^\gamma)|^2 \, dx\, dy \leq C(r,\varepsilon)\, c\, R^{t+2m-2|\gamma|-\varepsilon}.$$

But

$$\int_{y\in B(x_0,r)} \int_{x\in A(x_0,R)} |\nabla_x^{|\gamma|} \nabla_y^{v} (P_\gamma(y)\, x^\gamma)|^2 \, dx\, dy = C R^d \int_{B(x_0,r)} |\nabla_y^v P_\gamma(y)|^2 \, dy.$$

Because $m + t/2 - d/2 - \varepsilon/2 < |\gamma|$, we have that $2m + t - 2|\gamma| - \varepsilon < d$ and so R^d grows faster than $R^{2m+t-2|\gamma|-\varepsilon}$. Thus, the only way that both conditions can hold is if $\nabla_y^v P_\gamma(y) = 0$ almost everywhere in $B(x_0, r)$. Since x_0 and r were arbitrary this means that P_γ is a polynomial of order $v - 1$, as desired.

We now construct an appropriate array of functions $p_\gamma(y)$. We work by induction; notice that by assumption, the bound (55) is valid in the case $q = 0$.

Choose some $q > 0$ satisfying the conditions of the lemma, and suppose that we have renormalized E so that the bound (55) is valid if we replace q by $q - 1$. Choose some multiindices γ and ζ with $|\gamma| = m - q$ and $|\zeta| = v$.

Let $A_i = B(x_0, 2^i) \setminus B(x_0, 2^{i-1})$, and define

$$E_i(y) = \fint_{A_i} \partial_x^\gamma \partial_y^\zeta E(x,y)\, dx.$$

For any constant c_i we have the bound

$$\begin{aligned} |E_i(y) - E_{i+1}(y)| &= \left| \fint_{A_i} \partial_x^\gamma \partial_y^\zeta E(x,y)\, dx - \fint_{A_{i+1}} \partial_x^\gamma \partial_y^\zeta E(x,y)\, dx \right| \\ &\leq \left| \fint_{A_i} \partial_x^\gamma \partial_y^\zeta E(x,y)\, dx - c_i \right| + \left| \fint_{A_{i+1}} \partial_x^\gamma \partial_y^\zeta E(x,y)\, dx - c_i \right| \\ &\leq C \fint_{A_i \cup A_{i+1}} |\partial_x^\gamma \partial_y^\zeta E(x,y) - c_i|\, dx. \end{aligned}$$

Choosing c_i appropriately, by Poincaré's inequality,

$$|E_i(y) - E_{i+1}(y)| \leq C 2^{-i(d-1)} \int_{A_i \cup A_{i+1}} |\nabla_x^{m-q+1} \nabla_y^{v} E(x,y)|\, dx.$$

Thus by Hölder's inequality

$$\begin{aligned} &\int_{B(x_0,r)} |E_i(y) - E_{i+1}(y)|^2\, dy \\ &\leq \frac{C}{2^{2i(d-1)}} \int_{B(x_0,r)} \left(\int_{A_i \cup A_{i+1}} |\nabla_x^{m-q+1} \nabla_y^{v} E(x,y)|\, dx \right)^2 dy \\ &\leq \frac{C}{2^{i(d-2)}} \int_{B(x_0,r)} \int_{A_i \cup A_{i+1}} |\nabla_x^{m-q+1} \nabla_y^{v} E(x,y)|^2\, dx\, dy. \end{aligned}$$

Recall that we assumed that we had the desired decay estimates for $q-1$; this implies that

$$\int_{B(x_0,r)} |E_i(y) - E_{i+1}(y)|^2\, dy \leq C\, c\, 2^{i(t-d+2q-\varepsilon)} r^{\varepsilon}.$$

Thus, by our conditions on q, $E_\infty(y) = \lim_{i\to\infty} E_i(y)$ exists as an $L^2(B(x_0,r))$-function. As usual we may use Lemma 41 to see that there is some $p_\gamma(y)$ such that $E_\infty(y) = \gamma!\, \partial^\zeta p_\gamma(y)$, where $\gamma! = \gamma_1! \gamma_2! \ldots \gamma_d!$. Let $\widetilde{E}(x,y) = E(x,y) - p_\gamma(y)\, x^\gamma$.

We construct an $\widetilde{E}_i$ from $\widetilde{E}$, similar to our construction of E_i; then $\widetilde{E}_i$ satisfies the same bounds as above and converges to zero as $i \to \infty$. Because geometric series converge, we have that

$$\int_{B(x_0,r)} |\widetilde{E}_i(y)|^2\, dy \leq C(\varepsilon)\, c\, 2^{i(t-d+2q-\varepsilon)} r^{\varepsilon}.$$

By the Poincaré inequality

$$\begin{aligned} &\int_{B(x_0,r)} \int_{A_i} |\partial_x^\gamma \partial_y^\zeta \widetilde{E}(x,y)|^2\, dy\, dx \\ &\leq C \int_{B(x_0,r)} \int_{A_i} |\partial_x^\gamma \partial_y^\zeta \widetilde{E}(x,y) - \widetilde{E}_i(y)|^2\, dx\, dy + C(\varepsilon)\, c\, 2^{i(t+2q-\varepsilon)} r^{\varepsilon} \\ &\leq C 2^{2i} \int_{B(x_0,r)} \int_{A_i} |\nabla_x^{m-q+1} \partial_y^\zeta \widetilde{E}(x,y)|^2\, dx\, dy + C(\varepsilon)\, c\, 2^{i(t+2q-\varepsilon)} r^{\varepsilon} \\ &\leq C(\varepsilon)\, c\, 2^{i(t+2q-\varepsilon)} r^{\varepsilon} \end{aligned}$$

as desired. Repeating this construction for all γ with $|\gamma| = m - q$, we complete the proof. □

By Lemmas 53 and 54, there is a unique appropriately normalized representative of $\nabla_x^{m-q} \nabla_y^m \boldsymbol{E}^L(x,y)$. Recall that by formula (47), we have that $\boldsymbol{E}^L(x,y)$ satisfies the conclusion of Lemma 53 with the roles of x and y reversed. We may thus find a unique additive normalization of $\nabla_x^m \nabla_y^{m-q} \boldsymbol{E}^L(x,y)$. Also notice that by formula (47), applying the same procedure to $\boldsymbol{E}^{L^*}$, we see that this normalization preserves the relations

$$\nabla_x^{m-q}\nabla_y^m E_{j,k}^L(x,y) = \overline{\nabla_x^{m-q}\nabla_y^m E_{k,j}^{L^*}(y,x)},$$
$$\nabla_x^{m}\nabla_y^{m-q} E_{j,k}^L(x,y) = \overline{\nabla_x^{m}\nabla_y^{m-q} E_{k,j}^{L^*}(y,x)}.$$

We are now interested in the mixed derivatives, that is, in the case where we take fewer than m derivatives in both x and y.

Observe first that if $q < d(1-1/p^+)$ and if $x_0 \in \mathbb{R}^d$, $y_0 \in \mathbb{R}^d$, then

$$\int_{B(y_0,R)}\int_{B(x_0,R)} |\nabla_x^m \nabla_y^{m-q}\boldsymbol{E}^L(x,y)|^2\,dx\,dy \leq CR^{2q}, \quad R = |x_0-y_0|/3.$$

As in the proof of Lemma 53, we may use Hölder's inequality and Theorem 24 to see that

$$\int_{x\in A(x_0,R)}\int_{y\in B(x_0,r)} |\nabla_y^{m-q}\nabla_x^m \boldsymbol{E}^L(x,y)|^2\,dy\,dx \leq C(\varepsilon)R^{2q}\left(\frac{r}{R}\right)^\varepsilon$$

for all $0<\varepsilon<d(1-2/p_q^+)$, where $p_q^+ = p^+d/(d-q\,p^+)$ in the case $q<d/p^+$ and $p_q^+ = \infty$ if $d/p^+ \leq q < d/p^-$. We may rewrite this requirement as $0<\varepsilon<\min(d, d(1-2/p^+)+2q)$.

We may thus apply Lemma 54 with $v = m-q$ and $t = 2q$. Hence, if q and ε are as above, and if $s < d/2+\varepsilon/2-q$, then there is a unique additive normalization of $\nabla_y^{m-q}\nabla_x^{m-s}\boldsymbol{E}^L(x,y)$ such that

$$\int_{B(x_0,r)}\int_{A(x_0,R)} |\nabla_x^{m-s}\nabla_y^{m-q}\boldsymbol{E}^L(x,y)|^2\,dx\,dy \leq C(\varepsilon)R^{2q+2s}\left(\frac{r}{R}\right)^\varepsilon. \tag{56}$$

We remark that we may find an appropriate ε if and only if q and s satisfy the conditions $0\leq q\leq m$, $0\leq s\leq m$, $q<d/p^-$, $s<d/p^-$, and $q+s<d$.

We will establish one more bound on the fundamental solution. Specifically, notice that $\nabla_x^m\nabla_y^m\boldsymbol{E}^L(x,y)$ is only locally integrable away from the diagonal $\{(x,y) : x = y\}$. The lower-order derivatives, however, are locally integrable even near $x=y$.

Lemma 57. *Let q and s be such that $0<q+s<d$ and such that the bound (56) is valid for all $x_0\in\mathbb{R}^d$ and all $R>4r>0$.*

Suppose that $p<d/(d-(q+s))$ and that $p\leq 2$. We then have the local estimate

$$\int_{B(x_0,r)}\int_{B(x_0,r)} |\nabla_x^{m-s}\nabla_y^{m-q}\boldsymbol{E}^L(x,y)|^p\,dx\,dy \leq Cr^{2d-p(d-s-q)}.$$

Proof. Let Q_0 be the cube of sidelength $\ell(Q_0)=2r$ with $B(x_0,r)\subset Q_0$, so that

$$\int_{B(x_0,r)}\int_{B(x_0,r)} |\nabla_x^{m-s}\nabla_y^{m-q}\boldsymbol{E}^L(x,y)|^p\,dx\,dy$$
$$\leq \int_{Q_0}\int_{2Q_0} |\nabla_x^{m-s}\nabla_y^{m-q}\boldsymbol{E}^L(x,y)|^p\,dx\,dy.$$

We divide Q_0 as follows. Let $\mathcal{G}_j$ be a grid of dyadic subcubes of Q_0 of sidelength $2^{1-j}r$. Notice that $\mathcal{G}_0 = \{Q_0\}$ and that $\mathcal{G}_j$ contains 2^{jd} cubes.

If $y \in B(x_0, r)$, let $Q_j(y)$ be the cube that satisfies $y \in Q_j(y) \in \mathcal{G}_j$. If $Q \in \mathcal{G}_{j+1}$, let $P(Q)$ be the unique cube with $Q \subset P(Q) \in \mathcal{G}_j$. If Q is a cube, let $2Q$ be the concentric cube with side-length $\ell(2Q) = 2\ell(Q)$. Then

$$\begin{aligned}
&\int_{Q_0}\int_{2Q_0} |\nabla_x^{m-s}\nabla_y^{m-q}\boldsymbol{E}^L(x,y)|^p\,dx\,dy \\
&\quad= \int_{Q_0}\sum_{j=0}^{\infty}\int_{2Q_j(y)\setminus 2Q_{j+1}(y)} |\nabla_x^{m-s}\nabla_y^{m-q}\boldsymbol{E}^L(x,y)|^p\,dx\,dy \\
&\quad= \sum_{j=0}^{\infty}\sum_{Q\in\mathcal{G}_{j+1}}\int_Q\int_{2P(Q)\setminus 2Q} |\nabla_x^{m-s}\nabla_y^{m-q}\boldsymbol{E}^L(x,y)|^p\,dx\,dy.
\end{aligned}$$

We apply Hölder's inequality to see that

$$\begin{aligned}
&\int_Q\int_{2P(Q)\setminus 2Q} |\nabla_x^{m-s}\nabla_y^{m-q}\boldsymbol{E}^L(x,y)|^p\,dx\,dy \\
&\quad\le C\ell(Q)^{d(2-p)}\biggl(\int_Q\int_{2P(Q)\setminus 2Q} |\nabla_x^{m-s}\nabla_y^{m-q}\boldsymbol{E}^L(x,y)|^2\,dx\,dy\biggr)^{p/2}
\end{aligned}$$

and the bound (56) to see that

$$\int_Q\int_{2P(Q)\setminus 2Q} |\nabla_x^{m-s}\nabla_y^{m-q}\boldsymbol{E}^L(x,y)|^p\,dx\,dy \le C\ell(Q)^{d(2-p)+(q+s)p}.$$

Combining these estimates and recalling that there are 2^{jd} cubes $Q \in \mathcal{G}_j$, we see that

$$\int_{Q_0}\int_{2Q_0} |\nabla_x^{m-s}\nabla_y^{m-q}\boldsymbol{E}^L(x,y)|^p\,dx\,dy \le Cr^{2d-(d-q-s)p}\sum_{j=0}^{\infty}2^{-jd+j(d-q-s)p}.$$

If $p < d/(d-(q+s))$, then the geometric series converges, as desired. □

We have renormalized the fundamental solution so that we may bound its higher-order derivatives. This renormalization will not affect the bound (46), and because our renormalization is unique it maintains the symmetry condition (47).

Theorem 44 had one more conclusion, the formula (45). This states that

$$\Pi_j^L\dot{\boldsymbol{F}}(x) = \sum_{k=1}^{N}\sum_{|\beta|=m}\int_{\mathbb{R}^d}\partial_y^\beta E_{j,k,z_0,r}^L(x,y)\,F_{k,\beta}(y)\,dy \quad \text{as } \dot{W}_m^2(\mathbb{R}^d)\text{-functions.}$$

We would like to consider in what sense this equation is still true after renormalization. To address this, we will also need natural normalizations of the left-hand side $\boldsymbol{\Pi}^L\dot{\boldsymbol{F}}$ involving decay at infinity; this normalization is given by the following lemma.

Lemma 58. (The Gagliardo–Nirenberg–Sobolev inequality in $\mathbb{R}^d$). *Let u lie in the space $\dot{W}^p_m(\mathbb{R}^d)$ for some $1 \leq p < d$. Let $0 < k < d/p$ be an integer, and let $p_k = p\,d/(d - p\,k)$.*

Then there is a unique additive normalization of $\nabla^{m-k}u$ in $L^{p_k}(\mathbb{R}^d)$.

See, for example, Section 5.6.1 in [17]. We use this lemma to address the relation between the Newton potential and the renormalized fundamental solution.

Lemma 59. *Let $p^- < p < \min(d, p^+)$, let γ be a multiindex with $m - d/p < |\gamma| \leq m - 1$, and let $q > d/(m - |\gamma|)$. Let $1 \leq j \leq N$.*

Suppose that we have normalized $\boldsymbol{E}^L$ as above. We normalize the lower-order derivatives of $\boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$ as in Lemma 58. If $\dot{\boldsymbol{F}}$ lies in $L^p(\mathbb{R}^d)$ and in $L^q_{loc}(\mathbb{R}^d)$, then

$$\partial^\gamma \Pi^L_j \dot{\boldsymbol{F}}(x) = \sum_{k=1}^N \sum_{|\beta|=m} \int_{\mathbb{R}^d} \partial^\gamma_x \partial^\beta_y E^L_{j,k}(x, y)\, F_{k,\beta}(y)\, dy \tag{60}$$

for almost every $x \in \mathbb{R}^d$.

Proof. Let us define

$$\Pi^L_{j,\gamma} \dot{\boldsymbol{F}}(x) = \sum_{k=1}^N \sum_{|\beta|=m} \int_{\mathbb{R}^d} \partial^\gamma_x \partial^\beta_y E^L_{j,k}(x, y)\, F_{k,\beta}(y)\, dy$$

where $\boldsymbol{E}^L$ is the fundamental solution normalized to obey the bound (56). We begin by showing that $\Pi^L_{j,\gamma}$ is a bounded operator in some sense. Specifically, let $B(x_0, r) \subset \mathbb{R}^d$ be a ball. We will show that $\Pi^L_{j,\gamma}$ is bounded $L^q(B(x_0, 2r)) \cap L^p(\mathbb{R}^d) \mapsto L^1(B(x_0, r))$.

First, we see that

$$\begin{aligned}\int_{B(x_0,r)} |\Pi^L_{j,\gamma} \dot{\boldsymbol{F}}(x)|\, dx &\leq C \int_{B(x_0,r)} \int_{B(x_0,2r)} |\nabla^{|\gamma|}_x \nabla^m_y \boldsymbol{E}^L(x, y)|\, |\dot{\boldsymbol{F}}(y)|\, dy\, dx \\ &\quad + C \sum_{i=1}^\infty \int_{B(x_0,r)} \int_{A_i} |\nabla^{|\gamma|}_x \nabla^m_y \boldsymbol{E}^L(x, y)|\, |\dot{\boldsymbol{F}}(y)|\, dy\, dx\end{aligned}$$

where $A_i = B(x_0, 2^{i+1}r) \setminus B(x_0, 2^i r)$. If $1/q + 1/q' = 1$, then $q' < d/(d - (m - |\gamma|))$, and so by Lemma 57 and Hölder's inequality, the first integral is at most

$$C r^{d-d/q+m-|\gamma|} \|\dot{\boldsymbol{F}}\|_{L^q(B(x_0,2r))}.$$

We control the second integral as follows. Fix some $i \geq 1$. Then by Hölder's inequality,

$$\begin{aligned}&\int_{B(x_0,r)} \int_{A_i} |\nabla^{|\gamma|}_x \nabla^m_y \boldsymbol{E}^L(x, y)|\, |\dot{\boldsymbol{F}}(y)|\, dy\, dx \\ &\leq C r^{d/p} \left(\int_{B(x_0,r)} \int_{A_i} |\nabla^{|\gamma|}_x \nabla^m_y \boldsymbol{E}^L(x, y)|^{p'}\, dy\, dx \right)^{1/p'} \|\dot{\boldsymbol{F}}\|_{L^p(A(x_0,2^i r))}.\end{aligned}$$

Notice that $p' < p^+$. Arguing as in the proof of Lemma 53, we use Theorem 24 to show that

$$\left(\int_{B(x_0,r)} \int_{A_i} |\nabla_x^{|\gamma|} \nabla_y^m \boldsymbol{E}^L(x,y)|^{p'}\,dy\,dx\right)^{1/p'}$$
$$\leq C 2^{i(d/p'-d/2)} r^{2d/p'-d} \left(\int_{B(x_0,r)} \int_{\widetilde{A}(x_0,2^i r)} |\nabla_x^{|\gamma|} \nabla_y^m \boldsymbol{E}^L(x,y)|^2\,dy\,dx\right)^{1/2}$$

where $\widetilde{A}(x_0, 2^i r)$ is the enlarged annulus $B(x_0, 2^{i+2}r)\setminus B(x_0, (3/4)2^i r)$.

By the bound (56),

$$\left(\int_{B(x_0,r)} \int_{A_i} |\nabla_x^{|\gamma|} \nabla_y^m \boldsymbol{E}^L(x,y)|^{p'}\,dy\,dx\right)^{1/p'}$$
$$\leq C(\varepsilon) 2^{i(d/p'-d/2+m-|\gamma|-\varepsilon/2)} r^{2d/p'-d+m-|\gamma|}$$

for all $0 < \varepsilon < \min(d, d(1-1/p^+) + 2m - 2|\gamma|)$. Let $\theta = \theta(\varepsilon) = -d/p' + d/2 - m + |\gamma| + \varepsilon/2$. We remark that by our assumptions on γ and p, we may always find an ε that satisfies the above conditions and such that $\theta > 0$.

Thus,

$$\int_{B(x_0,r)} |\Pi^L_{j,\gamma} \dot{\boldsymbol{F}}(x)|\,dx \leq C r^{m-|\gamma|+d/q'} \|\dot{\boldsymbol{F}}\|_{L^q(B(x_0,2r))}$$
$$+ C(\theta) r^{m-|\gamma|+d/p'} \sum_{i=1}^{\infty} 2^{-i\theta} \|\dot{\boldsymbol{F}}\|_{L^p(A(x_0,2^i r))}$$

and by convergence of geometric series, we have that $\Pi^L_{j,\gamma}$ is bounded as an operator from $L^q(B(x_0,2r)) \cap L^p(\mathbb{R}^d)$ to $L^1(B(x_0,r))$, as desired.

We may now work in a dense subspace of $L^p(\mathbb{R}^d) \cap L^q_{loc}(\mathbb{R}^d)$; we will work with $\dot{\boldsymbol{F}}$ bounded and compactly supported.

In particular, suppose that $\dot{\boldsymbol{F}}$ is supported in some ball $B(y_0, r)$. Let z_0 be such that $|y_0 - z_0| = 3r$, and consider the fundamental solution $\boldsymbol{E}^L_{z_0,r}$ of Theorem 44; as in Sect. 5.2 we will let

$$\Pi^L_{j,z_0,r} \dot{\boldsymbol{F}}(x) = \sum_{k=1}^{N} \sum_{|\beta|=m} \int_{\mathbb{R}^d} \partial_y^\beta E^L_{j,k,z_0,r}(x,y)\, F_{k,\beta}(y)\,dy.$$

Begin with the case $|\gamma| = m-1$. We will show that there is some constant c such that $\Pi^L_{j,\gamma} \dot{\boldsymbol{F}}(x) = \partial^\gamma \Pi^L_{j,z_0,r} \dot{\boldsymbol{F}}(x) + c$ for almost every $x \in \mathbb{R}^d$; it will then be straightforward to establish that $\Pi^L_{j,\gamma} \dot{\boldsymbol{F}}$ decays and so must equal the normalization of Lemma 58.

Observe that our renormalization of $\boldsymbol{E}^L$ preserves the relation

$$\nabla_x^m \nabla_y^m \boldsymbol{E}^L(x,y) = \nabla_x^m \nabla_y^m \boldsymbol{E}^L_{z_0,r}(x,y).$$

Thus by Lemma 54, for every β with $|\beta| = m$ and every j, k, there is a unique function p such that

$$\partial_x^\gamma \partial_y^\beta E^L_{j,k}(x,y) = \partial_x^\gamma \partial_y^\beta E^L_{j,k,z_0,r}(x,y) + p(y).$$

In particular, while p may depend on γ, β, j, k, z_0 and r, once these parameters are fixed, p cannot depend on x. It will be convenient to write $p = p_{k,\beta}$ and leave the remaining dependencies implied.

Let x_0 satisfy $|x_0 - y_0| = |x_0 - z_0| = 3r$. Notice that

$$\begin{aligned}\int_{B(y_0,r)} |p_{k,\beta}|^2 &= \int_{B(y_0,r)} \left| \fint_{B(x_0,r)} p_{k,\beta}(y)\,dx\right|^2 dy \\ &\leq \int_{B(y_0,r)} \fint_{B(x_0,r)} \left|\partial_x^\gamma \partial_y^\beta E^L_{j,k}(x,y) - \partial_x^\gamma \partial_y^\beta E^L_{j,k,z_0,r}(x,y)\right|^2 dx\,dy\end{aligned}$$

and so, using the bounds (56) and (48), we see that $p_{k,\beta} \in L^2(B(y_0,r))$ with $\|p_{k,\beta}\|_{L^2(B(y_0,r))} \leq C r^{2-d}$.

Thus,

$$\Pi^L_{j,\gamma} \dot{\boldsymbol{F}}(x) = \sum_{k=1}^N \sum_{|\beta|=m} \int_{\mathbb{R}^d} \partial_x^\gamma \partial_y^\beta E^L_{j,k,z_0,r}(x,y)\, F_{k,\beta}(y) + p_{k,\beta}(y)\, F_{k,\beta}(y)\,dy.$$

Notice that, by Lemma 57, $\partial_x^\gamma \partial_y^\beta \boldsymbol{E}^L(x,y) \in L^1(U \times B(y_0,r))$ for any bounded set U. If $U = B(x_0,r)$, then the inclusion $\partial_x^\gamma \partial_y^\beta \boldsymbol{E}^L_{z_0,r}(x,y) \in L^1(U \times B(y_0,r))$ follows from the bound (48); because $\partial_x^\gamma \partial_y^\beta E^L_{j,k}(x,y) = \partial_x^\gamma \partial_y^\beta E^L_{j,k,z_0,r}(x,y) + p_{k,\beta}(y)$, we may extend this second inclusion to all bounded sets U. Thus

$$\begin{aligned}\Pi^L_{j,\gamma} \dot{\boldsymbol{F}}(x) = \sum_{k=1}^N \sum_{|\beta|=m} &\int_{\mathbb{R}^d} \partial_x^\gamma \partial_y^\beta E^L_{j,k,z_0,r}(x,y)\, F_{k,\beta}(y)\,dy \\ &+ \int_{\mathbb{R}^d} p_{k,\beta}(y)\, F_{k,\beta}(y)\,dy.\end{aligned}$$

Observe that the second integral is convergent and also is independent of x. Furthermore, we may apply Fatou's lemma to the first integral to see that

$$\begin{aligned}\Pi^L_{j,\gamma} \dot{\boldsymbol{F}}(x) &= c_1 + \partial_x^\gamma \sum_{k=1}^N \sum_{|\beta|=m} \int_{\mathbb{R}^d} \partial_y^\beta E^L_{j,k,z_0,r}(x,y)\, F_{k,\beta}(y)\,dy \\ &= c_1 + \partial_x^\gamma \Pi^L_{j,z_0,r} \dot{\boldsymbol{F}}(x).\end{aligned}$$

Because $\boldsymbol{\Pi}^L_{z_0,r}$ is an additive normalization of $\boldsymbol{\Pi}^L$, this means that $\Pi^L_{j,\gamma} \dot{\boldsymbol{F}}(x) = c_2 + \partial_x^\gamma \Pi^L_j \dot{\boldsymbol{F}}(x)$ where $\partial_x^\gamma \Pi^L_j \dot{\boldsymbol{F}}(x)$ is normalized as in Lemma 58. We must now establish that $c_2 = 0$, that is, that $\Pi^L_{j,\gamma} \dot{\boldsymbol{F}}$ decays at infinity. But by the bound (56), we have that

$$\lim_{R\to\infty} \fint_{A(y_0,R)} |\Pi^L_{j,\gamma} \dot{\boldsymbol{F}}(x)|^2\,dx = 0$$

and this can only be true for one additive normalization of $\partial^\gamma \Pi_j^L \dot{\boldsymbol{F}}$; it is this normalization that is chosen by Lemma 58, as desired.

We now consider $|\gamma| < m-1$; we still work only with bounded, compactly supported functions $\dot{\boldsymbol{F}}$. If $|\gamma+\zeta| \leq m-1$, then by Fatou's lemma $\boldsymbol{\Pi}^L_{\zeta+\gamma}\dot{\boldsymbol{F}} = \partial^\zeta \boldsymbol{\Pi}^L_\gamma \dot{\boldsymbol{F}}$, and if $|\gamma+\zeta| = m-1$ then by the above results $\boldsymbol{\Pi}^L_{\zeta+\gamma}\dot{\boldsymbol{F}} = \partial^{\zeta+\gamma}\boldsymbol{\Pi}^L\dot{\boldsymbol{F}}$. Thus $\partial^\gamma \boldsymbol{\Pi}^L \dot{\boldsymbol{F}} = \boldsymbol{\Pi}^L_\gamma \dot{\boldsymbol{F}}$ up to adding polynomials. But again by the bound (56), we have that

$$\lim_{R\to\infty} \fint_{A(y_0,R)} |\Pi^L_{j,\gamma}\dot{\boldsymbol{F}}(x)|^2\,dx = 0$$

whenever $m - d/p^- < |\gamma| \leq m$; thus, $\partial^\gamma \boldsymbol{\Pi}^L \dot{\boldsymbol{F}} = \boldsymbol{\Pi}^L_\gamma \dot{\boldsymbol{F}}$, as desired. □

Remark 61. We have established decay results and the relation (60) only for the higher-order derivatives. We expect the lower-order derivatives to be problematic. As an example, consider the case of the polyharmonic operator $L = (-\Delta)^m$; we may normalize the fundamental solution so that, for some constant $C_{m,d}$,

$$E^{(-\Delta)^m}(x,y) = \begin{cases} C_{m,d}|x-y|^{2m-d}, & d \text{ odd or } d > 2m, \\ C_{m,d}|x-y|^{2m-d}\log|x-y|, & d \text{ even and } d \leq 2m. \end{cases}$$

Notice that $\partial_x^\zeta \partial_y^\xi E^{(-\Delta)^m}(x,y)$ decays at infinity only if $|\zeta| + |\xi| > 2m-d$. Furthermore, if $|\zeta|+|\xi| = 2m-d$, then no natural normalization condition applies; the fundamental solution given above must be normalized using deeper symmetry properties of the Laplacian and a choice of length scale for the logarithm.

In the case of more general operators, these symmetry properties are not available, and it is not apparent whether dimensionally-appropriate decay estimates are valid unless $\min(|\zeta|,|\xi|) > m - d + d/p^+$. Thus, in general, we do not have a unique normalization of the fundamental solution for operators of higher order.

We will see that we can construct a fundamental solution for operators of lower order and retain the above decay estimates, and in that case we will have a unique normalization of $\boldsymbol{E}^L$ provided $2m < d$. (If $2m = d$ then we will have unique normalizations of $\nabla_x \boldsymbol{E}^L(x,y)$ and $\nabla_y \boldsymbol{E}^L(x,y)$, and hence a normalization of $\boldsymbol{E}^L$ that is unique up to additive constants.)

5.4. The fundamental solution for operators of lower order

Consider the following theorem. In the case where $2m > d$, validity of the following theorem was established in Sects. 5.2 and 5.3. In this section we will show that Theorem 62 is still valid even if $2m \leq d$.

Theorem 62. *Let L be an operator of order $2m$ that satisfies the bounds (6) and (7). Then there exists an array of functions $E^L_{j,k}(x,y)$ with the following properties.*

Let q and s be two integers that satisfy $q+s<d$ and the bounds $0 \leq q \leq \min(m,d/2)$, $0 \leq s \leq \min(m,d/2)$.

Then there is some $\varepsilon > 0$ such that if $x_0 \in \mathbb{R}^d$, if $0 < 4r < R$, if $A(x_0, R) = B(x_0, 2R)\setminus B(x_0, R)$, and if $q < d/2$ then

$$\int_{y\in B(x_0,r)} \int_{x\in A(x_0,R)} |\nabla_x^{m-s} \nabla_y^{m-q} \boldsymbol{E}^L(x, y)|^2 \, dx \, dy \le C r^{2q} R^{2s} \left(\frac{r}{R}\right)^{\varepsilon}. \tag{63}$$

If $q = d/2$ then we instead have the bound

$$\int_{y\in B(x_0,r)} \int_{x\in A(x_0,R)} |\nabla_x^{m-s} \nabla_y^{m-q} \boldsymbol{E}^L(x, y)|^2 \, dx \, dy \le C(\delta)\, r^{2q} R^{2s} \left(\frac{R}{r}\right)^{\delta} \tag{64}$$

for all $\delta > 0$ and some constant $C(\delta)$ depending on δ.

We also have the symmetry property

$$\partial_x^\gamma \partial_y^\delta E^L_{j,k}(x, y) = \overline{\partial_x^\gamma \partial_y^\delta E^{L^*}_{k,j}(y, x)} \tag{65}$$

as locally L^2 functions, for all multiindices γ, δ with $|\gamma| = m-q$ and $|\delta| = m-s$.

If in addition $q+s > 0$, then for all p with $1 \le p \le 2$ and $p < d/(d-(q+s))$, we have that

$$\int_{B(x_0,r)} \int_{B(x_0,r)} |\nabla_x^{m-s} \nabla_y^{m-q} \boldsymbol{E}^L(x, y)|^p \, dx \, dy \le C(p)\, r^{2d+p(s+q-d)} \tag{66}$$

for all $x_0 \in \mathbb{R}^d$ and all $r > 0$.

Finally, there is some $\varepsilon > 0$ such that if $2-\varepsilon < p < 2+\varepsilon$ then $\nabla^m \boldsymbol{\Pi}^L$ extends to a bounded operator $L^p(\mathbb{R}^d) \mapsto L^p(\mathbb{R}^d)$. If γ satisfies $m - d/p < |\gamma| \le m-1$ for some such p, then

$$\partial_x^\gamma \Pi_j^L \dot{\boldsymbol{F}}(x) = \sum_{k=1}^N \sum_{|\beta|=m} \int_{\mathbb{R}^d} \partial_x^\gamma \partial_y^\beta E^L_{j,k}(x, y)\, F_{k,\beta}(y)\, dy \quad \text{for a.e. } x \in \mathbb{R}^d \tag{67}$$

for all $\dot{\boldsymbol{F}} \in L^p(\mathbb{R}^d)$ that are also locally in $L^P(\mathbb{R}^d)$, for some $P > d/(m-|\gamma|)$. In the case of $|\alpha| = m$, we still have that

$$\partial^\alpha \Pi_j^L \dot{\boldsymbol{F}}(x) = \sum_{k=1}^N \sum_{|\beta|=m} \int_{\mathbb{R}^d} \partial_x^\alpha \partial_y^\beta E^L_{j,k}(x, y)\, F_{k,\beta}(y)\, dy \quad \text{for a.e. } x \notin \operatorname{supp} \dot{\boldsymbol{F}} \tag{68}$$

for all $\dot{\boldsymbol{F}} \in L^2(\mathbb{R}^d)$ whose support is not all of $\mathbb{R}^d$.

Validity of the condition (67) requires that we normalize $\boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$ by decay at infinity, as in Lemma 58.

Before proving Theorem 62 in the case $2m \le d$, we mention two important corollaries.

First, we have the following uniqueness result.

Lemma 69. *Let $E^L_{j,k}$ be the fundamental solution given by Theorem 62. Let $m - d/2 \leq |\gamma| \leq m$, let $|\beta| = m$, and let $1 \leq j \leq N$, $1 \leq k \leq N$. Let U and V be two bounded open sets with $\overline{U} \cap \overline{V} = \emptyset$. Suppose that for some $\widetilde{E}^L_{j,k,\gamma,\beta} \in L^2(U \times V)$,*

$$\partial^\gamma \Pi^L_j(\mathbf{1}_V F\, \dot{\boldsymbol{e}}_{k,\beta})(x) = \int_V \widetilde{E}^L_{j,k,\gamma,\beta}(x, y)\, F(y)\, dy \quad \textit{as } L^2(U)\textit{-functions}$$

for all $\dot{\boldsymbol{F}} \in L^2(V)$.

Then $\widetilde{E}^L_{j,k,\gamma,\beta}(x, y) = \partial^\gamma_x \partial^\beta_y E^L_{j,k}(x, y)$ as $L^2(U \times V)$-functions.

In particular, if $E^L_{j,k}$ and $\widetilde{E}^L_{j,k}$ both satisfy the conditions of Theorem 62, then

$$\widetilde{E}^L_{j,k}(x, y) = E^L_{j,k}(x, y) + \sum_{|\gamma|<m-d/2} \big(f_\gamma(x)\, y^\gamma + g_\gamma(y)\, x^\gamma\big) + \sum_{|\zeta|=|\xi|=m-d/2} c_{\zeta,\xi}\, x^\zeta\, y^\xi$$

for some functions f_γ and g_γ and some constants $c_{\zeta,\xi}$. (Notice that if the dimension d is odd, then the final sum is empty, and if $2m < d$ then $\widetilde{E}^L = E^L$ without modification.)

Second, recall that if $\boldsymbol{\varphi} \in \dot{W}^2_m(\mathbb{R}^d)$, then $\boldsymbol{\varphi} = \mathbf{\Pi}^L(\boldsymbol{A}\nabla^m \boldsymbol{\varphi})$ as $\dot{W}^2_m(\mathbb{R}^d)$-functions. Thus, if $\dot{\boldsymbol{F}} = \boldsymbol{A}\nabla^m \boldsymbol{\varphi}$ and γ satisfy the conditions of formula (67), then

$$\partial^\gamma \varphi_j(x) = \sum_{k=1}^N \sum_{\ell=1}^N \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^d} \partial^\gamma_x \partial^\alpha_y E^L_{j,k}(x, y)\, A^{\alpha\beta}_{k\ell}(y)\, \partial^\beta \varphi_\ell(y)\, dy \tag{70}$$

for almost every $x \in \mathbb{R}^d$.

Proof (Proof of Theorem 62).

Let L be an operator of order $2m$ for some $m \leq d/2$. Construct the operator $\widetilde{L}$ as follows. Let M be large enough that $\widetilde{m} = m + 2M > d/2$, and let $\widetilde{L} = \Delta^M L\, \Delta^M$. That is, if $\mathbf{u} \in \dot{W}^2_{\widetilde{m}}(\Omega)$, then

$$\big\langle \boldsymbol{\varphi}, \widetilde{L}\mathbf{u} \big\rangle_\Omega = \big\langle \Delta^M \boldsymbol{\varphi}, L\Delta^M \mathbf{u} \big\rangle_\Omega \quad \text{for all smooth } \boldsymbol{\varphi} \text{ supported in } \Omega.$$

Then $\widetilde{L}$ is a bounded and elliptic operator of order $2\widetilde{m}$, and so a fundamental solution $E^{\widetilde{L}}_{j,k}$ exists.

There exist constants a_ζ such that $\Delta^M \varphi = \sum_{|\zeta|=2M} a_\zeta \partial^\zeta \varphi$ for all smooth functions φ. Let

$$E^L_{j,k}(x, y) = \sum_{|\zeta|=2M} \sum_{|\xi|=2M} a_\zeta\, a_\xi\, \partial^\zeta_x \partial^\xi_y E^{\widetilde{L}}_{j,k}(x, y).$$

We claim that $E^L_{j,k}$ satisfies the conditions of Theorem 62.

First, notice that the symmetry formula (65) and the bounds (63), (64) and (66) follow immediately from the corresponding formulas for $E^{\widetilde{L}}$.

We are left with formulas (67) and (68); that is, we must now show that $\partial_x^\gamma \partial_y^\beta E^L_{j,k}(x,y)$ is the kernel of the Newton potential. Choose some bounded, compactly supported function $\dot{\boldsymbol{F}}$ and some multiindex γ with $m - d/2 \le |\gamma| \le m$, and let

$$\widetilde{F}_{k,\widetilde{\beta}} = \sum_{|\xi|=2M,\, \xi<\widetilde{\beta}} a_\xi\, F_{k,\widetilde{\beta}-\xi}, \quad \text{for all } |\widetilde{\beta}| = \widetilde{m}.$$

Let

$$v_j(x) = \sum_{k=1}^N \sum_{|\beta|=m} \int_{\mathbb{R}^d} \partial_x^\gamma \partial_y^\beta E^L_{j,k}(x,y)\, F_{k,\beta}(y)\, dy.$$

We have that

$$\begin{aligned} v_j(x) &= \sum_{|\zeta|=2M} a_\zeta \sum_{k=1}^N \sum_{|\beta|=m} \int_{\mathbb{R}^d} \sum_{|\xi|=2M} \partial_x^{\gamma+\zeta} \partial_y^{\beta+\xi} E^{\widetilde{L}}_{j,k}(x,y)\, a_\xi\, F_{k,\beta}(y)\, dy \\ &= \sum_{|\zeta|=2M} a_\zeta \sum_{k=1}^N \sum_{|\widetilde{\beta}|=\widetilde{m}} \int_{\mathbb{R}^d} \partial_x^{\gamma+\zeta} \partial_y^{\widetilde{\beta}} E^{\widetilde{L}}_{j,k}(x,y) \widetilde{F}_{k,\widetilde{\beta}}(y)\, dy. \end{aligned}$$

Formulas (67) and (68) are valid for $\boldsymbol{E}^{\widetilde{L}}$; thus we have that

$$v_j(x) = \sum_{|\zeta|=2M} a_\zeta\, \partial_x^{\gamma+\zeta} \boldsymbol{\Pi}_j^{\widetilde{L}} \dot{\widetilde{\boldsymbol{F}}}(x) = \partial_x^\gamma \Delta^M \boldsymbol{\Pi}_j^{\widetilde{L}} \dot{\widetilde{\boldsymbol{F}}}(x).$$

Thus, it suffices to show that $\Delta^M \boldsymbol{\Pi}^{\widetilde{L}} \dot{\widetilde{\boldsymbol{F}}} = \boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$.

Choose some $\boldsymbol{\varphi} \in \dot{W}^2_m(\mathbb{R}^d)$; then there is some $\widetilde{\boldsymbol{\varphi}} \in \dot{W}^2_{\widetilde{m}}(\mathbb{R}^d)$ with $\boldsymbol{\varphi} = \Delta^M \widetilde{\boldsymbol{\varphi}}$. Then

$$\begin{aligned} \langle \nabla^m \boldsymbol{\varphi}, \boldsymbol{A} \nabla^m (\Delta^M \boldsymbol{\Pi}^{\widetilde{L}} \dot{\widetilde{\boldsymbol{F}}}) \rangle_{\mathbb{R}^d} &= \langle \boldsymbol{\varphi}, L(\Delta^M \boldsymbol{\Pi}^{\widetilde{L}} \dot{\widetilde{\boldsymbol{F}}}) \rangle_{\mathbb{R}^d} \\ &= \langle \Delta^M \widetilde{\boldsymbol{\varphi}}, L(\Delta^M \boldsymbol{\Pi}^{\widetilde{L}} \dot{\widetilde{\boldsymbol{F}}}) \rangle_{\mathbb{R}^d}. \end{aligned}$$

But by definition of $\widetilde{L}$,

$$\langle \Delta^M \widetilde{\boldsymbol{\varphi}}, L(\Delta^M \boldsymbol{\Pi}^{\widetilde{L}} \dot{\widetilde{\boldsymbol{F}}}) \rangle_{\mathbb{R}^d} = \langle \widetilde{\boldsymbol{\varphi}}, \widetilde{L}(\boldsymbol{\Pi}^{\widetilde{L}} \dot{\widetilde{\boldsymbol{F}}}) \rangle_{\mathbb{R}^d}$$

and by definition of $\boldsymbol{\Pi}^{\widetilde{L}}$,

$$\langle \widetilde{\boldsymbol{\varphi}}, \widetilde{L}(\boldsymbol{\Pi}^{\widetilde{L}} \dot{\widetilde{\boldsymbol{F}}}) \rangle_{\mathbb{R}^d} = \langle \nabla^{\widetilde{m}} \widetilde{\boldsymbol{\varphi}}, \dot{\widetilde{\boldsymbol{F}}} \rangle_{\mathbb{R}^d}.$$

Writing out the sums in the inner product and using the definition of $\widetilde{F}$, we see that

$$\begin{aligned} \langle \nabla^{\widetilde{m}} \widetilde{\boldsymbol{\varphi}}, \dot{\widetilde{\boldsymbol{F}}} \rangle_{\mathbb{R}^d} &= \sum_{k=1}^N \sum_{|\widetilde{\beta}|=\widetilde{m}} \langle \partial^{\widetilde{\beta}} \widetilde{\varphi}_k, \widetilde{F}_{k,\widetilde{\beta}} \rangle_{\mathbb{R}^d} \\ &= \sum_{k=1}^N \sum_{|\widetilde{\beta}|=\widetilde{m}} \sum_{|\delta|=2M,\, \delta<\widetilde{\beta}} \langle \partial^{\widetilde{\beta}} \widetilde{\varphi}_k, a_\delta\, F_{k,\widetilde{\beta}-\delta} \rangle_{\mathbb{R}^d}. \end{aligned}$$

Interchanging the order of summation, we see that

$$\langle \nabla^{\widetilde{m}} \widetilde{\boldsymbol{\varphi}}, \dot{\widetilde{\boldsymbol{F}}} \rangle_{\mathbb{R}^d} = \sum_{k=1}^{N} \sum_{|\beta|=m} \sum_{|\delta|=2M} \langle a_\delta \partial^{\delta+\beta} \widetilde{\varphi}_k, F_{k,\beta} \rangle_{\mathbb{R}^d}$$

and recalling the definitions of a_δ and $\widetilde{\boldsymbol{\varphi}}$, we see that

$$\begin{aligned} \langle \nabla^{\widetilde{m}} \widetilde{\boldsymbol{\varphi}}, \dot{\widetilde{\boldsymbol{F}}} \rangle_{\mathbb{R}^d} &= \sum_{k=1}^{N} \sum_{|\beta|=m} \langle \partial^\beta \Delta^M \widetilde{\varphi}_k, F_{k,\beta} \rangle_{\mathbb{R}^d} \\ &= \sum_{k=1}^{N} \sum_{|\beta|=m} \langle \partial^\beta \varphi_k, F_{k,\beta} \rangle_{\mathbb{R}^d} = \langle \nabla^m \boldsymbol{\varphi}, \dot{\boldsymbol{F}} \rangle_{\mathbb{R}^d}. \end{aligned}$$

Thus,

$$\langle \nabla^m \boldsymbol{\varphi}, \boldsymbol{A} \nabla^m (\Delta^M \boldsymbol{\Pi}^{\widetilde{L}} \dot{\widetilde{\boldsymbol{F}}}) \rangle_{\mathbb{R}^d} = \langle \nabla^m \boldsymbol{\varphi}, \dot{\boldsymbol{F}} \rangle_{\mathbb{R}^d}.$$

By uniqueness of $\boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$, this implies that $\Delta^M \boldsymbol{\Pi}^{\widetilde{L}} \dot{\widetilde{\boldsymbol{F}}} = \boldsymbol{\Pi}^L \dot{\boldsymbol{F}}$, as desired. □

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