manuscripta math. 149, 179–203 (2016)

Boris Vertman

The biharmonic heat operator on edge manifolds and non-linear fourth order equations

Received: 15 January 2015 / Accepted: 18 June 2015
Published online: 5 August 2015

Abstract. We construct the biharmonic heat kernel for a suitable self-adjoint extension of the bi-Laplacian on a manifold with incomplete edge singularities. We employ a microlocal description of the biharmonic heat kernel to establish mapping properties of the corresponding biharmonic heat operator on certain Banach spaces. This yields short time existence for a class of semi-linear equations of fourth order, including for example the Cahn–Hilliard equation. We also obtain asymptotics of the solutions near the edge singularity.

1. Introduction

In this paper we provide a microlocal construction of the biharmonic heat kernel for a self-adjoint extension of the Laplace operator on a manifold with incomplete edge singularities. Such manifolds include spaces with isolated conical singularities, more precisely open manifolds (M, g) with a decomposition $M = K \cup_N \mathscr{U}$, where K is a compact manifold with boundary N, (N, g^N) is a closed Riemannian manifold, $\mathscr{U} = (0, 1] \times N$ and

$$g \restriction \mathscr{U} = dx^2 \oplus x^2 g^N, \ x \in (0, 1].$$

While the heat kernel for the Laplace operator on edge manifolds has been studied extensively before, compare for example joint work with Mazzeo [15], Bahuaud [4], Bahuaud and Dryden [2], as well as [18]; the present work seems to be the first step towards a microlocal analysis of the bi-Laplacian, its heat kernel and associated non-linear partial differential equations of fourth order on edge spaces. In the non-singular setting cf. Lamm [12].

Our construction of the biharmonic heat kernel yields a precise understanding of its asymptotic properties, which in turn allows to study the mapping properties of the corresponding biharmonic heat operator. In the present paper we concentrate on the mapping properties with respect to certain Banach spaces that yield short

B. Vertman (✉): Mathematisches Institut, University Münster, Einsteinstr. 62, Münster, Germany.
e-mail: vertman@uni-muenster.de
URL: http://wwwmath.uni-muenster.de/42/arbeitsgruppen/ag-differentialgeometrie/prof-dr-boris-vertman/

Mathematics Subject Classification: 53C44; 58J35; 35K08

DOI: 10.1007/s00229-015-0768-0

time existence for some semi-linear equations of fourth order, in analogy to [11] and [2].

We point out that several other interesting open questions are delegated to future research and not touched upon here. These include mapping properties of the biharmonic heat kernel with respect to certain Hölder spaces and applications to short time existence of quasi-linear equations, as in [4]. Further questions are concerned with elliptic boundary value problems for the bi-Laplacian and the corresponding biharmonic heat trace asymptotics, as in [18].

Our interest in non-linear fourth order equations of parabolic type stems from recent results on e.g. the Cahn–Hilliard equation on spaces with isolated conical singularities by Roidos and Schrohe [17], as well as results on higher order geometric flows on compact manifolds, generated by powers of the Laplacian applied to the Ricci tensor or by the ambient obstruction tensor, introduced by Fefferman and Graham [10], see Bahuaud and Helliwell [3].

The Cahn–Hilliard equation was proposed by Cahn and Hilliard in [6,7] as a simple model of the phase separation process, where at a fixed temperature the two components of a binary fluid spontaneously separate and form domains that are pure in each component. Let Δ denote the Laplace Beltrami operator. Then the Cahn–Hilliard equation may be stated in the following form

$$\partial_t u + \Delta^2 u + \Delta \left(u - u^3\right)) = 0, \; u(0) = u_0.$$

Global existence for solutions to the Cahn–Hilliard equation has been established by Elliott and Songmu [9], and Caffarelli and Muler [5]. In the setup of singular manifolds however, there is still a question of asymptotics of solutions at the singular strata. This aspect has been studied by the recent work of Roidos and Schrohe [17] in the context of manifolds with isolated conical singularities, which has partly motivated the present discussion here. Using the notion of maximal regularity, they establish short time existence of solutions to the Cahn–Hilliard equation in certain weighted Mellin-Sobolev spaces which then yields regularity and asymptotics of solutions near the conical point.

In this paper we study the semi-linear parabolic equations of fourth order in the geometric setup of spaces with incomplete edges, which generalizes the notion of isolated conical singularities. Our method is different from [17] and, as emphasized above, uses the microlocal construction of the heat kernel for the bi-Laplacian.

Another recent example of higher order geometric evolution equations has been studied by Bahuaud and Helliwell [3]. The authors consider geometric flows by powers of the Laplacian applied to the Ricci tensor or generated by the ambient obstruction tensor. The ambient obstruction tensor was introduced by Charles Fefferman and Robin Graham [10] as the obstruction to a formal expansion of an asymptotically hyperbolic Einstein metric with a given conformal infinity in dimension $n + 1$. When $n = 4$, the ambient obstruction tensor is the Bach tensor.

In both instances the geometric flows admit a strongly parabolic linearization after some de Turck like adjustment by the Lie derivative of the metric with respect to a suitable vector field.

The microlocal analysis of the biharmonic heat kernel on edge spaces, presented here, allows for derivation of Schauder estimates with respect to certain Hölder spaces and ultimately leads to short time existence results for fourth order PDE's, including the geometric flows studied in [3] in the setting of singular spaces. This will be the subject of forthcoming analysis.

We also point out that our approach is not limited to squares of Laplacians on functions, but yields similar results for general powers of Hodge Laplacians on differential forms along the same lines.

2. Preliminaries and statement of the main results

In this section we introduce the notion of manifolds with edge singularities, specify a self-adjoint extension of the bi-Laplacian and state our main results.

2.1. Manifolds with incomplete edge singularities

We introduce the fundamental geometric aspects of spaces with incomplete edge singularities, as described in detail in [14], compare also [15].

Let $\overline{M}$ be a compact stratified space with its open interior M as a single top-dimensional stratum, and a single lower dimensional stratum B, which is a smooth closed manifold by definition of stratified spaces. The singular stratum B admits an open neighborhood $U \subset \overline{M}$ and a radial function $x : U \cap M \to \mathbb{R}$, such that $U \cap M$ is a smooth fibre bundle over B with fibre $\mathscr{C}(F) = (0, 1) \times F$, a finite open cone over a compact smooth manifold F. The restriction of x to each fibre defines a radial function of that cone.

The singular stratum B in $\overline{M}$ may be *resolved* and defines a compact manifold $\widetilde{M}$ with boundary ∂M, where ∂M is the total space of a fibration $\phi : \partial M \to B$ with the fibre F. The resolution process is described in detail for instance in [14]. The neighborhood U lifts to a collar neighborhood $\mathscr{U}$ of the boundary, which is again a smooth fibration over B with fibre $[0, 1) \times F$, a cylinder with the radial function x.

Definition 2.1. A Riemannian manifold (M, g) with an incomplete edge singularity at B is the open stratum of a stratified space with a single lower dimensional stratum B, and the Riemannian metric g, such that $g = g_0 + h$ over $\mathscr{U}$, where $|h|_{g_0} = O(x)$ as $x \to 0$ and

$$g_0 \restriction \mathscr{U} \backslash \partial M = dx^2 + x^2 g^F + \phi^* g^B,$$

with g^B being a Riemannian metric on B, and g^F a symmetric 2-tensor on the fibration ∂M which restricts to a Riemannian metric on each fibre F.

We set $m = \dim M$, $b = \dim B$ and $f = \dim F$. Clearly, $m = 1 + b + f$. We assume henceforth $f = \dim F \geq 1$. Otherwise M reduces to a compact manifold with boundary, where our discussion below is no longer applicable.

Similarly to other discussions in the singular edge setup, see [1,2,4] and [15], we additionally require $\phi : (\partial M, g^F + \phi^* g^B) \to (B, g^B)$ to be a Riemannian submersion. If $p \in \partial M$, then the tangent bundle $T_p \partial M$ splits into vertical and horizontal subspaces as $T_p^V \partial M \oplus T_p^H \partial M$, where $T_p^V \partial M$ is the tangent space to the fibre of

ϕ through p and $T_p^H \partial M$ is the annihilator of the subbundle $T_p^V \partial M \lrcorner g^F \subset T^* \partial M$ ($\lrcorner$ meaning contraction). The requirement for ϕ to be a Riemannian submersion is the condition that the restriction of the tensor g^F to $T_p^H \partial M$ vanishes.

We summarize the necessary assumptions on g in the following definition.

Definition 2.2. Let (M, g) be a Riemannian manifold with an incomplete edge singularity. The Riemannian metric $g = g_0 + h$ is said to be admissible if

(i) the fibration $\phi : (\partial M, g^F + \phi^* g^B) \to (B, g^B)$ is a Riemannian submersion,
(ii) the Laplace Beltrami operators $\Delta_{F,y}$ associated to $(F, g^F|_{\phi^{-1}(y)})$ for any $y \in B$ are isospectral.
(iii) the lowest non-zero eigenvalue $\lambda_0 > 0$ of the Laplace Beltrami operators $\Delta_{F,y}$ satisfies $\lambda_0 > \dim F$.
(iv) h vanishes to second order at $x = 0$, i.e. $|h|_{g_0} = O(x^2)$ as $x \to 0$.

The reasons behind the feasibility assumptions are as follows. Let $y = (y_1, ..., y_b)$ be the local coordinates on B lifted to ∂M and then extended inwards. Let $z = (z_1, ..., z_f)$ restrict to local coordinates on F along each fibre. Then (x, y, z) are the local coordinates on M near the boundary. Consider the Laplace Beltrami operator Δ associated to (M, g) and its normal operator $N(x^2\Delta)_{y_0}$, defined as the limiting operator with respect to the local family of dilatations $(x, y, z) \to (\lambda x, \lambda(y - y_0), z)$ and acting on functions on the model edge $\mathbb{R}^+_s \times F \times \mathbb{R}^b_u$. Under the first admissibility assumption, $N(x^2\Delta)_{y_0}$ is naturally identified with s^2 times the Laplace Beltrami operator on the model edge $\mathbb{R}^+_s \times F \times \mathbb{R}^b_u$ with incomplete edge metric $g_{\mathrm{ie}} = ds^2 + s^2 g^F + |du|^2$. This is key for constructing the initial crude approximation of the biharmonic heat kernel.

The second condition on isospectrality is severe, but has to be imposed to ensure polyhomogeneity of the associated heat kernels when lifted to the corresponding parabolic blowup space. More precisely we actually only need that the eigenvalues of the Laplacians on fibres are constant in a fixed range [0, 1], though we still make the stronger assumption for a clear and convenient representation.

The reasons behind the last two admissibility assumptions are of technical rather than geometric nature and somewhat less straightforward to explain. However we point out that condition $\lambda_0 > \dim F$ is easily satisfied by a rescaling of g^F. Condition $|h|_{g_0} = O(x^2)$ in particular holds for *even* metrics which depend on x^2 instead of x. Altogether the admissibility assumptions yield precise information on the heat kernel expansion, which is then used in Proposition 3.3.

2.2. Edge vector fields and Banach spaces

An important ingredient in the analysis of singular edge spaces is the vector space $\mathcal{V}_{\mathrm{e}}$ of *edge* vector fields smooth in the interior of $\widetilde{M}$ and tangent at the boundary ∂M to the fibres of the fibration. This space $\mathcal{V}_{\mathrm{e}}$ is closed under the ordinary Lie bracket of vector fields, hence defines a Lie algebra. Its description in local coordinates is as follows. Consider the local coordinates (x, y, z) on $\overline{M}$ near the boundary. Then the edge vector fields $\mathcal{V}_{\mathrm{e}}$ are locally generated by

$$\left\{ x\partial_x, x\partial_{y_1}, \ldots, x\partial_{y_b}, \partial_{z_1}, \ldots, \partial_{z_f} \right\}.$$

We may now define the Banach space of continuous sections $\mathscr{C}^0_{\mathrm{ie}}(M)$, continuous on $\widetilde{M}$ up to the boundary and fibrewise constant at $x = 0$. This is precisely the space of continuous sections with respect to the topology on M induced by the Riemannian metric g. The standard space of $2k$-times continuously differentiable functions in the open interior M is denoted by $C^{2k}(M)$. Banach spaces of higher order are now defined as follows, compare [2].

Definition 2.3. Let (M, g) be a Riemannian manifold with an incomplete edge metric. Let $\mathcal{D}$ denote a subspace generated by a finite collection $\widehat{\mathcal{D}}$ of derivatives in $\{\Delta, x^{-1}\mathcal{V}_{\mathrm{e}}^2, x^{-1}\mathcal{V}_{\mathrm{e}}, \mathcal{V}_{\mathrm{e}}\}$, which will be specified later. Then for each $k \in \mathbb{N}$ we define

$$\mathscr{C}^{2\mathrm{k}}_{\mathrm{ie}}(\mathrm{M}, \mathcal{D}) := \left\{u \in C^{2k}(M) \cap \mathscr{C}^0_{\mathrm{ie}}(M) \mid X \circ \Delta^j u \in \mathscr{C}^0_{\mathrm{ie}}(M), X \in \mathcal{D},\ j < k\right\},$$

with the norm $\|u\|_{2k} := \|u\|_\infty + \sum_{j=0}^{k} \sum_{X \in \widehat{\mathcal{D}}} \|X \circ \Delta^j u\|_\infty.$

2.3. *Self-adjoint extension of the bi-Laplacian*

Let Δ denote the Laplace Beltrami operator acting on functions on an incomplete edge space (M, g) with an admissible incomplete edge metric g. Consider the space of square-integrable forms $L^2(M, g)$, with respect to g. The *maximal* and *minimal* closed extensions of Δ are defined by the domains

$$\begin{aligned}\mathcal{D}_{\max}(\Delta) &:= \{u \in L^2(M, g) \mid \Delta u \in L^2(M, g)\},\\ \mathcal{D}_{\min}(\Delta) &:= \{u \in \mathcal{D}_{\max}(\Delta) \mid \exists\, u_j \in C_0^\infty(M) \text{ such that}\\ &\qquad u_j \to u \text{ and } \Delta u_j \to \Delta u \text{ both in } L^2(M, g)\}.\end{aligned} \tag{2.1}$$

where $\Delta u \in L^2$ is a priori understood in the distributional sense. Under the unitary rescaling transformation in the singular edge neighborhood

$$\Phi : L^2(\mathscr{U}, \mathrm{dvol}(g)) \to L^2(\mathscr{U}, x^{-f}\mathrm{dvol}(g)),\ u \mapsto x^{f/2}u, \tag{2.2}$$

the rescaled Laplacian $\Delta^\Phi = \Phi \circ \Delta \circ \Phi^{-1}$ is a perturbation of

$$\Delta_0^\Phi = -\frac{\partial^2}{\partial x^2} + \frac{1}{x^2}\left(\Delta_{F,y} + \left(\frac{f-1}{2}\right)^2 - \frac{1}{4}\right),$$

with higher order terms coming from the curvature of the Riemannian submersion ϕ and the second fundamental forms of the fibres F.

The following lemma is a straightforward reformulation of [15, Lemma 2.2] for the simpler case of Laplace Beltrami operators.

Lemma 2.4. ([15]) *Let (M, g) be an incomplete edge space with an admissible edge metric. Consider the increasing sequence of eigenvalues $(\sigma_j)_{j\in\mathbb{N}}$ of $\Delta_{F,y}$, counted with their multiplicities, and put $\nu_j^2 := \sigma_j + (f-1)^2/4$. The associated* *indicial roots are given by* $\gamma_j^\pm = \pm\nu_j + 1/2$. *Let* $p \in \mathbb{N}$ *be the largest index such that* $\nu_p \in [0, 1)$. *Then any* $u \in \mathcal{D}_{\max}(\Delta)$ *admits a weak expansion as* $x \to 0$, *in the sense that for any test function* $\chi \in C^\infty(B)$ *there is an expansion of the pairing*

$$\int_B \Phi u(x, y, z)\chi(y)\, dy \sim \sum_{j=1}^{p} \Big(\psi_j^+(x)c_j^+[u, \chi](z) + \psi_j^-(x)c_j^-[u, \chi](z)\Big) \\ + \tilde{u},\ x \to 0,$$

where the higher order term $\tilde{u} = O(x^{3/2})$ *as* $x \to 0$, *and the coefficients* $c_j^\pm[u, \chi]$ *are constant for* $j = 1, \ldots, \dim H^0(F)$. *Moreover*

$$\begin{aligned}
&\psi_j^+(x, z; y) = x^{\gamma_j^+}, \text{ and}\\
&\psi_j^-(x, z; y) = x^{1/2}(\log x), \quad \text{if } \nu_j = 0,\\
&\psi_j^-(x, z; y) = x^{\gamma_j^-}(1 + a_j x) \quad \text{if } \nu_j > 0,
\end{aligned}$$

with $a_j \in \mathbb{R}$ *uniquely determined by* Δ.

The Friedrichs self-adjoint extension of Δ has been identified in [15] as

$$\mathcal{D}(\Delta_{\mathscr{F}}) = \{u \in \mathcal{D}_{\max}(\Delta) \mid \forall_{j=1,..,p} : c_j^-[u] = 0\}. \tag{2.3}$$

Note that the sequence of eigenvalues $(\sigma_j)_{j=1}^p$ of $\Delta_{F,y}$ starts with $\sigma_j = 0$ for $j = 1, .., \dim H^0(F)$. The corresponding indicial roots compute to $\gamma_j = 1/2 \pm (f-1)/2$ and the coefficients are constant in $z \in F$, being simply the harmonic functions of fibres F. Consequently, the Friedrichs domain contains precisely those elements in the maximal domain whose leading term in the weak expansion as $x \to 0$ is given by x^0 with fibrewise constant coefficients. In particular

$$\mathcal{D}_{\max}(\Delta) \cap \mathscr{C}^0_{\text{ie}}(M) \subset \mathcal{D}(\Delta_{\mathscr{F}}). \tag{2.4}$$

We fix a self-adjoint extension of the bi-Laplacian as the square of $\Delta_{\mathscr{F}}$

$$\mathcal{D}(\Delta^2_{\mathscr{F}}) = \{u \in \Delta_{\mathscr{F}} \mid \Delta u \in \Delta_{\mathscr{F}}\}. \tag{2.5}$$

2.4. The biharmonic heat space blowup

Consider $\Delta^2_{\mathscr{F}}$ and the corresponding heat operator $e^{-t\Delta^2_{\mathscr{F}}}$. Let H be the biharmonic heat kernel, the Schwartz kernel of $e^{-t\Delta^2_{\mathscr{F}}}$. H is a priori a function on $M^2_h = \mathbb{R}^+ \times \widetilde{M}^2$. Let (x, y, z) and $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ be the coordinates on the two copies of M near the edge. Then the local coordinates near the corner in M^2_h are given by $(t, (x, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z}))$. The kernel $H(t, (x, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z}))$ has a non-uniform behaviour at the submanifolds

$$\begin{aligned}
A &= \left\{(t = 0, (0, y, z), (0, \widetilde{y}, \widetilde{z})) \in \mathbb{R}^+ \times \partial M^2 \mid y = \widetilde{y}\right\},\\
D &= \left\{(t = 0, (x, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z})) \in \mathbb{R}^+ \times \widetilde{M}^2 \mid x = \widetilde{x},\ y = \widetilde{y},\ z = \widetilde{z}\right\}.
\end{aligned}$$

Exactly as in the case of the Hodge Laplacian on edges, see [15], we introduce an appropriate blowup of the heat space M_h^2, such that the corresponding heat kernel lifts to a polyhomogeneous distribution in the sense of the definition below. This procedure has been introduced by Melrose in [16]. For self-containment of the paper we repeat the definition of polyhomogeneity as well as the blowup process here.

Definition 2.5. Let $\mathfrak{W}$ be a manifold with corners, with all boundary faces embedded, and $\{(H_i, \rho_i)\}_{i=1}^N$ an enumeration of its boundaries and the corresponding defining functions. For any multi-index $b = (b_1, \ldots, b_N) \in \mathbb{C}^N$ we write $\rho^b = \rho_1^{b_1} \ldots \rho_N^{b_N}$. Denote by $\mathcal{V}_b(\mathfrak{W})$ the space of smooth vector fields on $\mathfrak{W}$ which lie tangent to all boundary faces. A distribution ω on $\mathfrak{W}$ is said to be conormal, if $\omega \in \rho^b L^\infty(\mathfrak{W})$ for some $b \in \mathbb{C}^N$ and $V_1 \ldots V_\ell \omega \in \rho^b L^\infty(\mathfrak{W})$ for all $V_j \in \mathcal{V}_b(\mathfrak{W})$ and for every $\ell \geq 0$. An index set $E_i = \{(\gamma, p)\} \subset \mathbb{C} \times \mathbb{N}$ satisfies the following hypotheses:

(i) $\mathrm{Re}(\gamma)$ accumulates only at plus infinity,
(ii) For each γ there is $P_\gamma \in \mathbb{N}_0$, such that $(\gamma, p) \in E_i$ for every $0 \leq p \leq P_\gamma$,
(iii) If $(\gamma, p) \in E_i$, then $(\gamma + j, p') \in E_i$ for all $j \in \mathbb{N}$ and $0 \leq p' \leq p$.

An index family $E = (E_1, \ldots, E_N)$ is an N-tuple of index sets. Finally, we say that a conormal distribution ω is polyhomogeneous on $\mathfrak{W}$ with index family E, we write $\omega \in \mathscr{A}_{\mathrm{phg}}^E(\mathfrak{W})$, if ω is conormal and if in addition, near each H_i,

$$\omega \sim \sum_{(\gamma,p)\in E_i} a_{\gamma,p} \rho_i^\gamma (\log \rho_i)^p, \text{ as } \rho_i \to 0,$$

with coefficients $a_{\gamma,p}$ conormal on H_i, polyhomogeneous with index E_j at any $H_i \cap H_j$.

Our analysis of the biharmonic heat kernel will start with a discussion of an explicitly solvable model situation, which leads to a homogeneity property (3.2). That property contains the information how precisely the submanifolds $A, D \subset M_h^2$ need to be blown up such that the heat kernel H becomes polyhomogeneous. To get the correct blowup of M_h^2 we first bi-parabolically ($t^{1/4}$ is viewed as a coordinate function) blow up the submanifold

$$A = \left\{(t, (0, y, z), (0, \widetilde{y}, \widetilde{z})) \in \mathbb{R}^+ \times \partial M^2 : t = 0, y = \widetilde{y}\right\} \subset M_h^2.$$

The resulting heat-space $[M_h^2, A]$ is defined as the union of $M_h^2 \backslash A$ with the interior spherical normal bundle of A in M_h^2. The blowup $[M_h^2, A]$ is endowed with the unique minimal differential structure with respect to which smooth functions in the interior of M_h^2 and polar coordinates on M_h^2 around A are smooth. As in [15], this blowup introduces four new boundary hypersurfaces; we denote these by ff (the front face), rf (the right face), lf (the left face) and tf (the temporal face).

The actual heat-space blowup $\mathscr{M}_h^2$ is obtained by a bi-parabolic blowup of $[M_h^2, A]$ along the diagonal D, lifted to a submanifold of $[M_h^2, A]$. The resulting blowup $\mathscr{M}_h^2$ is defined as before by cutting out the submanifold and replacing it

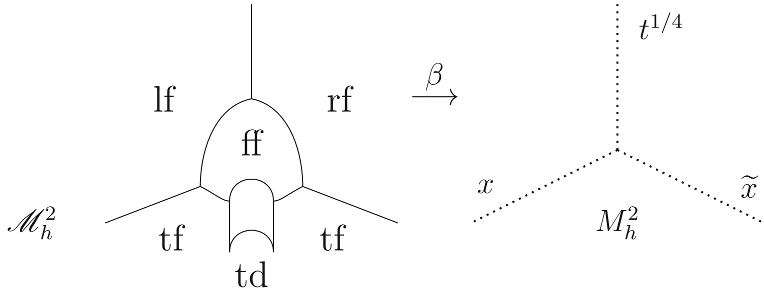

Fig. 1. The biharmonic heat-space blowup $\mathscr{M}_h^2$

with its spherical normal bundle. It is a manifold with boundaries and corners, visualized in Figure below.

The projective coordinates on $\mathscr{M}_h^2$ are then given as follows. Near the top corner of the front face ff, the projective coordinates are given by

$$\rho = t^{1/4},\ \xi = \frac{x}{\rho},\ \widetilde{\xi} = \frac{\widetilde{x}}{\rho},\ u = \frac{y - \widetilde{y}}{\rho}, \quad z,\ \widetilde{y},\ \widetilde{z}, \tag{2.6}$$

where in these coordinates $\rho, \xi, \widetilde{\xi}$ are the defining functions of the boundary faces ff, rf and lf respectively. For the bottom corner of the front face near the right hand side projective coordinates are given by

$$\tau = \frac{t}{\widetilde{x}^4},\ s = \frac{x}{\widetilde{x}},\ u = \frac{y - \widetilde{y}}{\widetilde{x}}, \quad z,\ \widetilde{x},\ \widetilde{y},\ \widetilde{z}, \tag{2.7}$$

where in these coordinates $\tau, s, \widetilde{x}$ are the defining functions of tf, rf and ff respectively. For the bottom corner of the front face near the left hand side projective coordinates are obtained by interchanging the roles of x and $\widetilde{x}$. Projective coordinates on $\mathscr{M}_h^2$ near temporal diagonal are given by

$$\eta = \frac{t^{1/4}}{\widetilde{x}},\ S = \frac{(x - \widetilde{x})}{t^{1/4}},\ U = \frac{y - \widetilde{y}}{t^{1/4}},\ Z = \frac{\widetilde{x}(z - \widetilde{z})}{t^{1/4}}, \quad \widetilde{x},\ \widetilde{y},\ \widetilde{z}. \tag{2.8}$$

In these coordinates tf is the face in the limit $|(S, U, Z)| \to \infty$, ff and td are defined by $\widetilde{x}, \eta$, respectively. The blowdown map $\beta : \mathscr{M}_h^2 \to M_h^2$ is in local coordinates simply the coordinate change back to $(t, (x, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z}))$.

2.5. Statement of the main results

Our first main result is concerned with the asymptotic properties of the biharmonic heat kernel as a polyhomogeneous function on the biharmonic heat space blowup.

Theorem 2.6. *Let (M^m, g) be an incomplete edge space with an admissible edge metric g, and let $\Delta_{\mathscr{F}}$ denote the Friedrichs extension of the corresponding Laplace–Beltrami operator. Let H be the Schwartz kernel of the heat operator $e^{-t\Delta^2_{\mathscr{F}}}$ associated to the bi-Laplacian $\Delta^2_{\mathscr{F}}$. Then the lift $\beta^* H$ is polyhomogeneous on $\mathscr{M}_h^2$ of* *order* $(-\dim M)$ *at* ff *and* td, *vanishing to infinite order at* tf, *and with the index set at* rf *and* lf *given by* $E + \mathbb{N}_0$ *where*

$$E = \left\{ \gamma \geq 0 \mid \gamma = -\frac{(f-1)}{2} + \sqrt{\frac{(f-1)^2}{4} + \sigma^2}, \quad \sigma^2 \in \operatorname{Spec} \Delta_{F,y} \right\}.$$

More precisely, if s *denotes the boundary defining function of* rf, *we obtain*

$$\beta^* H \sim \sum_{\gamma \in E} \left(\sum_{j=0}^{\infty} s^{\gamma+2j} a_{\gamma,j}(\beta^* H) + \sum_{j=0}^{\infty} s^{\gamma+2+j} a'_{\gamma,j}(\beta^* H) \right) \quad \text{as } s \to 0,$$

where the coefficients $a_{\gamma,j}(H)$ *are of order* $(-m)$ *at the front face and lie in their corresponding* $\Delta_{F,y}$ *eigenspaces. The higher coefficients* $a'_{\gamma,j}(\beta^* H)$ *are of order* $(-m+1)$ *at ff.*

We employ this microlocal heat kernel description to establish our next main theorem on the mapping properties of the corresponding biharmonic heat operator.

Theorem 2.7. *Let* (M^m, g) *be an incomplete edge space with an admissible edge metric* g *and* Δ *the corresponding Laplace Beltrami operator. Put* $\mathcal{D}_0 = \langle \Delta \rangle$ *and* $\mathcal{D} = \langle \Delta, x^{-1}\,\mathcal{V}_{\mathrm{e}}^2, x^{-1}\mathcal{V}'_e, \mathcal{V}_{\mathrm{e}} \rangle$, *where* $\mathcal{V}'_e \subset \mathcal{V}_{\mathrm{e}}$ *consists locally of all edge vector fields where* $x\partial_y$ *is weighted with functions that are fibrewise constant. Then the biharmonic heat operator* $e^{-t\Delta^2_{\mathscr{F}}}$ *is a bounded map between the (weighted) Banach spaces*

$$e^{-t\Delta^2_{\mathscr{F}}} : \mathscr{C}^{2k}_{\mathrm{ie}}(M, \mathcal{D}_0) \to t^{-1/4}\mathscr{C}^{2(k+1)}_{\mathrm{ie}}(M, \mathcal{D}).$$

We should point out that in the theorem above the target Banach space $\mathscr{C}^{2(k+1)}_{\mathrm{ie}}(M, \mathcal{D})$ is defined with respect to the set $\mathcal{D}$ that includes a large variety of higher order derivatives. In fact, from the perspective of the presented proof, this is the largest possible variety of derivatives with respect to which boundedness of the biharmonic heat operator $e^{-t\Delta^2_{\mathscr{F}}}$ persists. On the other hand, the initial space $\mathscr{C}^{2k}_{\mathrm{ie}}(M, \mathcal{D}_0)$ poses significantly less regularity assumptions, since it is defined with respect to a very restricted set of derivatives $\mathcal{D}_0 = \langle \Delta \rangle$. In that respect, the biharmonic heat operator indeed improves regularity.

An important aspect of the statement is that regularity is not defined with respect to $x^{-1}\,\mathcal{V}_{\mathrm{e}}$ but rather $x^{-1}\mathcal{V}'_e$, i.e. we require the generators to be weighted with functions that are constant on fibres F when restricted to $x = 0$. This is due to the fact that we consider spaces of continuously differentiable functions with the continuity defined with respect to the Riemannian metric g. Such continuous functions are constant on fibres at ∂M. For this aspect also note the Remark 4.3.

Our final result is concerned with local existence of solutions to certain semi-linear parabolic equations of fourth order.

Theorem 2.8. *Let* (M, g) *be an incomplete edge space with an admissible edge metric* g. *Put* $\mathcal{D}' = \langle \Delta, \mathcal{V}_{\mathrm{e}}^2, \mathcal{V}_{\mathrm{e}} \rangle$ *and* $\mathcal{D} = \langle \Delta, x^{-1}\,\mathcal{V}_{\mathrm{e}}^2, x^{-1}\mathcal{V}'_e, \mathcal{V}_{\mathrm{e}} \rangle$, *where* $\mathcal{V}'_e \subset \mathcal{V}_{\mathrm{e}}$

consists locally of linear combinations of $\{x\partial_x, x\partial_y, \partial_z\}$*, where* $x\partial_y$ *is weighted with functions that are fibrewise constant. Assume* $Q : \mathscr{C}^{2(k+1)}_{\mathrm{ie}}(M, \mathcal{D}') \to \mathscr{C}^{2k}_{\mathrm{ie}}(M, \mathcal{D}')$ *is locally Lipschitz. Then the semilinear equation*

$$\partial_t u + \Delta^2 u = Q(u),\ u(0) = u_0 \in \mathscr{C}^{2(k+1)}_{\mathrm{ie}}(M, \mathcal{D}')$$

has a unique solution $u \in C([0,T], \mathscr{C}^{2(k+1)}_{\mathrm{ie}}(M, \mathcal{D})) \cap C^\infty((0,T] \times M)$*, for some* $T > 0$*, where* T *may be estimated from below in terms of* $||u_0||_{2(k+1)}$.

As an application we arrive at a statement on existence and regularity of solutions to the Cahn–Hilliard equation.

Corollary 2.9. *Let* (M, g) *be an incomplete edge space with an admissible edge metric* g*. Put* $\mathcal{D}' = \langle \Delta, \mathcal{V}_{\mathrm{e}}^2, \mathcal{V}_{\mathrm{e}} \rangle$ *and* $\mathcal{D} = \langle \Delta, x^{-1}\mathcal{V}_{\mathrm{e}}^2, x^{-1}\mathcal{V}'_e, \mathcal{V}_{\mathrm{e}} \rangle$*, where* $\mathcal{V}'_e \subset \mathcal{V}_{\mathrm{e}}$ *consists locally of linear combinations of* $\{x\partial_x, x\partial_y, \partial_z\}$*, where* $x\partial_y$ *is weighted with functions that are fibrewise constant. Then the Cahn–Hilliard equation*

$$\partial_t u + \Delta^2 u + \Delta(u - u^3) = 0, \quad u(0) = u_0 \in \mathscr{C}^{2(k+1)}_{\mathrm{ie}}(M, \mathcal{D}')$$

has a unique solution $u \in C([0,T], \mathscr{C}^{2\mathrm{k}+2}_{\mathrm{ie}}(\mathrm{M}, \mathcal{D})) \cap C^\infty((0,T] \times M)$*, for some* $T > 0$.

It should be noted that in correspondence with [17] our approach leads to an explicit identification of the asymptotics of the Cahn–Hilliard solution at $x = 0$. Indeed, $u \in \mathscr{C}^{2\mathrm{k}+2}_{\mathrm{ie}}(\mathrm{M}, \mathcal{D}) \subset \mathscr{D}(\Delta^{k+1}_{\mathscr{F}})$, which yields a partial asymptotics of u to higher and higher order, depending on $k \in \mathbb{N}$, by an iterative application of Lemma 2.4 for k steps.

3. Microlocal heat kernel construction

3.1. Biharmonic heat kernel on a model edge

In this section we construct the heat kernel for $\Delta^2_{\mathscr{F}}$ explicitly. We begin with studying the homogeneity properties of the heat kernel for the bi-Laplacian in the model case of an exact edge $(\mathcal{E} = \mathbb{R}^b \times \mathscr{C}(F), dy^2 + g)$ where $(\mathscr{C}(F) = (0,\infty) \times F, g = ds^2 + s^2 g^F)$ is an exact unbounded cone over a closed Riemannian manifold (F, g^F). The Laplacian $\Delta_{\mathcal{E}}$ on the exact edge is then a sum of the Laplacian on $(\mathscr{C}(F), g)$ and the Euclidean Laplacian on $\mathbb{R}^b$. Consider the scaling operation $(\lambda > 0)$

$$\begin{aligned}&\Psi_\lambda : C^\infty(\mathbb{R}^+ \times \mathcal{E}^2) \to C^\infty(\mathbb{R}^+ \times \mathcal{E}^2),\\ &(\Psi_\lambda u)(t, (s, y, z), (\widetilde{s}, \widetilde{y}, \widetilde{z})) = u(\lambda^4 t, (\lambda s, \lambda(y - \widetilde{y}), z), (\lambda\widetilde{s}, \lambda\widetilde{y}, \widetilde{z})).\end{aligned}$$

Under the scaling operation we find

$$(\partial_t + \Delta^2_{\mathcal{E}})\Psi_\lambda u = \lambda^4 \Psi_\lambda(\partial_t + \Delta^2_{\mathcal{E}})u. \tag{3.1}$$

Consequently, given the heat kernel $H_{\mathcal{E}}$ for the Friedrichs extension of $\Delta^2_{\mathcal{E}}$ (or at that stage any other self-adjoint extension), any multiple of $\Psi_\lambda H_{\mathcal{E}}$ still solves the heat equation and also maps into the domain of $\Delta^2_{\mathcal{E}}$. For the initial condition we obtain substituting $\widetilde{Y} = \lambda\widetilde{y}$, $\widetilde{S} = \lambda\widetilde{s}$

$$\begin{aligned}
&\lim_{t\to 0}\int_{\mathcal{E}} (\Psi_\lambda H_{\mathcal{E}})(t,\, s,\, y,\, z,\, \widetilde{s},\, \widetilde{y},\, \widetilde{z}) u(\widetilde{s},\, \widetilde{y},\, \widetilde{z})\, \widetilde{s}^{f}\, d\widetilde{s}\, d\widetilde{y}\, d\widetilde{z} \\
&= \lim_{t\to 0} \lambda^{-1-b-f} \int_{\mathcal{E}} H_{\mathcal{E}}(\lambda^4 t,\, \lambda s, \lambda y - \widetilde{Y}, z, \widetilde{S}, \widetilde{Y},\, \widetilde{z})\, u(\widetilde{S}/\lambda,\, \widetilde{Y}/\lambda,\, \widetilde{z}) \widetilde{S}^f\, d\widetilde{S}\, d\widetilde{Y} d\widetilde{z} \\
&= \lambda^{-1-b-f} u(\lambda s/\lambda,\, \lambda y/\lambda,\, z) = \lambda^{-1-b-f} u(s,\, y,\, z).
\end{aligned}$$

By uniqueness of the heat kernel we obtain

$$\Psi_\lambda H_{\mathcal{E}} = \lambda^{1-b-f} H_{\mathcal{E}}. \tag{3.2}$$

In addition to the homogeneity properties of $H_{\mathcal{E}}$, we also require a full asymptotic expansion of the biharmonic heat kernel as $(s, \widetilde{s}) \to 0$. We accomplish this by establishing an explicit integral representation of $H_{\mathcal{E}}$. Under the unitary rescaling (2.2) and a spectral decomposition of $L^2(F, g^F)$ into σ^2-eigenspaces of Δ_F, we may write for the rescaled model edge Laplacian

$$\begin{aligned}
\Delta^{\Phi}_{\mathcal{E}} &= -\partial_s^2 + s^{-2}\left(\Delta_F + \left(\frac{f-1}{2}\right)^2 - \frac{1}{4}\right) + \Delta_{\mathbb{R}^b} \\
&= \bigoplus_{\sigma} -\partial_s^2 + s^{-2}\left(\sigma^2 + \left(\frac{f-1}{2}\right)^2 - \frac{1}{4}\right) + \Delta_{\mathbb{R}^b} =: \bigoplus_{\sigma} l_{\nu(\sigma)} + \Delta_{\mathbb{R}^b},
\end{aligned}$$

where $\nu(\sigma) := \sqrt{\sigma^2 + (f-1)^2/4}$ and $l_{\nu(\sigma)}$ is defined on $C_0^\infty(0,\infty)$. The Friedrichs extension of $\Delta_{\mathcal{E}}$ is compatible with the decomposition, compare a similar discussion in ([19], Proposition 4.9). As a special case of Lemma 2.4, l_ν has unique self-adjoint extension L_ν in $L^2(\mathbb{R}^+)$ for $\nu \geq 1$, and in case of $\nu \in [0, 1)$, solutions $u \in \mathscr{D}(l_{\nu,\max})$ admit a partial asymptotic expansion as $s \to 0$

$$u(s) = \widetilde{u} + c^+[u]\, s^{\nu+1/2} + c^-[u] \begin{cases} s^{-\nu+1/2}, & \nu \in (0, 1), \\ \sqrt{s}\log(s), & \nu = 0, \end{cases} \qquad \widetilde{u} \in \mathscr{D}(l_{\nu,\min}).$$

Then the Friedrichs extension L_ν of l_ν is defined, similar to (2.3), by requiring $c^-[u] = 0$, and moreover, identifying $\Delta_{\mathbb{R}^b}$ with its unique self-adjoint extension in $L^2(\mathbb{R}^b)$, we may write

$$\Delta^{\Phi}_{\mathcal{E},\mathscr{F}} = \bigoplus_{\sigma} L_{\nu(\sigma)} + \Delta_{\mathbb{R}^b}. \tag{3.3}$$

Consequently, it suffices to construct the biharmonic heat kernel for $L_\nu + \Delta_{\mathbb{R}^b}$ in $L^2(\mathbb{R}^+ \times \mathbb{R}^b)$. Denote by J_ν the ν-th Bessel function of first kind and consider the Hankel transform of order $\nu \geq 0$

$$(\mathscr{H}_\nu u)(s) := \int_0^\infty \sqrt{ss'}\, J_\nu(ss') u(s') ds',\ u \in C_0^\infty(0,\infty). \tag{3.4}$$

By ([8], Chapter III) and also by ([13], Proposition 2.3.4), the Hankel transform extends to a self-adjoint isometry on $L^2(\mathbb{R}^+)$. We denote by

$$(\mathscr{F}u)(\xi) := (2\pi)^{-b/2} \int_{\mathbb{R}^b} u(y) e^{-iy\cdot\xi} dy, \quad u \in C_0^\infty(\mathbb{R}^b),$$

the Fourier transform on $\mathbb{R}^b$, which extends to an isometric automorphism of $L^2(\mathbb{R}^b)$. Consequently, $\mathscr{G}_\nu := \mathscr{H}_\nu \circ \mathscr{F}$ defines an isometric automorphism of $L^2(\mathbb{R}^+ \times \mathbb{R}^b)$ such that $\mathscr{G}_\nu^{-1} = \mathscr{H}_\nu \circ \mathscr{F}^{-1}$. Applying ([13], Proposition 2.3.5), we arrive at the following

Proposition 3.1. *The isometric automorphism $\mathscr{G}_\nu$ diagonalizes $L_\nu + \Delta_{\mathbb{R}^b}$. More precisely,*

$$\mathscr{D}(L_\nu + \Delta_{\mathbb{R}^b}) = \left\{ u \in L^2(\mathbb{R}^+ \times \mathbb{R}^b) \mid (S^2 + |\Xi|^2)\, \mathscr{G}_\nu u \in L^2(\mathbb{R}^+ \times \mathbb{R}^b) \right\},$$
$$\mathscr{G}_\nu\, (L_\nu + \Delta_{\mathbb{R}^b})\, \mathscr{G}_\nu^{-1} = S^2 + |\Xi|^2,$$

where X, Ξ denote multiplication operators by $x \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^b$, respectively.

Similary, the isometry $\mathscr{G}_\nu$ diagonalizes the squared operator $(L_\nu + \Delta_{\mathbb{R}^b})^2$, identifying its action with $(S^2 + |\Xi|^2)^2$. Consequently we may express the biharmonic heat kernel of $(L_\nu + \Delta_{\mathbb{R}^b})^2$ as an integral in terms of Bessel functions. For $u \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^b)$ we find

$$\begin{aligned}
&\left(e^{-t(L_\nu + \Delta_{\mathbb{R}^b})^2} u\right)(s, y) = \left(\mathscr{G}_\nu e^{-t(S^2 + |\Xi|^2)^2} \mathscr{G}_\nu^{-1} u\right)(s, y) \\
&\quad = (2\pi)^{-b/2} \int_{\mathbb{R}^b} \int_0^\infty \left(\int_{\mathbb{R}^b} \int_0^\infty e^{i(y - \widetilde{y})\xi} \sqrt{s\widetilde{s}}\, J_\nu(s\rho) J_\nu(\widetilde{s}\rho)\, \rho\, e^{-t(\rho^2 + |\xi|^2)^2} d\rho\, d\xi \right) \\
&\qquad \times u(\widetilde{s}, \widetilde{y})\, d\widetilde{s}\, d\widetilde{y}.
\end{aligned}$$

Denote by ϕ_σ the normalized σ^2-eigenfunction of Δ_F, where we count the eigenvalues $\sigma^2 \in \mathrm{Spec}(\Delta_F)$ with their multiplicities. Then, as a consequence of (3.3), we finally obtain for the Φ-rescaled biharmonic heat kernel on a model edge

$$\begin{aligned}
H_{\mathcal{E}}^\Phi &= (2\pi)^{-b/2} \bigoplus_\sigma \int_{\mathbb{R}^b} \int_0^\infty e^{i(y - \widetilde{y})\xi} \sqrt{s\widetilde{s}}\, J_{\nu(\sigma)}(s\rho) J_{\nu(\sigma)}(\widetilde{s}\rho) \rho e^{-t(\rho^2 + |\xi|^2)^2} d\rho\, d\xi \\
&\quad \cdot \phi_\sigma(z) \otimes \phi_\sigma(\widetilde{z}).
\end{aligned}$$

The ν-th Bessel function of first kind admits an asymptotic expansion for small arguments $J_\nu(\zeta) \sim \sum_{j=0}^\infty a_j \zeta^{\nu + 2j}$, as $\zeta \to 0$. This yields an asymptotic expansion of $H_{\mathcal{E}}^\Phi$ as $(s, \widetilde{s}) \to 0$ and consequently, rescaling back, we obtain as $s \to 0$

$$H_{\mathcal{E}}(t, s, y, z, \widetilde{s}, \widetilde{y}, \widetilde{z}) \sim \sum_\gamma a_{\nu,j}(t, \widetilde{s}, y, \widetilde{y}, z, \widetilde{z}) s^{\gamma + 2j}, \tag{3.5}$$

where the summation is over all $\gamma = -(f-1)/2 + \sqrt{\sigma^2 + (f-1)^2/4}$ with $\sigma^2 \in \mathrm{Spec}\Delta_F$, counted with multiplicity, and each coefficient $a_{\nu,j}$ lies in the corresponding σ^2-eigenspace. We summarize the properties of $H_{\mathcal{E}}$, established above, in a single proposition for later reference.

Proposition 3.2. *Consider the model edge* ($\mathcal{E} = \mathbb{R}^b \times \mathscr{C}(F), dy^2 + g$), *where* ($\mathscr{C}(F) = (0, \infty) \times F, g = ds^2 + s^2 g^F$) *is an exact unbounded cone over a closed Riemannian manifold* (F^f, g^F). *Fix the Friedrichs self-adjoint extension of the associated Laplace Beltrami operator* $\Delta_{\mathcal{E}}$. *Then the biharmonic heat kernel* $H_{\mathcal{E}}$ *of* $\Delta^2_{\mathcal{E}}$ *is homogeneous of order* $(-1 - b - f)$ *under the scaling operation* $(\lambda > 0)$

$$\Psi_\lambda : C^\infty(\mathbb{R}^+ \times \mathcal{E}^2) \to C^\infty(\mathbb{R}^+ \times \mathcal{E}^2),$$
$$(\Psi_\lambda u)(t, (s, y, z), (\widetilde{s}, \widetilde{y}, \widetilde{z})) = u(\lambda^4 t, (\lambda s, \lambda(y - \widetilde{y}), z), (\lambda\widetilde{s}, \lambda\widetilde{y}, \widetilde{z})).$$

Moreover, $H_{\mathcal{E}}$ *admits an asymptotic expansion as* $(s, \widetilde{s}) \to 0$ *with the index set given by* $E + 2\mathbb{N}_0$, *where*

$$E = \left\{ \gamma \geq 0 \mid \gamma = -\frac{(f-1)}{2} + \sqrt{\frac{(f-1)^2}{4} + \sigma^2},\ \sigma^2 \in \operatorname{Spec} \Delta_F \right\},$$

uniformly in other variables and with coefficients taking value in the corresponding σ^2*-eigenspace.*

3.2. Construction of the biharmonic heat kernel

We can now proceed from the analysis of the heat kernel on the model edge to the construction of the heat kernel H for the bi-Laplacian on a space (M, g) with an incomplete admissible edge metric. The heat kernel construction here follows ad verbatim the discussion in [15] for the edge Laplacian, with the only difference that for the bi-Laplacian now rather $t^{1/4}$ instead of $\sqrt{t}$ is treated as a smooth variable.

In case the edge manifold is an exact edge ($\mathcal{E} = \mathbb{R}^b \times \mathscr{C}(F), dy^2 + g$) where ($\mathscr{C}(F) = (0, \infty) \times F^f, g = ds^2 + s^2 g^F$), Proposition 3.2 implies that $H_{\mathcal{E}}$ lifts to a polyhomogeneous conormal distribution on the biharmonic heat space blowup, of order $(-m)$ at the front and the temporal diagonal faces, vanishing to infinite order at tf, and with the index set at rf and lf given by $E + 2\mathbb{N}_0$, where

$$E = \left\{ \gamma \geq 0 \mid \gamma = -\frac{(f-1)}{2} + \sqrt{\frac{(f-1)^2}{4} + \sigma^2}, \quad \sigma^2 \in \operatorname{Spec} \Delta_F \right\}.$$

In the general case of an admissible edge space (M, g), $H_{\mathcal{E}}$ is only an initial parametrix, defines a polyhomogeneous function on the front face of $\mathscr{M}^2_h$ and solves the heat equation only to first order. Repeating almost ad verbatim the heat kernel construction in case of the edge Laplacian in [15], we arrive at the following

Proposition 3.3. *Let* (M^m, g) *be an incomplete edge space with an admissible edge metric* g. *Then the lift* $\beta^* H$ *is polyhomogeneous on* $\mathscr{M}^2_h$ *of order* $(-\dim M)$ *at* ff *and* td, *vanishing ot infinite order at* tf, *and with the index set at* rf *and* lf *given by* $E + \mathbb{N}_0$ *where*

$$E = \left\{ \gamma \geq 0 \mid \gamma = -\frac{(f-1)}{2} + \sqrt{\frac{(f-1)^2}{4} + \sigma^2},\ \sigma^2 \in \operatorname{Spec} \Delta_{F,y} \right\}.$$

More precisely, if s denotes the boundary defining function of rf, *we obtain*

$$\beta^* H \sim \sum_{\gamma\in E}\left(\sum_{j=0}^{\infty} s^{\gamma+2j} a_{\gamma,j}(\beta^* H) + \sum_{j=0}^{\infty} s^{\gamma+2+j} a'_{\gamma,j}(\beta^* H)\right) \quad \text{as } s\to 0,$$

where the coefficients $a_{\gamma,j}(H)$ are of order $(-m)$ at the front face and lie in their corresponding $\Delta_{F,y}$ eigenspaces. The higher coefficients $a'_{\gamma,j}(\beta^* H)$ are of order $(-m+1)$ at ff.

Proof. Recall the heat kernel construction in [15], which we basically follow here. Denote by Δ the Laplace Beltrami operator on (M,g). We write $\mathcal{L} := \partial_t + \Delta^2$ for the heat operator. The restriction of the lift $\beta^*(t\mathcal{L})$ to ff is called the normal operator $N_{\mathrm{ff}}(t\mathcal{L})_{y_0}$ at the front face (at the fibre over $y_0 \in B$) and is given in projective coordinates (2.7) explicitly as follows

$$\begin{aligned} N_{\mathrm{ff}}(t\mathcal{L})_{y_0} &= \tau\left(\partial_\tau + \left(-\partial_s^2 - f s^{-1}\partial_s + s^{-2}\Delta_{F,y_0} + \Delta_u^{\mathbb{R}^b}\right)^2\right) \\ &=: \tau\left(\partial_\tau + \left(\Delta_{s,y_0}^{\mathscr{C}(F)} + \Delta_u^{\mathbb{R}^b}\right)^2\right). \end{aligned}$$

$N_{\mathrm{ff}}(t\mathcal{L})$ does not involve derivatives with respect to $(y_0, \widetilde{x}, \widetilde{y}, \widetilde{z})$ and hence acts tangentially to the fibres of the front face. Consequently in our choice of an initial parametrix H_0 we note that the equation

$$N_{\mathrm{ff}}(t\mathcal{L}) \circ N_{\mathrm{ff}}(H_0) = 0$$

is the heat equation for the bi-Laplace operator on the model edge $\mathscr{C}(F)\times\mathbb{R}^b$. Consequently, the initial parametrix H_0 is defined by choosing $N_{\mathrm{ff}}(H_0)$ to equal the fundamental solution for the heat operator $N_{\mathrm{ff}}(t\mathcal{L})$, and extending $N_{\mathrm{ff}}(H_0)$ trivially to a neighborhood of the front face. More precisely, consider the biharmonic heat kernel $H_{\mathcal{E},y}$ on the model edge $(\mathscr{C}(F)\times\mathbb{R}^b, ds^2 + s^2 g^F_{y_0} + du^2$ with the parameter $y_0 \in B$. Then in projective coordinates $(\tau, s, y_0, z, \widetilde{x}, u, \widetilde{z})$ near the right corner of ff, where $\widetilde{x}$ is the defining function of the front face, we set

$$H_0(\tau, s, u, y_0, z, \widetilde{z}) := \widetilde{x}^{-m}\phi(\widetilde{x}) H_{\mathcal{E},y_0}(\tau, s, u, z, \widetilde{s}=1, \widetilde{u}=0, \widetilde{z}), \tag{3.6}$$

where ϕ is a smooth cutoff function, $\phi \equiv 1$ in an open neighborhood of $\widetilde{x}=0$, and with compact support in $[0,1)$. By Proposition 3.2, our initial parametrix H_0 is polyhomogeneous on M_h^2 and solves the heat equation to first order at the front face ff of $\mathscr{M}_h^2$. Moreover Proposition 3.2 asserts

$$H_0 \sim \sum_{\gamma\in E}\sum_{j=0}^{\infty} s^{\gamma+2j} a_{\gamma,j}(H_0), \quad s\to 0, \tag{3.7}$$

with each coefficient $a_{\gamma,j}(H_0)$ lying in the corresponding Δ_{F,y_0} eigenspace. The error of the initial parametrix H_0 is given by

$$\beta^*(t\mathcal{L})H_0 = \left(\beta^*(t\Delta^2) - \tau(\Delta_{s,y_0}^{\mathscr{C}(F)} + \Delta_u^{\mathbb{R}^b})^2\right) H_0 =: P_0.$$

The leading order term in the expansion of $\beta^*(t\mathcal{L})$ at td does not depend on the edge geometry and corresponds to the bi-Laplacian on a closed manifold. Consequently, classical arguments allow to refine the initial parametrix such that the error term P_0 is vanishing to infinite order at td, compare the corresponding discussion in ([15], Section 3.2). We need to understand the explicit structure of the asymptotic expansion of P_0 at ff and rf. By Definition 2.2 (iv) we find

$$\beta^*\Delta = \widetilde{x}^{-2}\left(\Delta_{s,y_0}^{\mathscr{C}(F)} + \Delta_u^{\mathbb{R}^b}\right) + \widetilde{x}^{-1}L_1 + L_2, \tag{3.8}$$

where L_1 is comprised of the derivatives $\partial_u\partial_z$ and L_2 consists of edge derivatives $\mathcal{V}_e^2$. Consequently, we obtain after taking squares

$$\begin{aligned}\beta^*(t\Delta^2) - \tau\left(\Delta_{s,y_0}^{\mathscr{C}(F)} + \Delta_u^{\mathbb{R}^b}\right)^2 &= \tau\widetilde{x}\left(\left(\Delta_{s,y_0}^{\mathscr{C}(F)} + \Delta_u^{\mathbb{R}^b}\right)L_1 + L_1\left(\Delta_{s,y_0}^{\mathscr{C}(F)} + \Delta_u^{\mathbb{R}^b}\right)\right)\\ &+ \tau\widetilde{x}^2\left(\left(\Delta_{s,y_0}^{\mathscr{C}(F)} + \Delta_u^{\mathbb{R}^b}\right)L_2 + L_2\left(\Delta_{s,y_0}^{\mathscr{C}(F)} + \Delta_u^{\mathbb{R}^b}\right) + L_1^2\right) + \tau\widetilde{x}^4 L_2^2.\end{aligned}$$

We now apply each of the summands above to the asymptotic expansion (3.7) of H_0. Note that $\Delta_{s,y_0}^{\mathscr{C}(F)}$ annihilates each $s^\gamma a_{\gamma,0}(H_0)$, $\gamma \in E$, and lowers the s-order of $s^{\gamma+2j}a_{\gamma,j}(H_0)$ by 2, if $j \geq 1$. Consequently, we obtain as $s \to 0$

$$P_0 = \beta^*(t\mathcal{L})H_0 \sim \widetilde{x}^{-m+1}\sum_{\gamma\in E}\sum_{j=0}^{\infty} s^{\gamma+j-2}c_{\gamma,j}.$$

The next step in the construction of the heat kernel involves adding a kernel H_0' to H_0, such that the new error term is vanishing to infinite order at rf. In order to eliminate the term $s^k a$ in the asymptotic expansion of P_0 at rf, we only need to solve

$$(-\partial_s^2 - f s^{-1}\partial_s + s^{-2}\Delta_{F,y_0})^2 u = s^k(\tau^{-1}a). \tag{3.9}$$

This is because all other terms in the expansion of $t\mathcal{L}$ at rf lower the exponent in s by at most one, while the indicial part lowers the exponent by two. The variables $(\tau, u, \widetilde{x}, y_0, \widetilde{y}, \widetilde{z})$ enter the equation only as parameters. The equation is solved by Mellin transform as well as spectral decomposition over F. The solution is polyhomogeneous in all variables, including parameters and is of leading order $(k+4)$. Consequently, the correcting kernel H_0' must be of leading order 2 at rf and of leading order $(-m+1)$ at ff, since P_0 is of order (-2) at rf and $(-m+1)$ at ff and the defining function $\widetilde{x}$ of the front face enters (3.9) only as a parameter. Hence

$$H_1 := H_0 + H_0' \sim \sum_{\gamma\in E}\left(\sum_{j=0}^{\infty} s^{\gamma+2j}a_{\gamma,j}(H_1) + \sum_{j=0}^{\infty} s^{\gamma+2+j}a'_{\gamma,j}(H_1)\right) \quad \text{as } s \to 0,$$

where the coefficients $a_{\gamma,j}(H_1)$ each lie in the corresponding Δ_{F,y_0} eigenspace. In the following correction steps the exact heat kernel is obtained from H_1 by an iterative correction procedure, adding terms of the form $H_1 \circ (P_1)^k$, where $P_1 := t\mathcal{L}H_1$ is vanishing to infinite order at rf and td. This leads to an expansion

$$\beta^* H \sim \sum_{\gamma \in E} \left(\sum_{j=0}^{\infty} s^{\gamma+2j} a_{\gamma,j}(\beta^* H) + \sum_{j=0}^{\infty} s^{\gamma+2+j} a'_{\gamma,j}(\beta^* H) \right) \text{ as } s \to 0, \tag{3.10}$$

where the coefficients $a_{\gamma,j}(H)$ still lie in their corresponding Δ_{F,y_0} eigenspaces, and are of order $(-m)$ at the front face. The higher coefficients $a'_{\gamma,j}(\beta^* H)$ are of order $(-m+1)$ at ff.

Note that in the construction above, we have only used Definition 2.2 (i) and (iv). The assumption of Definition 2.2 (iii) for admissible edge metrics is not required there, but plays an essential role in the argument that H indeed takes values in $\mathscr{D}(\Delta^2_{\mathscr{F}})$. First note that H indeed takes values in $\mathscr{D}(\Delta_{\mathscr{F}})$, since the expansion (3.10) satisfies the characterization of maximal solutions in Lemma 2.4 and also the condition in (2.3) under the rescaling Φ.

The conclusion that ΔH also takes values in $\mathscr{D}(\Delta_{\mathscr{F}})$ is more intricate. Recall (3.8). It is easily checked from (3.10) that $\widetilde{x}^{-2}\left(\Delta^{\mathscr{C}(F)}_{s,y_0} + \Delta^{\mathbb{R}^b}_u\right) H$ indeed takes values in $\mathscr{D}(\Delta_{\mathscr{F}})$ without any further assumptions. Application of $(\widetilde{x}^{-1}L_1 + L_2)$ to H preserves the expansion (3.10), however the coefficients in the expansion need not lie in the correct Δ_{F,y_0} eigenspaces, and hence we cannot deduce that $(\widetilde{x}^{-1}L_1 + L_2)H$ maps into $\mathscr{D}(\Delta_{\mathscr{F}})$ in general.

By condition (iii) of Definition 2.2, any $\gamma \neq 0$ is automatically $\gamma > 1$, and hence it then suffices to check whether the s^0 coefficient in the expansion of $(\widetilde{x}^{-1}L_1 + L_2)H$ lies in the zero-eigenspace of Δ_F, in other words is harmonic on fibres and hence constant in z. The leading term $s^0 a_{0,0}(\beta^* H)$ in the expansion of $\beta^* H$ is annihilated by $(s\partial_s)$, ∂_z and is increased by $\beta^* x\partial_y = s\partial_u + \widetilde{x}s\partial_y$. Consequently, $(\widetilde{x}^{-1}L_1 + L_2)H$ admits no s^0 term and hence trivially maps into $\mathscr{D}(\Delta_{\mathscr{F}})$.

The kernels H and ΔH thus both map into $\mathscr{D}(\Delta_{\mathscr{F}})$ and hence by definition, H indeed maps into $\mathscr{D}(\Delta^2_{\mathscr{F}})$ and thus is the biharmonic heat kernel associated to $\Delta^2_{\mathscr{F}}$. □

4. Mapping properties of the biharmonic heat operator

In this section we prove boundedness and strong continuity of the biharmonic heat operator with respect to Banach spaces introduced in Definition 2.3.

Theorem 4.1. *Let (M^m, g) be an incomplete edge space with an admissible edge metric g. Fix the Friedrichs extension $\Delta_{\mathscr{F}}$ of the corresponding Laplace Beltrami operator. Put $\mathcal{D}_0 = \langle \Delta \rangle$ and $\mathcal{D} = \langle \Delta, x^{-1}\mathcal{V}^2_{\mathrm{e}}, x^{-1}\mathcal{V}'_e, \mathcal{V}_{\mathrm{e}} \rangle$, where $\mathcal{V}'_e \subset \mathcal{V}_{\mathrm{e}}$ consists locally of all edge vector fields where $x\partial_y$ is weighted with functions that are fibre-wise constant. Then the associated biharmonic heat operator $e^{-t\Delta^2_{\mathscr{F}}}$ is a bounded map between the (weighted) Banach spaces*

$$e^{-t\Delta^2_{\mathscr{F}}} : \mathscr{C}^{2k}_{\mathrm{ie}}(M, \mathcal{D}_0) \to t^{-1/4}\mathscr{C}^{2(k+1)}_{\mathrm{ie}}(M, \mathcal{D}).$$

Proof. First we prove the statement for $k = 0$. The explicit structure of the heat kernel expansion in Proposition 3.3 implies that for any $X \in \mathcal{D}$ applied to the biharmonic heat kernel H, the lift $\beta^*(XH)$ admits the following behaviour near the front face of the heat space $\mathscr{M}_h^2$

$$\beta^*(XH) = O\left((\rho_{\mathrm{rf}}\rho_{\mathrm{lf}})^0(\rho_{\mathrm{ff}}\rho_{\mathrm{td}})^{-m-2}\rho_{\mathrm{tf}}^{\infty}\right), \tag{4.1}$$

where ρ_* denotes a defining function of a boundary face $*$, $* \in \{\mathrm{rf}, \mathrm{lf}, \mathrm{ff}, \mathrm{td}, \mathrm{tf}\}$.

Consider the lift of the volume form in the various projective coordinates near ff. We explify the transformation rules for the volume form near the lower left, lower right and the top corner of the front face

$$\begin{aligned}
\text{near left corner:}\quad & \tau = \frac{t}{x^4} = \rho_{\mathrm{tf}},\ s = \frac{\widetilde{x}}{x} = \rho_{\mathrm{lf}},\ u = \frac{y-\widetilde{y}}{x},\ x = \rho_{\mathrm{ff}},\ y,\ z,\ \widetilde{z}, \\
& \beta^*(\widetilde{x}^f\, d\widetilde{x}\ \mathrm{dvol}_{\partial\mathrm{M}}(\widetilde{x})) = h\cdot x^m\, s^f\, ds\, du\, d\widetilde{z}, \\
\text{near right corner:}\quad & \widetilde{\tau} = \frac{t}{\widetilde{x}^4} = \rho_{\mathrm{tf}},\ \widetilde{s} = \frac{x}{\widetilde{x}} = \rho_{\mathrm{rf}},\ \widetilde{u} = \frac{y-\widetilde{y}}{\widetilde{x}},\ z,\ \widetilde{x} = \rho_{\mathrm{ff}},\ \widetilde{y},\ \widetilde{z}, \\
& \beta^*(\widetilde{x}^f\, d\widetilde{x}\ \mathrm{dvol}_{\partial\mathrm{M}}(\widetilde{x})) = h\cdot \widetilde{x}^m\, \widetilde{\tau}^{-1}\, d\widetilde{\tau}\, d\widetilde{u}\, d\widetilde{z}, \\
\text{near top corner:}\quad & \rho = t^{1/4} = \rho_{\mathrm{ff}},\ \xi = \frac{x}{\rho} = \rho_{\mathrm{rf}},\ \widetilde{\xi} = \frac{\widetilde{x}}{\rho} = \rho_{\mathrm{lf}},\ u \\
& \quad = \frac{y-\widetilde{y}}{\rho},\ y,\ z,\ \widetilde{z}, \\
& \beta^*(\widetilde{x}^f\, d\widetilde{x}\ \mathrm{dvol}_{\partial\mathrm{M}}(\widetilde{x})) = h\cdot \rho^m\, \widetilde{\xi}^f\, d\widetilde{\xi}\, du\, d\widetilde{z}.
\end{aligned} \tag{4.2}$$

The projective coordinates and the transformation rule for the volume form where the front and the temporal diagonal faces meet, is as follows

$$\begin{aligned}
&\eta = \frac{t^{1/4}}{x} = \rho_{\mathrm{td}},\ S = \frac{x-\widetilde{x}}{t^{1/4}},\ U = \frac{y-\widetilde{y}}{t^{1/4}},\ Z = \frac{x(z-\widetilde{z})}{t^{1/4}},\ x = \rho_{\mathrm{ff}},\ y,\ z, \\
&\beta^*(\widetilde{x}^f\, d\widetilde{x}\, \mathrm{dvol}_{\partial\mathrm{M}}(\widetilde{x})) = h\cdot x^m\eta^m dS\, dU\, dZ.
\end{aligned} \tag{4.3}$$

When we combine the asymptotics of $\beta^*(XH)$ in (4.1) with the behaviour of the volume form in the various projective coordinates (4.2) and (4.3), we find that $\beta^*(XH\widetilde{x}^f d\widetilde{x}\, \mathrm{dvol}_{\partial\mathrm{M}}(\widetilde{x}))$ has a singular behaviour of $(\rho_{\mathrm{ff}}\rho_{\mathrm{td}})^{-2} \le ct^{-1/4}$. Consequently, we may estimate for $X \in \mathcal{D}$ and any continuous function, in particular for any $u \in \mathscr{C}^0_{\mathrm{ie}}(M)$

$$\|Xe^{-t\Delta^2_{\mathscr{F}}}u\|_\infty \le Ct^{-1/4}\|u\|_\infty,$$

for some constant $C > 0$ independent of u. Furthermore, by Proposition 3.3, $\beta^*XH \sim a_0\rho^0_{\mathrm{rf}}$, as $\rho_{\mathrm{rf}} \to 0$ for $X \in \mathcal{D}$, where a_0 is fibrewise constant, i.e. independent of $z \in F$. Here the fact that for $X \in x^{-1}\mathcal{V}'_e$ its ∂_y component is weighted with a fibrewise constant function, is essential. Hence, indeed $Xe^{-t\Delta^2_{\mathscr{F}}}u \in t^{-1/4}\mathscr{C}^0_{\mathrm{ie}}(M)$. This proves the statement for $k = 0$. The general statement follows from the fact that due to (2.4), $\mathscr{C}^{2\mathrm{k}}_{\mathrm{ie}}(\mathrm{M}, \mathcal{D}) \subset \mathscr{D}(\Delta^k_{\mathscr{F}})$ and for any $u \in \mathscr{D}(\Delta^k_{\mathscr{F}})$, $\Delta^k e^{-t\Delta^2_{\mathscr{F}}} u = e^{-t\Delta^2_{\mathscr{F}}} \Delta^k u$, by uniqueness of solutions to the biharmonic heat equation with fixed initial condition.

Finally note that while $t^{1/4} X e^{-t\Delta^2_{\mathscr{F}}} u$ is continuous even for $X \in x^{-1}\mathcal{V}_e$, it need not remain fibrewise constant at $x = 0$ since in general X may include vector fields weighted with z-dependent functions. Hence $x^{-1}\mathcal{V}_e$ is replaced by $x^1\mathcal{V}'_e$ in $\mathcal{D}$, where $\mathcal{V}'_e \subset \mathcal{V}_e$ consists locally of linear combinations of $\{x\partial_x, x\partial_y, \partial_z\}$, weighted with functions that are fibrewise constant. □

Remark 4.2. A particular property[1] of $u \in \mathscr{C}^2_{\mathrm{ie}}(M, \mathcal{D})$ with $\{\partial_x, \partial_y, x^{-1}\partial_z\} \subset \mathcal{D}$ is worth noticing. In the singular neighborhood of the edge, the distance defined by the Riemannian edge metric g is equivalent to

$$d((x, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z})) = \left(|x - \widetilde{x}|^2 + |y - \widetilde{y}|^2 + (x + \widetilde{x})^2 |z - \widetilde{z}|^2\right)^{1/2}.$$

In local coordinates near the edge we find

$$\begin{aligned} &u(x, y, z) - u(\widetilde{x}, \widetilde{y}, \widetilde{z}) \\ &\quad = u(x, y, z) - u(\widetilde{x}, y, z) + u(\widetilde{x}, y, z) - u(\widetilde{x}, \widetilde{y}, z) + u(\widetilde{x}, \widetilde{y}, z) - u(\widetilde{x}, \widetilde{y}, \widetilde{z}) \\ &\quad = \partial_x u(\xi, y, z)(x - \widetilde{x}) + \partial_y u(\widetilde{x}, \gamma, z)(y - \widetilde{y}) + \widetilde{x}^{-1} \partial_z u(\widetilde{x}, \widetilde{y}, \zeta)\, \widetilde{x}\, (z - \widetilde{z}). \end{aligned}$$

Consequently we obtain

$$\begin{aligned} |u(x, y, z) - u(\widetilde{x}, \widetilde{y}, \widetilde{z})| &\leq ||u||_2 \left(|x - \widetilde{x}| + |y - \widetilde{y}| + (x + \widetilde{x})|z - \widetilde{z}|\right) \\ &\leq \sqrt{2}\, ||u||_2\, d((x, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z})). \end{aligned}$$

In other words, $u \in \mathscr{C}^2_{\mathrm{ie}}(M, \mathcal{D})$ is automatically Lipschitz with respect to d.

Remark 4.3. We point out that Theorem 4.1 holds also when the basic space $\mathscr{C}^0_{\mathrm{ie}}(M)$ is replaced by the Banach space of sections continuous up to $x = 0$, without the requirement of being fibrewise constant at ∂M. Also, we may set $\mathcal{D} = \langle \Delta, x^{-1}\mathcal{V}^2_e, x^{-1}\mathcal{V}_e, \mathcal{V}_e \rangle$. The use of the refined space $\mathscr{C}^0_{\mathrm{ie}}(M)$ and the restriction of $x^{-1}\mathcal{V}_e$ to $x^{-1}\mathcal{V}'_e$ in $\mathcal{D}$ becomes however crucial in Theorem 5.3.

5. Short time existence of semi-linear equations of fourth order

In this section we explain how the mapping properties of the biharmonic heat operator and its strong continuity yields short-time existence of solutions to certain semilinear equations of fourth order.

[1] It suffices that u is continuously differentiable to first order.

5.1. The Banach fixed point argument

The underlying idea is based on a fixed point argument in the following theorem.

Theorem 5.1. [20, Proposition 1.1 in 15] *Let P be some, possibly unbounded, linear operator in a Hilbert space H, bounded from below. Suppose that $V, W \subset H$ are Banach spaces, such that $P : V \to W$ is bounded and moreover*

(i) $e^{-tP} : V \longrightarrow V$ is a strongly continuous semigroup, for $t \geq 0$.
(ii) $Q : V \longrightarrow W$ is locally Lipschitz,
(iii) $e^{-tP} : W \longrightarrow t^{-\gamma} V$ bounded for some $\gamma < 1$.

Then for any $u_0 \in W$, the initial value problem

$$\partial_t u - Pu = Q(u),\ u(0) = u_0 \in W$$

has a unique solution $u \in C([0, T], V)$[2]*, for some $T > 0$, where T may be estimated from below in terms of $||u_0||_V$. The solution u is the fixed point of the operator $F : V \to V$ with*

$$F(u) = e^{-tP} u_0 + \int_0^t e^{-(t-s)P} Q(u) ds.$$

5.2. Strong continuity of the biharmonic heat operator

As seen from Theorem 5.1, existence of solutions to certain semi-linear fourth order equations crucially depends on the strong continuity property. Strong continuity of the biharmonic heat operator with respect to the Banach space $\mathscr{C}^{2k}_{\mathrm{ie}}(\mathrm{M}, \mathcal{D})$ is the content of the next theorem. Note that for strong continuity we will choose a different space $\mathcal{D}'$ of allowable operators, smaller than in Theorem 4.1. Beforehand we note the following well-known functional analytic result.

Lemma 5.2. *Let D be a self-adjoint non-negative unbounded operator in a Hilbert space H. Then the following is true.*

(i) A solution to $(\partial_t + D^2)u = 0$, that is continuously differentiable in $t > 0$, with $u(t) \in \mathscr{D}(D^2)$ for $t > 0$ and $\lim_{t \to 0} u(t) = u_0 \in H$, is unique, for any fixed $u_0 \in H$.
(ii) For any $u_0 \in \mathscr{D}(D)$, we have $De^{-tD^2} u_0 = e^{-tD^2} Du_0$.

Proof. (i) For $s \in (0, t]$ we compute

$$\begin{aligned}\partial_s\, e^{-(t-s)D^2} u(s) &= -\partial_t\, e^{-(t-s)D^2} u(s) + e^{-(t-s)D^2} \partial_s\, u(s) \\ &= e^{-(t-s)D^2} D^2 u(s) - e^{-(t-s)D^2} D^2 u(s) = 0.\end{aligned}$$

Consequently, $e^{-(t-s)D^2} u(s)$ is constant for $s \in (0, t]$. Since e^{-tD^2} converges to Id in the Hilbert space norm as $t \to 0$ and, by assumption, $u(t)$ is continuous at

[2] Moreover, by [20, Proposition 1.2 in 15], $u \in C^\infty((0, T] \times M)$.

$t = 0$, we find that $e^{-(t-s)D^2}u(s)$ is constant for $s \in [0, t]$. Considering the limit of $e^{-(t-s)D^2}u(s)$ as $s \to t$ and as $s \to 0$ proves for any $u_0 \in H$

$$u(t) = e^{-tD^2}u_0.$$

(ii) Consider $\lambda \in \mathrm{Res}(D^2)$ in the resolvent set of D^2. Then for any $u_0 \in \mathscr{D}(D)$, the resolvent $(D^2 - \lambda)^{-1}u_0 \in \mathscr{D}(D^2)$ and we compute

$$\begin{aligned}(D^2 - \lambda)D(D^2 - \lambda)^{-1}u_0 =& D(D^2 - \lambda)(D^2 - \lambda)^{-1}u_0 = Du_0, \\ &\Rightarrow D(D^2 - \lambda)^{-1}u_0 = (D^2 - \lambda)^{-1}Du_0.\end{aligned}$$

The statement now follows by closedness of D and definition of the heat operator as the strong limit $e^{-tD^2} := \lim_{n\to\infty} \left(I + tD^2/n\right)^{-n}$. $\square$

Theorem 5.3. *Let (M^m, g) be an incomplete edge space with an admissible edge metric g. Fix the Friedrichs extension $\Delta_{\mathscr{F}}$ of the corresponding Laplace–Beltrami operator. Put $\mathcal{D}' = \langle \Delta, \mathcal{V}_e^2, \mathcal{V}_e\rangle$. Then the associated biharmonic heat operator $e^{-t\Delta^2_{\mathscr{F}}}$ is a strongly continuous bounded map between Banach spaces*

$$e^{-t\Delta^2_{\mathscr{F}}} : \mathscr{C}^{2k}_{\mathrm{ie}}(M, \mathcal{D}') \to \mathscr{C}^{2k}_{\mathrm{ie}}(M, \mathcal{D}').$$

Proof. By (2.4), we have $\mathscr{C}^{2k}_{\mathrm{ie}}(M, \mathcal{D}') \subset \mathscr{D}(\Delta^k_{\mathscr{F}})$ and hence for any $u \in \mathscr{C}^{2k}_{\mathrm{ie}}(M, \mathcal{D}')$ we infer by the previous Lemma 5.2, $\Delta^k e^{-t\Delta^2_{\mathscr{F}}}u = e^{-t\Delta^2_{\mathscr{F}}}\Delta^k u$. This reduces the statement to $k = 1$ and $k = 0$. Proof of both cases requires stochastic completeness of the biharmonic heat kernel, which we explain below. Solutions to the initial value problem

$$\partial_t u + \Delta^2 u = 0,\ u(0) = u_0,\ u(t) \in \mathscr{D}(\Delta^2_{\mathscr{F}}),\ t > 0,$$

are unique and in fact given by $u(t) = e^{-t\Delta^2_{\mathscr{F}}}u_0 \in \mathscr{D}(\Delta^2_{\mathscr{F}})$. We have observed in subsection 2.3 that, reversing eventual rescaling, the Friedrichs domain contains precisely those elements in the maximal domain whose leading term in the weak expansion as $x \to 0$ is given by x^0, with a fibrewise constant coefficient, cf. (2.4). Consequently, the constant function $\mathbf{1} \in \mathscr{D}(\Delta_{\mathscr{F}})$. Moreover, $\Delta\mathbf{1} = 0 \in \mathscr{D}(\Delta_{\mathscr{F}})$ and consequently $\mathbf{1} \in \mathscr{D}(\Delta^2_{\mathscr{F}})$. The constant function $\mathbf{1}$ solves the heat equation and hence we deduce by uniqueness of solutions the *stochastic completeness*

$$e^{-t\Delta^2_{\mathscr{F}}}\mathbf{1} \equiv \int_M H(t, p, \widetilde{p})\,\mathrm{dvol_g}(\widetilde{p}) = \mathbf{1}, \quad \text{for all } p \in M, t > 0. \tag{5.1}$$

This reduces the case to $k = 0, 1$. We can now prove the statement for $k = 0$, basically repeating the arguments in ([2]) where the classical proof of strong continuity of the heat operator on closed (non-singular) manifolds is adapted to the present setup. Using stochastic completeness we find

$$(e^{-t\Delta^2_{\mathscr{F}}}u)(p, t) - u(p) = \int_M H\,(t, p, \widetilde{p})\,(u(\widetilde{p}) - u(p))\,\mathrm{dvol_g}(\widetilde{p}).$$

Consider the distance function $d(p, \widetilde{p})$ induced by the incomplete edge metric g. In the singular neighborhood of the edge, the distance is equivalent to

$$d((x, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z})) = \left(|x - \widetilde{x}|^2 + |y - \widetilde{y}|^2 + (x + \widetilde{x})^2 |z - \widetilde{z}|^2\right)^{1/2}.$$

Note that $u \in \mathscr{C}^0_{\mathrm{ie}}(M)$ is continuous with respect to the topology induced by the Riemannian metric g and hence by the distance function d. Hence for any $\epsilon > 0$ there exists some $\delta(\epsilon) > 0$, such that for $d(p, \widetilde{p}) \leq \delta(\epsilon)$ one has $|u(p) - u(\widetilde{p})| \leq \epsilon$. For any given $\epsilon > 0$ we separate the integration region into

$$\begin{aligned} M^+_\epsilon &:= \{\widetilde{p} \mid d(p, \widetilde{p}) \geq \delta(\epsilon)\}, \\ M^-_\epsilon &:= \{\widetilde{p} \mid d(p, \widetilde{p}) \leq \delta(\epsilon)\}. \end{aligned} \tag{5.2}$$

Employing continuity of u we find

$$\begin{aligned} |e^{-t\Delta^2_{\mathscr{F}}} u - u| &= \left| \int_M H\,(t, p, \widetilde{p})\,(u(\widetilde{p}) - u(p))\,\mathrm{dvol}_{\mathrm{g}}(\widetilde{p}) \right| \\ &\leq \int_{M^+} |H\,(t, p, \widetilde{p})\,| \cdot |u(\widetilde{p}) - u(p)|\,\mathrm{dvol}_{\mathrm{g}}(\widetilde{p}) \\ &\quad + \int_{M^-} |H\,(t, p, \widetilde{p})\,| \cdot |u(\widetilde{p}) - u(p)|\,\mathrm{dvol}_{\mathrm{g}}(\widetilde{p}) \\ &\leq 2\,\frac{t^{1/4}}{\delta(\epsilon)}\,\|u\|_0 \int_{M^+} |H\,(t, p, \widetilde{p})\,|\,\frac{d(p, \widetilde{p})}{t^{1/4}}\,\mathrm{dvol}_{\mathrm{g}}(\widetilde{p}) \\ &\quad + \epsilon \int_{M^-} |H\,(t, p, \widetilde{p})\,|\,\mathrm{dvol}_{\mathrm{g}}(\widetilde{p}). \end{aligned}$$

It may be checked in the various projective coordinates around the front face in the heat space $\mathscr{M}^2_h$, that $\beta^*(|H|\,\mathrm{dvol}_{\mathrm{g}})$ and $\beta^*(d(p, \widetilde{p})t^{-1/4})\rho_{\mathrm{tf}}$ is bounded. Since $\beta^*|H|$ is vanishing to infinite order at tf, we find that both integrals above are bounded uniformly in (t, p, ϵ). Therefore we obtain

$$\|e^{-t\Delta} u - u\|_0 \leq C\left(\frac{t^{1/4}}{\delta(\epsilon)}\,\|u\|_0 + \epsilon\right).$$

Thus, for any given $\epsilon > 0$ we can estimate $\|e^{-t\Delta} u - u\|_0 \leq 2\epsilon C$ for $t^{1/4} < \epsilon\delta(\epsilon)/\|u\|_0$. This proves strong continuity of the biharmonic heat operator on $\mathscr{C}^0_{\mathrm{ie}}(M)$. It remains to prove the case $k = 1$. Strong continuity of the biharmonic heat operator with respect to $\mathscr{C}^2_{\mathrm{ie}}(M)$ means $\|X(e^{-t\Delta^2_{\mathscr{F}}} u - u)\|_0 \to 0$ as $t \to 0$, for $u \in \mathscr{C}^2_{\mathrm{ie}}(M)$ and $X \in \mathcal{D}'$. If $X = \Delta$, this follows from the case $k = 0$, since $\Delta e^{-t\Delta^2_{\mathscr{F}}} u = e^{-t\Delta^2_{\mathscr{F}}} \Delta u$ for $u \in \mathscr{C}^2_{\mathrm{ie}}(M) \subset \mathscr{D}(\Delta_{\mathscr{F}})$. For $X \in \{\mathcal{V}^2_{\mathrm{e}}, \mathcal{V}_{\mathrm{e}}\}$ the leading order of $\beta^* H$ at the front face is preserved under X, so that away from td, the estimates reduce to the case $k = 0$.

Near td, a priori XH admits ρ_{td}^{-2} singular behaviour at the temporal diagonal. However, integration by parts, exactly as worked out in detail in [2] allows to pass derivatives X to u, so that the estimates again reduce to the case $k = 0$. We write down the argument for completeness. The edge vector fields obey the following transformation rules in projective coordinates (2.8) near the temporal diagonal

$$\beta^*(x\partial_x) = -\eta\partial_\eta + \frac{1}{\eta}\partial_S + Z\partial_Z + x\partial_x,\ \beta^*(x\partial_y) = \frac{1}{\eta}\partial_U + x\partial_y,\ \beta^*(\partial_z) = \frac{1}{\eta}\partial_Z + \partial_z.$$

By Proposition 3.3

$$\beta^* H(\eta, S, U, Z, x, y, z) = x^{-m}\eta^{-m} G(\eta, S, U, Z, x, y, z),$$
$$\beta^*(\widetilde{x}^f d\widetilde{x}\, \mathrm{dvol}_{\partial \mathrm{M}}(\widetilde{x})) = h(x\eta)^m (1-\eta S)^f dS\, dU\, dZ,$$

where G is bounded in its entries, and in fact infinitely vanishing as $|(S, U, Z)| \to \infty$, and $h = h\,(\eta, x(1-\eta S), y - x\eta U, z - \eta Z, x, y, z)$ is a bounded distribution on $\mathscr{M}^2_h$. We consider $||x\partial_x(e^{-t\Delta}u - u)||_0$. Using stochastic completeness of the heat kernel, we find

$$F := x\partial_x(e^{-t\Delta}u - u) = \int (x\partial_x H)u(\widetilde{x}, \widetilde{y}, \widetilde{z})\widetilde{x}^f d\widetilde{x}\, \mathrm{dvol}_{\partial \mathrm{M}}(\widetilde{x})$$
$$- \int (x\partial_x)[Hu(x, y, z)\widetilde{x}^f d\widetilde{x}\, \mathrm{dvol}_{\partial \mathrm{M}}(\widetilde{x})] =: F_1 - F_2.$$

Next we transform to projective coordinates and integrate by parts in S, where the boundary terms lie away from the diagonal and hence are vanishing to infinite order for $t \to 0$ by the asymptotic behaviour of the heat kernel. Omitting these irrelevant terms, we obtain

$$\begin{aligned} F_1 &= \int \left(-\eta\partial_\eta + \frac{1}{\eta}\partial_S + Z\partial_Z + x\partial_x\right)\left[(x\eta)^{-m} G(\eta, S, U, Z, x, y, z)\right] \\ &\quad \times u\,(x(1-\eta S), y - x\eta U, z - \eta Z)\, h(x\eta)^m (1-\eta S)^f dS\, dU\, dZ \\ &= \int \left[(-\eta\partial_\eta + Z\partial_Z + x\partial_x)(x\eta)^{-m} G\right] \cdot u\, h(x\eta)^m (1-\eta S)^f dS\, dU\, dZ \\ &\quad - \int G\left[\left(\frac{1}{\eta}\partial_S\right) u\right] h(1-\eta S)^f dS\, dU\, dZ \\ &\quad - \int (x\eta)^{-m} G \cdot u \left[\left(\frac{1}{\eta}\partial_S\right) h(x\eta)^m (1-\eta S)^f\right] dS\, dU\, dZ. \end{aligned}$$

We perform similar computations for F_2:

$$\begin{aligned} F_2 &= \int \left[(x\partial_x H)u(x, y, z) + H(x\partial_x u)\right] \widetilde{x}^f d\widetilde{x}\, \mathrm{dvol}_{\partial \mathrm{M}}(\widetilde{x}) \\ &= \int \left(\left[-\eta\partial_\eta + \frac{1}{\eta}\partial_S + Z\partial_Z + x\partial_x\right](x\eta)^{-m} G\right) u \cdot h(x\eta)^m (1-\eta S)^f dS\, dU\, dZ \\ &\quad + \int G(\eta, S, U, Z, x, y, z)(x\partial_x u(x, y, z)) h(1-\eta S)^f dS\, dU\, dZ \end{aligned}$$

$$= \int \left[(-\eta\partial_\eta + Z\partial_Z + x\partial_x)(x\eta)^{-m} G\right] \cdot u\, h(x\eta)^m (1-\eta S)^f \, dS\, dU\, dZ$$
$$- \int (x\eta)^{-m} G \cdot u \left[\left(\frac{1}{\eta}\partial_S\right) h(x\eta)^m (1-\eta S)^f\right] dS\, dU\, dZ$$
$$+ \int G(\eta, S, U, Z, x, y, z)(x\partial_x u(x, y, z)) h (1-\eta S)^f \, dS\, dU\, dZ.$$

Thus $F = F_1 - F_2$ becomes

$$F = \int \left[(-\eta\partial_\eta + Z\partial_Z + x\partial_x)(x\eta)^{-m} G(\eta, S, U, Z, x, y, z)\right] h(x\eta)^m (1-\eta S)^f$$
$$\times (u(x(1-\eta S), y - x\eta U, z - \eta Z) - u(x, y, z)) dS\, dU\, dZ$$
$$- \int G(\eta, S, U, Z, x, y, z) \left[\left(\frac{1}{\eta}\partial_S\right) h \cdot (1-\eta S)^f\right]$$
$$\times (u(x(1-\eta S), y - x\eta U, z - \eta Z) - u(x, y, z)) dS\, dU\, dZ$$
$$- \int G \left[\frac{1}{\eta}\partial_S u(x(1-\eta S), y - x\eta U, z - \eta Z) + x\partial_x u(x, y, z)\right]$$
$$\times h(1-\eta S)^f \, dS\, dU\, dZ.$$

Now, each of the three integrals is estimated as above for $k = 0$ by separating the integration region into M_ϵ^+ and M_ϵ^- for any given $\epsilon > 0$. Note that in the final integral we use the fact that $u \in \mathscr{C}^2_{\mathrm{ie}}(M, \mathcal{D}')$ so that $\eta^{-1}\partial_S u$ and $x\partial_x u$ are bounded. Higher order and other edge derivatives may be estimated in a similar way. This proves strong continuity in general and as a trivial consequence boundedness of the biharmonic heat operator. □

5.3. Existence and regularity of solutions

We can now establish our final existence and regularity results.

Corollary 5.4. *Let (M, g) be an incomplete edge space with an admissible edge metric g. Put $\mathcal{D}' = \langle \Delta, \mathcal{V}_{\mathrm{e}}^2, \mathcal{V}_{\mathrm{e}} \rangle$ and $\mathcal{D} = \langle \Delta, x^{-1}\mathcal{V}_{\mathrm{e}}^2, x^{-1}\mathcal{V}_e', \mathcal{V}_{\mathrm{e}} \rangle$, where $\mathcal{V}_e' \subset \mathcal{V}_{\mathrm{e}}$ consists locally of linear combinations of $\{x\partial_x, x\partial_y, \partial_z\}$, where $x\partial_y$ is weighted with functions that are fibrewise constant. Assume $Q : \mathscr{C}^{2(k+1)}_{\mathrm{ie}}(M, \mathcal{D}') \to \mathscr{C}^{2k}_{\mathrm{ie}}(M, \mathcal{D}')$ is locally Lipschitz. Then the semilinear equation*

$$\partial_t u + \Delta^2 u = Q(u), \ u(0) = u_0 \in \mathscr{C}^{2(k+1)}_{\mathrm{ie}}(M, \mathcal{D}')$$

has a unique solution $u \in C([0, T], \mathscr{C}^{2(k+1)}_{\mathrm{ie}}(M, \mathcal{D})) \cap C^\infty((0, T] \times M)$, for some $T > 0$, where T may be estimated from below in terms of $||u_0||_{2(k+1)}$.

Proof. Consider first a slightly smaller set of operators $\mathcal{D}' = \langle \Delta, \mathcal{V}_{\mathrm{e}}^2, \mathcal{V}_{\mathrm{e}} \rangle$ and set $W = \mathscr{C}^{2k}_{\mathrm{ie}}(M, \mathcal{D}')$, $V = \mathscr{C}^{2(k+1)}_{\mathrm{ie}}(M, \mathcal{D}')$. In view of Theorem 4.1 and Theorem 5.3, the heat operator associated to $\Delta^2_{\mathscr{F}}$ satisfies the conditions of Theorem 5.1 with $\gamma = 1/4$. Consequently, by Theorem 5.1 the unique solution u exists and lies in

$\mathscr{C}^{2(k+1)}_{\text{ie}}(M, \mathcal{D}')$. This solution is the fixed point of the map $F : \mathscr{C}^{2(k+1)}_{\text{ie}}(M, \mathcal{D}') \to \mathscr{C}^{2(k+1)}_{\text{ie}}(M, \mathcal{D}')$ with

$$F(u) = e^{-t\Delta^2_{\mathscr{F}}} u_0 + \int_0^t e^{-(t-s)\Delta^2_{\mathscr{F}}} Q(u) ds.$$

However, by Theorem 4.1, F actually maps $\mathscr{C}^{2(k+1)}_{\text{ie}}(M, \mathcal{D}') \subset \mathscr{C}^{2(k+1)}_{\text{ie}}(M, \mathcal{D}_0)$ to $\mathscr{C}^{2(k+1)}_{\text{ie}}(M, \mathcal{D})$. Consequently, $u \in \mathscr{C}^{2(k+1)}_{\text{ie}}(M, \mathcal{D})$, with a slightly better regularity, as claimed. □

We now apply this general existence result to the example of the Cahn–Hilliard equation on an incomplete edge manifold. We define

$$Q : \mathscr{C}^{2\text{k}+2}_{\text{ie}}(\text{M}, \mathcal{D}) \to \mathscr{C}^{2\text{k}}_{\text{ie}}(\text{M}, \mathcal{D}), \quad Q(u) := \Delta(u - u^3).$$

The mapping Q is indeed locally Lipschitz, since for any $u, v \in \mathscr{C}^{2\text{k}+2}_{\text{ie}}(\text{M}, \mathcal{D})$

$$\begin{aligned} \|Q(u - v)\|_{2k} &\leq \|\Delta(u - v)\|_{2k} + \|\Delta(u - v)^3\|_{2k} \\ &\leq \|u - v\|_{2(k+1)} \left(1 + \|u - v\|^2_{2(k+1)}\right). \end{aligned}$$

We hence arrive at our final result.

Corollary 5.5. *Let (M, g) be an incomplete edge space with an admissible edge metric g. Put $\mathcal{D}' = \langle \Delta, \mathcal{V}^2_{\text{e}}, \mathcal{V}_{\text{e}} \rangle$ and $\mathcal{D} = \langle \Delta, x^{-1}\mathcal{V}^2_{\text{e}}, x^{-1}\mathcal{V}'_e, \mathcal{V}_{\text{e}} \rangle$, where $\mathcal{V}'_e \subset \mathcal{V}_{\text{e}}$ consists locally of linear combinations of $\{x\partial_x, x\partial_y, \partial_z\}$, where $x\partial_y$ is weighted with functions that are fibrewise constant. Then the Cahn–Hilliard equation*

$$\partial_t u + \Delta^2 u + \Delta(u - u^3) = 0, \; u(0) = u_0 \in \mathscr{C}^{2(k+1)}_{\text{ie}}(M, \mathcal{D}')$$

has a unique solution $u \in C([0, T], \mathscr{C}^{2\text{k}+2}_{\text{ie}}(\text{M}, \mathcal{D})) \cap C^\infty((0, T] \times M)$, for some $T > 0$.

Acknowledgments The author would like to thank Elmar Schrohe, Rafe Mazzeo and Eric Bahuaud for insightful discussions and gratefull acknowledges the support by the Hausdorff Research Institute at the University of Bonn.

References

[1] Albin, P.: A renormalized index theorem for some complete asymptotically regular metrics: the Gauss Bonnet theorem. Adv. Math. **213**(1), 1–52 (2007)
[2] Bahuaud, E., Dryden, E., Vertman, B.: Mapping properties of the heat operator on edge manifolds. Math. Nachr. **288**(2), 126–361 (2015)
[3] Bahuaud, E., Helliwell, D.: Short time existence for some higher order geometric flows. To appear in Comm. PDE, arXiv:1010.4287 [math.DG] (2010)
[4] Bahuaud, E., Vertman, B.: Yamabe flow on edge manifolds. Math. Nachr. **287**(23), 127–159 (2014)
[5] Caffarelli, L.A., Muler, N.E.: An L^1 bound for solutions of the Cahn–Hilliard equation. Arch. Ration. Mech. Anal. **133**, 129–144 (1995)

[6] Cahn, J.W.: On spinodal decomposition. Acta Metall. **9**, 795–801 (1961)
[7] Cahn, J.W., Hilliard, J.E.: Free energy of a non-uniform system I. Interfacial free energy. J. Chem. Phys. **28**, 258–267 (1958)
[8] Colombo, S.: Les transformations de Mellin et de Hankel. Centre National de la Recherche Scientifique, Paris (1959)
[9] Elliott, C., Songmu, Z.: On the Cahn-Hilliard equation. Arch. Ration. Mech. Anal. **96**, 339–357 (1986)
[10] Fefferman, C., Graham, C.R.: Conformal invariants. Asterisque, no. Numero Hors Serie, pp. 95–116, The mathematical heritage of Elie Cartan, Lyon (1984)
[11] Jeffres, T., Loya, P.: Regularity of solutions of the heat equation on a cone. Int. Math. Res. Not. **3**, 161–178 (2003)
[12] Lamm, T.: Biharmonischer Wärmefluss, Diplomarbeit. Albert-Ludwigs-Universität Freiburg, Freiburg (2001)
[13] Lesch, M.: Operators of Fuchs type, conical singularities, and asymptotic methods, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 136, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997
[14] Mazzeo, R.: Elliptic theory of differential edge operators. I. Comm. Partial Differential Equations **16**(10), 1615–1664 (1991)
[15] Mazzeo, R., Vertman, B.: Analytic torsion on manifolds with edges. A. Adv. Math. **231**(2), 1000–1040 (2012)
[16] Melrose, R.: The Atiyah-Patodi-Singer index theorem, Research Notes in Math., Vol. 4. A K Peters, Massachusetts (1993)
[17] Roidos, N., Schrohe, E.: The Cahn–Hilliard Equation and the Allen-Cahn Equation on Manifolds with Conical Singularities. preprint on arXiv:1203.3111v2 [math.AP]
[18] Vertman, B.: Heat-trace asymptotics for edge Laplacians with algebraic boundary conditions. J. d'Analyse Math. **125**(1), 285–318 (2015)
[19] Vertman, B.: Zeta determinants for regular-singular Laplace-type operators. J. Math. Phy. **50**(8) (2009)
[20] Taylor, M.E.: Partial differential equations. III. Nonlinear Equations, Applied Mathematical Sciences, vol. 117. Springer, New York (1997)