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Evolution of curvatures on a surface with boundary to prescribed functions

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Abstract. Let (M, g_0) be a compact Riemann surface with boundary and with negative Euler characteristic. Let $f(x)$ be a strictly negative smooth function on $\bar{M}$ and denote by $\sigma(x)$ the value of f in the interior and $\zeta(x)$ the value of f on the boundary. By studying the evolution of curvatures on M, we prove that there exist a constant λ_∞ and a conformal metric g_∞ such that $\lambda_\infty\sigma(x)$ and $\lambda_\infty\zeta(x)$ can be realized as the Gaussian curvature and boundary geodesic curvature of g_∞ respectively.

1. Introduction

Let M be a closed Riemann surface with Riemannian metric g_0. Given a smooth function f on M, is it possible to find a metric g which is pointwise conformal to g_0, i.e. $g = e^{2u}g_0$, such that f can be realized as the Gaussian curvature of g? This prescribing Gaussian curvature problem is equivalent to solving the following equation

$$-\Delta_{g_0}u + K_0 = fe^{2u}, \quad u \in C^\infty(M), \tag{1.1}$$

where Δ_{g_0} and K_0 are, respectively, the Laplace–Beltrami operator and Gaussian curvature of g_0. The solvability of the Eq. (1.1) depends on the sign of Euler characteristic $\chi(M)$ of M. When $\chi(M) > 0$, i.e., M is the sphere, the problem is called Nirenberg's problem which has been extensively studied by Morser [14, 15], Kazdan–Warner [13], Chang–Yang [8,9] and many others; when $\chi(M) = 0$, the problem was completed solved by Berger [3] and Kazdan–Warner [13]; when $\chi(M) < 0$, this problem was studied by Chen–Li [10] and Kazdan–Warner [13]. Particularly, Kazdan–Warner, by using the method of upper and lower solutions, showed in [13] the result below

Theorem 1.1. *Assume that $f \leq 0$ and $f \not\equiv 0$, then the Eq. (1.1) possesses a solution.*

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Later on, Ho [12] partially recovered this result by using the curvature flow introduced by Struwe [17]. Baird–Fardoun–Regbaoui [2] extended this result to sign-changing functions by studying an abstract gradient flow.

A natural analogue of prescribing Gaussian curvature problem for Riemann surface with boundary is as follows. Let M be a compact Riemann surface with boundary with metric g_0. Assume that $f(x)$ is a smooth function on $\bar{M}$ and denote by $\sigma(x)$ the value of f in the interior and $\zeta(x)$ the value of f on the boundary. It is natural to ask if it is possible to find a metric g which is pointwise conformal to g_0 such that $\sigma(x)$ and $\zeta(x)$ can be realized as the Gaussian curvature and boundary geodesic curvature of g, respectively. This problem is equivalent to solving the following boundary value problem

$$\begin{cases} -\Delta_{g_0} u + K_0 = \sigma(x)e^{2u}, & \text{in } M, \\ \frac{\partial}{\partial \eta_{g_0}} u + \kappa_0 = \zeta(x)e^{u}, & \text{on } \partial M, \end{cases} \tag{1.2}$$

where κ_0 and $\partial/\partial\eta_{g_0}$ are geodesic curvature and out normal derivative of g_0, respectively.

In this paper, we partially generalize Theorem 1.1 to problem (1.2) through studying the evolution of curvatures on M. The main result of the paper is stated as the following

Theorem 1.2. *Suppose that $f(x) < 0$ and $\sigma(x)$ and $\zeta(x)$ are defined as before, then there exist a constant λ_∞ and a conformal metric g_∞ such that $\lambda_\infty\sigma(x)$ and $\lambda_\infty\zeta(x)$ can be realized as the Gaussian curvature and boundary geodesic curvature of g_∞ respectively, i.e., Eq. (1.2), with $\sigma(x)$ and $\zeta(x)$ replaced respectively by $\lambda_\infty\sigma(x)$ and $\lambda_\infty\zeta(x)$, has a solution.*

2. The flow equation and long time existence

2.1. The flow equation and its energy

Let M be a compact Riemann surface with boundary with background metric g_0 and the Euler characteristic $\chi(M) < 0$. Assume that f is a strictly negative smooth function on $\bar{M}$ and $\sigma(x)$ and $\zeta(x)$ are defined the same as above. Also, without loss of generality, we may assume

$$\int_M \sigma(x)\, dA_{g_0} + \int_{\partial M} 2\zeta(x)\, ds_{g_0} = -2\pi. \tag{2.1}$$

Inspired by Brendle's work [4], we consider the following evolution equation

$$\begin{cases} \frac{\partial g}{\partial t} = \frac{2}{\sigma(x)}(K - \lambda\sigma(x))g, & \text{in } M, \\ \frac{\partial g}{\partial t} = \frac{2}{\zeta(x)}(\kappa - \lambda\zeta(x))g, & \text{on } \partial M, \end{cases} \tag{2.2}$$

with the initial condition

$$g(0) = g_0,$$

where K and κ are, respectively, Gaussian curvature and boundary geodesic curvature of g, and λ is a constant defined in (2.7). If we write $g = e^{2u} g_0$, then the evolution Eq. (2.2) implies that the evolution equation for u is given by

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{K}{\sigma(x)} - \lambda, & \text{in } M, \\ \frac{\partial u}{\partial t} = \frac{\kappa}{\zeta(x)} - \lambda, & \text{on } \partial M, \end{cases} \tag{2.3}$$

with the initial condition

$$u(\cdot, 0) = 0.$$

Moreover, K and κ can be formulated by

$$\begin{cases} K = e^{-2u}(-\Delta_{g_0} u + K_0), & \text{in } M, \\ \kappa = e^{-u} \left(\frac{\partial u}{\partial \eta_{g_0}} + \kappa_0 \right), & \text{on } \partial M, \end{cases} \tag{2.4}$$

Notice that (1.2) can be viewed, differing by a constant, as Euler–Lagrange equation of the following energy functional

$$E[u] = \int_M \frac{1}{2} |\nabla u|_{g_0}^2 + K_0 u \, dA_{g_0} + \int_{\partial M} \kappa_0 u \, ds_{g_0} \tag{2.5}$$

with the constraint

$$L(u) := \int_M \sigma(x) e^{2u} \, dA_{g_0} + \int_{\partial M} 2\zeta(x) e^u \, ds_{g_0} = -2\pi. \tag{2.6}$$

The constraint (2.6) above reminds us that we should keep L unchanged along the flow (2.2). That is

$$\begin{aligned} \frac{dL}{dt} &= \frac{d}{dt} \left(\int_M \sigma(x) e^{2u} \, dA_{g_0} + \int_{\partial M} 2\zeta(x) e^u \, ds_{g_0} \right) \\ &= 2 \int_M \sigma(x) u_t \, dA_g + 2 \int_{\partial M} \zeta(x) u_t \, ds_g \\ &= 2 \left[\lambda \left(\int_M \sigma(x) \, dA_g + \int_{\partial M} \zeta(x) \, ds_g \right) - \left(\int_M K \, dA_g + \int_{\partial M} \kappa \, ds_g \right) \right] \\ &= 2 \left[\lambda \left(\int_M \sigma(x) \, dA_g + \int_{\partial M} \zeta(x) \, ds_g \right) - 2\pi \chi(M) \right] \\ &= 0, \end{aligned}$$

where we have used the Gauss–Bonnet formula in the second last equality above. Hence, the natural choice of λ will be

$$\lambda = \frac{2\pi \chi(M)}{\int_M \sigma(x) \, dA_g + \int_{\partial M} \zeta(x) \, ds_g}. \tag{2.7}$$

From the choice of λ and (2.1), it follows that $L(u)(t) = L(u)(0) = L(0) = -2\pi$. Hence, $u(t)$ satisfies the constraint (2.6) for all $t \geq 0$.

An important characterization of the flow (2.2) is that the energy functional $E[u]$ is decay during the evolution. In fact, we have the following lemma

Lemma 2.1. *For any smooth solution $u(x,t)$ of (2.3), there holds*

$$\frac{dE[u]}{dt} = \int_M \frac{1}{\sigma(x)}(K - \lambda\sigma(x))^2\, dA_g + \int_{\partial M} \frac{1}{\zeta(x)}(\kappa - \lambda\zeta(x))^2\, ds_g.$$

In particular, $E[u]$ is decreasing.

Proof. It follows from (2.3), (2.4) and (2.7) that

$$\begin{aligned}
\frac{dE[u]}{dt} &= \int_M \langle\nabla u, \nabla u_t\rangle_{g_0} + K_0 u_t\, dA_{g_0} + \int_{\partial M} \kappa_0 u_t\, ds_{g_0}\\
&= \int_M -\Delta_{g_0} u u_t + K_0 u_t\, dA_{g_0} + \int_{\partial M} \frac{\partial u}{\partial \eta_{g_0}} u_t + \kappa_0 u_t\, ds_{g_0}\\
&= \int_M \left(\frac{K}{\sigma(x)} - \lambda\right) K\, dA_g + \int_{\partial M} \left(\frac{\kappa}{\zeta(x)} - \lambda\right) \kappa\, ds_g\\
&= \int_M \left(\frac{K}{\sigma(x)} - \lambda\right)(K - \lambda\sigma(x))\, dA_g + \int_{\partial M} \left(\frac{\kappa}{\zeta(x)} - \lambda\right)(\kappa - \lambda\zeta(x))\, ds_g\\
&\quad + \lambda\left[2\pi\chi(M) - \lambda\left(\int_M \sigma(x)\, dA_g + \int_{\partial M} \zeta(x)\, ds_g\right)\right]\\
&= \int_M \frac{1}{\sigma(x)}(K - \lambda\sigma(x))^2\, dA_g + \int_{\partial M} \frac{1}{\zeta(x)}(\kappa - \lambda\zeta(x))^2\, ds_g.
\end{aligned}$$

Since $\sigma(x)$ and $\zeta(x)$ are negative, $E[u]$ is decreasing. □

2.2. Long time existence of the flow

In order to bound the conformal factor u, as a initial step, one may need to show that the normalized coefficient $\lambda(t)$ keeps bounded during the evolution. This is the following lemma

Lemma 2.2. *There exist two positive constants λ_1 and λ_2 such that*

$$\lambda_1 \le \lambda(t) \le \lambda_2.$$

Proof. Since $\sigma(x)$ and $\zeta(x)$ are negative, it follows that

$$\int_M \sigma(x)\, dA_g + \int_{\partial M} \zeta(x)\, ds_g \ge \int_M \sigma(x)\, dA_g + 2\int_{\partial M} \zeta(x)\, ds_g.$$

and

$$2\left(\int_M \sigma(x)\, dA_g + \int_{\partial M} \zeta(x)\, ds_g\right) \le \int_M \sigma(x)\, dA_g + 2\int_{\partial M} \zeta(x)\, ds_g.$$

Since $u(t)$ satisfies the constraint (2.6), we obtain

$$-2\pi \le \int_M \sigma(x)\, dA_g + \int_{\partial M} \zeta(x)\, ds_g \le -\pi. \tag{2.8}$$

Combining (2.7) and (2.8), one can obtain

$$\lambda_1 := -\chi(M) \le \lambda(t) \le -2\chi(M) := \lambda_2,$$

□

The following lemma allows us to choose the background metric g_0 with the Gauss curvature $K_0 < 0$ and geodesic curvature $\kappa_0 < 0$. This fact will simplify our argument later.

Lemma 2.3. *There exists a metric g_* in the conformal class of the metric g_0 such that the Gauss curvature K_* and geodesic curvature κ_* of g keep the same sign.*

Proof. Consider the following boundary value problem

$$\begin{cases} \Delta_{g_0} u = K_0 - \frac{2\pi\chi(M)}{A(g_0)} & \text{in } M, \\ \frac{\partial u}{\partial \eta_{g_0}} = -\kappa_0 + \frac{2\pi\chi(M)}{\mathcal{L}(g_0)} & \text{on } \partial M. \end{cases} \tag{2.9}$$

Since

$$\int_M K_0 - \frac{2\pi\chi(M)}{A(g_0)}\, dA_{g_0} = \int_{\partial M} -\kappa_0 + \frac{2\pi\chi(M)}{\mathcal{L}(g_0)}\, ds_{g_0},$$

the Eq. (2.9) has a solution u_*. Now, define $g = e^{2u_*} g_0$. Then

$$K_* = e^{-2u}\left(-\Delta_{g_0} u + K_0\right) = e^{-2u}\frac{2\pi\chi(M)}{A(g_0)}$$

and

$$\kappa_* = e^{-u}\left(-\frac{\partial u}{\partial \eta_{g_0}} + \kappa_0\right) = e^{-u}\frac{2\pi\chi(M)}{\mathcal{L}(g_0)}.$$

□

Since $\chi(M) < 0$, we may assume that $K_0 < 0$ and $\kappa_0 < 0$ by the two equalities above. Otherwise, we let g_* be the initial metric of evolution Eq. (2.2). If this evolution equation has a limit metric $g_\infty^* = e^{2u_\infty^*} g_*$, then $g_\infty = e^{2(u_\infty^* + u_*)} g_0$ will be our desired metric. Using this fact and Lemma 2.2, we are able to show that the conformal factor u is uniformly bounded.

Lemma 2.4. *There exists a positive constant C independent of time t such that*

$$||u(t)||_\infty \le C.$$

Proof. From (2.3) and (2.4), it follows that the conformal factor u will satisfy

$$\begin{cases} u_t = -\frac{1}{\sigma(x)} e^{-2u} \Delta_{g_0} u + \frac{1}{\sigma(x)} e^{-2u} K_0 - \lambda, & \text{in } M, \\ u_t = \frac{1}{\zeta(x)} e^{-u} \frac{\partial u}{\partial \eta_{g_0}} + \frac{1}{\zeta(x)} e^{-u} \kappa_0 - \lambda, & \text{on } \partial M. \end{cases} \tag{2.10}$$

Firstly, we show that u is uniformly bounded from above. To see this, let $x' \in \bar{M}$ be a point where the function u attains its maximum. Assume that $u(x')$ is sufficiently large. Then we consider the following two cases.

(i) $x' \in M$. Using maximum principle and the fact that $\sigma(x) < 0$, we have $\left(-\frac{1}{\sigma} e^{-2u} \Delta_{g_0} u\right)(x') \le 0$. Since $\lambda > 0$ by Lemma 2.2 and $u(x')$ is sufficiently large by our assumption, it follows that $\left(\frac{1}{\sigma} e^{-2u} K_0\right)(x') - \lambda < 0$. Hence, the first equation in (2.10) implies that $\frac{\partial u}{\partial t}(x') < 0$.

(ii) $x' \in \partial M$. It follows from maximum principle and the fact that $\zeta(x) < 0$ that $\left(\frac{1}{\zeta} e^{-u} \frac{\partial u}{\partial \eta_{g_0}}\right)(x') \leq 0$. The same reason as in (i) yields $\left(\frac{1}{\zeta} e^{-u} \kappa_0\right)(x') - \lambda < 0$. Then the second equation in (2.10) implies that $\frac{\partial u}{\partial t}(x') < 0$.

Therefore, the time derivative of u at the maximum point where u is sufficiently large is negative and hence the maximum of u must be uniformly bounded from above.

Next, we show that u is also uniformly bounded from below. Let $x^* \in \bar{M}$ be a point where the function u attains its minimum. Suppose that $u(x^*)$ is sufficiently negative. Similarly, we split the argument into two cases.

(a) $x^* \in M$. Using maximum principle and the fact that $\sigma(x) < 0$, we have $\left(-\frac{1}{\sigma} e^{-2u} \Delta_{g_0} u\right)(x^*) \geq 0$. Since λ is bounded by Lemma 2.2, $K_0 < 0$ and $u(x^*)$ is sufficiently negative by our assumption, it follows that $\left(\frac{1}{\sigma} e^{-2u} K_0\right)(x') - \lambda > 0$. Hence, the first equation in (2.10) implies that $\frac{\partial u}{\partial t}(x^*) > 0$.

(b) $x^* \in \partial M$. It follows from maximum principle and $\zeta(x) < 0$ that $\left(\frac{1}{\zeta} e^{-u} \frac{\partial u}{\partial \eta_{g_0}}\right)(x^*) \geq 0$. From the boundedness of λ, $\kappa_0 < 0$ and sufficiently negative $u(x^*)$, it follows tha $\left(\frac{1}{\zeta} e^{-u} \kappa_0\right)(x^*) - \lambda > 0$. Then the second equation in (2.10) implies that $\frac{\partial u}{\partial t}(x^*) > 0$.

Therefore, the time derivative of u at the minimum point where u is sufficiently negative is positive and hence the minimum of u must be uniformly bounded from below. □

Next step, we need to bound all higher derivatives of the conformal factor u. To do so, we will show that the curvatures are bounded in L^p for all $p \geq 2$. For this purpose, deriving the curvatures evolution equations seems necessary which is the following lemma.

Lemma 2.5.

$$\begin{cases} K_t = -\Delta_{g(t)}\left(\frac{K}{\sigma(x)}\right) - 2K\left(\frac{K}{\sigma(x)} - \lambda\right), & \textit{in } M, \\ \kappa_t = \frac{\partial}{\partial \eta_{g(t)}}\left(\frac{\kappa}{\zeta(x)}\right) - \kappa\left(\frac{\kappa}{\zeta(x)} - \lambda\right), & \textit{on } \partial M. \end{cases}$$

Proof. From (2.4) and (2.3), it follows that

$$\begin{aligned} K_t &= -2e^{-2u}(-\Delta_{g_0} u + K_0)u_t - e^{-2u}\Delta_{g_0} u_t \\ &= -\Delta_{g(t)} u_t - 2K u_t \\ &= -\Delta_{g(t)}\left(\frac{K}{\sigma(x)}\right) - 2K\left(\frac{K}{\sigma(x)} - \lambda\right) \end{aligned}$$

and

$$\begin{aligned} \kappa_t &= -e^{-u}(-\frac{\partial u}{\partial \eta_{g_0}} + \kappa_0)u_t + e^{-u}\frac{\partial u_t}{\partial \eta_{g_0}} \\ &= \frac{\partial u_t}{\partial \eta_{g_t}} - \kappa u_t \\ &= \frac{\partial}{\partial \eta_{g_t}}\left(\frac{\kappa}{\zeta(x)}\right) - \kappa\left(\frac{\kappa}{\zeta(x)} - \lambda\right), \end{aligned}$$

where we have used the relations: $\Delta_g = e^{-2u}\Delta_{g_0}$ and $\partial/\partial\eta_g = e^{-u}\partial/\partial\eta_{g_0}$. □

Now, observing (2.3), we may rewrite K and κ as

$$\begin{cases} K = \sigma(x)(\frac{\partial u}{\partial t} + \lambda) \\ \kappa = \zeta(x)(\frac{\partial u}{\partial t} + \lambda) \end{cases}$$

Hence, by setting

$$h = \frac{\partial u}{\partial t} + \lambda, \tag{2.11}$$

we then have the following relation

$$\begin{cases} h = \frac{K}{\sigma(x)}, & \text{in } M, \\ h = \frac{\kappa}{\zeta(x)}, & \text{on } \partial M. \end{cases} \tag{2.12}$$

By using (2.12), it is easy to obtain the evolution equation for h.

Lemma 2.6. *The function h satisfies the following evolution equation*

$$\begin{cases} h_t = -\frac{1}{\sigma(x)}\Delta_{g(t)}h - 2h(h-\lambda), & \textit{in } M, \\ h_t = \frac{1}{\zeta(x)}\frac{\partial}{\partial\eta_{g(t)}}h - h(h-\lambda), & \textit{on } \partial M \end{cases}$$

Proof. In view of Lemma 2.5 and (2.12), it is just a routine calculation. We omit the detail. □

Since $\sigma(x)$ and $\zeta(x)$ are smooth functions, it suffices to bound h in L^p instead of bounding K and κ in L^p in view of (2.12). The next lemma shows that h belongs to $L^2(M, g) \cap L^2(\partial M, g)$ for $t \in [0, T]$ with $T < \infty$.

Lemma 2.7. *For any fixed $T < \infty$, the function h is bounded in $L^2(M, g) \cap L^2(\partial M, g)$ for $t \in [0, T]$.*

Proof. From Lemma 2.1 and (2.12), we have

$$\frac{d}{dt}E[u] = \int_M \sigma(x)(h-\lambda)^2\, dA_g + \int_{\partial M} \zeta(x)(h-\lambda)^2\, ds_g. \tag{2.13}$$

Since the conformal factor u is bounded by Lemma 2.4, it follows from the energy functional formula (2.5) that

$$E[u] \geq -C. \tag{2.14}$$

Hence, by integrating (2.13) and using (2.14) and the fact that $\sigma(x) < 0$ and $\zeta(x) < 0$, we have

$$\int_0^T \left(\int_M -\sigma(x)(h-\lambda)^2\, dA_g + \int_{\partial M} -\zeta(x)(h-\lambda)^2\, ds_g \right) dt \leq C.$$

From Lemma 2.2, it follows that

$$\int_0^T \left(\int_M -\sigma(x)h^2\, dA_g + \int_{\partial M} -\zeta(x)h^2\, ds_g \right) dt \leq C. \tag{2.15}$$

For simplicity, we set

$$F_p(t) = \int_M -\sigma(x)|h|^p \, dA_g + \int_{\partial M} -\zeta(x)|h|^p \, ds_g.$$

Then, by Lemma 2.6 and (2.3), we have

$$\begin{aligned}
\frac{d}{dt}F_p(t) &= p\int_M -\sigma(x)|h|^{p-2}hh_t \, dA_g + 2\int_M -\sigma(x)|h|^p u_t \, dA_g \\
&\quad + p\int_{\partial M} -\zeta(x)|h|^{p-2}hh_t \, ds_g + \int_{\partial M} -\zeta(x)|h|^p u_t \, ds_g \\
&= p\int_M -\sigma(x)|h|^{p-2}h\Big[-\frac{1}{\sigma(x)}\Delta_g h - 2h(h-\lambda)\Big] \, dA_g \\
&\quad + p\int_{\partial M} -\zeta(x)|h|^{p-2}h\Big[\frac{1}{\zeta(x)}\frac{\partial h}{\partial \eta_g} - h(h-\lambda)\Big] \, ds_g \\
&\quad + 2\int_M -\sigma(x)|h|^p(h-\lambda) \, dA_g + \int_{\partial M} -\zeta(x)|h|^p(h-\lambda) \, ds_g \\
&= -p\int_M \langle \nabla(|h|^{p-2}h), \nabla h\rangle_g \, dA_g \\
&\quad + 2(1-p)\int_M -\sigma(x)|h|^p(h-\lambda) \, dA_g \\
&\quad + (1-p)\int_{\partial M} -\zeta(x)|h|^p(h-\lambda) \, ds_g \\
&\le -\frac{4(p-1)}{p}\int_M |\nabla|h|^{\frac{p}{2}}|^2 \, dA_g + C\Big(\int_M |h|^p \, dA_g + \int_{\partial M} |h|^p \, ds_g\Big) \\
&\quad + C\Big(\int_M |h|^{p+1} \, dA_g + \int_{\partial M} |h|^{p+1} \, ds_g\Big), \qquad (2.16)
\end{aligned}$$

for $p \ge 2$. Using the Gagliardo–Nirenberg inequality (see [1], p. 60), we obtain

$$\int_M |h|^{p+1} \, dA_g \le C\Big(\int_M |h|^q \, dA_g\Big)\Big(\int_M |\nabla|h|^{\frac{p}{2}}|^2 \, dA_g\Big)^{1-\frac{q-1}{p}}$$

and

$$\int_{\partial M} |h|^{p+1} \, dA_g \le C\Big(\int_{\partial M} |h|^q \, ds_g\Big)\Big(\int_M |\nabla|h|^{\frac{p}{2}}|^2 \, dA_g\Big)^{1-\frac{q-1}{p}}.$$

Plugging the two inequalities above into (2.16) and applying Young's inequality, we have

$$\begin{aligned}
\frac{d}{dt}F_p(t) &\le -C\int_M |\nabla|h|^{\frac{p}{2}}|^2 \, dA_g + C\Big(\int_M |h|^p \, dA_g + \int_{\partial M} |h|^p \, ds_g\Big) \\
&\quad + C\Big(\int_M |h|^q \, dA_g\Big)^{\frac{p}{q-1}} + C\Big(\int_{\partial M} |h|^q \, ds_g\Big)^{\frac{p}{q-1}} \\
&\le CF_p(t) + CF_q(t)^{\frac{p}{q-1}}, \qquad (2.17)
\end{aligned}$$

for all $q \leq p \leq 2q$. Now, set $p = q = 2$ in (2.17). Then

$$\frac{d}{dt} F_2(t) \leq C F_2(t) + C F_2(t)^2,$$

which implies that

$$\frac{d}{dt} \log(F_2(t) + 1) \leq C F_2(t) + C. \tag{2.18}$$

From (2.15), it follows that

$$\int_0^t F_2(t)\, dt \leq C,$$

for $t \in [0, T]$. Hence, by integrating (2.18) with respect to t and using the estimate above, we obtain

$$F_2(t) \leq C, \tag{2.19}$$

for $t \in [0, T]$. Since $\sigma(x)$ and $\zeta(x)$ are bounded functions, it follows that

$$\int_M h^2\, dA_g + \int_{\partial M} h^2\, ds_g \leq C F_2(t) \leq C.$$

This proves the assertion. □

With the help of Lemma 2.7, we are able to show that the function h is bounded in L^p for all $p \geq 2$.

Lemma 2.8. *The function h is bounded in $L^p(M, g) \cap L^p(\partial M, g)$ for $t \in [0, T]$.*

Proof. It suffices to prove that the function h is bounded in $L^{2^n}(M, g) \cap L^{2^n}(\partial M, g)$ for all integers $n \geq 1$. To see this, we only need to show that $F_{2^n}(t)$ is bounded for all $n \geq 1$.

When $n = 1$, it follows from (2.19) that $F_2(t)$ is bounded for $t \in [0, T]$.

Now, assume that $F_{2^k}(t)$ is bounded for $t \in [0, T]$. By setting $p = 2^{k+1}$, $q = 2^k$ in (2.17) and applying the induction assumption, we obtain

$$\begin{aligned} \frac{d}{dt} F_{2^{k+1}}(t) &\leq C F_{2^{k+1}}(t) + C F_{2^k}(t)^{\frac{2^{k+1}}{2^k - 1}} \\ &\leq C F_{2^{k+1}}(t) + C, \end{aligned}$$

which implies that $F_{2^{k+1}}(t)$ is bounded. Hence, by mathematical induction, $F_{2^n}(t)$ is bounded for all $n \geq 1$. □

Lemma 2.9. *The function $\frac{\partial}{\partial t} u$ is bounded in $L^p(M, g) \cap L^p(\partial M, g)$ for all $p \geq 2$ and $t \in [0, T]$. Moreover, the function u is bounded in $W^{2,p}(M, g_0) \cap W^{1,p}(\partial M, g_0)$ for all $p \geq 2$ and $t \in [0, T]$.*

Proof. From (2.3) and (2.12), it follows that

$$\frac{\partial}{\partial t}u = h - \lambda.$$

Hence, by Lemma 2.8, the function $\frac{\partial}{\partial t}u$ is bounded in $L^p(M, g) \cap L^p(\partial M, g)$ for all $p \geq 2$ and $t \in [0, T]$. Moreover, we have by (2.4) and (2.12)

$$\begin{cases} -\Delta_{g_0} u = e^{2u}\sigma(x)h - K_0 & \text{in } M, \\ \frac{\partial}{\partial \eta_{g_0}} u = e^{u}\zeta(x)h - \kappa_0 & \text{on } \partial M \end{cases}$$

Hence, it follows from Lemma 2.8 that

$$\int_M |\Delta_{g_0} u|^p \, dA_{g_0} \leq C \quad \text{and} \quad \int_{\partial M} |\frac{\partial}{\partial \eta_{g_0}} u|^p \, ds_{g_0} \leq C.$$

Now, from Lemma 2.4 and the estimate (A.1), it follows that u is bounded in $W^{2,p}(M, g_0) \cap W^{1,p}(\partial M, g_0)$ for all $p \geq 2$ and $t \in [0, T]$. □

At this point, we are able to show that all higher order derivatives of function u are bounded for $t \in [0, T]$.

Lemma 2.10. *The function u is bounded in $C^k(M\times[0, T], g_0) \cap C^k(\partial M\times[0, T], g_0)$ for all $k \geq 0$.*

Proof. We firstly show the interior regularity of the function u. For this, we will follow the idea in [7, Proposition 3.8]. Hence, it suffices to show that for some $0 < s < 1$, there exists a constant $C(T) > 0$ such that

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C(T)((t_1 - t_2)^{\frac{s}{2}} + d(x_1, x_2)^s).$$

for all $x_1, x_2 \in M$ and all $t_1, t_2 \in [0, T]$ with $0 < t_1 - t_2 < 1$.

Let $s = 1 - 2/p$, where $p > 2$. Using Lemma 2.9, we have

$$\int_M |\Delta_{g_0} u|^p \, dA_{g_0} \leq C(T)$$

and

$$\int_M \left|\frac{\partial}{\partial t}u\right|^p \, dA_{g_0} \leq C(T)$$

for all $t \in [0, T]$. It follows from the first inequality above that

$$|u(x_1, t) - u(x_2, t)| \leq C(T)d(x_1, x_2)^s$$

for all $x_1, x_2 \in M$ and $t \in [0, T]$. Now, applying the second inequality above yields

$$\begin{aligned}
&|u(x,t_1) - u(x,t_2)| \\
&\leq C(t_1 - t_2)^{-1} \int_{B_{\sqrt{t_1-t_2}}(x)} |u(x,t_1) - u(x,t_2)|\, dA_{g_0} \\
&\leq C(t_1 - t_2)^{-1} \int_{B_{\sqrt{t_1-t_2}}(x)} |u(t_1) - u(t_2)|\, dA_{g_0} + C(T)(t_1 - t_2)^{\frac{s}{2}} \\
&\leq C \sup_{t_2 \leq t \leq t_1} \int_{B_{\sqrt{t_1-t_2}}(x)} |\frac{\partial}{\partial t} u|\, dA_{g_0} + C(T)(t_1 - t_2)^{\frac{s}{2}} \\
&\leq C(t_1 - t_2)^{\frac{s}{2}} \sup_{t_2 \leq t \leq t_1} \left(\int_M |\frac{\partial}{\partial t} u|^p \, dA_{g_0} \right)^{\frac{1}{p}} + C(T)(t_1 - t_2)^{\frac{s}{2}} \\
&\leq C(T)(t_1 - t_2)^{\frac{s}{2}}.
\end{aligned}$$

Therefore, we can now apply the standard regularity theory for parabolic equations (see [11, Theorem 5], on p. 64) to obtain that u is bounded in $C^k(M \times [0, T], g_0)$ for all $k \geq 0$.

Next, we show the boundary regularity of the function u. It follows from Lemmas 2.4 and 2.9 and the interior regularity we just proved above that u is bounded in $L^\infty(\bar{M} \times [0, T])$, in $C^\infty(M \times [0, T]) \cap W^{1,4}(\partial M \times [0, T])$. Hence, we can apply the recurrence estimate for the linear equation from Lemma A.2 to obtain that u is bounded in $C^k(\partial M \times [0, T], g_0)$ for all $k \geq 0$. □

Notice that we may follow the proof of [5, Theorem 2.5] to get the local existence of our flow. Then, using Lemma 2.10, we immediately obtain

Corollary 2.11. *The evolution Eq. (2.2) has a unique smooth solution on* $[0, \infty)$.

3. Existence of conformal metrics

In this section, we devote ourselves to proving the Theorem 1.2. To do so, we need the following L^2 convergence of the curvatures.

Lemma 3.1.

$$\int_M -\sigma(x)(h - \lambda)^2 \, dA_g + \int_{\partial M} -\zeta(x)(h - \lambda)^2 \, ds_g \to 0, \quad \text{as } t \to \infty.$$

Proof. For abbreviation, we set

$$y(t) = \int_M -\sigma(x)(h - \lambda)^2 \, dA_g + \int_{\partial M} -\zeta(x)(h - \lambda)^2 \, ds_g.$$

Then by Lemma 2.6, (2.7) and a direct computation, we obtain

$$\begin{aligned}
\frac{d}{dt} y(t) \leq -2 \int_M |\nabla h|^2 \, dA_g &+ C\left(\int_M (h - \lambda)^2 \, dA_g + \int_{\partial M} (h - \lambda)^2 \, ds_g \right) \\
&+ C\left(\int_M (h - \lambda)^3 \, dA_g + \int_{\partial M} (h - \lambda)^3 \, ds_g \right)
\end{aligned}$$

Using the Gagliardo-Nirenberg inequality as before, we obtain

$$\int_M |h-\lambda|^3\, dA_g \le C\Big(\int_M (h-\lambda)^2\, dA_g\Big)\Big(\int_M |\nabla h|^2\, dA_g\Big)^{\frac{1}{2}}$$

and

$$\int_{\partial M} |h-\lambda|^3\, dA_g \le C\Big(\int_{\partial M} (h-\lambda)^2\, ds_g\Big)\Big(\int_M |\nabla h|^2\, dA_g\Big)^{\frac{1}{2}}.$$

Here, the constant C can be chosen independent of t, since the conformal factor u is uniformly bounded by Lemma 2.4. From this and Young's inequality, it follows that

$$\begin{aligned}\frac{d}{dt}y(t) \le C\Big(\int_M -\sigma(x)(h-\lambda)^2\, dA_g + \int_{\partial M} -\zeta(x)(h-\lambda)^2\, ds_g\Big)^2 \\ +C\Big(\int_M -\sigma(x)(h-\lambda)^2\, dA_g + \int_{\partial M} \zeta(x)(h-\lambda)^2\, ds_g\Big).\end{aligned}$$

Hence, the function $y(t)$ satisfies

$$\frac{d}{dt}y(t) \le Cy(t)^2 + Cy(t). \tag{3.1}$$

By (2.13) and (2.14), we have

$$\int_0^\infty y(t)\, dt \le C.$$

This implies that there exists a sequence $t_j \to \infty$ as $j \to \infty$ such that

$$\lim_{j\to\infty} y(t_j) \to 0.$$

Observing that (3.1) is equivalent to the following differential inequality

$$\frac{d}{dt}(\log(y(t)+1)) \le Cy(t).$$

Integrating this inequality with respective to t from t_j to t with $t > t_j$ yields

$$y(t) \le (y(t_j)+1)\exp\left\{C\int_{t_j}^\infty y(t)\, dt\right\} - 1.$$

Letting $j \to \infty$ gives

$$\lim_{t\to\infty} y(t) = 0.$$

□

With the help of Lemma (3.1), we can show that $u(t)$ is uniformly bounded in $W^{2,2}(M, g_0) \cap W^{1,2}(\partial M, g_0)$ for all $t \ge 0$. From this, the main theorem thus follows.

Proof of Theorem 1.2. By (2.4) and (2.12), we have

$$\begin{aligned}-\Delta_{g_0} u &= K e^{2u} - K_0 \\ &= \sigma(x)(h-\lambda)e^{2u} + \lambda\sigma(x)e^{2u} - K_0\end{aligned}$$

in M and

$$\begin{aligned}\frac{\partial}{\partial\eta_{g_0}} u &= \kappa e^{u} - \kappa_0 \\ &= \zeta(x)(h-\lambda)e^{u} + \lambda\zeta(x)e^{u} - \kappa_0\end{aligned}$$

on ∂M. From Lemmas 2.4 and 3.1, it follows that there exists a constant C independent of t such that

$$\int_M |\Delta_{g_0} u|^2 \, dA_{g_0} \leq C \text{ and } \int_{\partial M} \left|\frac{\partial}{\partial\eta_{g_0}} u\right|^2 \, ds_{g_0} \leq C.$$

From this and the estimate (A.1), we conclude that $u(t)$ is uniformly bounded in $W^{2,2}(M, g_0) \cap W^{1,2}(\partial M, g_0)$ for all $t \geq 0$. Hence, there exists a function $u_\infty \in W^{2,2}(M, g_0) \cap W^{1,2}(\partial M, g_0)$ such that, up to a subsequence $(t_j)_j$, we have

$$\begin{cases} u(t_j) \rightharpoonup u_\infty, & \text{weakly in } W^{2,2}(M, g_0) \cap W^{1,2}(\partial M, g_0), \\ u(t_j) \to u_\infty, & \text{strongly in } C^s(M, g_0) \cap C^s(\partial M, g_0) \text{ for } 0 < s < 1. \end{cases}$$

It follows from Lemma 3.1 that u_∞ weakly solves the following equation

$$\begin{cases} -\Delta_{g_0} u_\infty + K_0 = \lambda_\infty \sigma(x) e^{2u_\infty}, & \text{in } M \\ \frac{\partial}{\partial\eta_{g_0}} u_\infty + \kappa_0 = \lambda_\infty \zeta(x) e^{u_\infty}, & \text{on } \partial M. \end{cases}$$

The standard regularity theory implies that u_∞ is smooth since $\sigma(x)$ and $\zeta(x)$ are smooth. Hence, if we let $g_\infty = e^{2u_\infty} g_0$, then under the metric g_∞, the Gaussian curvature $K_\infty(x) = \lambda_\infty \sigma(x)$ and the boundary geodesic curvature $\kappa_\infty(x) = \lambda_\infty \zeta(x)$. This completes the proof of Theorem 1.2. □

A. Appendix

In this appendix, we will provide some estimates which are needed in the proof of long time existence and convergence of the flow. We will firstly borrow the idea in [4, Lemma 2.1] and [5, Lemma 3.2] to show the following result. In fact, the result below is a generalization of [5, Lemma 3.2] where only harmonica functions were considered.

Proposition A.1. *Let (M, g_0) be a compact Riemannian manifold with boundary. Also let ϕ be a smooth function on $\bar{M}$. Then we have the following estimate*

$$||\nabla\phi||_{L^p(\partial M, g_0)} \leq C||\frac{\partial}{\partial\eta_{g_0}}\phi||_{L^p(\partial M, g_0)} + C||\Delta_{g_0}\phi||_{L^p(M, g_0)}. \tag{A.1}$$

Proof. Firstly, we will give a proof of (A.1) for the model problem on the half space $\{x_n \geq 0\}$. Hence, let $(x', x_n) = (x_1, \ldots, x_{n-1}, x_n)$ be the coordinate of the half space. For abbreviation, we set $\alpha(x) := \sum_i^n \frac{\partial^2}{\partial x_i^2}\phi$ and $\beta(x') := -\frac{\partial}{\partial x_n}\phi(x', 0)$. Taking the Fourier transformation in the first $n-1$ variables, we obtain

$$\begin{cases} \frac{\partial^2}{\partial x_n^2}\hat{\phi}(\xi, x_n) - |\xi|^2\hat{\phi}(\xi, x_n) = \hat{\alpha}(\xi, x_n) & \text{in } x_n > 0, \\ \frac{\partial}{\partial x_n}\hat{\phi}(\xi, 0) = \hat{\beta}(\xi) & \text{on } x_n = 0. \end{cases}$$

This is an second order linear ordinary differential equation with respect to x_n whose solution is given by

$$\begin{aligned} &\hat{\phi}(\xi, x_n) \\ &\quad = -\frac{1}{2|\xi|}\int_0^\infty e^{-|\xi||x_n - y_n|}\hat{\alpha}(\xi, x_n)\, dy_n \\ &\qquad -\frac{1}{2|\xi|}\int_0^\infty e^{-|\xi|(x_n + y_n)}\hat{\alpha}(\xi, x_n)\, dy_n + \frac{1}{|\xi|}e^{-|\xi|x_n}\hat{\beta}(\xi). \end{aligned}$$

Therefore, it follows from the expression above that

$$\widehat{\nabla_{tan}\,\phi}(\xi, 0) = -\frac{i\xi}{|\xi|}\int_0^\infty e^{-|\xi|y_n}\hat{\alpha}(\xi, x_n)\, dy_n + \frac{i\xi}{|\xi|}\hat{\beta}(\xi),$$

where $\nabla_{tan} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}})$. Using Mikhlin's theorm (see [16], p. 109), we conclude that

$$\begin{aligned} &\int_{\{x_n=0\}} |\nabla_{tan}\,\phi(x', 0)|^p\, dx' \\ &\leq C\left(\int_{\{x_n>0\}} |\alpha(x', x_n)|^p\, dx'dx_n + \int_{\{x_n=0\}} |\beta(x', 0)|^p\, dx'\right). \\ &= C\left(\int_{\{x_n>0\}} \left|\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\phi(x', x_n)\right|^p dx'dx_n + \int_{\{x_n=0\}} \left|\frac{\partial}{\partial x_n}\phi(x', 0)\right|^p dx'\right). \end{aligned}$$

From this, it follows that

$$\begin{aligned} &\int_{\{x_n=0\}} |\nabla\phi(x', 0)|^p\, dx' \\ &\leq C\left(\int_{\{x_n=0\}} |\nabla_{tan}\,\phi(x', 0)|^p\, dx' + \int_{\{x_n=0\}} \left|\frac{\partial}{\partial x_n}\phi(x', 0)\right|^p\right) \\ &= C\left(\int_{\{x_n>0\}} \left|\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\phi(x', x_n)\right|^p dx'dx_n + \int_{\{x_n=0\}} \left|\frac{\partial}{\partial x_n}\phi(x', 0)\right|^p dx'\right). \end{aligned}$$

Hence, the estimate (A.1) holds for the half space. Next, using the standard unitary partition argument and the estimate above, we conclude that (A.1) holds on M. □

Next, we will prove a recurrence estimate for a linear evolution equation. The proof of this estimate is basically the same as [6, Lemma 3.2]. For simplicity, all norms below are considered with respective to the metric g_0.

Lemma A.2. *Let $\gamma(x)$ be a strictly negative smooth function on $\bar{M}$. Let ϕ be a solution of the linear evolution equation*

$$\begin{cases} \frac{\partial}{\partial t}\phi = -\gamma(x)e^{-2u}\Delta_{g_0}\phi + f_1 & \text{in } M \\ \frac{\partial}{\partial t}\phi = \gamma(x)e^{-u}\frac{\partial}{\partial \eta_{g_0}}\phi + f_2 & \text{on } \partial M \end{cases} \tag{A.2}$$

with the initial data

$$\phi(\cdot,0) = 0. \tag{A.3}$$

Suppose that u is bounded. Then we have the estimate

$$||\phi||_{W^{1,2}(M\times[0,T])} + ||\phi||_{W^{1,2}(\partial M\times[0,T])} \leq C\big(||f_1||_{L^2(M\times[0,T])} + ||f_2||_{L^2(\partial M\times[0,T])}\big).$$

Proof. Notice that

$$\begin{aligned} &2\left(\int_0^T\int_M \frac{\partial\phi}{\partial t}\Delta_{g(t)}\phi\, dA_g dt - \int_0^T\int_{\partial M}\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial\eta_{g(t)}}\, ds_g dt\right) \\ &= -2\int_0^T\int_M \frac{\partial}{\partial t}(\nabla_{g_0}\phi)\cdot\nabla_{g_0}\phi\, dA_{g_0}dt \\ &= -\int_0^T\int_M \frac{\partial}{\partial t}|\nabla_{g_0}\phi|^2\, dA_{g_0}dt \\ &= -\int_M |\nabla_{g_0}\phi|^2\, dA_{g_0}\Big|_{t=T} \leq 0. \end{aligned}$$

From this it follows that

$$\begin{aligned} &\left(\int_0^T\int_M -\frac{1}{\gamma(x)}\left(\frac{\partial}{\partial t}\phi\right)^2 dA_g dt + \int_0^T\int_M -\gamma(x)(\Delta_{g(t)}\phi)^2\, dA_g dt\right) \\ &\quad+\left(\int_0^T\int_{\partial M} -\frac{1}{\gamma(x)}\left(\frac{\partial}{\partial t}\phi\right)^2 ds_g dt + \int_0^T\int_{\partial M} -\gamma(x)\left(\frac{\partial}{\partial\eta_{g(t)}}\phi\right)^2 ds_g dt\right) \\ &\leq \int_0^T\int_M\left(\frac{1}{\sqrt{-\gamma(x)}}\frac{\partial}{\partial t}\phi - \sqrt{-\gamma(x)}\Delta_{g(t)}\phi\right)^2 dA_g dt \\ &\quad+\int_0^T\int_M\left(\frac{1}{\sqrt{-\gamma(x)}}\frac{\partial}{\partial t}\phi + \sqrt{-\gamma(x)}\frac{\partial}{\partial\eta_{g(t)}}\phi\right)^2 ds_g dt \\ &= \int_0^T\int_M \frac{1}{-\gamma(x)}(f_1)^2\, dA_g dt + \int_0^T\int_M \frac{1}{-\gamma(x)}(f_2)^2\, ds_g dt \end{aligned}$$

Since u and $\gamma(x)$ are bounded, it follows from the estimate above that

$$\begin{aligned} &\left(\int_0^T\int_M\left(\frac{\partial}{\partial t}\phi\right)^2 dA_{g_0}dt + \int_0^T\int_M(\Delta_{g_0}\phi)^2\, dA_{g_0}dt\right) \\ &\quad+\left(\int_0^T\int_{\partial M}\left(\frac{\partial}{\partial t}\phi\right)^2 ds_{g_0}dt + \int_0^T\int_{\partial M}\left(\frac{\partial}{\partial\eta_{g_0}}\phi\right)^2 ds_{g_0}dt\right) \\ &\leq C\left(\int_0^T\int_M(f_1)^2\, dA_{g_0}dt + \int_0^T\int_M(f_2)^2\, ds_{g_0}dt\right). \end{aligned}$$

Using the estimate A.1, we obtain

$$\left(\int_0^T\int_M\left(\frac{\partial}{\partial t}\phi\right)^2 dA_{g_0}dt+\int_0^T\int_M(\Delta_{g_0}\phi)^2\, dA_{g_0}dt\right)$$
$$+\left(\int_0^T\int_{\partial M}\left(\frac{\partial}{\partial t}\phi\right)^2 ds_{g_0}dt+\int_0^T\int_{\partial M}|\nabla_{g_0}\phi|^2\, ds_{g_0}dt\right)$$
$$\leq C\left(\int_0^T\int_M(f_1)^2\, dA_{g_0}dt+\int_0^T\int_M(f_2)^2\, ds_{g_0}dt\right). \qquad \text{(A.4)}$$

Moreover

$$\frac{d}{dt}\left(\int_M\phi^2\, dA_{g_0}+\int_{\partial M}\phi^2\, ds_{g_0}\right)=2\int_M\frac{\partial\phi}{\partial t}\phi\, dA_{g_0}+2\int_{\partial M}\frac{\partial\phi}{\partial t}\phi\, ds_{g_0}$$
$$\leq\left(\int_M\phi^2\, dA_{g_0}+\int_{\partial M}\phi^2\, ds_{g_0}\right)+\left(\int_M\left(\frac{\partial\phi}{\partial t}\right)^2 dA_{g_0}+\int_{\partial M}\left(\frac{\partial\phi}{\partial t}\right)^2 ds_{g_0}\right).$$

Therefore

$$\int_M\phi^2\, dA_{g_0}+\int_{\partial M}\phi^2\, ds_{g_0}\leq e^t\int_0^T e^{-t}\left(\int_M\left(\frac{\partial\phi}{\partial t}\right)^2 dA_{g_0}+\int_{\partial M}\left(\frac{\partial\phi}{\partial t}\right)^2 ds_{g_0}\right)dt.$$

Integrating the inequality above with respect to t and using (A.4), we obtain

$$\int_0^T\int_M\phi^2\, dA_{g_0}dt+\int_0^T\int_{\partial M}\phi^2\, ds_{g_0}dt$$
$$\leq C\left(\int_0^T\int_M\left(\frac{\partial\phi}{\partial t}\right)^2 dA_{g_0}dt+\int_0^T\int_{\partial M}\left(\frac{\partial\phi}{\partial t}\right)^2 ds_{g_0}dt\right)$$
$$\leq C\left(\int_0^T\int_M(f_1)^2\, dA_{g_0}dt+\int_0^T\int_M(f_2)^2\, ds_{g_0}dt\right). \qquad \text{(A.5)}$$

The assertion now follows from the estimate (A.4) and (A.5). □

Lemma A.3. *Let ϕ be a solution of* (A.2) *and* (A.3). *Suppose that u is bounded in $L^\infty(\bar{M}\times[0,T])$, in $C^\infty(M\times[0,T])\cap W^{1,4}(\partial M\times[0,T])$ and in $W^{m,2}(\partial M\times[0,T])$. Then we have*

$$||\phi||_{W^{m+1,2}(M\times[0,T])}+||\phi||_{W^{m+1,2}(\partial M\times[0,T])}$$
$$\leq C\big(||f_1||_{W^{m,2}(M\times[0,T])}+||f_2||_{W^{m,2}(\partial M\times[0,T])}\big).$$

Proof. Assume that the assertion holds for $m-1$. Observe that the function $\nabla_{g_0}\phi$ satisfies the evolution equation

$$\begin{cases}\frac{\partial}{\partial t}(\nabla_{g_0}\phi)=-\gamma(x)e^{-2u}\Delta_{g_0}(\nabla_{g_0}\phi)+\tilde{f}_1 & \text{in } M\\ \frac{\partial}{\partial t}(\nabla_{g_0}\phi)=-\gamma(x)e^{-u}\frac{\partial}{\partial\eta_{g_0}}(\nabla_{g_0}\phi)+\tilde{f}_2 & \text{on } \partial M\end{cases}$$

with the initial data

$$\nabla_{g_0}\phi(\cdot,0)=0,$$

where

$$\tilde{f}_1 = \nabla_{g_0} f_1 - e^{-2u}\big(2\gamma(x)\nabla_{g_0} u - \nabla_{g_0}\gamma(x)\big)\Delta_{g_0}\phi$$

and

$$\tilde{f}_2 = \nabla_{g_0} f_2 - e^{-u}\big(\gamma(x)\nabla_{g_0} u - \nabla_{g_0}\gamma(x)\big)\frac{\partial}{\partial\eta_{g_0}}\phi.$$

Using the Lemma A.2, Gagliado–Nirenberg inequality, Sobolev embedding theorem and induction assumption, we obtain

$$\begin{aligned}
&||\nabla_{g_0}\phi||_{W^{m,2}(M\times[0,T])} + ||\nabla_{g_0}\phi||_{W^{m,2}(\partial M\times[0,T])}\\
&\quad\leq C\big(||\tilde{f}_1||_{W^{m-1,2}(M\times[0,T])} + ||\tilde{f}_2||_{W^{m-1,2}(\partial M\times[0,T])}\big)\\
&\quad= C\Big[||\nabla_{g_0} f_1 - e^{-2u}\big(2\gamma(x)\nabla_{g_0} u - \nabla_{g_0}\gamma(x)\big)\Delta_{g_0}\phi||_{W^{m-1,2}(M\times[0,T])}\\
&\qquad+||\nabla_{g_0} f_2 - e^{-u}\big(\gamma(x)\nabla_{g_0} u - \nabla_{g_0}\gamma(x)\big)\frac{\partial}{\partial\eta_{g_0}}\phi||_{W^{m-1,2}(\partial M\times[0,T])}\Big]\\
&\quad\leq C\Big(||\Delta_{g_0}\phi||_{W^{m-1,4}(M\times[0,T])} + ||\frac{\partial}{\partial\eta_{g_0}}\phi||_{W^{m-1,4}(\partial M\times[0,T])}\\
&\qquad+||\nabla_{g_0} f_1||_{W^{m-1,2}(M\times[0,T])} + ||\nabla_{g_0} f_2||_{W^{m-1,2}(\partial M\times[0,T])}\Big)\\
&\quad\leq C\Big(||-\frac{1}{\gamma(x)}e^{2u}\left(\frac{\partial}{\partial t}\phi - f_1\right)||_{W^{m-1,4}(M\times[0,T])} + ||\phi||_{W^{m,4}(\partial M\times[0,T])}\\
&\qquad+||f_1||_{W^{m,2}(M\times[0,T])} + ||f_2||_{W^{m,2}(\partial M\times[0,T])}\Big)\\
&\quad\leq C\Big(||\phi||_{W^{m,4}(M\times[0,T])} + ||\phi||_{W^{m,4}(\partial M\times[0,T])}\\
&\qquad+||f_1||_{W^{m,2}(M\times[0,T])} + ||f_2||_{W^{m,2}(\partial M\times[0,T])}\Big)\\
&\quad\leq C\Big(||\phi||^{\frac{1}{4}}_{W^{m,2}(M\times[0,T])}||\phi||^{\frac{3}{4}}_{W^{m+1,2}(M\times[0,T])} + ||f_1||_{W^{m,2}(M\times[0,T])}\\
&\qquad+||\phi||^{\frac{1}{2}}_{W^{m,2}(\partial M\times[0,T])}||\phi||^{\frac{1}{2}}_{W^{m+1,2}(\partial M\times[0,T])} + ||f_2||_{W^{m,2}(\partial M\times[0,T])}\Big)\\
&\quad\leq C\Big(\big(||f_1||_{W^{m,2}(M\times[0,T])} + ||f_2||_{W^{m,2}(\partial M\times[0,T])}\big)^{\frac{1}{4}}||\phi||^{\frac{3}{4}}_{W^{m+1,2}(M\times[0,T])}\\
&\qquad+\big(||f_1||_{W^{m,2}(M\times[0,T])} + ||f_2||_{W^{m,2}(\partial M\times[0,T])}\big)^{\frac{1}{2}}||\phi||^{\frac{1}{2}}_{W^{m+1,2}(\partial M\times[0,T])}\\
&\qquad+||f_1||_{W^{m,2}(M\times[0,T])} + ||f_2||_{W^{m,2}(\partial M\times[0,T])}\Big)
\end{aligned}$$

Using Lemma A.2 again, we conclude that

$$\begin{aligned}
&||\phi||_{W^{m+1,2}(M\times[0,T])} + ||\phi||_{W^{m+1,2}(\partial M\times[0,T])}\\
&\quad\leq C\Big(\big(||f_1||_{W^{m,2}(M\times[0,T])} + ||f_2||_{W^{m,2}(\partial M\times[0,T])}\big)^{\frac{1}{4}}||\phi||^{\frac{3}{4}}_{W^{m+1,2}(M\times[0,T])}\\
&\qquad+\big(||f_1||_{W^{m,2}(M\times[0,T])} + ||f_2||_{W^{m,2}(\partial M\times[0,T])}\big)^{\frac{1}{2}}||\phi||^{\frac{1}{2}}_{W^{m+1,2}(\partial M\times[0,T])}\\
&\qquad+||f_1||_{W^{m,2}(M\times[0,T])} + ||f_2||_{W^{m,2}(\partial M\times[0,T])}\Big)
\end{aligned}$$

From Young's inequality, it follows that

$$\begin{aligned} & ||\phi||_{W^{m+1,2}(M\times[0,T])} + ||\phi||_{W^{m+1,2}(\partial M\times[0,T])} \\ & \leq C\big(||f_1||_{W^{m,2}(M\times[0,T])} + ||f_2||_{W^{m,2}(\partial M\times[0,T])}\big). \end{aligned}$$

We thus complete the proof. □

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