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Existence of weak solutions of parabolic systems with p, q-growth

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Abstract. We consider evolutionary problems associated with a convex integrand $f\colon \Omega_T \times \mathbb{R}^{Nn} \to [0, \infty)$, which is α-Hölder continuous with respect to the x-variable and satisfies a non-standard p, q-growth condition. We prove the existence of weak solutions $u\colon \Omega_T \to \mathbb{R}^N$, which solve

$$\partial_t u - \operatorname{div} \partial_\zeta f(x, t, Du) = 0$$

weakly in Ω_T. Therefore, we use the concept of variational solutions, which exist under a mild assumption on the gap $q - p$, namely

$$\frac{2n}{n+2} < p \le q < p + 1.$$

For

$$\frac{2n}{n+2} < p \le q < p + \frac{\min\{2, p\}\,\alpha}{n+2},$$

we prove that the spatial derivative Du of a variational solution u admits a higher integrability and is accordingly a weak solution.

1. Introduction and statement of the results

In this paper we are interested in the existence of solutions of parabolic systems with p, q-growth of the type

$$\partial_t u - \operatorname{div} \partial_\zeta f(x, t, Du) = 0 \ \text{ in } \Omega_T. \tag{1.1}$$

The corresponding stationary problem has been studied extensively in the past, where the papers [18,19] of Marcellini have been the starting point. In these papers a $W^{1,\infty}$-bound for minimizers or respectively weak solutions is shown. The strategy therein is, to regularize the problem by adding the q-energy $\varepsilon|Du|^q$. The regularized problem, which satisfies a standard q-growth condition, exhibits solutions $u_\varepsilon \in$

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$W^{1,q}_{\mathrm{loc}}$, that possess an uniform $W^{1,\infty}$-bound and sub-converge to a $W^{1,\infty}_{\mathrm{loc}}$-solution of the original p, q-growth problem. For more details we refer to [5,8,9,18–21].

In the elliptic setting, a second approach was introduced in [12]. Therein, regularity results for functionals of the form

$$F(u) := \int_\Omega f(Du)\,\mathrm{d}x$$

with a convex integrand $f : \mathbb{R}^{Nn} \to [0, \infty)$, satisfying a non-standard p, q-growth, are established. Due to the coercivity of the integrand, the gradient of the minimizer u lies in L^p. The aim is to establish, that minimizers admit a gradient in L^q_{loc}. Therefore, one tests the corresponding Euler–Lagrange system with a finite difference of u and obtains a fractional differentiability of Du. At this stage it is not clear that the minimizer is also a solution to the Euler–Lagrange system. Hence one has to perform an approximation procedure. By using fractional Sobolev embeddings and a finite iteration, the desired higher integrability of Du can be deduced. Initially, this results holds only for the regularized problem, but it is possible to show, that these minimizers sub-converge to a minimizer of the original functional.

In [13], this method was extended for functionals, where the integrand f can additionally depend on x. It is assumed that f is α-Hölder continuous with respect to the x-variable, but not differentiable. Again fractional Sobolev spaces are used, to obtain a higher integrability for the gradient of the minimizer. Although it is not possible to differentiate the integrand, the Hölder continuity of f provides a certain kind of fractional differentiability for the gradient of minimizers u. Of course, a stronger assumption, depending on α, on the difference between p and q as in [12] is needed to show the desired higher integrability of Du. For more information to this topic we refer to [7,16,24,25].

Here we are interested in existence and regularity results for parabolic systems with p, q-growth. In this setting a variational approach was developed in [4]. Therein, the notion of variational solutions, which was introduced by Lichnevsky and Temam in [15] in the context of evolutionary minimal surface equations, is adapted. The advantage of these solutions is, that the existence can be established under mild assumptions on the convex integrand, which is independent on x and t, and on p and q, namely

$$\frac{2n}{n+2} < p \le q < p + 1.$$

After having the existence at hand, a parabolic version of fractional Sobolev spaces is used to achieve the higher integrability property $Du \in L^q_{\mathrm{loc}}$ in the case $p \ge 2$.

Moreover, higher integrability results via differentiability and interpolation in the parabolic case are obtained in [2]. Therein, Lipschitz regular integrands $f(x, Du)$ with $p(x, t)$-growth are considered and the a priori estimates are proven only by using the fact that the vector fields satisfies a p, q-growth condition. The method of fractional differentiability for parabolic systems has also been used in [11].

The aim of this paper is to establish the existence of weak solutions to parabolic systems of the form (1.1), where the integrand f satisfies a non-standard p, q-growth condition and is only Hölder continuous with respect to the x-variable. Note that besides measurability, we do not need any other assumption for the time variable. Existence results for variational solutions, can be gained in the same way as in [4]. The main effort of this works persists in proving a higher integrability result for the spatial gradient. This is accomplished by proving a suitable Caccioppoli type inequality for the regularized problem, and afterwards the higher integrability is gained by the parabolic fractional Sobolev embedding, where the condition

$$\frac{2n}{n+2} < p \leq q < p + \frac{\min\{2, p\}\,\alpha}{n+2}$$

is required. We also treat the singular case $\frac{2n}{n+2} < p < 2$. To deal with it, we have to overcome some problems. First, the Caccioppoli inequality can not directly be applied to the spatial gradient. Here, we have to make use of the V-function, which interpolates between quadratic and p-growth. The second problem is the appearance of quadratic terms of Du. Since $p < 2$, it is not clear that $Du \in L^2_{\mathrm{loc}}$ holds. But with the help of an interpolation argument it is possible to absorb the quadratic term and to handle these problems.

1.1. The setting

We consider Cauchy–Dirichlet problems of the type

$$\begin{cases} \partial_t u - \operatorname{div} \partial_\zeta f(x, t, Du) = 0 & \text{in } \Omega_T \\ u = g & \text{on } \partial_{\mathcal{P}}\Omega_T, \end{cases} \tag{1.2}$$

where $u\colon \Omega_T \subset \mathbb{R}^{n+1} \to \mathbb{R}^N$ with $n \geq 2$ and $N \geq 1$, can be a vector valued function. Here, Ω denotes a bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and for $T > 0$, $\Omega_T := \Omega \times (0, T)$ denotes the space-time cylinder. The parabolic boundary of Ω_T is denoted by $\partial_{\mathcal{P}}\Omega_T := [\partial\Omega \times (0, T)] \cup [\overline{\Omega} \times \{0\}]$. Points in $\mathbb{R}^{n+1}$ are termed $z = (x, t)$. With Du we mean the spatial gradient, and $\partial_t u$ stands for the differentiation with respect to the time variable. The function $f : \Omega_T \times \mathbb{R}^{Nn} \to \mathbb{R}$ is supposed to be a Carathéodory-function with

$$\begin{cases} \nu|\zeta|^p \leq f(x, t, \zeta) \leq L\,(1 + |\zeta|)^q\,, \\ \left|\partial_\zeta^2 f(x, t, \zeta)\right| \leq L(1 + |\zeta|)^{q-2}; \\ \left\langle \partial_\zeta^2 f(x, t, \zeta)\eta, \eta \right\rangle \geq \nu|\zeta|^{p-2}|\eta|^2, \\ |f(x_1, t, \zeta) - f(x_2, t, \zeta)| \leq L|x_1 - x_2|^\alpha (1 + |\zeta|)^q \end{cases} \tag{1.3}$$

for almost every $(x, t) \in \Omega_T$, $\zeta, \eta \in \mathbb{R}^{Nn}$, $\alpha \in (0, 1)$ and $0 < \nu \leq 1 \leq L < \infty$ with

$$\frac{2n}{n+2} < p < q < p + \frac{\min\{2, p\}\,\alpha}{n+2}. \tag{1.4}$$

It is an easy consequence, that also

$$\begin{cases} |\partial_\zeta f(x,t,\zeta)| \le c(q)L(1+|\zeta|)^{q-1}; \\ |\partial_\zeta f(x_1,t,\zeta) - \partial_\zeta f(x_2,t,\zeta)| \le c(q)L|x_1-x_2|^\alpha(1+|\zeta|)^{q-1} \end{cases} \tag{1.5}$$

holds (c.f. [18, Lemma 2.1]). For the boundary data g we assume that the following regularity assumptions hold true:

$$\begin{cases} g \in L^{\frac{p}{p-1+q}}\big(0,T;W^{1,\frac{p}{p-1+q}}(\Omega_T,\mathbb{R}^N)\big) \cap C^0\big([0,T];L^2(\Omega,\mathbb{R}^N)\big) \\ \partial_t g \in L^{p'}\big(0,T;W^{-1,p'}(\Omega_T,\mathbb{R}^N)\big), \end{cases} \tag{1.6}$$

where $p' = \frac{p}{p-1}$ denotes the Hölder conjugate of p. Note that $p'(q-1) > q$. In the following we will use the notation $u \in L^p(0,T;W_g^{1,p}(\Omega,\mathbb{R}^N))$, if $u - g \in L^p(0,T;W_0^{1,p}(\Omega,\mathbb{R}^N))$ holds.

1.2. The main result

Now, we state our existence result for the parabolic Cauchy–Dirichlet problem (1.2) and start with the definition of a weak solution, which has been already used in a similar way in [4]:

Definition 1.1. A function

$$u \in L^p\big(0,T;W_g^{1,p}(\Omega,\mathbb{R}^N)\big) \cap L^q_{\mathrm{loc}}\big(0,T;W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\big) \cap L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big),$$

with $u(\cdot,0) = g(\cdot,0)$, is called a weak solution of the parabolic system (1.2) if and only if

$$\int_{\Omega_T} u \cdot \partial_t\varphi - \big\langle \partial_\zeta f(x,t,Du), D\varphi\big\rangle \,\mathrm{d}z = 0$$

holds true whenever $\varphi \in C_0^\infty(\Omega_T,\mathbb{R}^N)$.

For weak solutions in the sense of Definition 1.1, we prove the following existence result:

Theorem 1.2. *Suppose that the integrand $f : \Omega_T \times \mathbb{R}^{Nn} \to [0,\infty)$ satisfies (1.3) and (1.4) and further, that g is as in (1.6). Then there exists a weak solution*

$$u \in L^p\big(0,T;W_g^{1,p}(\Omega,\mathbb{R}^N)\big) \cap L^q_{\mathrm{loc}}\big(0,T;W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\big) \cap L^\infty\big(0,T;L^2(\Omega,\mathbb{R}^N)\big),$$

with $u(\cdot,0) = g(\cdot,0)$ of the parabolic system (1.2). Moreover, there exist constants $\chi = \chi(n,p,q,\alpha)$ and $\tilde\chi = \tilde\chi(n,p,q,\alpha)$ such that for any cylinder $Q_R(z_0) \Subset \Omega_T$ the quantitative estimate

$$\int_{Q_{\frac{R}{2}}(z_0)} |Du|^q \,\mathrm{d}z \le c\left(M_{z_0,R}\right)^\chi,$$

holds for $p \geq 2$ and for $p < 2$ we have

$$\int_{Q_{\frac{R}{2}}(z_0)} |Du|^q \,\mathrm{d}z \leq c \left(M_{z_0,R} + N_{z_0,R}\right)^{\tilde{\chi}}$$

with

$$M_{z_0,R} := \sup_{t \in (t_0 - R^2, t_0)} \int_{B_R(x_0)} |u(\cdot,t)|^2 \mathrm{d}x + \int_{Q_R(z_0)} 1 + |u|^p + |Du|^p \,\mathrm{d}z$$

and

$$N_{z_0,R} := \left(\int_{Q_R(z_0)} 1 + |Du|^p \,\mathrm{d}z\right)^{\frac{4-(2-p)n}{p(n+2)-2n}}$$

and a constant $c = c(n, q, p, L, \nu, \alpha, R)$.

Remark 1.3. Now, we compare the elliptic bound for the difference between p and q with the parabolic bound for $p \geq 2$. In the stationary case in [13], the assumption

$$q < p\frac{n+\alpha}{n} = p + \frac{\alpha p}{n}$$

is needed for all $p > 1$, while for evolutionary problems

$$q < p + \frac{2\alpha}{n+2} = p + \frac{\alpha p}{n+2} \cdot \frac{2}{p}$$

is required for $p \geq 2$. This seems to be the natural bound, since one must replace n by $n+2$ and must take the parabolic deficit $\frac{2}{p}$ for $p \geq 2$ into account. However, for $p < 2$ an interesting phenomena appears. If we take the scaling deficit $2p/(p(n+2) - 2n)$, the maximal difference would be $\alpha(pn + 2p - 2n)/(2(n+2))$, which is smaller than $\alpha p/(n+2)$. However, we can prove the better bound, which is also stable for $p \nearrow 2$.

The first step of the proof is to show that there exists a variational solution $u \in L^\infty(0, T; L^2(\Omega, \mathbb{R}^n)) \cap L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N))$ in the sense of Definition 3.2. The existence of such solutions can be established under the assumption

$$\frac{2n}{n+2} < p \leq q < p + 1,$$

cf. Theorem 3.4. For more details, we refer to Sect. 3.

1.3. Model examples

Here, we give some examples for integrands f, which are discussed in this paper. For instance, we can consider functions

$$f(x,\zeta) = a(x)h(\zeta),$$

where

$$\nu|\zeta|^p \le h(\zeta) \le L(1+|\zeta|)^q$$

holds and $a(x)$ is α-Hölder continuous with $0 < \nu \le a(x) \le L$. For the function h, we can take for example

$$h(\zeta) = |\zeta|^p \log(1+|\zeta|).$$

We could also consider functions with anisotropic growth, i.e.

$$f(x,\zeta) = \sum_{i=1}^{n} a_i(x)|\zeta_i|^{p_i},$$

where $0 < \nu \le a_i(x) \le L$ is Hölder continuous and $p = p_1 \le p_2 \le \cdots \le p_n = q$ holds.

For integrands of the form

$$f(x,\zeta) = |\zeta|^p + a(x)|\zeta|^q,$$

we require only the weaker assumption $0 \le a(x) \le L$, where a is again a α-Hölder continuous function.

2. Preliminaries

Here, we state some usefull tools, that will be needed throughout the paper.

2.1. Auxiliary tools and notations

With

$$B_\rho(x_0) := \left\{x \in \mathbb{R}^n : |x - x_0| < \rho\right\}$$

we denote the open ball in $\mathbb{R}^n$ with centre x_0 and radius ρ, and

$$Q_\rho(z_0) := B_\rho(x_0) \times (t_0 - \rho^2, t_0)$$

is the parabolic standard cylinder.

In order to absorb certain terms, we will use the following iteration Lemma, which can be found for instance in [14, Lemma 6.1].

Lemma 2.1. *Let $\phi(\rho)$ be a bounded, non-negative function on $0 \le R_0 \le \rho \le R_1$ and assume for $R_0 \le \rho < r \le R_1$ there holds*

$$\phi(\rho) \le \vartheta\phi(r) + \frac{A}{(r-\rho)^\alpha} + \frac{B}{(r-\rho)^\beta} + C$$

for some fixed non-negative constants $A, B, C, \alpha \ge \beta \ge 0$ and $\vartheta \in (0,1)$. Then there exists a constant $c = c(\vartheta, \alpha)$ such that for all $R_0 \le \rho_0 < r_0 \le R_1$ we have

$$\phi(\rho_0) \le c\left(\frac{A}{(r_0-\rho_0)^\alpha} + \frac{B}{(r_0-\rho_0)^\beta} + C\right).$$

The next Lemma can be found in [1, Lemma 2.1] and [22, Lemma 2.1].

Lemma 2.2. *Let $k \in \mathbb{N}$. For every $\sigma \in (-1/2, \infty)$ there exists a constant $c = c(\sigma) \geq 1$ such that the following estimate holds true:*

$$\begin{aligned} c^{-1}(\mu^2 + |A|^+|B|^2)^\sigma &\leq \int_0^1 (\mu^2 + |A + s(B - A)|^2)^\sigma \, \mathrm{d}s \\ &\leq c(\mu^2 + |A|^2 + |B|^2)^\sigma \end{aligned}$$

for any $\mu \geq 0$ and $A, B \in \mathbb{R}^k$, not both zero if $\mu = 0$ and $\sigma < 0$.

We need the auxiliary function $V : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$.

$$V(\zeta) := |\zeta|^{\frac{p-2}{2}} \zeta,$$

and from the last Lemma, we can conclude (cf. [1, Lemma 2.2]):

Lemma 2.3. *For any $p \in (1, 2)$ and $\mu \in [0, 1]$ there holds*

$$\begin{aligned} \frac{1}{c(p,n)} \frac{|\zeta - \eta|}{\left(\mu^2 + |\zeta|^2 + |\eta|\right)^{\frac{2-p}{4}}} &\leq |V(\zeta) - V(\eta)| \\ &\leq c(p,n) \frac{|\zeta - \eta|}{\left(\mu^2 + |\zeta|^2 + |\eta|^2\right)^{\frac{2-p}{4}}} \end{aligned}$$

for arbitrary $\zeta, \eta \in \mathbb{R}^n$, not both zero if $\mu = 0$.

Another important tool is the next interpolation inequality, which is a consequence of Gagliardo-Nirenberg's inequality (see [23, Lemma 3.2]).

Lemma 2.4. *Assume that the function $v : Q_r(z_0) \to \mathbb{R}^k$ satisfies*

$$v \in L^\infty\big(t_0 - r^2, t_0; L^2(B_r(x_0), \mathbb{R}^k)\big) \cap L^p\big(t_0 - r^2, t_0; W^{1,p}(B_r(x_0), \mathbb{R}^k)\big)$$

for some exponents $1 \leq p < \infty$. Then there holds, for every radius $\rho \in (r/2, r)$,

$$\begin{aligned} &\int_{Q_\rho(z_0)} |v|^{p\frac{n+2}{n}} \, \mathrm{d}z \\ &\quad \leq c r^p \left(\sup_{s \in (t_0 - r^2, t_0)} \int_{B_r(x_0)} |v(x,s)|^q \, \mathrm{d}x \right)^{p/n} \int_{Q_r(z_0)} \left(|Dv|^p + \frac{|v|^p}{(r-\rho)^p} \right) \mathrm{d}z \end{aligned}$$

with a constant c depending on n, k, p and q.

2.2. Fractional Sobolev spaces

Now we state some results for parabolic fractional Sobolev spaces. The embedding for such spaces will play a crucial part in the proof, since it provides higher integrability properties. We will only be concerned with the parabolic case, for more information for elliptic fractional Sobolev spaces see for instance [3,6].

We say that $u \in L^p(0, T; W^{k,p}(\Omega, \mathbb{R}^N))$ with $1 \leq p < \infty, k \in \mathbb{N}_0, \mu \in (0, 1)$ belongs to the parabolic fractional Sobolev space $L^p(0, T; W^{k+\mu,p}(\Omega, \mathbb{R}^N))$ if the parabolic Gagliardo semi-norm

$$[D^\beta u]^p_{\mu,0,p;\Omega_T} := \int_0^T \int_\Omega \int_\Omega \frac{|D^\beta u(x,t) - D^\beta u(y,t)|^p}{|x-y|^{n+\mu p}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

is finite for any multiindex $\beta \in \mathbb{N}_0$ with $|\beta| = k$. Analogous to the elliptic setting we define the norm

$$\|u\|_{k+\mu,0,p;\Omega_T} := \|u\|_{L^p(0,T;W^{k,p}(\Omega,\mathbb{R}^N))} + \sum_{|\beta|=k} [D^\beta f]_{\mu,0,p;\Omega_T},$$

which makes $L^p(0, T; W^{k+\mu,p}(\Omega, \mathbb{R}^N))$ to a Banach-space.

The next Lemma provides an embedding result for fractional parabolic Sobolev spaces and is proved in [4, Lemma 6.5].

Lemma 2.5. *Let $B_\rho(x_0) \times (t_1, t_2) \subset \mathbb{R}^{n+1}$ be a general space-time cylinder with $0 < \rho \leq 1$ and $\theta, \mu \in (0, 1)$, $1 < p, r < s < \infty$ parameters such that*

$$(s-p)\left(1 - \mu + \frac{n}{r}\right) \leq \theta p.$$

Further assume that $u \in L^p(t_1, t_2; W^{1+\theta,p}(B_\rho(x_0))) \cap L^\infty(t_1, t_2; W^{\mu,r}(B_\rho(x_0)))$. Then $Du \in L^s(B_\vartheta(x_0) \times (t_1, t_2))$ for any $0 < \vartheta < \rho$ and moreover, the quantitative estimate

$$\|Du\|_{L^s(B_\vartheta(x_0)\times(t_1,t_2))} \leq c\|u\|^{\frac{p}{s}}_{L^p(t_1,t_2;W^{1+\theta,p}(B_\rho(x_0)))} \sup_{t\in(t_1,t_2)} \|u(\cdot,t)\|^{\frac{s-p}{s}}_{W^{\mu,r}(B_\rho(x_0))}$$

holds true with a constant $c = c(n, \mu, \theta, r, p, s, 1/(\rho - \vartheta))$.

Finally we need an elliptic and parabolic version of the embedding of Nikolskii spaces into fractional Sobolev spaces (cf. [4, Lemma 6.6]).

Lemma 2.6. *Let $k \in \mathbb{N}$, $\tilde{\Omega} \Subset \Omega$, $\theta \in (0, 1)$ and $0 \leq t_1 < t_2 \leq T$.*

(1) *Assume that $u \in L^\infty(0, T, L^2(\Omega, \mathbb{R}^k))$ satisfies*

$$\sup_{t\in(t_1,t_2)} \int_{\tilde{\Omega}} |u(x + he_i, t) - u(x,t)|^2 \mathrm{d}x \leq M|h|^{2\theta}$$

for every $i \in \{1, \ldots, n\}$ and $h \in \mathbb{R}$ with $|h| \leq \min\left\{\mathrm{dist}(\tilde{\Omega}, \partial\Omega), A\right\}$, where $A, M > 0$. Then for every $\alpha \in (0, \theta)$ and $\mathcal{O} \Subset \tilde{\Omega}$ there exists a constant $c = c(n, \theta, \alpha, A, \mathrm{dist}(\mathcal{O}, \partial\tilde{\Omega}), \mathrm{dist}(\Omega, \tilde{\Omega}))$ such that

$$\sup_{t\in(t_1,t_2)} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|u(x,t) - u(y,t)|^2}{|x-y|^{n+2\alpha}} \mathrm{d}x\,\mathrm{d}y \leq cM.$$

(2) *Assume that* $u \in L^p(\Omega_T, \mathbb{R}^k)$ *satisfies*

$$\int_{t_1}^{t_2} \int_{\tilde{\Omega}} |u(x + he_i, t) - u(x, t)|^p \, \mathrm{d}x \, \mathrm{d}t \leq M|h|^{\theta p}$$

for every $i \in \{1, \ldots, n\}$ *and* $h \in \mathbb{R}$ *with* $|h| \leq \min\big\{\mathrm{dist}(\tilde{\Omega}, \partial\Omega), A\big\}$, *where* $A, M > 0$. *Then for every* $\gamma \in (0, \theta)$ *and* $\mathcal{O} \Subset \tilde{\Omega}$, *there exists a constant* $c = c(n, \theta, \gamma, A, \mathrm{dist}(\mathcal{O}, \partial\tilde{\Omega}), \mathrm{dist}(\Omega, \tilde{\Omega}))$ *such that*

$$\int_{t_1}^{t_2} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|u(x, t) - u(y, t)|^p}{|x - y|^{n+p\gamma}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \leq cM.$$

3. Existence of variational solutions

In this section, we prove the existence of variational solutions. In [4] such a result has already been shown for integrands, which do not depend on x or t. But the techniques are applicable in our case, too. Thus we will only describe the notion of variational solutions and give a sketch of the proof.

The existence of variational solutions can be shown under much weaker assumptions, than the existence of weak solutions. Here, the integrand f must only fulfil the following growth conditions:

$$\begin{cases} 0 \leq f(x, t, \zeta) \leq L\,(1 + |\zeta|)^q\,, \\ \big\langle \partial_\zeta f(x, t, \zeta) - \partial_\zeta f(x, t, \eta), \zeta - \eta \big\rangle \geq \nu \left(\mu^2 + |\zeta|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\zeta - \eta|^2, \end{cases} \tag{3.1}$$

whenever $\zeta, \eta \in \mathbb{R}^{Nn}$ and for some $0 < \nu \leq 1 \leq L$ and $\mu \in [0, 1]$.

To give the precise definition of variational solutions, we introduce a notion of weaker continuity with respect to time.

Definition 3.1. Let X be a Banach space. A function $u \in L^\infty(0, T; X)$ belongs to the function space $C_\omega([0, T]; X)$ of weakly continuous functions from $[0, T]$ to X if $u(\cdot, t) \in X$ for any $t \in [0, T]$ and

$$t \mapsto \langle \psi, u(t) \rangle_X \quad \text{is continuous for any } \psi \in X'.$$

Here $\langle \cdot, \cdot, \rangle$ denotes the duality pairing between X' and X.

Now we are able to give a definition of variational solutions to the Cauchy–Dirichlet problem (1.2) (cf. [4, Definition 2.2.]).

Definition 3.2. Suppose $f : \Omega_T \times \mathbb{R}^{Nn} \to [0, \infty)$ is an integrand satisfying (3.1). Furthermore, assume that the Cauchy–Dirichlet datum g fulfils (1.6). We identify a map

$$u \in L^p\big(0, T; W_g^{1,p}(\Omega, \mathbb{R}^N)\big) \cap C_\omega\big([0, T]; L^2(\Omega, \mathbb{R}^N)\big)$$

as a variational solution of the Cauchy–Dirichlet problem (1.2) if and only if $u(\cdot, 0) = g(\cdot, 0)$ and, further, the variational inequality

$$\int_0^\tau \langle \partial_t v, v - u\rangle_{W^{1,p}(\Omega,\mathbb{R}^N)} \mathrm{d}t + \int_{\Omega_\tau} [f(x,t,Dv) - f(x,t,Du)]\mathrm{d}z$$
$$\geq \frac{1}{2}\|(v-u)(\cdot,\tau)\|^2_{L^2(\Omega,\mathbb{R}^N)} - \frac{1}{2}\|(v-g)(\cdot,0)\|^2_{L^2(\Omega,\mathbb{R}^N)} \tag{3.2}$$

holds true, whenever $v \in L^p(0,T; W^{1,p}_g(\Omega,\mathbb{R}^N))$ with $\partial_t v \in L^{p'}(0,T; W^{-1,p'}(\Omega,\mathbb{R}^N))$ and $\tau \in (0,T]$.

Remark 3.3. A variational solution belonging to the space $L^q_{\mathrm{loc}}(0,T; W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^n))$ is in fact already a weak solution. Hence, if the higher integrability is established, the two concepts coincide.

First, we show an existence result for variational solutions.

Theorem 3.4. *Suppose that* $f : \Omega_T \times \mathbb{R}^{Nn} \to [0,\infty)$ *satisfies* (3.1) *and*

$$\frac{2n}{n+2} < p < q < p+1$$

and g *satisfies* (1.6). *Then, there exists a unique variational solution*

$$u \in L^p\big(0,T; W^{1,p}_g(\Omega,\mathbb{R}^N)\big) \cap C_\omega\big([0,T]; L^2(\Omega,\mathbb{R}^N)\big)$$

with $u(\cdot,0) = g(0,\cdot)$.

Proof. Since the proof is essentially the same as the one of Theorem 2.4 in [4], we will only give a sketch of the proof.
Step 1: First, we consider the regularized integrand

$$f_\varepsilon(x,t,\zeta) := f(x,t,\zeta) + \varepsilon|\zeta|^p$$

for $\varepsilon \in (0,1]$. Then, f_ε satisfies a standard q-growth condition and [17] ensures the existence of a unique weak solution

$$u_\varepsilon \in L^q\big(0,T; W^{1,q}(\Omega,\mathbb{R}^N)\big) \cap C^0\big([0,T], L^2(\Omega,\mathbb{R}^N)\big)$$

to the Cauchy–Dirichlet problem

$$\begin{cases} \partial_t u_\varepsilon - \operatorname{div} \partial_\zeta f_\varepsilon(x,t,Du_\varepsilon) = 0 & \text{in } \Omega_T \\ u_\varepsilon = g & \text{on } \partial_{\mathcal{P}}\Omega_T. \end{cases}$$

Step 2: Next, we prove a suitable energy bound for u_ε. Therefore, we take $\varphi = (u_\varepsilon - g)$ as testing function in the weak formulation (note, that this is only possible on a formal level) and with help of the growth conditions (3.1), we get the following energy bound

$$\sup_{t\in(0,T)} \int_\Omega |u_\varepsilon(u(\cdot,t))|^2 \mathrm{d}x + \int_{\Omega_T} |u_\varepsilon|^p + |Du_\varepsilon|^p \mathrm{d}z \leq c(\nu, L, q, p, \Omega, g). \tag{3.3}$$

Step 3: Using the energy bound (3.3) and the fact, that u_ε is a weak solution, we get for any $0 < t_1 < t_2 < T$ and $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$

$$\left| \int_{\Omega\times(t_1,t_2)} u_\varepsilon \cdot \partial_t \varphi \, dz \right| \leq c|\operatorname{spt}\varphi|^{\frac{p+1-q}{p}} \|D\varphi\|_{L^\infty(\Omega\times(t_1,t_2))}.$$

This and a density argument guarantees that

$$\|u_\varepsilon(\cdot, s_1) - u_\varepsilon(\cdot, s_2)\|_{W^{-\ell,2}(\Omega)} \leq c|s_1 - s_2|^{\frac{p+1-q}{p}} \tag{3.4}$$

holds true for any $s_1, s_2 \in (t_1, t_2)$ and $\ell > \frac{n+2}{n}$. But this is the desired weak continuity property with respect to the time variable for u_ε.
Step 4: The weak solutions are also variational solutions, which can be easily deduced by testing the weak formulation with $\varphi = v - u_\varepsilon$.
Step 5: In order to prove, that there exists a variational solution, we have to pass the limit $\varepsilon \downarrow 0$. The energy bound (3.3) and (3.4) ensure the existence of a function $u \in L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N))$ such that

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{weakly in } L^p\big(0, T; W^{1,p}(\Omega, \mathbb{R}^N)\big) \\ u_\varepsilon(\cdot, t) \rightharpoonup u(\cdot, t) & \text{weakly in } L^2\big(\Omega, \mathbb{R}^N\big) \text{ for any } t \in [0, T] \end{cases}$$

for a (not re-labelled) subsequence. Since u_ε is already a variational solution for every $\varepsilon > 0$, we can pass to the limit $\varepsilon \downarrow 0$ in (3.2). Note, that the functions u_ε belong to space $C^0([0, T]; L^2(\Omega, \mathbb{R}^N))$, but they loose this property in the limit $\varepsilon \downarrow 0$ and u belongs only to the space $C_\omega([0, T]; L^2(\Omega, \mathbb{R}^N))$.

Step 6: It remains to show, that there exists only one variational solution. To this end, we assume that there exist two different solutions u_1 and u_2. If we choose $v = (u_1 + u_2)/2$ as comparison map in (3.2), we get a contradiction and the desired claim follows. Note, that this choice for the comparison map is only possible on a formal level, because the functions u_1 and u_2 do not possess a time derivative in $L^{p'}(0, T; W^{-1,p'}(\Omega, \mathbb{R}^N))$. Therefore, one has to use a mollification in time to make the calculations rigorous. □

4. A local L^q-estimate for the Spatial gradient

This section contains the main effort of this work. Here, we show the higher integrability for the spatial gradient Du. To be more precisely, we first assume that $Du \in L^q_{\text{loc}}(\Omega_T, \mathbb{R}^N)$ holds, and prove that the $L^q_{\text{loc}}(\Omega_T, \mathbb{R}^N)$-norm of Du can be estimated only in terms of the $L^p(\Omega_T, \mathbb{R}^N)$ of Du. This result can later on be used in an approximation scheme. For the approximating sequence, the higher integrability is known and the results from this section, ensures the higher integrability of variational solutions.

First we define

$$\tau_{h,i}[v](x, t) := v(x + he_i, t) - v(x, t) \quad \text{and} \quad \Delta_{h,i}[v](x, t) := \frac{\tau_{h,i}[v](x, t)}{h},$$

and start with a Caccioppoli-type inequality.

Lemma 4.1. *Let* $p > \frac{2n}{n+2}$ *and*

$$u \in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N)) \cap L^q_{\mathrm{loc}}(0,T;W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)) \cap C^0([0,T];L^2(\Omega,\mathbb{R}^N))$$

be a weak solution to (1.2)*, where* (1.3) *holds. Then for every parabolic cylinder* $Q_R(z_0) \Subset \Omega_T$*, any* $0 < r \le \rho_1 < \rho_2 \le R$*, any* $0 < |h| < \frac{\rho_2-\rho_1}{8}$ *and any* $i \in \{1,\dots,n\}$ *there holds*

$$\begin{aligned}
&\sup_{t\in(t_0-\rho_1^2,t_0)} \int_{B_{\rho_1}(x_0)} |\tau_{h,i}[u](\cdot,t)|^2 \mathrm{d}x \\
&\quad + \int_{Q_{\rho_1}(z_0)} \Big(|Du(x,t)|^2 + |Du(x+he_i,t)|^2\Big)^{\frac{p-2}{2}} |\tau_{h,i}[Du]|^2 \mathrm{d}z \\
&\le c\frac{|h|^\alpha}{(\rho_2-\rho_1)^2} \int_{Q_{\rho_2}(z_0)} (1+|Du|)^q \mathrm{d}z + \frac{c}{(\rho_2-\rho_1)^2} \int_{Q_{\frac{\rho_1+\rho_2}{2}}(z_0)} |\tau_{h,i}u|^2 \mathrm{d}z
\end{aligned} \tag{4.1}$$

with a constant $c = c(\nu, L, p, q)$.

Proof. Without loss of generality, we assume that $z_0 = 0$ and write Q_ρ instead of $Q_\rho(0)$. In the weak formulation

$$\int_{\Omega_T} u \cdot \varphi_t - \big\langle \partial_\zeta f(x,t,Du), D\varphi \big\rangle \mathrm{d}z = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega_T,\mathbb{R}^n)$$

we replace φ by $\tau_{-h,i}[\varphi]$ with $0 < |h| \ll 1$ and obtain after an integration by parts for finite differences

$$\int_{\Omega_T} \tau_{h,i}[u] \cdot \varphi_t - \big\langle \tau_{h,i}[\partial_\zeta f(x,t,Du)], D\varphi \big\rangle \mathrm{d}z = 0 \tag{4.2}$$

for all $\varphi \in C_0^\infty(\Omega_T,\mathbb{R}^N)$ and $|h|$ small enough. In this formulation we choose the testing function $\varphi(x,t) = \tau_{h,i}[u](x,t)\eta^2(x)\zeta(t)\chi_\theta(t)$, where $\eta \in C_0^1(B_{(\rho_1+\rho_2)/2}, [0,1])$ and $\zeta, \chi_\theta \in W^{1,\infty}(\mathbb{R},[0,1])$ are cut-off functions. The spatial cut-off function η satisfies $\eta \equiv 1$ in B_{ρ_1} and $|D\eta| \le 4/(\rho_2-\rho_1)$, while ζ is defined by

$$\zeta(t) := \begin{cases} 0 & \text{for } t \in \Big(-\infty, -\big(\frac{\rho_1+\rho_2}{2}\big)^2\Big] \\ \frac{1}{\big(\frac{\rho_1+\rho_2}{2}\big)^2-\rho_1^2}\Big(t+\big(\frac{\rho_1+\rho_2}{2}\big)\Big)^2 & \text{for } t \in \Big[-\big(\frac{\rho_1+\rho_2}{2}\big)^2, -\rho_1^2\Big) \\ 1 & \text{for } t \in [-\rho_1^2, \infty) \end{cases}$$

and χ_θ is given by

$$\chi_\theta(t) := \begin{cases} 1 & \text{for } t \in (-\infty, \tau-\theta] \\ \frac{1}{\theta}(\tau - t) & \text{for } t \in (\tau-\theta, \tau] \\ 0 & \text{for } t \in (\tau, 0], \end{cases}$$

for some $\tau \in (-\rho_1^2, 0)$ and $\theta \in (0, \rho_1^2 + \tau)$. With this choice, (4.2) turns into

$$-\int_{Q_{\rho_2}} \tau_{h,i}[u] \cdot \partial_t(\tau_{h,i}[u]\zeta\chi_\theta)\eta^2 \mathrm{d}z + \int_{Q_{\rho_2}} \left\langle \tau_{h,i}[\partial_\zeta f(x,t,Du)], \tau_{h,i}[Du]\right\rangle \eta^2\zeta\chi_\theta \mathrm{d}z$$
$$= -\int_{Q_{\rho_2}} \left\langle \tau_{h,i}[\partial_\zeta f(x,t,Du)], \nabla\eta^2 \otimes \tau_{h,i}[u]\right\rangle \zeta\chi_\theta \mathrm{d}z. \tag{4.3}$$

For the first term on the left-hand side in (4.3), we obtain for almost every $\tau \in (-\rho_1^2, 0)$

$$\begin{aligned}
-\int_{Q_{\rho_2}} \tau_{h,i}[u] \cdot \partial_t(\tau_{h,i}[u]\zeta\chi_\theta)\eta^2 \mathrm{d}z &= \int_{Q_{\rho_2}} \partial_t \tau_{h,i}[u] \cdot (\tau_{h,i}[u]\zeta\chi_\theta)\eta^2 \mathrm{d}z \\
&= \frac{1}{2}\int_{Q_{\rho_2^2}} \partial_t|\tau_{h,i}[u]|^2\eta^2\zeta\chi_\theta \mathrm{d}z = -\frac{1}{2}\int_{Q_{\rho_2^2}} |\tau_{h,i}[u]|^2\eta^2\partial_t(\zeta\chi_\theta)\mathrm{d}z \\
&= -\frac{1}{2\left(\left(\frac{\rho_1+\rho_2}{2}\right)^2 - \rho_1^2\right)} \int_{-\left(\frac{\rho_1+\rho_2}{2}\right)^2}^{-\rho_1^2} \int_{B_{\frac{\rho_1+\rho_2}{2}}} |\tau_{h,i}[u]|^2\eta^2\chi_\theta \mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{2\theta}\int_{\tau-\theta}^{\tau}\int_{B_{\rho_2}} |\tau_{h,i}[u]|^2\eta^2\zeta \mathrm{d}x\,\mathrm{d}t \\
&\xrightarrow{\theta\downarrow 0} -\frac{1}{2\left(\left(\frac{\rho_1+\rho_2}{2}\right)^2 - \rho_1^2\right)} \int_{-\left(\frac{\rho_1+\rho_2}{2}\right)^2}^{-\rho_1^2} \int_{B_{\frac{\rho_1+\rho_2}{2}}} |\tau_{h,i}[u]|^2\eta^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{2}\int_{B_{\rho_2}} |\tau_{h,i}[u](\cdot,\tau)|^2\eta^2 \mathrm{d}x.
\end{aligned}$$

Passing also the limit $\theta \downarrow 0$ in (4.3) and using the estimate $1/((\frac{\rho_1+\rho_2}{2})^2 - \rho_1^2) \le 4/(\rho_2-\rho_1)^2$, we get

$$\begin{aligned}
\frac{1}{2}\mathrm{I} + \mathrm{II} &:= \frac{1}{2}\int_{B_{\rho_2}} |\tau_{h,i}[u](\cdot,\tau)|^2\eta^2 \mathrm{d}x + \int_{Q^\tau_{\rho_2}} \left\langle \tau_{h,i}[\partial_\zeta f(x,t,Du)], \tau_{h,i}[Du]\right\rangle \eta^2\zeta \mathrm{d}z \\
&\le -2\int_{Q^\tau_{\rho_2}} \left\langle \tau_{h,i}[\partial_\zeta f(x,t,Du)], \nabla\eta \otimes \tau_{h,i}[u]\right\rangle \eta\zeta \mathrm{d}z \\
&\quad + \frac{2}{(\rho_2-\rho_1)^2}\int_{Q_{\frac{\rho_1+\rho_2}{2}}} |\tau_{h,i}[u]|^2 \mathrm{d}z \\
&=: \mathrm{III} + \mathrm{IV},
\end{aligned}$$

where we used the abbreviation $Q^\tau_\rho := B_\rho \times (-\rho^2, \tau)$. Now, we rewrite the term II

$$\begin{aligned}
\mathrm{II} &= \int_{Q^\tau_{\rho_2}} \eta^2\zeta \left\langle \partial_\zeta f(x+he_i,t,Du(x+he_i,t)) \right. \\
&\quad \left. - \partial_\zeta f(x,t,Du(x,t)), \tau_{h,i}[Du](x,t)\right\rangle \mathrm{d}z \\
&= \int_{Q^\tau_{\rho_2}} \eta^2\zeta \Big[\left\langle \partial_\zeta f(x+he_i,t,Du(x+he_i,t)) \right.
\end{aligned}$$

$$
\begin{aligned}
&- \partial_\zeta f(x,t,Du(x+he_i,t)),\tau_{h,i}[Du](x,t)\big\rangle\Big]\mathrm{d}z\\
&+\int_{Q^\tau_{\rho_2}} \eta^2\zeta\left\langle\partial_\zeta f(x,t,Du(x+he_i,t)) - \partial_\zeta f(x,t,Du(x,t)),\tau_{h,i}[Du](x,t)\right\rangle \mathrm{d}z\\
&=: \mathrm{II}_1+\mathrm{II}_2
\end{aligned}
$$

and estimate II_1 with help of $(1.5)_2$

$$
\begin{aligned}
|\mathrm{II}_1| &\le c(q)|h|^\alpha L\int_{-\rho_2^2}^0\int_{B_{\frac{\rho_2+\rho_1}{2}}}(1+|Du(x+he_i,t)|)^{q-1}|\tau_{h,i}[Du](x,t)|\mathrm{d}x\,\mathrm{d}t\\
&\le c|h|^\alpha\int_{Q_{\rho_2}}(1+|Du|)^q\,\mathrm{d}z,
\end{aligned}
$$

since $|h|<(\rho_2-\rho_1)/8$. For the other term we use $(1.3)_3$ and Lemma 2.2 to obtain

$$
\begin{aligned}
\mathrm{II}_2 &= \int_{Q^\tau_{\rho_2}}\eta^2\zeta\int_0^1\left\langle\partial_\zeta^2 f(x,t,Du+s\tau_{h,i}[Du])\tau_{h,i}[Du],\tau_{h,i}[Du]\right\rangle\mathrm{d}s\,\mathrm{d}z\\
&\ge \nu\int_{Q^\tau_{\rho_2}}\eta^2\zeta\int_0^1|Du+s\tau_{h,i}[Du]|^{p-2}|\tau_{h,i}[Du]|^2\mathrm{d}s\,\mathrm{d}z\\
&\ge \frac{\nu}{c}\int_{Q^\tau_{\rho_2}}\eta^2\zeta\left(|Du(x,t)|^2+|Du(x+he_i,t)|^2\right)^{\frac{p-2}{2}}|\tau_{h,i}[Du]|^2\mathrm{d}z.
\end{aligned}
$$

It remains to estimate III. If we use

$$
\int_{B_{\frac{\rho_1+\rho_2}{2}}}|\tau_{h,i}[u]|^q\,\mathrm{d}x \le |h|^q\int_{B_{\rho_2}}|Du|^q\,dx
$$

and the condition $(1.5)_1$, we get

$$
\begin{aligned}
|\mathrm{III}| &\le c\int_{-\rho_2^2}^0\int_{B_{\frac{\rho_1+\rho_2}{2}}}|D\eta|\left[(1+|Du(x,t)|)^{q-1}+(1+|Du(x+he_i,t)|)^{q-1}\right]\\
&\qquad\cdot|\tau_{h,i}[u]|\mathrm{d}x\,\mathrm{d}t\\
&\le \frac{c}{\rho_2-\rho_1}\int_{-\rho_2^2}^0\left(\int_{B_{\rho_2}}(1+|Du|)^q\,\mathrm{d}x\right)^{1-\frac{1}{q}}\left(\int_{B_{\frac{\rho_1+\rho_2}{2}}}|\tau_{h,i}[u]|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\mathrm{d}t\\
&\le c\frac{|h|}{\rho_2-\rho_1}\int_{Q_{\rho_2}}(1+|Du|)^q\,\mathrm{d}z\\
&\le c\frac{|h|^\alpha}{\rho_2-\rho_1}\int_{Q_{\rho_2}}(1+|Du|)^q\,\mathrm{d}z.
\end{aligned}
$$

Combing the previous estimates, we can conclude that

$$\begin{aligned}
&\int_{B_{\rho_2}} |\tau_{h,i}[u](\cdot,\tau)|^2\eta^2\,\mathrm{d}x \\
&\qquad + \int_{Q^\tau_{\rho_2}} \eta^2\zeta\left(|Du(x,t)|^2 + |Du(x+he_i,t)|^2\right)^{\frac{p-2}{2}} |\tau_{h,i}[Du]|^2\,\mathrm{d}z \\
&\qquad \le c\frac{|h|^\alpha}{(\rho_2-\rho_1)^2}\int_{Q_{\rho_2}} (1+|Du|)^q\,\mathrm{d}z + \frac{c}{(\rho_2-\rho_1)^2}\int_{Q_{\frac{\rho_1+\rho_2}{2}}(z_0)} |\tau_{h,i}[u]|^2\,\mathrm{d}z
\end{aligned}$$

holds for every $\tau \in (-\rho_2, 0)$. Taking the supremum over $\tau \in (-\rho_2, 0)$ in the first term and letting $\tau \to 0$ in the second term on the left-hand side, completes the proof of the Lemma. □

With the Caccioppoli type inequality at hand, we can prove the desired higher integrability for the spatial gradient. To this end, we will make use of the fractional Gagliardo-Nirenberg inequality (Lemma 2.5).

Lemma 4.2. *Let*

$$\frac{2n}{n+2} < p < q < p + \frac{\min\{2,\, p\}\,\alpha}{n+2} \tag{4.4}$$

and

$$u \in L^p\big(0,T;\, W^{1,p}(\Omega,\mathbb{R}^N)\big) \cap L^q_{\mathrm{loc}}\big(0,T;\, W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\big) \cap C^0\big([0,T];\, L^2(\Omega,\mathbb{R}^N)\big)$$

be a weak solution to (1.2)*, where* (1.3) *holds. If $p \ge 2$, then there exists a constant $\chi = \chi(n,q,p,\alpha)$ such that for every parabolic cylinder $Q_R(x_0) \Subset \Omega_T$ there holds*

$$\int_{Q_{\frac{R}{2}}(z_0)} |Du|^q\,\mathrm{d}z \le c\left(M_{z_0,R}\right)^\chi,$$

and for $p < 2$ there exists a constant $\tilde\chi = \tilde\chi(n,q,p,\alpha)$ such that for every parabolic cylinder $Q_R(x_0) \Subset \Omega_T$ there holds

$$\int_{Q_{\frac{R}{2}}(z_0)} |Du|^q\,\mathrm{d}z \le c\left(M_{z_0,R} + N_{z_0,R}\right)^{\tilde\chi}$$

with

$$M_{z_0,R} := \sup_{t\in(t_0-R^2,t_0)} \int_{B_R(x_0)} |u(\cdot,t)|^2\,\mathrm{d}x + \int_{Q_R(z_0)} 1 + |u|^p + |Du|^p\,\mathrm{d}z \tag{4.5}$$

and

$$N_{z_0,R} := \left(\int_{Q_R(z_0)} 1 + |Du|^p\,\mathrm{d}z\right)^{\frac{4-(2-p)n}{p(n+2)-2n}} \tag{4.6}$$

and a constant $c =$ depending only on n, q, p, L, ν, α and R.

Proof. We start with the case $p \geq 2$, where (4.4) can be written as

$$q < p + \frac{2\alpha}{n+2}. \tag{4.7}$$

Consider $0 < \rho_1 < \rho_2 \leq R$. Making use of Lemma 4.1 and the fact that $p \geq 2$, we get for $0 < |h| < (\rho_2 - \rho_1)/8$

$$\begin{aligned}
&\sup_{t\in(t_0-\rho_1^2,t_0)} \int_{B_{\rho_1}(x_0)} |\tau_{h,i}[u](\cdot,t)|^2 \mathrm{d}x + \int_{Q_{\rho_1}(z_0)} |\tau_{h,i}[Du]|^p \mathrm{d}z \\
&\quad \leq \sup_{t\in(t_0-\rho_1^2,t_0)} \int_{B_{\rho_1}(x_0)} |\tau_{h,i}[u](\cdot,t)|^2 \mathrm{d}x \\
&\qquad + \int_{Q_{\rho_1}(z_0)} \Big(|Du(x,t)|^2 + |Du(x+he_i,t)|^2\Big)^{\frac{p-2}{2}} |\tau_{h,i}[Du]|^2 \mathrm{d}z \\
&\quad \leq c\frac{|h|^\alpha}{(\rho_2-\rho_1)^2} \int_{Q_{\rho_2}(z_0)} (1+|Du|)^q \mathrm{d}z \\
&\qquad + \frac{c}{(\rho_2-\rho_1)^2} \int_{Q_{\frac{\rho_1+\rho_2}{2}}(z_0)} |\tau_{h,i}u|^2 \mathrm{d}z \\
&\quad \leq c\frac{|h|^\alpha}{(\rho_2-\rho_1)^2} \int_{Q_{\rho_2}(z_0)} (1+|Du|)^q \mathrm{d}z,
\end{aligned}$$

where we used

$$\int_{Q_{\frac{\rho_1+\rho_2}{2}}(z_0)} |\tau_{h,i}u|^2 \mathrm{d}z \leq |h|^2 \int_{Q_{\rho_2}(z_0)} |Du|^2 \mathrm{d}z.$$

For $r_1, r_2 > 0$ with $\frac{\rho_1+\rho_2}{2} \leq r_1 < r_2 \leq \frac{\rho_1+3\rho_2}{4}$, Lemma 2.6 and the last inequality imply that

$$\begin{cases} u \in L^\infty\left(t_0 - r_2^2, t_0; W^{\mu,2}(B_{r_2}(x_0), \mathbb{R}^N)\right) & \text{for all } \mu \in \left(0, \frac{\alpha}{2}\right) \\ u \in L^p\left(t_0 - r_2^2, t_0; W^{1+\theta,p}(B_{r_2}(x_0), \mathbb{R}^N)\right) & \text{for all } \theta \in \left(0, \frac{\alpha}{p}\right) \end{cases}$$

holds. Moreover, Lemma 2.5 ensures the existence of constants $\beta = \beta(p, \mu, \theta, \alpha, s) > 0$ and $c = c(n, p, s, \alpha, \mu, \theta, 1/(\rho_2 - \rho_1))$ such that

$$\begin{aligned}
\int_{Q_{r_1}(z_0)} |Du|^s \mathrm{d}z \leq \frac{c}{(r_2-r_1)^\beta} &\left(\int_{Q_{r_2}(z_0)} 1 + |u|^p + |Du|^p + |Du|^q \mathrm{d}z\right) \\
&\cdot \left(\sup_{t\in(t_0-r_2^2,t_0)} \int_{B_{r_2}(x_0)} |u(\cdot,t)|^2 \mathrm{d}x + \int_{Q_{r_2}(z_0)} 1 + |Du|^q \mathrm{d}z\right)^{\frac{s-p}{2}},
\end{aligned} \tag{4.8}$$

for all s with

$$(s-p)\left(1 - \mu + \frac{n}{2}\right) \leq \theta p.$$

Using the maximal range for the variables θ and μ, we can conclude that the last estimate holds for all

$$s < p + \frac{2\alpha}{n+2-\alpha}.$$

Since

$$q < p + \frac{2\alpha}{n+2} < p + \frac{2\alpha}{n+2-\alpha}$$

by assumption we can choose $s \in (q, p + \frac{2\alpha}{n+2-\alpha})$. For $\delta > \frac{s}{q} > 1$, where δ will be chosen later, we infer from (4.8)

$$\begin{aligned}
&\int_{Q_{r_1}(z_0)} |Du|^s \,\mathrm{d}z \\
&\quad \le \frac{c}{(r_2-r_1)^\beta} \left(\int_{Q_{r_2}(z_0)} |Du|^q \,\mathrm{d}z + M_{z_0,R} \right)^{1+\frac{s-p}{2}} \\
&\quad \le \frac{c}{(r_2-r_1)^\beta} \left(\int_{Q_{r_2}(z_0)} |Du|^{\frac{s}{\delta}} |Du|^{q-\frac{s}{\delta}} \,\mathrm{d}z \right)^{1+\frac{s-p}{2}} + \frac{c}{(r_2-r_1)^\beta} M_{z_0,R}^{1+\frac{s-p}{2}} \\
&\quad \le \frac{c}{(r_2-r_1)^\beta} \left(\int_{Q_{r_2}(z_0)} |Du|^s \,\mathrm{d}z \right)^{\frac{1}{\delta}(1+\frac{s-p}{2})} \\
&\qquad \cdot \left(\int_{Q_{r_2}(z_0)} |Du|^{\frac{q\delta-s}{\delta-1}} \,\mathrm{d}z \right)^{\frac{\delta-1}{\delta}(1+\frac{s-p}{2})} + \frac{c}{(r_2-r_1)^\beta} M_{z_0,R}^{1+\frac{s-p}{2}}.
\end{aligned} \tag{4.9}$$

Next, we want to absorb the term involving the L^s-norm of Du from the right-hand side into the left, so that there remain only terms with the L^p-norm of Du on the right-hand side. Therefore, we have to choose δ and s in such a way that

$$\frac{1}{\delta}\left(1 + \frac{s-p}{2}\right) < 1 \quad \text{and} \quad \frac{\delta q - s}{\delta - 1} \le p \tag{4.10}$$

holds, but this is equivalent to

$$1 + \frac{s-p}{2} < \delta \quad \text{and} \quad \delta \le \frac{s-p}{q-p}.$$

If we choose

$$\delta = \frac{s-p}{q-p},$$

it is sufficient to show that we can find $s \in (q, p + \frac{2\alpha}{n+2-\alpha})$ satisfying

$$1 + \frac{s-p}{2} < \delta \quad \Leftrightarrow \quad q - p < \frac{2(s-p)}{2+s-p},$$

where we note that $\delta > \frac{s}{q}$ holds. Since

$$q - p < \frac{2\alpha}{n+2},$$

there exists $\varepsilon = \varepsilon(\alpha, n) > 0$ such that

$$q - p < \frac{2\alpha - 2\varepsilon}{n+2-\varepsilon} \tag{4.11}$$

is true. Moreover, there exists $s_0 \in (q, p + \frac{2\alpha}{n+2-\alpha})$ such that (4.9) is true for all $s \in [s_0, p + \frac{2\alpha}{n+2-\alpha})$ with

$$s - p \geq \frac{2\alpha}{n+2-\alpha} - \frac{2\varepsilon}{n+2-\alpha}.$$

Hence, by (4.11), we obtain

$$\frac{2(s-p)}{2+s-p} \geq \frac{2\frac{2\alpha-2\varepsilon}{n+2-\alpha}}{2+\frac{2\alpha-2\varepsilon}{n+2-\alpha}} = \frac{2\alpha-2\varepsilon}{n+2-\varepsilon} > q - p,$$

which means, we can choose δ and s such that (4.10) and $s > \frac{s}{q}$ holds. With this choice in (4.9), we can apply Young's inequality and obtain

$$\begin{aligned}
&\int_{Q_{r_1}(z_0)} |Du|^s \,\mathrm{d}z \\
&\quad \leq \frac{1}{2} \int_{Q_{r_2}(z_0)} |Du|^s \,\mathrm{d}z + \frac{c}{(r_2 - r_1)^\beta} \left(\int_{Q_R(z_0)} |Du|^p \,\mathrm{d}z \right)^\chi \\
&\quad\quad + \frac{c}{(r_2 - r_1)^\beta} M_{z_0,R}^{1+\frac{s-p}{2}},
\end{aligned}$$

for some exponent χ depending on n, p, q, and α. Lemma 2.1 allows to absorb the term involving the L^s-norm of Du from the left-hand side into the right and yields that

$$\int_{Q_{\frac{R}{2}}(z_0)} |Du|^s \,\mathrm{d}z \leq c \left(M_{z_0,R}\right)^\chi$$

for some $s \in [s_0, p + \frac{2\alpha}{n+2-\alpha})$. But this implies already the claim of the Lemma for $p \geq 2$

Now we consider the case $p < 2$, where

$$q < p + \frac{\alpha p}{n+2} \tag{4.12}$$

holds. Lemma 2.3 and 4.1 imply the following estimate

$$\begin{aligned}
&\sup_{t \in (t_0 - \rho_1^2, t_0)} \int_{B_{\rho_1}(x_0)} |\tau_{h,i}[u](\cdot, t)|^2 \,\mathrm{d}x + \int_{Q_{\rho_1}(z_0)} |\tau_{h,i}[V(Du)]|^2 \,\mathrm{d}z \\
&\quad \leq c \frac{|h|^\alpha}{(\rho_2 - \rho_1)^2} \int_{Q_{\rho_2}(z_0)} (1 + |Du|)^q \,\mathrm{d}z + \frac{c}{(\rho_2 - \rho_1)^2} \int_{Q_{\frac{\rho_1+\rho_2}{2}}(z_0)} |\tau_{h,i}[u]|^2 \,\mathrm{d}z
\end{aligned}$$

and dividing the last inequality by $|h|^2$ leads to

$$\begin{aligned}\sup_{t\in(t_0-\rho_1^2,t_0)}\int_{B_{\rho_1}(x_0)}|\Delta_{h,i}[u](\cdot,t)|^2\,\mathrm{d}x+\int_{Q_{\rho_1}(z_0)}|\Delta_{h,i}[V(Du)]|^2\,\mathrm{d}z\\ \le c\frac{|h|^{\alpha-2}}{(\rho_2-\rho_1)^2}\int_{Q_{\rho_2}(z_0)}(1+|Du|)^q\,\mathrm{d}z\\ +\frac{c}{(\rho_2-\rho_1)^2}\int_{Q_{\frac{\rho_1+\rho_2}{2}}(z_0)}|\Delta_{h,i}[u]|^2\,\mathrm{d}z. \qquad (4.13)\end{aligned}$$

Since $p<2$ holds, we do not know, whether $Du\in L^2_{\mathrm{loc}}(\Omega_T)$ is satisfied or not. Thus, we want to absorb the quadratic term on the right-hand side and proceed in a similar way as in [23,27]. To this end, we estimate with Hölder's inequality

$$\begin{aligned}&\int_{Q_{\frac{\rho_1+\rho_2}{2}}(z_0)}|\Delta_{h,i}[u]|^2\,\mathrm{d}z\\ &=\int_{Q_{\frac{\rho_1+\rho_2}{2}}(z_0)}|\Delta_{h,i}[u]|^{\frac{p(n+2)-2n}{2}}|\Delta_{h,i}[u]|^{\frac{(n+2)(2-p)}{2}}\,\mathrm{d}z\\ &\le\left(\int_{Q_{\frac{\rho_1+\rho_2}{2}}(z_0)}|\Delta_{h,i}[u]|^p\,\mathrm{d}z\right)^{\frac{p(n+2)-2n}{2p}}\\ &\quad\cdot\left(\int_{Q_{\frac{\rho_1+\rho_2}{2}}(z_0)}|\Delta_{h,i}[u]|^{p\frac{n+2}{n}}\,\mathrm{d}z\right)^{\frac{(2-p)n}{2p}}.\end{aligned}$$

Next, we use Lemma 2.4 to estimate the second term, where we note that $(\rho_1+\rho_2)/2+|h|\le(\rho_1+3\rho_2)/4$ holds, and get

$$\begin{aligned}&\int_{Q_{\frac{\rho_1+\rho_2}{2}}(z_0)}|\Delta_{h,i}[u]|^2\,\mathrm{d}z\\ &\le c\left(\int_{Q_{\rho_2}(z_0)}|Du|^p\,\mathrm{d}z\right)^{\frac{p(n+2)-2n}{2p}}\\ &\quad\cdot\left(\int_{Q_{\frac{\rho_1+3\rho_2}{4}}(z_0)}\rho_2^p|\Delta_{h,i}[Du]|^p+\frac{\rho_2^p}{(\rho_2-\rho_1)^p}|\Delta_{h,i}[u]|^p\,\mathrm{d}z\right)^{\frac{(2-p)n}{2p}}\\ &\quad\cdot\left(\sup_{t\in(t_0-\rho_2^2,t_0)}\int_{B_{\rho_2}(x_0)}|\Delta_{h,i}[u](\cdot,t)|^2\,\mathrm{d}x\right)^{\frac{2-p}{2}}. \qquad (4.14)\end{aligned}$$

Lemma 2.3 implies

$$|\Delta_{h,i}[Du]|^p\le c|\Delta_{h,i}[V(Du)]|^p\left(|Du(x+he_i,t)|^2+|Du(x,t)|^2\right)^{\frac{(2-p)p}{4}},$$

which leads to

$$
\begin{aligned}
&\left(\int_{Q_{\frac{\rho_1+3\rho_2}{4}}(z_0)} \rho_2^p|\Delta_{h,i}[Du]|^p + \frac{\rho_2^p}{(\rho_2-\rho_1)^p}|\Delta_{h,i}[u]|^p \mathrm{d}z\right)^{\frac{(2-p)n}{2p}} \\
&\le c\left[\left(\int_{Q_{\rho_2}} \rho_2^p|\Delta_{h,i}[V(Du)]|^2\mathrm{d}z\right)^{\frac{p}{2}}\left(\int_{Q_{\rho_2}}(1+|Du|)^p\mathrm{d}z\right)^{\frac{2-p}{2}}\right. \\
&\quad\left.+\frac{\rho_2^p}{(\rho_2-\rho_1)^p}\int_{Q_{\rho_2}(z_0)}|Du|^p\mathrm{d}z\right]^{\frac{(2-p)n}{2p}} \\
&\le c\left[\left(\int_{Q_{\rho_2}} \rho_2^p|\Delta_{h,i}[V(Du)]|^2\mathrm{d}z\right)\left(\int_{Q_{\rho_2}}(1+|Du|)^p\mathrm{d}z\right)^{\frac{2-p}{p}}\right. \\
&\quad\left.+\left(\frac{\rho_2^p}{(\rho_2-\rho_1)^p}\int_{Q_{\rho_2}(z_0)}|Du|^p\mathrm{d}z\right)^{\frac{2}{p}}\right]^{\frac{(2-p)n}{4}} \\
&\le c\left(\frac{1}{(\rho_2-\rho_1)^p}\int_{Q_{\rho_2}(z_0)}(1+|Du|^p)\mathrm{d}z\right)^{\frac{(2-p)^2n}{4p}} \\
&\quad\cdot\left(\int_{Q_{\rho_2}(z_0)}\rho_2^2|\Delta_{h,i}[V(Du)]|^2\mathrm{d}z + \frac{\rho_2^p}{(\rho_2-\rho_1)^p}\int_{Q_{\rho_2}(z_0)}|Du|^p\mathrm{d}z\right)^{\frac{(2-p)n}{4}}.
\end{aligned}
$$

Combining the last estimate with (4.14), yields

$$
\begin{aligned}
&\frac{c}{(\rho_2-\rho_1)^2}\int_{Q_{\frac{\rho_1+\rho_2}{2}}(z_0)}|\Delta_{h,i}[u]|^2\mathrm{d}z \\
&\le c\left(\left(\frac{1}{(\rho_2-\rho_1)^2}\right)^{\frac{4}{4-(2-p)n}}\frac{1}{(\rho_2-\rho_1)^p}\int_{Q_{\rho_2}(z_0)}(1+|Du|)^p\mathrm{d}z\right)^{\frac{4-(2-p)n}{4}} \\
&\quad\cdot\left(\sup_{t\in(t_0-\rho_2^2,t_0)}\int_{B_{\rho_2}(x_0)}|\Delta_{h,i}[u](\cdot,t)|^2\mathrm{d}x\right)^{\frac{2-p}{2}} \\
&\quad\cdot\left(\int_{Q_{\rho_2}(z_0)}\rho_2^2|\Delta_{h,i}[V(Du)]|^2 + \frac{\rho_2^p}{(\rho_2-\rho_1)^p}|Du|^p\mathrm{d}z\right)^{\frac{(2-p)n}{4}} \\
&\le c\left(\frac{1}{(\rho_2-\rho_1)^2}\right)^{\frac{4}{p(n+2)-2n}}\left(\frac{1}{(\rho_2-\rho_1)^p}\int_{Q_{\rho_2}(z_0)}(1+|Du|)^p\right)^{\frac{4-(2-p)n}{p(n+2)-2n}} \\
&\quad+\frac{1}{2}\sup_{t\in(t_0-\rho_2^2,t_0)}\int_{B_{\rho_2}(x_0)}|\Delta_{h,i}[u](\cdot,t)|^2\mathrm{d}x + \frac{1}{2}\int_{Q_{\rho_2}(z_0)}|\Delta_{h,i}[V(Du)]|^2\mathrm{d}z,
\end{aligned}
$$

where we used Young's inequality with $4/(p(n+2)-2n)$, $2/(2-p)$ and $4/(2-p)n$. If we insert this in (4.13) and use Lemma 2.1 to absorb the last two terms on the right-hand side, we get

$$\begin{aligned}
&\sup_{t\in(t_0-\rho_1^2,t_0)}\int_{B_{\rho_1}(x_0)}|\Delta_{h,i}[u](\cdot,t)|^2\,\mathrm{d}x+\int_{Q_{\rho_1}(z_0)}|\Delta_{h,i}[V(Du)]|^2\,\mathrm{d}z\\
&\quad\le c\frac{|h|^{\alpha-2}}{(\rho_2-\rho_1)^2}\int_{Q_{\rho_2}(z_0)}(1+|Du|)^q\,\mathrm{d}z\\
&\qquad+c\left(\frac{1}{(\rho_2-\rho_1)^2}\right)^{\frac{4}{p(n+2)-2n}}\left(\int_{Q_{\rho_2}(z_0)}(1+|Du|)^p\,\mathrm{d}z\right)^{\frac{4-(2-p)n}{p(n+2)-2n}},
\end{aligned}$$

or respectively

$$\begin{aligned}
&\sup_{t\in(t_0-\rho_1^2,t_0)}\int_{B_{\rho_1}(x_0)}|\tau_{h,i}[u](\cdot,t)|^2\,\mathrm{d}x+\int_{Q_{\rho_1}(z_0)}|\tau_{h,i}[V(Du)]|^2\,\mathrm{d}z\\
&\quad\le c\frac{|h|^{\alpha}}{(\rho_2-\rho_1)^2}\int_{Q_{\rho_2}(z_0)}(1+|Du|)^q\,\mathrm{d}z\\
&\qquad+c|h|^{\alpha}\left(\frac{1}{(\rho_2-\rho_1)^2}\right)^{\frac{4}{p(n+2)-2n}}\left(\int_{Q_{\rho_2}(z_0)}(1+|Du|)^p\,\mathrm{d}z\right)^{\frac{4-(2-p)n}{p(n+2)-2n}}.
\end{aligned}$$

On the one hand, the last inequality implies

$$\begin{aligned}
&\sup_{t\in(t_0-\rho_1^2,t_0)}\int_{B_{\rho_1}(x_0)}|\tau_{h,i}[u](\cdot,t)|^2\,\mathrm{d}x\\
&\quad\le c\frac{|h|^{\alpha}}{(\rho_2-\rho_1)^2}\int_{Q_{\rho_2}(z_0)}(1+|Du|)^q\,\mathrm{d}z\\
&\qquad+c|h|^{\alpha}\left(\frac{1}{(\rho_2-\rho_1)^2}\right)^{\frac{4}{p(n+2)-2n}}\left(\int_{Q_{\rho_2}(z_0)}(1+|Du|)^p\,\mathrm{d}z\right)^{\frac{4-(2-p)n}{p(n+2)-2n}},
\end{aligned}\tag{4.15}$$

on the other hand, if we additionally use Lemma 2.3, we obtain

$$\begin{aligned}
&\int_{Q_{\rho_1}(z_0)}|\tau_{h,i}[Du]|^p\,\mathrm{d}z\\
&\quad\le c\int_{Q_{\rho_1}(z_0)}|\tau_{h,i}[V(Du)]|^p\left(|Du(x+he_i,t)|^2+|Du(x,t)|^2\right)^{\frac{(2-p)p}{4}}\\
&\quad\le c\left(\int_{Q_{\rho_1}(z_0)}|\tau_{h,i}[V(Du)]|^2\,\mathrm{d}z\right)^{\frac{p}{2}}\left(\int_{Q_{\rho_2}(z_0)}|Du|^p\,\mathrm{d}z\right)^{\frac{2-p}{2}}\\
&\quad\le c|h|^{\alpha\frac{p}{2}}\left(\int_{Q_{\rho_2}(z_0)}|Du|^p\,\mathrm{d}z\right)^{\frac{2-p}{2}}\left[\frac{1}{(\rho_2-\rho_1)^2}\int_{Q_{\rho_2}(z_0)}(1+|Du|)^q\,\mathrm{d}z\right.
\end{aligned}$$

$$+\left(\frac{1}{(\rho_2-\rho_1)^2}\right)^{\frac{4}{p(n+2)-2n}}\left(\int_{Q_{\rho_2}(z_0)}(1+|Du|)^p\,\mathrm{d}z\right)^{\frac{4-(2-p)n}{p(n+2)-2n}}\Bigg]^{\frac{p}{2}} \tag{4.16}$$

If we choose $r_1, r_2 > 0$ with $\frac{\rho_1+\rho_2}{2} \le r_1 < r_2 \le \frac{\rho_1+3\rho_2}{4}$, Lemma 2.6, (4.15) and (4.16) imply

$$\begin{cases} u \in L^\infty\left(t_0-r_2^2, t_0; W^{\mu,2}(B_{r_2}(x_0),\mathbb{R}^N)\right) & \text{for all } \mu \in \left(0, \frac{\alpha}{2}\right) \\ u \in L^p\left(t_0-r_2^2, t_0; W^{1+\theta,p}(B_{r_2}(x_0),\mathbb{R}^N)\right) & \text{for all } \theta \in \left(0, \frac{\alpha}{2}\right). \end{cases}$$

Moreover, Lemma 2.5, combined with the estimates (4.15) and (4.16), ensures the existence of a constants $\tilde{\beta} = \tilde{\beta}(p, \mu, \theta, \alpha, s) > 0$ and $c = c(n, p, q, \nu, L, s)$ such that

$$\begin{aligned}
&\int_{Q_{r_1}(z_0)} |Du|^s\,\mathrm{d}z \\
&\quad\le \frac{c}{(r_2-r_1)^{\tilde{\beta}}}\left(\int_{Q_{r_2}(z_0)}|Du|^q\,\mathrm{d}z + M_{z_0,R} + N_{z_0,R}\right)^{\frac{p}{2}}\left(M_{z_0,R}+N_{z_0,R}\right)^{\frac{2-p}{2}} \\
&\qquad\cdot\left(\int_{Q_{r_2}(z_0)}|Du|^q\,\mathrm{d}z + M_{z_0,R} + N_{z_0,R}\right)^{\frac{s-p}{2}} \\
&\quad= \frac{c}{(r_2-r_1)^{\tilde{\beta}}}\left(\int_{Q_{r_2}(z_0)}|Du|^q\,\mathrm{d}z + M_{z_0,R} + N_{z_0,R}\right)^{\frac{s}{2}}\left(M_{z_0,R}+N_{z_0,R}\right)^{\frac{2-p}{2}} \\
&\quad\le \frac{c}{(r_2-r_1)^{\tilde{\beta}}}\left(\int_{Q_{r_2}(z_0)}|Du|^q\,\mathrm{d}z\right)^{\frac{s}{2}}\left(M_{z_0,R}+N_{z_0,R}\right)^{\frac{2-p}{2}} \\
&\qquad+\left(M_{z_0,R}+N_{z_0,R}\right)^{\frac{s+2-p}{2}}
\end{aligned} \tag{4.17}$$

holds for all s with

$$(s-p)\left(1-\mu+\frac{n}{2}\right) \le \theta p,$$

where $M_{z_0,R}$ and $N_{z_0,R}$ are defined in (4.5) and (4.6). Using the maximal range for the variables θ and μ, we can conclude that the last estimate holds for all

$$s < p + \frac{\alpha p}{n+2-\alpha}.$$

Since

$$q < p + \frac{\alpha p}{n+2} < p + \frac{\alpha p}{n+2-\alpha},$$

by assumption, we can choose $s \in (q, p + \frac{\alpha p}{n+2-\alpha})$. For $\delta > \frac{s}{q} > 1$, where δ will be chosen later, we estimate (4.17) in the following way

$$
\begin{aligned}
&\int_{Q_{r_1}(z_0)} |Du|^s \,\mathrm{d}z \\
&\quad \le \frac{c}{(r_2-r_1)^\beta}\left(\int_{Q_{r_2}(z_0)} |Du|^{\frac{s}{\delta}}|Du|^{q-\frac{s}{\delta}}\,\mathrm{d}z\right)^{\frac{s}{2}}\left(M_{z_0,R}+N_{z_0,R}\right)^{\frac{2-p}{2}} \\
&\qquad + \left(M_{z_0,R}+N_{z_0,R}\right)^{\frac{s+2-p}{2}} \\
&\quad \le \frac{c}{(r_2-r_1)^\beta}\left(\int_{Q_{r_2}(z_0)} |Du|^{s}\,\mathrm{d}z\right)^{\frac{1}{\delta}\frac{s}{2}}\left(\int_{Q_{r_2}(z_0)} |Du|^{\frac{q\delta-s}{\delta-1}}\,\mathrm{d}z\right)^{\frac{\delta-1}{\delta}\frac{s}{2}} \\
&\qquad \cdot \left(M_{z_0,R}+N_{z_0,R}\right)^{\frac{2-p}{2}} + \left(M_{z_0,R}+N_{z_0,R}\right)^{\frac{s+2-p}{2}}. \qquad (4.18)
\end{aligned}
$$

As in the case $p \ge 2$, we want to absorb the term involving the L^s-norm of Du on the right-hand side. Therefore, we have to choose δ and s in such a way that

$$
\frac{s}{2\delta} < 1 \quad \text{and} \quad \frac{\delta q - s}{\delta - 1} \le p \qquad (4.19)
$$

holds, but this is equivalent to

$$
\frac{s}{2} < \delta \quad \text{and} \quad \delta \le \frac{s-p}{q-p}.
$$

If we choose $\delta = \frac{s-p}{q-p}$, we only need to find $s \in (q, p + \frac{\alpha p}{n+2-\alpha})$ satisfying

$$
q - p < 2\frac{s-p}{s},
$$

where we note that $\delta > \frac{s}{q}$ holds. Since

$$
q - p < \frac{\alpha p}{n+2},
$$

there exists $\varepsilon = \varepsilon(\alpha, n, p) > 0$ such that

$$
q - p < \frac{\alpha p - 2\varepsilon}{n+2-\varepsilon} \qquad (4.20)
$$

is true. Moreover, there exists $s_0 \in (q, p + \frac{\alpha p}{n+2-\alpha})$ such that (4.18) is true for all $s \in [s_0, p + \frac{\alpha p}{n+2-\alpha})$ with

$$
s \ge \frac{\alpha p}{n+2-\alpha} + p - \frac{\varepsilon p}{n+2-\alpha}.
$$

Hence, by (4.20), we obtain

$$
2\frac{s-p}{s} \ge 2\frac{\frac{\alpha p - p\varepsilon}{n+2-\alpha}}{\frac{\alpha p - p\varepsilon}{n+2-\alpha} + p} = \frac{2\alpha - 2\varepsilon}{n+2-\varepsilon} > \frac{p\alpha - 2\varepsilon}{n+2-\varepsilon} > q - p,
$$

which means, we can choose δ and s such that (4.19) and $\delta > \frac{s}{q}$ holds. With this choice, (4.18) turns into

$$\int_{Q_{r_1}(z_0)} |Du|^s \,\mathrm{d}z \leq \frac{1}{2}\int_{Q_{r_2}(z_0)} |Du|^s \,\mathrm{d}z + \frac{c}{(r_2-r_1)^{\tilde{\beta}}}\left(M_{z_0,R}+N_{z_0,R}\right)^{\tilde{\chi}},$$

for some exponent $\tilde{\chi}$ depending only on n, p, q and α. Lemma 2.1 gives

$$\int_{Q_{\frac{R}{2}}(z_0)} |Du|^s \,\mathrm{d}z \leq c\left(M_{z_0,R}+N_{z_0,R}\right)^{\tilde{\chi}}$$

for some $s \in [s_0, p+\frac{\alpha p}{n+2-\alpha})$. This finishes the proof of the Lemma. □

5. Proof of Theorem 1.2

In this section we complete the proof of Theorem 1.2. Therefore, we regularize the functional and obtain variational solutions u_ε, which also solve the associated parabolic system. Lemma 4.2 guarantees an L^q_{loc}-bound for the spatial gradient of u_ε. In the limit $\varepsilon \downarrow 0$, this property can be transferred to the variational solution u and hence it also a weak solution.

Proof of Theorem 1.2. The procedure will be the same as in Sect. 7 of [4], so we will only give a sketch of the proof.
Step 1: For $\varepsilon \in (0, 1]$ we define

$$f_\varepsilon(x,t,\zeta) := f(x,t,\zeta) + \varepsilon|\zeta|^q.$$

For every fixed ε, $\partial_\zeta f_\varepsilon$ satisfies a standard q-growth condition and we obtain a unique weak solution

$$u_\varepsilon \in L^q\big(0,T;W^{1,q}(\Omega,\mathbb{R}^N)\big) \cap C^0\big([0,T],L^2(\Omega,\mathbb{R}^N)\big)$$

to the parabolic Cauchy–Dirichlet problem

$$\begin{cases} \partial_t u_\varepsilon - \operatorname{div}\partial_\zeta f_\varepsilon(x,t,Du_\varepsilon) = 0 & \text{in } \Omega_T \\ u_\varepsilon = g & \text{on } \partial_{\mathcal{P}}\Omega_T. \end{cases}$$

Step 2: In the following we want to pass to the limit $\varepsilon \downarrow 0$. Since we perform the same approximation schema as in Sect. 3, we gain the energy bound (3.3). From [10, Chapter I, Proposition 3.1] we conclude that

$$\int_{\Omega_T} |u_\varepsilon|^{\frac{p(n+2)}{n}} \,\mathrm{d}z \leq c\int_{\Omega_T}\left(|u_\varepsilon|^p + |Du_\varepsilon|^p\right)\mathrm{d}z \left(\sup_{t\in(0,T)}\int_\Omega |u_\varepsilon(\cdot,t)|^2 \,\mathrm{d}x\right)^{\frac{p}{n}}$$

holds for a constant $c = c(c, p, \Omega)$. Since $q < p + \frac{\min\{2,p\}\alpha}{n+2} \leq p\frac{n+2}{n}$ holds, the energy bound implies that u_ε is uniformly bounded in $L^q(\Omega_T,\mathbb{R}^N)$. Moreover, if

we combine the energy bound (3.3) with Lemma 4.2, we infer that Du is uniformly bounded in $L^q_{\mathrm{loc}}(\Omega_T, \mathbb{R}^{Nn})$. This ensures the existence of a function

$$u \in L^p\big(0, T; W^{1,p}(\Omega, \mathbb{R}^N)\big) \cap L^q_{\mathrm{loc}}\big(0, T; W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)\big),$$

with

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{weakly in } L^q_{\mathrm{loc}}(\Omega_T, \mathbb{R}^N) \\ Du_\varepsilon \rightharpoonup Du & \text{weakly in } L^q_{\mathrm{loc}}(\Omega_T, \mathbb{R}^{Nn}). \end{cases}$$

As in Sect. 3, we can only show that $u \in C_\omega([0, T]; L^2(\Omega, \mathbb{R}^N))$ holds, although $u_\varepsilon \in C^0([0, T]; L^2(\Omega, \mathbb{R}^N))$ is true for every $\varepsilon > 0$.
Step 3: With help of [26, Theorem 6], we obtain

$$\begin{cases} u_\varepsilon \to u & \text{strongly in } L^2(\Omega_T, \mathbb{R}^N) \text{ and } L^q(Q_0, \mathbb{R}^N) \text{ for any } Q_0 \Subset \Omega_T, \\ Du_\varepsilon \to Du & \text{strongly in } L^p(Q_0, \mathbb{R}^{Nn}) \text{ for any } Q_0 \Subset \Omega_T, \\ u_\varepsilon(\cdot, t) \to u(\cdot, t) & \text{strongly in } L^2(\mathcal{O}, \mathbb{R}^N) \text{ for any } \mathcal{O} \Subset \Omega \text{ and any } t \in (t_1, t_2). \end{cases}$$

Step 4: The convergence results allow us to pass to the limit $\varepsilon \downarrow 0$ in the weak formulation for u_ε, which implies that u is also a weak solution to the Cauchy–Dirichlet problem (1.2). The bounds of Lemma 4.2 can be transferred from u_ε to u, which completes the proof. □

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References

[1] Acerbi, E., Fusco, N.: Regularity for minimizers of non-quadratic functionals: the case $1 < p < 2$. Math. Anal. Appl. **140**, 115–135 (1989)
[2] Acerbi, E., Mingione, G., Seregin, G.A.: Regularity results for parabolic systems related to a class of non-Newtonian fluids. Ann. Inst. H. Poincaré Anal. Non Linéaire **21**(1), 25–60 (2004)
[3] Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
[4] Bögelein, V., Duzaar, F., Marcellini, P.: Parabolic systems with p, q-growth: a variational approach. Arch. Ration. Mech. Anal. **210**(1), 219–267 (2013)
[5] Bögelein, V., Duzaar, F., Marcellini, P.: Parabolic equations with p, q-growth. J. Math. Pures Appl. **100**(4), 535–563 (2013)
[6] Brezis, H., Mironescu, P.: Gagliardo–Nirenberg composition and products in fractional Sobolev spaces. J. Evol. Equ. **1**(4), 387–404 (2001)

[7] Carozza, M., Kristensen, J., Passarelli di Napoli, A.: Higher differentiability of minimizers of convex variational integrals. Ann. Inst. H. Poincaré Anal. Non Linéaire **28**(3), 395–411 (2011)
[8] Cupini, G., Marcellini, P., Mascolo, E.: Regularity under sharp anisotropic general growth conditions. Discret. Contin. Dyn. Syst. Ser. B **11**, 66–86 (2009)
[9] Cupini, G., Marcellini, P., Mascolo, E.: Local boundedness of solutions to quasilin-ear elliptic systems. Manuscr. Math. **137**, 287–315 (2012)
[10] DiBenedetto, E.: Degenerate Parabolic Equations, Universitytext xv, 387. Springer, New York, NY (1993)
[11] Duzaar, F., Mingione, G., Steffen, K.: Parabolic systems with polynomial growth and regularity. Mem. Am. Math. Soc. **214**(1005), x+118 (2011)
[12] Esposito, L., Leonetti, F., Mingione, G.: Higher Integrability for minimizers of integral functionals with p, q-growth. J. Differ. Equ. **157**(2), 414–438 (1999)
[13] Esposito, L., Leonetti, F., Mingione, G.: Sharp regularity for functionals with p, q growth. J. Differ. Equ. **204**(1), 5–55 (2004)
[14] Giusti, E.: Direct methods in the calculus of variations. World Scientific Publishing Company, Tuck Link, Singapore (2003)
[15] Lichnewsky, A., Temam, R.: Pseudosolutions of the time-dependent minimal surface problem. J. Differ. Equ. **30**(3), 340–364 (1978)
[16] Lieberman, G.M.: Gradient estimates for anisotropic elliptic equations. Adv. Differ. Equ. **10**(7), 767–812 (2005)
[17] Lions, J.L.: Quelques méthodes de résolution des problémes aux limites non linéaires. Dunod, Paris (1969)
[18] Marcellini, P.: Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. Arch. Math. Mech. Anal. **105**, 267–284 (1989)
[19] Marcellini, P.: Regularity and existence of solutions of elliptic equations with p, q-growth conditions. J. Differ. Equ. **90**(1), 1–30 (1991)
[20] Marcellini, P.: Regularity for elliptic equations with general growth conditions. J. Differ. Equ. **105**, 296–333 (1993)
[21] Marcellini, P.: Everywhere regularity for a class of elliptic systems without growth conditions. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **23**(1), 1–25 (1996)
[22] Giaquinta, M., Modica, G.: Partial regularity of minimizers of quasi- convex integrals. Annales de l'I. H. P., Section C, Tome **3**(3), 185–208 (1986)
[23] Scheven, C.: Regularity for subquadratic parabolic systems: higher integrability and dimension estimates. Proc. R. Soc. Edinb. **140A**, 1269–1308 (2010)
[24] Schmidt, T.: Regularity theorems for degenerate quasiconvex energies with p, q-growth. Adv. Calc. Var. **1**(3), 241–270 (2008)
[25] Schmidt, T.: Regularity of relaxed minimizers of quasiconvex variational integrals with p, q-growth. Arch. Ration. Mech. Anal. **193**(2), 311–337 (2009)
[26] Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Math. Pura Appl. IV. **146**, 65–96 (1987)
[27] Singer, T.: Parabolic equations with p, q-growth: the subquadratic case. Q. J. Math. **66**(2), 707–742 (2015)