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## Hypersurfaces with $H_{r+1} = 0$ in $\mathbb{H}^n \times \mathbb{R}$

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**Abstract.** We prove the existence of rotational hypersurfaces in  $\mathbb{H}^n \times \mathbb{R}$  with  $H_{r+1} = 0$  ( $r$ -minimal hypersurfaces) and we classify them. Then we prove some uniqueness theorems for  $r$ -minimal hypersurfaces with a given (finite or asymptotic) boundary. In particular, we obtain a Schoen-type theorem for two ended complete hypersurfaces.

### Introduction

In this article we deal with  $r$ -minimal hypersurfaces in  $\mathbb{H}^n \times \mathbb{R}$ , that is hypersurfaces in  $\mathbb{H}^n \times \mathbb{R}$  with  $H_{r+1} = 0$ .

First we address the problem of finding all  $r$ -minimal hypersurfaces in  $\mathbb{H}^n \times \mathbb{R}$  invariant by rotation with respect to a vertical axis. We prove that there is a one parameter family of them and that their behavior is very similar to that of catenoids in  $\mathbb{H}^n \times \mathbb{R}$ , obtained in Pierre Bérard and Ricardo Sa Earp [3] (Theorem 2.1).

Once proved the existence of this family of examples, we prove some rigidity results for  $r$ -minimal hypersurfaces spanning a fixed boundary or asymptotic boundary. In particular, we obtain classification results provided either the boundary or the asymptotic boundary is contained in two parallel slices (Theorems 3.1 and 3.2). For the precise definition of asymptotic boundary, see the end of Sect. 1. Theorem 3.1 is inspired by the results of Jorge Hounie and Maria Luiza Leite [9] for  $r$ -minimal hypersurfaces in Euclidean space. Theorem 3.2 is what we call a Schoen-type result. In his pioneer paper [14], Schoen characterizes the minimal complete hypersurfaces which are regular at infinity and have two ends. This result was generalized for  $r$ -minimal hypersurfaces of Euclidean space by Levi Lopes de Lima and Antonio Sousa [10] and also by Maria Luiza Leite and Henrique Araújo [1]. A Schoen-type result for minimal hypersurfaces in  $\mathbb{H}^n \times \mathbb{R}$  was obtained by the second author, Ricardo Sa Earp and Eric Toubiana in [11]. Our Theorem 3.2 is a generalization of the latter. In Euclidean space, the proofs in [1, 9, 10, 14] use

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the invariance of the minimality (or  $r$ -minimality) condition under ambient space scaling. The lack of such invariance in  $\mathbb{H}^n \times \mathbb{R}$  obliges one to look for reasonable geometrically analogous results and suitable strategies to obtain them. The reader will find more details and comments throughout the text.

We recall that, when working with  $H_{r+1} = 0$ , we are led to use a version of the maximum principle different from the one used for classical minimal hypersurfaces. In fact, here, ellipticity is not for free and one has to add some hypothesis on the principal curvatures vector (see Sect. 3). One of the consequence of this fact is that we must assume embeddedness in Theorem 3.2, that is for free in the mean curvature case.

Hypersurfaces with  $H_{r+1} = 0$  in  $\mathbb{R}^{n+1}$  have been broached in several papers. We refer the reader to [1, 7, 8, 10] and the references therein.

The paper is organized as follows. In the first section we fix notations. The second section is devoted to the classification of  $r$ -minimal hypersurfaces invariant by rotations and to the establishment of their properties. In the third section, we establish our uniqueness results for  $r$ -minimal hypersurfaces with either (finite) boundary or asymptotic boundary contained in two parallel slices.

### 1. Preliminaries

Let  $M^n, \bar{M}^{n+1}$  be oriented Riemannian manifolds of dimension  $n$  and  $n + 1$  respectively and let  $X : M^n \rightarrow \bar{M}^{n+1}$  be an isometric immersion. Let  $A$  be the linear operator associated to the second fundamental form of  $X$  and  $k_1, \dots, k_n$  be its eigenvalues. The  $r$ -mean curvature  $H_{r+1}$  of  $X$  is given by

$$\binom{n}{r+1} H_{r+1} = \sum_{i_1 < \dots < i_{r+1}} k_{i_1}, \dots, k_{i_{r+1}}, \quad 1 \leq r + 1 \leq n.$$

We recall that  $H_1$  ( $r = 0$ ) is the mean curvature of the immersion and that  $H_n$  ( $r + 1 = n$ ) is the Gauss–Kronecker curvature. The Newton tensors associated to  $X$  are inductively defined by

$$P_0 = I, \\ P_{r+1} = \binom{n}{r+1} H_{r+1} I - A \circ P_r, \quad r > 0.$$

For further details about the Newton tensors, see [12, 13]. We are interested in the case where  $\bar{M}^{n+1} = \mathbb{H}^n \times \mathbb{R}$ , where  $\mathbb{H}^n$  denotes the hyperbolic  $n$ -space and  $H_{r+1} = 0$ , for some  $r$ .

We use the ball model of the hyperbolic space  $\mathbb{H}^n$  ( $n \geq 2$ ), i.e.

$$\mathbb{H}^n = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1 \right\}$$

endowed with the metric

$$g_{\mathbb{H}} := \frac{dx_1^2 + \dots + dx_n^2}{\left(\frac{1-|x|^2}{2}\right)^2}.$$

In  $\mathbb{H}^n \times \mathbb{R}$ , with coordinates  $(x_1, \dots, x_n, t)$ , we consider the product metric

$$g_{\mathbb{H}} + dt^2.$$

For later use, we briefly recall the notion of asymptotic boundary of a hypersurface. We denote the ideal boundary of  $\mathbb{H}^n \times \mathbb{R}$  by  $\partial_{\infty}(\mathbb{H}^n \times \mathbb{R})$ .

Since we are using the ball model for  $\mathbb{H}^n$ ,  $\partial_{\infty}(\mathbb{H}^n \times \mathbb{R})$  is naturally identified with the cylinder  $S^{n-1} \times \mathbb{R}$  joined with the endpoints of all the non horizontal geodesic of  $\mathbb{H}^n \times \mathbb{R}$ . Here,  $S^{n-1}$  denotes the unitary  $(n - 1)$ -dimensional sphere.

The *asymptotic boundary* of a hypersurface  $M$  in  $\mathbb{H}^n \times \mathbb{R}$  is the set of the limit points of  $M$  in  $\partial_{\infty}(\mathbb{H}^n \times \mathbb{R})$  with respect to the Euclidean topology of  $S^{n-1} \times \mathbb{R}$ . The asymptotic boundary of the surface  $M$  will be denoted by  $\partial_{\infty}M$ , while the usual (finite) boundary of  $M$  will be denoted by  $\partial M$ .

### 2. $r$ -Minimal rotational hypersurfaces

Our aim in this section is to classify the  $r$ -minimal hypersurfaces in  $\mathbb{H}^n \times \mathbb{R}$  invariant by rotation about a vertical axis. In  $\mathbb{H}^n \times \mathbb{R}$ , we consider the coordinates  $(x_1, \dots, x_n, t)$  and, up to isometry, we can assume the rotation axis to be  $\{0\} \times \mathbb{R}$ . Notice that the slices  $t = const$  are  $r$ -minimal hypersurfaces invariant by rotation for any  $r$ .

We consider a hypersurface obtained by the rotation of a regular curve in the vertical plane  $V := \{(x_1, \dots, x_n, t) \in \mathbb{H}^n \times \mathbb{R} | x_1 = \dots = x_{n-1} = 0\}$ , parametrized by  $(\tanh(\frac{f(t)}{2}), t)$ , where  $f$  is a positive function.

We define a *rotational hypersurface* in  $\mathbb{H}^n \times \mathbb{R}$  by the parametrization

$$X : \begin{cases} \mathbb{R} \times S^{n-1} \rightarrow \mathbb{H}^n \times \mathbb{R} \\ (t, \zeta) \rightarrow (\tanh(f(t)/2)\zeta, t). \end{cases}$$

The normal field to the immersion can be chosen to be

$$N = \left(1 + f_t^2(t)\right)^{-1/2} \left(\frac{-1}{2 \cosh^2(f(t)/2)}\zeta, f_t(t)\right) \tag{1}$$

and the principal curvatures associated to  $X$  are then given by (see [3])

$$k_1 = k_2 = \dots = k_{n-1} = \operatorname{cotgh}(f(t)) \left(1 + f_t^2(t)\right)^{-1/2} \quad \text{and} \\ k_n = -f_{tt}(t) \left(1 + f_t^2(t)\right)^{-3/2}.$$

We set  $q = \frac{n-r-1}{r+1}$  and a straightforward computation yields

$$(q + 1)H_{r+1} = -\operatorname{cotgh}^r(f(t))f_{tt}(t) \left(1 + f_t^2(t)\right)^{-\frac{r+3}{2}} \\ + q \operatorname{cotgh}^{r+1}(f(t)) \left(1 + f_t^2(t)\right)^{-\frac{r+1}{2}} \tag{2}$$

or, equivalently,

$$(q+1)f_t(t) \left(1+f_t^2(t)\right)^{\frac{r}{2}} \frac{\sinh^{q+r}(f(t))}{\cosh^r(f(t))} H_{r+1} = \frac{\partial}{\partial t} \left[ \sinh^q(f(t)) \left(1+f_t^2(t)\right)^{-\frac{1}{2}} \right]. \tag{3}$$

The solutions of either (2) or (3) with  $H_{r+1} = 0$  will be the profile of the  $r$ -minimal hypersurfaces invariant by rotation.

We state below our classification result. We point out that in the statement we discard the slices, that are  $r$ -minimal for each  $r$ .

**Theorem 2.1.** *The  $r$ -minimal complete hypersurfaces invariant by rotation in  $\mathbb{H}^n \times \mathbb{R}$  are the following:*

- (a) For  $n = r + 1$  : right cylinders above spheres of dimension  $n - 1$ .
- (b) For  $r + 1 < n$  : a one parameter family  $\{\mathcal{M}_a(r)\}_{a>0}$  of hypersurfaces with the following properties. Any  $\mathcal{M}_a(r)$  is embedded and homeomorphic to an annulus symmetric with respect to the slice  $t = 0$ . The distance between the rotational axis and the “neck” of  $\mathcal{M}_a(r)$  is  $a$ . The asymptotic boundary of  $\mathcal{M}_a(r)$  is composed by two horizontal circles in  $\partial_\infty(\mathbb{H}) \times \mathbb{R}$  whose vertical distance is an increasing function of  $a$ , taking values in  $\left(0, \frac{(r+1)\pi}{(n-r-1)}\right)$ . Moreover, if  $a \neq b$  then the generating curves of  $\mathcal{M}_a(r)$  and  $\mathcal{M}_b(r)$  intersect exactly at two symmetric points.

*Proof.* For  $n = r + 1$ , it is easy to see that the solutions of Eq. (2) for  $H_{r+1} = 0$  satisfy  $f_t(t) = \text{const}$ , that is, they are part of cones or right cylinders. Since we search for complete hypersurfaces, (a) is proved.

We now prove (b). We first notice that, in order to solve (2) with  $H_{r+1} = 0$ , it is enough to solve the following Cauchy problem

$$\begin{cases} f_{tt} = q \operatorname{cotgh}(f(t)) (1 + f_t^2(t)) \\ f(0) = a \\ f_t(0) = 0, \end{cases} \tag{4}$$

for any  $a > 0$ .

In fact, we only have to realize that the condition  $f_t(0) = 0$  is not restrictive. We recall that the Cauchy–Lipschitz theorem guarantees the existence of a unique maximal solution for given initial data. Since we are considering  $f(t) > 0$ , a solution of the equation in (4) satisfies  $f_{tt} \geq q > 0$ . Then, the maximal solution attains a minimum at some point of the corresponding interval. We can, w.l.g., suppose it attains a minimum at  $t = 0$  and we are done.

Let  $(I_a, f(a, t))$  be the maximal solution of (4). Since  $f(a, -t)$  also solves the equation, we conclude that  $f(a, t)$  is an even function of  $t$ , and we can write  $I_a = (-L(a), L(a))$  for some  $L(a) \in \mathbb{R}^+ \cup \{\infty\}$ .

By imposing  $H_{r+1} = 0$  in the Eq. (3), integrating and using the initial conditions of the Cauchy problem we obtain

$$\frac{\sinh^q(f(a, t))}{(1 + f_t^2(a, t))^{\frac{1}{2}}} = \sinh^q(a) \quad \text{for all } t \in I_a. \tag{5}$$

In order to obtain the result, we explore the geometric properties of the solutions  $(I_a, f(a, t))$ , that can be deduced from (4) and (5). Our analysis is inspired by the one in [3] and [5].

Since  $f_{tt}(t) > 0$ , the profile curve is strictly convex. Moreover,  $f(a, \cdot)$  is greater or equal to  $a$  and is increasing on  $(0, L(a))$ . As it is a maximal solution of (4) (and (5)),  $f(a, \cdot)$  must go to infinity for  $t \rightarrow \pm L(a)$ . Then, we can define the inverse function  $\lambda(a, \rho)$  for  $\rho \in [a, \infty)$  onto  $[0, L(a)]$  that satisfies  $\lambda_\rho(a, f(a, t))f_t(a, t) = 1$ . Hence we have

$$\lambda(a, \rho) = \sinh^q(a) \int_a^\rho \frac{1}{\sqrt{\sinh^{2q}(u) - \sinh^{2q}(a)}} du. \tag{6}$$

Setting  $v = \frac{\sinh(u)}{\sinh(a)}$ , we obtain

$$\lambda(a, \rho) = \int_1^{\frac{\sinh(\rho)}{\sinh(a)}} (v^{2q} - 1)^{-1/2} \sinh(a) \left(1 + v^2 \sinh^2(a)\right)^{-1/2} dv. \tag{7}$$

Now, we notice that

$$\begin{aligned} \sinh(a) \left(1 + v^2 \sinh^2(a)\right)^{-1/2} &\leq v^{-1} \\ \lim_{a \rightarrow \infty} \sinh(a) \left(1 + v^2 \sinh^2(a)\right)^{-1/2} &= v^{-1} \end{aligned} \tag{8}$$

and that

$$\int v^{-1} (v^{2q} - 1)^{-1/2} dv = \frac{1}{q} \arctan(v^{2q} - 1)^{\frac{1}{2}} + \text{const.} \tag{9}$$

From the relations above, we obtain that  $\lambda(\rho, a)$  converges at  $\rho = a$  and also when  $\rho \rightarrow \infty$ . Thus we can write

$$L(a) = \int_1^\infty (v^{2q} - 1)^{-1/2} \sinh(a) \left(1 + v^2 \sinh^2(a)\right)^{-1/2} dv. \tag{10}$$

Moreover the limit when  $a \rightarrow \infty$  can be taken under the integral and

$$\lim_{a \rightarrow \infty} L(a) = \int_1^\infty v^{-1} (v^{2q} - 1)^{-1/2} dv = \frac{\pi}{2q} = \frac{\pi(r+1)}{2(n-r-1)}. \tag{11}$$

Finally, since

$$\frac{dL}{da} = \cosh(a) \int_1^\infty (v^{2q} - 1)^{-1/2} \left(1 + v^2 \sinh^2(a)\right)^{-3/2} dv > 0, \tag{12}$$

we conclude that the function  $a \rightarrow L(a)$  increases from 0 to  $\frac{\pi(r+1)}{2(n-r-1)}$  when  $a$  increases from 0 to  $\infty$ . Since  $f(a, t)$  is an even function of  $t$ , we can make a reflection of the graph of the function  $\lambda(\rho, a)$  with respect to the horizontal slice  $t = 0$  and we obtain a *catenary like* curve with finite height.

The fact that two generating curves intersect exactly at two symmetric points follow by considering the function  $\lambda(b, \rho) - \lambda(a, \rho)$  for  $a \neq b$  and by using the monotonicity of  $L(a)$  (see Fig. 1).

With this method we have then found all the complete rotational hypersurfaces that are local graphs over the vertical axis and we are then able to conclude that no immersed examples will appear.  $\square$

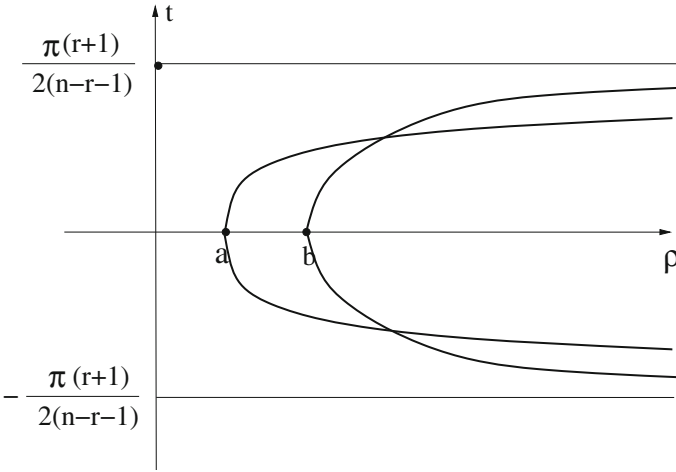


Fig. 1. The profile curves of  $\mathcal{M}_a(r)$  and  $\mathcal{M}_b(r)$

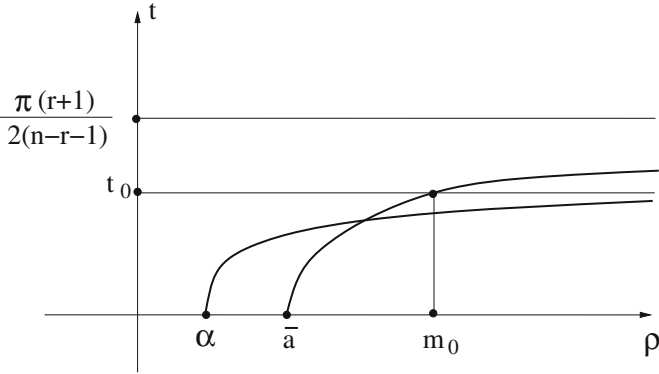


Fig. 2. The minimum for  $f$

**Definition 2.2.** The elements of the one parameter family  $\{\mathcal{M}_a(r)\}_{a>0}$  of  $r$ -minimal complete hypersurfaces invariant by rotation in  $\mathbb{H}^n \times \mathbb{R}$  are called  $r$ -catenoids.

In the rest of this section we explore further properties of the family of  $r$ -catenoids  $\mathcal{M}_a(r)$  (see Fig. 2).

Let us fix  $t_0$  in  $(0, \frac{\pi(r+1)}{2(n-r-1)})$  and let  $\alpha$  be such that  $L(\alpha) = t_0$ . This means that  $\lim_{t \rightarrow t_0^-} f(\alpha, t) = \infty$ . Let  $\phi^{t_0}$  be the positive continuous function defined by  $\phi^{t_0}(a) = f(a, t_0)$ . Since, by (12),  $\frac{dL}{da} > 0$ , we have that  $L(a) > L(\alpha) = t_0$ , for any  $a > \alpha$ . Then  $\phi^{t_0}$  is defined on  $(\alpha, \infty)$ . Moreover,  $\lim_{a \rightarrow \alpha^+} \phi^{t_0}(a) = \lim_{a \rightarrow \infty} \phi^{t_0}(a) = \infty$ . It is then clear that  $\phi^{t_0}$  has a minimum value  $m_0$  in  $(\alpha, \infty)$ . Let  $\bar{a} \in (\alpha, \infty)$  be such that  $\phi^{t_0}(\bar{a}) = m_0$ . Notice that  $f(\bar{a}, t_0) = m_0$  is a minimum of  $f$  with respect to the variable  $a$ .

**Claim.**  $f(\bar{a}, t_0) = m_0$  is a minimum of  $f$  with respect to the variable  $a$  if, and only if,  $\lambda(\bar{a}, m_0)$  is a maximum of  $\lambda$ , with respect to the variable  $a$ .

*Proof of claim.* Assume that  $f(\bar{a}, t_0) = m_0$  is a minimum of  $f$  and that there exists  $\tilde{a}$  such that  $\lambda(\tilde{a}, m_0) > \lambda(\bar{a}, m_0)$ . Then, the graph of  $\lambda(\tilde{a}, \rho)$  intersects  $t = t_0$  at a point  $(\tilde{\rho}, t_0)$  with  $\tilde{a} < \tilde{\rho} < m_0$ . Then  $f(\tilde{a}, t_0) = \tilde{\rho} < m_0 = f(\bar{a}, t_0)$ . Contradiction. The proof of the “only if” part is analogous.

We now state a technical lemma that will be useful in what follows.

**Lemma 2.1.** *Let  $\lambda(a, \rho)$  be given by (7). Then we have  $\lambda_{aa}(a, \rho) < 0$  for  $a \in (0, \rho)$  and:*

- $\rho \in (a, \infty)$ , if  $q \geq 1$ .
- $\rho \in (0, M]$ , where  $M = \operatorname{arcosh}\left(\sqrt{\frac{1}{1-q}}\right)$ , if  $q < 1$ .

*Proof.* By a straightforward computation, we obtain

$$\begin{aligned} \lambda_a(a, \rho) = & -\operatorname{tgh}(\rho) \operatorname{cotgh}(a) \left( \left( \frac{\sinh(\rho)}{\sinh(a)} \right)^{2q} - 1 \right)^{-\frac{1}{2}} \\ & + \cosh(a) \int_1^{\left(\frac{\sinh(\rho)}{\sinh(a)}\right)} (v^{2q} - 1)^{-\frac{1}{2}} (1 + v^2 \sinh^2(a))^{-\frac{3}{2}} dv \end{aligned}$$

and

$$\begin{aligned} \lambda_{aa}(a, \rho) = & \frac{\operatorname{tgh}(\rho)}{\sinh^2(a)} \left( \left( \frac{\sinh(\rho)}{\sinh(a)} \right)^{2q} - 1 \right)^{-\frac{3}{2}} \\ & \cdot \left[ \left( \frac{\sinh(\rho)}{\sinh(a)} \right)^{2q} \left( 1 - q \cosh^2(a) - \frac{\cosh^2(a)}{\cosh^2(\rho)} \right) + \left( \frac{\cosh^2(a)}{\cosh^2(\rho)} - 1 \right) \right] \\ & + \sinh(a) \int_1^{\left(\frac{\sinh(\rho)}{\sinh(a)}\right)} (v^{2q} - 1)^{-\frac{1}{2}} (1 + v^2 \sinh^2(a))^{-\frac{5}{2}} \\ & \cdot (1 - v^2 - 2v^2 \cosh^2(a)) dv. \end{aligned}$$

It is easy to see that, under the assumptions, the term  $\left(1 - q \cosh^2(a) - \frac{\cosh^2(a)}{\cosh^2(\rho)}\right)$  is negative. The remainder terms are clearly negative for  $a \in (0, \rho)$ .  $\square$

For any fixed  $\rho$ , let  $\gamma^\rho(a) = \lambda(a, \rho)$ .

We can easily see that  $\gamma^\rho$  is defined, positive and continuous in  $(0, \rho)$ . Moreover, we can see that  $\lim_{a \rightarrow 0^+} \gamma^\rho(a) = 0$  and that  $\gamma^\rho(\rho) = 0$ . Hence,  $\gamma^\rho$  reaches a maximum at some  $a$  in  $(0, \rho)$ . Set

- $J_q = (0, \infty)$  if  $q \geq 1$
- $J_q = (0, M]$  if  $q < 1$ .

Lemma 2.1 guarantees that for each  $\rho \in J_q$ ,  $\gamma^\rho(a) = \lambda(a, \rho)$  has a unique point of maximum.

When  $q < 1$ , let  $A$  be the unique point of maximum of  $\lambda(a, M)$ , for  $a \in (0, M)$ . We set

- $T = \frac{\pi(r+1)}{2(n-r-1)}$  if  $q \geq 1$
- $T = \lambda(A, M)$  if  $q < 1$ .

**Corollary 2.1.** *For each  $t_0 \in (0, T)$ , there exists a unique  $a_0 \in (\alpha, \infty)$  such that  $m_0 = \phi^{t_0}(a_0)$  is the minimum of  $\phi^{t_0}$ . Moreover, for each  $\rho > m_0$ , there exists at least a pair  $(a_1, a_2)$ , with  $a_1 < a_0 < a_2$ , such that  $\phi^{t_0}(a_1) = \phi^{t_0}(a_2) = \rho$ .*

*Proof.* For  $q < 1$ , it is clear that for each value of  $t \in (0, T)$ , the minimum value of  $\phi^t(a) = f(a, t)$  is less than  $M$ . Then, taking into account the last claim, we can conclude that  $\phi^t(a)$  has a unique point of minimum since, for each  $\rho \in J_q$ ,  $\gamma_\rho$  has a unique point of maximum. In particular,  $\phi^{t_0}$  reaches the minimum value  $m_0$  at a unique point, say  $a_0$ .

Now, we take  $\rho \in J_q$ ,  $\rho > m_0$ . By analysing the behavior of the profile curves  $\lambda(a_0, \rho)$ , we see that  $\gamma^\rho(a_0) = \lambda(a_0, \rho) > \lambda(a_0, m_0) = t_0$ . Since  $\lim_{a \rightarrow 0^+} \gamma^\rho(a) = 0$  and that  $\gamma^\rho(\rho) = 0$ , then  $\gamma^\rho$  reaches the height  $t_0$  twice for two values  $a_1$  and  $a_2$  such that  $a_1 < a_0 < a_2$ . The proof of the Corollary is now complete.  $\square$

The following Proposition follows easily from the previous results. Here,  $t_0 \in (0, T)$  and  $m_0$  and  $a_i, i = 0, 1, 2$ , are the numbers given in Corollary 2.1.

**Proposition 2.1.** *Let  $D_+(R), D_-(R)$  two  $(n - 1)$ -spheres of radius  $R$ , contained in the slices  $t = t_0$  and  $t = -t_0$ , respectively, with center on the axis  $t$ . We have*

- (1) *If  $R < m_0$ , there exist no  $r$ -minimal rotational hypersurfaces with boundary  $D_+(R) \cup D_-(R)$ .*
- (2) *If  $R = m_0$ , there exists a unique  $r$ -minimal rotational hypersurfaces with boundary  $D_+(m_0) \cup D_-(m_0)$ , namely,  $\mathcal{M}_{a_0}(r)$ .*
- (3) *If  $R \in J_q, R > m_0$ , there exist at least two  $r$ -minimal rotational hypersurfaces with boundary  $D_+(R) \cup D_-(R)$ . Two of them are  $\mathcal{M}_{a_1}(r)$  and  $\mathcal{M}_{a_2}(r)$ .*

The study of the  $r$ -catenoids in Euclidean space was addressed in [9]. There, we can see that the vertical heights of the  $r$ -catenoids are bounded for  $q > 1$  ( $n > 2(r + 1)$ ) and unbounded for  $q \leq 1$  ( $n \leq 2(r + 1)$ ). In  $\mathbb{H}^n \times \mathbb{R}$ , the heights are bounded in both cases. On the other hand, for each admissible value of  $t$ , the authors in [9] were able to prove the uniqueness of the minimum point of  $\phi^t$  by using ambient scaling. Here, by means of geometric arguments and of Lemma 2.1, we were able to prove the uniqueness for  $q \geq 1$ , but we fail to prove in the case  $q < 1$ . For  $q < 1$ , we have to restrict the values of  $t$  in order to obtain uniqueness. This is, possibly, a technical restriction and we ask the following.

**Question.** For any fixed  $t_0 \in (0, \frac{\pi(r+1)}{2(n-r-1)})$ , we know that there is an  $r$ -catenoid,  $\mathcal{C}$ , passing through  $(m_0, t_0)$  in the  $(\rho, t)$ -plane and that all  $r$ -catenoids passing through  $(m, t_0)$  satisfy  $m \geq m_0$ . Is  $\mathcal{C}$  unique?



We are able to give a positive answer for  $q \geq 1$ . For  $q < 1$ , we have to consider  $t_0 \in (0, T)$ . In other words, we ask: can we consider  $T = \frac{\pi(r+1)}{2(n-r-1)}$  for all values of  $q$  in Corollary 2.1 and in Proposition 2.1?

*Remark 2.1.* For any  $t$  in  $(0, \frac{\pi(r+1)}{2(n-r-1)})$ , we define  $m(t)$  as the minimum value of the function  $\phi^t = f(a, t)$ .

Then, the set (using the ball model for  $\mathbb{H}^n$ )

$$\mathcal{E}_{r+1} = \left\{ \left( \tanh \left( \frac{m(t)}{2} \right) \zeta, t \right) \in \mathbb{H}^n \times \mathbb{R} \mid \zeta \in \mathbb{S}^{n-1}, t \in (-T, T) \right\}$$

is the *envelope* of the family  $\mathcal{M}_a(r)$ , that satisfies  $H_{r+1} = 0$  (see Definition 5.16 in [2]).

Let us state a property of the family  $\mathcal{M}_a(r)$  that will be useful in the following section.

**Proposition 2.2.** *For a fixed  $r$ ,  $1 \leq r < n - 1$ , each rotational  $r$ -minimal hypersurface of the family  $\mathcal{M}_a(r)$  satisfies*

- (1)  $H_j > 0$ , for  $j < r + 1$ .
- (2)  $H_{r+1} = 0$ .
- (3)  $H_j < 0$ , for  $r + 1 < j \leq n$ .

*Proof.* By taking (4) into account we see that  $k_1 = \dots = k_{n-1}$  and that  $k_n = -\frac{n-r-1}{r+1} k_1$ . Then, a straightforward computation yields

$$H_j = k_1^j [(r + 1) - j], \quad j = 1, \dots, n,$$

that gives the result. □

### 3. Uniqueness results

In this section we obtain two classification results. The first one deals with compact  $r$ -minimal hypersurfaces with boundary on two slices and the second one deals with non compact  $r$ -minimal hypersurfaces with asymptotic boundary spanned by two copies of  $\partial_\infty \mathbb{H}^n$ .

Before stating the results of this section, we establish some notation. We denote the slice  $\mathbb{H}^n \times \{s\}$ ,  $s \in \mathbb{R}$ , by  $\Pi_s$  and a (closed) slab between two slices by  $S$ , say  $S = \{(p, t) \mid p \in \mathbb{H}^n, t_0 \leq t \leq t_1\}$ . The asymptotic boundary of  $S$  is given by  $\partial_\infty S = \partial_\infty \mathbb{H}^n \times [t_0, t_1]$ . We set  $\Pi_s^+ = \{(p, t) \mid p \in \mathbb{H}^n, t > s\}$ ,  $\Pi_s^- = \{(p, t) \mid p \in \mathbb{H}^n, t < s\}$  and, for notational convenience, we write  $\Pi = \Pi_0$ . Also, we set  $\sigma$  for the origin of the slice  $\Pi$ .

The complete totally geodesic hypersurface  $\mathcal{P} = \pi \times \mathbb{R}$ , where  $\pi$  is any totally geodesic  $(n - 1)$ -dimensional complete hypersurface of  $\mathbb{H}^n$ , is called a *vertical hyperplane*.

We will use suitable versions of the interior and boundary maximum principles for vanishing higher order mean curvatures. We believe it is worthwhile to recall

them here and to point out the important differences between the classical maximum principles for minimal hypersurfaces and these for  $r$ -minimal hypersurfaces. For further details about such generalized maximum principles, see [7, 8] for hypersurfaces of Euclidean space and [6] for hypersurfaces of a general Riemannian manifold.

Let  $\vec{\kappa} = (\kappa_1, \dots, \kappa_n)$  be the principal curvature vectors of  $M$ . Roughly speaking, for  $r \geq 1$ , the maximum principle requires, as extra hypotheses, that:

- (1) the principal curvature vectors of the two compared hypersurfaces belong to the same leaf of  $H_{r+1} = 0$ .
- (2) the rank of the Gauss map (the rank of  $\vec{\kappa}$ ) of one of the compared hypersurfaces at the contact point is greater than  $r$ . This hypothesis guarantees the ellipticity of the equation  $H_{r+1} = 0$  and it is satisfied if  $H_{r+2} \neq 0$ .

Let  $M, M'$  be two oriented  $r$ -minimal hypersurfaces of  $\mathbb{H}^n \times \mathbb{R}$ . Let  $\vec{\kappa}$  (respectively  $\vec{\kappa}'$ ) be the principal curvature vector of  $M$  (respectively  $M'$ ).

**Theorem A.** (Corollary 1.a [6]) *Let  $M$  and  $M'$  two  $r$ -minimal oriented hypersurfaces, tangent at a point  $p$ , with normal vector pointing in the same direction. Suppose that  $M$  remains on one side of  $M'$  in a neighborhood of  $p$ . Suppose that  $\vec{\kappa}(p)$  and  $\vec{\kappa}'(p)$  belong to the same leaf of  $H_{r+1} = 0$  and that the rank of either  $\vec{\kappa}$  or  $\vec{\kappa}'$  is at least  $r$ . Then  $M$  and  $M'$  coincide in a neighborhood of  $p$ .*

**Theorem B.** (Theorem 2.a [6]) *Let  $M$  and  $M'$  two  $r$ -minimal oriented hypersurfaces, tangent at a point  $p$ , with normal vector pointing in the same direction. Suppose that  $M$  remains on one side of  $M'$  in a neighborhood of  $p$ . Suppose further that  $H'_j(p) \geq 0$ ,  $1 \leq j \leq r$  and either  $H_{r+2} \neq 0$  or  $H'_{r+2} \neq 0$ . Then  $M$  and  $M'$  coincide in a neighborhood of  $p$ .*

The analogues of both Theorem A and B hold for hypersurfaces tangent at boundary points (see Corollary 1.b and Theorem 2.b [6]).

For the reader's convenience, we explain here in which cases either Theorem A or Theorem B (and their boundary versions) can be used. Then, it will be clear in the following when we use either the first or the second one.

- Theorem A will be used for the comparison of an  $r$ -minimal hypersurface with a reflection of the hypersurface itself. The assumption of Theorem A are satisfied by a hypersurface and its reflection because of the following two facts:

**Fact 1** Due to properties of hyperbolic polynomials, the principal curvature vector of a connected hypersurface with  $H_{r+1} = 0$  and  $H_{r+2} \neq 0$  does not change of leaf (see [7] for details).

**Fact 2** Let  $\tau$  be an isometry of  $\mathbb{H}^n \times \mathbb{R}$  that preserves the orientation of either  $\mathbb{H}^n$  or  $\mathbb{R}$  and reverses the other. Let  $f : M \rightarrow \mathbb{H}^n \times \mathbb{R}$  be an immersion and set  $\hat{f} = \tau \circ f$ . Then, we have  $\hat{N} = -\tau \circ N$ , where  $\hat{N}$  is the normal vector to  $\hat{f}$  (see [4], Proposition (3.8)). As a consequence, the second fundamental forms of  $f$  and  $\hat{f}$  have opposite sign.

- Theorem B will be used for the comparison of an  $r$ -minimal hypersurface with one of the  $\mathcal{M}_a(r)$  that, by Proposition 2.2, satisfy  $H_j > 0$  for  $j < r + 1$  and  $H_{r+2} < 0$ .

Now, we recall the description of a family of hypersurfaces found by the first author and Ricardo Sa Earp in [5], that will be crucial in the proof of Proposition 3.1. There, the authors proved the existence of a family  $\mathcal{F}_\sigma$  of entire rotational strictly convex graphs with constant  $H_{r+1} \in (0, \frac{n-r-1}{n}]$  that satisfy the following properties (see [5, Propositions (6.4) and (6.5)]):

- (1) The graphs of the family  $\mathcal{F}_\sigma$  intersect each other only at the point  $\sigma$ . Moreover, they are tangent to the slice  $\Pi$  at  $\sigma$  and have normal vector pointing upwards.
- (2) The graphs of the family  $\mathcal{F}_\sigma$  converge to  $\Pi$  uniformly on compact sets as  $H_{r+1}$  goes to zero.

By an isometry of the ambient space, we can produce a new family with an arbitrary common point  $q$  and with normal pointing either upward or downward. We denote by  $\mathcal{F}_q$  the family with common point  $q$  and upward normal vector and by  $\tilde{\mathcal{F}}_q$  the one with downward normal vector.

**Proposition 3.1.** *Let  $M$  be an  $r$ -minimal hypersurface in  $\mathbb{H}^n \times \mathbb{R}$  such that  $\partial M$  and  $\partial_\infty M$ , one of them possibly empty, are contained in  $S \cup \partial_\infty S$ , for a given slab  $S$ . Then  $M$  is contained in  $S$ .*

*Proof.* Suppose that  $M$  is not contained in the slab  $S$ . Without loss of generality, we can assume that  $S = \{(p, t) | p \in \mathbb{H}^n, s \leq t \leq 0\}$  and that there is a subset of  $M$  in  $\Pi^+$ . Now, we choose  $\varepsilon > 0$  such that  $M_\varepsilon^+ = M \cap \Pi_\varepsilon^+$  is not empty. Since  $\partial M$  and  $\partial_\infty M$  are in the slab,  $M_\varepsilon$  is compact with boundary in  $\Pi_\varepsilon$ . Let  $q$  be a point above  $M_\varepsilon^+$  and let  $\{\Sigma_i\}_{i \in \mathbb{N}}$  be a sequence of graphs with constant  $(r + 1)$ -mean curvature in the family  $\tilde{\mathcal{F}}_q$  that converges to the slice passing through  $q$  when  $i$  tends to infinity. Since  $M_\varepsilon^+$  is compact, we can suppose that  $M_\varepsilon^+$  is contained in the convex side of  $\Sigma_i$ , for large  $i$ . Let  $l$  be the vertical line passing through  $q$ . Now, we let  $q$  move downwards along  $l$  and simultaneously we let  $i$  increase. We do this process keeping  $M_\varepsilon^+$  in the convex side of the translated  $\Sigma_i$ , by choosing a subsequence, if necessary. We do this until one of the translated  $\Sigma_i$  touches  $M_\varepsilon^+$ . Such contact point must be interior and a strictly convex point of  $M$ . This is a contradiction since  $M$  is  $r$ -minimal. □

**Corollary 3.1.** *Let  $M \subset \mathbb{H}^n \times \mathbb{R}$  be a compact embedded  $r$ -minimal hypersurface with boundary contained in  $\Pi_s \cup \Pi_t$ ,  $s < t$ , and assume that  $\partial M_s = \partial M \cap \Pi_s \neq \emptyset$  and  $\partial M_t = \partial M \cap \Pi_t \neq \emptyset$ . Then,  $M$  can be oriented by a continuous normal pointing into the interior of a closed domain  $U$  in  $\mathbb{H}^n \times \mathbb{R}$ , with  $M \subset \partial U$ .*

*Proof.* By the last proposition, we have that  $M$  is contained in the slab between  $\Pi_s$  and  $\Pi_t$ . Let  $D_s \subset \Pi_s$  and  $D_t \subset \Pi_t$  be the bounded regions such that  $\partial D_s = \partial M_s$  and  $\partial D_t = \partial M_t$ . Then,  $M \cup D_s \cap D_t$  is an orientable homological boundary of an  $(n + 1)$ -dimensional chain in  $\mathbb{H}^n \times \mathbb{R}$ . We choose the inwards normal to  $M \cup D_s \cap D_t$ . □

Next Theorem is a uniqueness result for compact  $r$ -minimal hypersurfaces with boundary in two parallel slices. The analogous result in Euclidean space is Theorem 3.2 in [9].

In the next statement,  $t_0$ ,  $m_0$  and  $a_0$  are as in Corollary 2.1. Also, we recall that  $D_+(R)$  and  $D_-(R)$  are two  $(n - 1)$ -spheres of radius  $R$ , contained in the slices  $t = t_0$  and  $t = -t_0$ , respectively, with center on the  $t$ -axis.

**Theorem 3.1.** *Let  $M$  be a compact, connected and embedded  $r$ -minimal hypersurface in  $\mathbb{H}^n \times \mathbb{R}$ ,  $1 \leq r < n - 1$ , with boundary contained in  $\Pi_{t_0} \cup \Pi_{-t_0}$  with  $\partial M_+ = \partial M \cap \Pi_{t_0} \neq \emptyset$  and  $\partial M_- = \partial M \cap \Pi_{-t_0} \neq \emptyset$ . We suppose that  $\partial M_+ \subset D_+(m_0)$  and that  $\partial M_- \subset D_-(m_0)$ . Then,  $M$  coincides with the unique rotational hypersurface  $\mathcal{M}_{a_0}(r)$  with boundary  $D_+(m_0) \cup D_-(m_0)$ .*

*Proof.* By Proposition 3.1,  $M$  is contained in the slab  $S$  between  $\Pi_{t_0}$  and  $\Pi_{-t_0}$ . We orient  $M$  as in Corollary 3.1. As  $M$  is compact, for  $a$  large enough, there exists a rotational hypersurface  $\mathcal{M}_a(r)$  such that  $M$  is contained in the compact component determined by  $\mathcal{M}_a(r) \cap S$ . Now, we let  $a$  decrease. It is clear that there exists  $\alpha > 0$  such that  $\mathcal{M}_\alpha(r)$  has a first contact point with  $M$ . We notice that  $\alpha \neq 0$  because the waist of  $\mathcal{M}_a(r)$  shrinks to zero as  $a \rightarrow 0$  and the absence of a contact point before  $a = 0$  would contradict the connectedness of  $M$ .

If the first contact point  $p$  between  $M$  and  $\mathcal{M}_\alpha(r)$  is an interior point of  $M$ , then  $M$  and  $\mathcal{M}_\alpha(r)$  are tangent at  $p$ , both have normal vectors pointing into the compact region determined by  $M \cap S$  and  $M$  lies above  $\mathcal{M}_\alpha(r)$  with respect to the normal vector [recall that  $\mathcal{M}_\alpha(r)$  is oriented as in (1)]. By Proposition 2.2,  $\mathcal{M}_\alpha(r)$  is such that  $H_{r+2} < 0$  and  $H_j > 0$ , for  $j < r + 1$ , hence by the maximum principle (see Theorem B),  $M$  and  $\mathcal{M}_\alpha(r)$  coincide in a neighborhood of  $p$ . Then, they coincide everywhere. Moreover, since  $\partial M \subset D_+(m_0) \cup D_-(m_0)$ , Proposition 2.1 gives the result.

Now, let us analyse the case where the first contact point between  $M$  and  $\mathcal{M}_\alpha(r)$  is on  $\partial M$ . Let  $q \in \partial M$  be a first contact point between  $M$  and  $\mathcal{M}_\alpha(r)$ . As  $\partial M \subset D_+(m_0) \cup D_-(m_0)$ , again by Proposition 2.1,  $\alpha = a_0$  and  $q$  belongs to  $\partial D_+(m_0) \cup \partial D_-(m_0)$ .

If the tangent planes at  $q$  to  $M$  and  $\mathcal{M}_{a_0}(r)$  coincide, then, by the boundary maximum principle (see Theorem 2.b [6], that is the boundary version of Theorem B),  $M$  and  $\mathcal{M}_\alpha(r)$  coincide as well and the result is proved.

Otherwise, the slope of  $T_q M$  is strictly smaller than the slope of  $T_q \mathcal{M}_{a_0}(r)$ . We will get a contradiction in this case. By Proposition 2.1, for  $\varepsilon$  small,  $\mathcal{M}_{a_0-\varepsilon}(r)$  is such that  $\mathcal{M}_{a_0-\varepsilon}(r) \cap (\Pi_{t_0} \cup \Pi_{-t_0})$  contains  $D_+(m_0) \cup D_-(m_0)$  in its interior. This last fact, joint with the fact that the slope of  $T_q M$  is strictly smaller than the slope of  $T_q \mathcal{M}_{a_0}(r)$  yield that  $\mathcal{M}_{a_0-\varepsilon}(r) \cap S$  bounds a region containing  $M$ . Now, if we continue decreasing  $a$ ,  $\partial \mathcal{M}_a(r)$  can not touch  $\partial M$  again (because of Proposition 2.1), but for  $a \rightarrow 0$ , the waist of  $\mathcal{M}_a(r)$  shrink to zero, so there must be an interior contact point between  $M$  and  $\mathcal{M}_{\bar{a}}(r)$ , for some  $\bar{a} < a_0$ . Then, as before,  $M$  and  $\mathcal{M}_{\bar{a}}(r)$  must coincide. This is a contradiction because they have disjoint boundaries.  $\square$

Theorem 3.1 implies the following result (with the same notation as there).

**Corollary 3.2.** *There is no compact, connected and embedded  $r$ -minimal hypersurface in  $\mathbb{H}^n \times \mathbb{R}$ ,  $1 \leq r < n - 1$ , with  $\partial M_+ \subset D_+(R)$  and  $\partial M_- \subset D_-(R)$ , for  $R < m_0$ .*

Theorem 3.2 below is inspired by the classical result of Schoen [14, Theorem 3]. As in the proof of Theorem 3.1, we have to deal with the restrictions imposed by the geometry of  $\mathbb{H}^n \times \mathbb{R}$ . Here, based on the ideas contained in [11], we change the assumption of regular ends at infinity in Euclidean space by that of asymptotic boundary in two parallel slices in  $\mathbb{H}^n \times \mathbb{R}$ . The proofs of Lemma 3.1 and Theorem 3.2 are very similar to that of [11, Lemma 2.1, Theorem 4.2]. The differences are essentially due to the differences in the hypothesis of the maximum principle for minimal and for  $r$ -minimal hypersurfaces. Also, we point out that arguments used to prove embeddedness of the minimal immersion in [11, 14] can not be carried out here. The obstruction is the requirement in the maximum principle, for the  $r$ -minimal case, that the principle curvature vectors belong to the same leaf. Then, here, embeddedness is a hypothesis.

**Lemma 3.1.** *Let  $\Gamma^+$  and  $\Gamma^-$  be two  $(n - 1)$ -manifolds in  $\partial_\infty \mathbb{H}^n \times \mathbb{R}$  which are vertical graphs over  $\partial_\infty \mathbb{H}^n \times \{0\}$  and such that  $\Gamma^+ \subset \partial_\infty \Pi^+$  and  $\Gamma^- \subset \partial_\infty \Pi^-$ . Assume that  $\Gamma^-$  is the symmetric of  $\Gamma^+$  with respect to  $\Pi$ . Let  $M \subset \mathbb{H}^n \times \mathbb{R}$  be an embedded, connected, complete  $r$ -minimal hypersurface,  $1 \leq r < n - 1$ , with two ends  $E^+$  and  $E^-$ . Assume that each end is a vertical graph and that  $\partial_\infty M = \Gamma^+ \cup \Gamma^-$ , that is  $\partial_\infty E^+ = \Gamma^+$  and  $\partial_\infty E^- = \Gamma^-$ . Moreover, assume that  $H_{r+2} \neq 0$ . Then  $M$  is symmetric with respect to  $\Pi$ . Furthermore, each part  $M \cap \Pi^\pm$  is a vertical graph.*

*Proof.* We denote by  $t^+$  the highest  $t$ -coordinate of  $\Gamma^+$ . Since  $\partial_\infty M = \Gamma^+ \cup \Gamma^-$ , then Proposition 3.1 imply that  $M$  is contained in the slab between  $\Pi_{t^+}$  and  $\Pi_{-t^+}$ .

We now notice that since each end of  $M$  is a vertical graph, we can obtain a compact domain  $\Omega \in \mathbb{H}^n \times \{0\}$  such that  $E^+$  and  $E^-$  are graphs over  $(\mathbb{H}^n \times \{0\}) \setminus \Omega$ . We consider the cylinder  $C$  over  $\Omega$  and we see that  $M_C = M \cap C$  is compact and embedded, so it bounds a compact domain  $B$ . Then an argument similar to that used in the Corollary 3.1 gives that we can orient  $M_C$  towards  $B$ . Since,  $M \setminus M_C$  is a graph, we can extend the normal vector continuously to  $M$ . In this case, we will say that the whole  $M$  is oriented towards the interior.

For any  $t > 0$  we set  $M_t^+ = M \cap \Pi_t^+$ . We denote by  $M_t^{+*}$  the symmetry of  $M_t^+$  with respect to the slice  $\Pi_t$ . As  $E^+$  is a vertical graph, there exists  $\varepsilon > 0$  such that  $M_{t^+-\varepsilon}^+$  is a vertical graph, then we can start Alexandrov reflection. We keep doing the Alexandrov reflection with respect to  $\Pi_t$ , doing  $t \searrow 0$ . Here, we recall that reflection with respect to a slice preserves the orientation of  $\mathbb{H}^n$  and reverses that of  $\mathbb{R}$ . Then, taking Fact 2 into account, the principal curvature vector of  $M_t^{+*}$  with respect to the suitable orientation  $-\hat{N}$ , is equal to the principal curvature vector of  $M_t^+$ . By Fact 1, the principal curvature vectors of  $M_t^{+*}$  and  $M_t^-$  belong to the same leaf, hence we can apply the maximum principle for comparing them. Theorem A or its corresponding boundary version (Corollaries 1a and 1b of [6]), gives, for  $t > 0$ , that the surface  $M_t^{+*}$  stays above  $M_t^-$  and that both,  $M_t^+$  and  $M_t^-$ , are vertical graphs. By doing  $t \searrow 0$ , we obtain that  $M_0^+$  is a vertical graph and that  $M_0^{+*}$  stays above  $M_0^-$ .

Doing Alexandrov reflection with slices coming from below, one has that  $M_0^-$  is a vertical graph and that  $M_0^{-*}$  stays below  $M_0^+$ , henceforth we get  $M_0^{+*} = M_0^-$ .

Thus  $M$  is symmetric with respect to  $\Pi$  and each component of  $M \setminus \Pi$  is a graph. This completes the proof.  $\square$

**Theorem 3.2.** *Let  $M$  be a complete connected  $r$ -minimal hypersurface embedded in  $\mathbb{H}^n \times \mathbb{R}$ ,  $1 \leq r < n - 1$ , with  $H_{r+2} \neq 0$ . Assume that  $M$  has two ends and that each end is a vertical graph whose asymptotic boundary is a copy of  $\partial_\infty \mathbb{H}^n$ . Then  $M$  is isometric, by an ambient isometry, to one of the  $\mathcal{M}_a(r)$ .*

*Proof.* Up to a vertical translation, we can assume that the asymptotic boundary of  $M$  is symmetric with respect to  $\Pi := \mathbb{H}^n \times \{0\}$ , say  $\partial_\infty M = \partial_\infty \mathbb{H}^n \times \{t_0, -t_0\}$  for some  $t_0 > 0$ . Then  $\Gamma^+ := \partial_\infty M \times \{t_0\}$  and  $\Gamma^- := \partial_\infty M \times \{-t_0\}$ . By Proposition 3.1,  $M$  is contained in the slab between  $\Gamma^+$  and  $\Gamma^-$ . By Lemma 3.1,  $M$  is symmetric about  $\Pi$ , and each connected component of  $M \setminus \Pi$  is a vertical graph. Moreover, at any point of  $M \cap \Pi$ , the tangent hyperplane to  $M$  is orthogonal to  $\Pi$ .

Since  $M$  is embedded,  $M$  separates  $\mathbb{H}^n \times [-t_0, t_0]$  into two connected components. We denote by  $U_1$  the component whose asymptotic boundary is  $\partial_\infty \mathbb{H}^n \times [-t_0, t_0]$  and by  $U_2$  the component such that  $\partial_\infty U_2 = \partial_\infty \mathbb{H}^n \times \{t_0, -t_0\}$ .

Let  $q_\infty \in \partial_\infty \mathbb{H}^n$  and let  $\gamma \subset \mathbb{H}^n$  be an oriented geodesic issuing from  $q_\infty$ , that is  $q_\infty \in \partial_\infty \gamma$ . Let  $q_0 \in \gamma$  be any fixed point. For any  $s \in \mathbb{R}$ , we denote by  $P_s$  the vertical hyperplane orthogonal to  $\gamma$  passing through the point of  $\gamma$  whose oriented distance from  $q_0$  is  $s$ . We suppose that  $s < 0$  for any point in the geodesic segment  $(q_0, q_\infty)$ . For any  $s \in \mathbb{R}$ , we call  $M_s(e)$  the part of  $M \setminus P_s$  such that  $(q_\infty, t_0), (q_\infty, -t_0) \in \partial_\infty M_s(e)$  and let  $M_s^*(e)$  be the reflection of  $M_s(e)$  about  $P_s$ . We denote by  $M_s(d)$  the other part of  $M \setminus P_s$  and by  $M_s^*(d)$  its reflection about  $P_s$ . We recall that this reflection preserves the orientation of  $\mathbb{R}$  and reverses that of  $\mathbb{H}^n$ . This enable us to use Theorem A, as we did in Lemma 3.1.

By assumption there exists  $s_1 < 0$  such that for any  $s < s_1$  the part  $M_s(e)$  has two connected components and both of them are vertical graphs. We deduce that  $\partial M_s(e)$  has two (symmetric) connected components, each one being a vertical graph.

**Claim 1.** For any  $s < s_1$ , we have that  $M_s^*(e) \cap \Pi^+$  stays above  $M_s(d)$  and  $M_s^*(e) \cap \Pi^-$  stays below  $M_s(d)$ . Consequently  $M_s^*(e) \subset U_2$  for any  $s < s_1$ .

**Claim 2.** Given a geodesic  $\gamma \subset \mathbb{H}^n$ , there exists a vertical hyperplane  $P_\beta$  orthogonal to  $\gamma$  such that  $M_\beta^*(e) = M_\beta(d)$ , that is  $M$  is symmetric with respect to  $P_\beta$ .

The reader can find analogous claims joint with their proofs in [11, Theorem 2.3]. The proofs go exactly in the same way. There, the authors use the classical maximum principle and here we should use Theorem A or its corresponding boundary version.

By Claim 2, one has that  $M \cap \Pi$  satisfies the assumptions of [11, Proposition 4.2]. Then  $M \cap \Pi$  is a  $(n - 1)$ -geodesic sphere of  $\Pi$ . Let  $a$  be such that  $\mathcal{M}_a(r)$  is the rotational  $r$ -minimal hypersurface through  $M \cap \Pi$  and orthogonal to  $\Pi$ . We set  $\mathcal{M}_a(r)^+ := \mathcal{M}_a(r) \cap \{t > 0\}$ . Both  $\mathcal{M}_a(r)^+$  and  $M^+$  are vertical along their common finite boundary  $\Sigma$ , hence they are tangent along  $\Sigma$ . We want to show that they coincide. Let  $t(\mathcal{M}_a(r))$  (resp.  $t(M)$ ) be the height of the asymptotic boundary of  $\mathcal{M}_a(r)^+$  (resp.  $M^+$ ). Suppose, for example, that  $t(\mathcal{M}_a(r)) \leq t(M)$ .

We translate  $M^+$  upward so that it stays above  $\mathcal{M}_a(r)^+$ . Then we translate it down till we find the first point of contact. By using Theorem B, or its corresponding boundary version, we conclude that  $M^+ = \mathcal{M}_a(r)^+$ .

The case  $t(M) \leq t(\mathcal{M}_a(r))$  is analogous. We then conclude that  $M = \mathcal{M}_a(r)$  and the proof is completed.  $\square$

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