manuscripta math. 149, 45–62 (2016)

Nam Q. Le

Remarks on the Green's function of the linearized Monge–Ampère operator

Received: 15 May 2015
Published online: 24 June 2015

Abstract. In this note, we obtain sharp bounds for the Green's function of the linearized Monge–Ampère operators associated to convex functions with either Hessian determinant bounded away from zero and infinity or Monge–Ampère measure satisfying a doubling condition. Our result is an affine invariant version of the classical result of Littman–Stampacchia–Weinberger for uniformly elliptic operators in divergence form. We also obtain the L^p integrability for the gradient of the Green's function in two dimensions. As an application, we obtain a removable singularity result for the linearized Monge–Ampère equation.

1. Introduction and statement of the main result

In [7], Littman–Stampacchia–Weinberger established the fundamental sharp bounds for the Green's function of linear, uniformly elliptic operator in divergence form $L = -\partial_j(a^{ij}\partial_i)$ on a smooth, bounded domain $V \subset \mathbb{R}^n$. Here the coefficient matrix (a^{ij}) is symmetric with real, bounded measurable entries and uniformly elliptic, that is, there are positive constants λ, Λ such that

$$\lambda I_n \leq (a^{ij}) \leq \Lambda I_n. \tag{1.1}$$

This condition is invariant under the orthogonal transformation of coordinates. Let $g(x, y)$ be the Green's function of the operator L on V, that is, for each $y \in V$, $g(\cdot, y)$ is a positive solution of

$$Lg(\cdot, y) = \delta_y \text{ in } V, \quad \text{and } g(\cdot, y) = 0 \text{ on } \partial V.$$

Then, it was shown in [7] that g is comparable to the Green's function of the Laplace operator $-\Delta$. In particular, g satisfies the following sharp bounds in dimensions $n \geq 3$:

$$C^{-1}|x-y|^{-(n-2)} \leq g(x, y) \leq C|x-y|^{-(n-2)} \; \forall y \in V \tag{1.2}$$

where $C = C(n, \lambda, \Lambda, V, dist(y, \partial V))$. Other important properties of g such as integrability and continuity of its gradient were studied by Grüter–Widman in [4].

This note is concerned with estimates, analogous to (1.2), for the Green's function of the linearized Monge–Ampère equation, an affine invariant version of (1.1).

N. Q. Le (✉): Department of Mathematics, Indiana University, 831 E 3rd St, Bloomington, IN 47405, USA. e-mail: nqle@indiana.edu

Mathematics Subject Classification: 35J08, 35J96, 35J70, 35J75, 31B15

DOI: 10.1007/s00229-015-0766-2

Let Ω be a bounded, smooth, uniformly convex domain in $\mathbb{R}^n$ and μ a Borel measure in Ω with $\mu(\Omega) < \infty$. Let u be a convex function satisfying the following Monge–Ampère equation in the sense of Aleksandrov (see [5])

$$\det D^2 u = \mu \quad \text{in } \Omega, \; u = 0 \text{ on } \partial\Omega. \tag{1.3}$$

We consider two typical cases. The first case is when $\mu = f dx$ where f is bounded from below and above by some positive constants λ, Λ:

$$\lambda \leq f \leq \Lambda \quad \text{in } \Omega. \tag{1.4}$$

The second case is when μ is *doubling with respect to the center of mass.* This will be made more precise later. We assume throughout the note that u is smooth but our estimates do not depend on the smoothness of u.

Denote by $U = (U^{ij}) \equiv (\det D^2 u)(D^2 u)^{-1}$ the cofactor matrix of the Hessian matrix $D^2 u$. Then, the linearized operator of the Monge–Ampère equation (1.3) is given by

$$L_u v := -U^{ij} v_{ij} \equiv -(U^{ij} v)_{ij}.$$

The last equation is due to the fact that $U = (U^{ij})$ is divergence-free. The reader is referred to [2,12] and the references therein for more information on the theory of linearized Monge–Ampère equation and its applications to fluid mechanics and differential geometry.

The Monge–Ampère and linearized Monge–Ampère equations are invariant under unimodular transformation of coordinates. Indeed, let T be a linear transformation with $\det T = 1$. Then, the rescaled functions

$$\tilde{u}(x) = u(Tx), \quad \tilde{v}(x) = v(Tx),$$

satisfy

$$\det D^2 \tilde{u}(x) = \det D^2 u(x), \quad \tilde{U}^{ij} \tilde{v}_{ij}(x) = U^{ij} v_{ij}(Tx).$$

The linearized Monge–Ampère operator L_u is in general not uniformly elliptic. Under (1.3) and (1.4), the eigenvalues of $U = (U^{ij})$ are not necessarily bounded away from 0 and ∞. The degeneracy is the main difficulty in establishing our affine invariant analogue of (1.2). As in [2], we handle the degeneracy of L_u by working with sections of solutions to the Monge–Ampère equations. These sections have the same role as Euclidean balls have in the classical theory. The section of u with center x_0 and height t is defined by

$$S_u(x_0, t) = \{x \in \overline{\Omega} : u(x) < u(x_0) + \nabla u(x_0)(x - x_0) + t\}.$$

We say that the Borel measure μ is *doubling with respect to the center of mass* on the sections of u if there exist constants $\beta > 1$ and $0 < \alpha < 1$ such that for all sections $S_u(x_0, t)$,

$$\mu(S_u(x_0, t)) \leq \beta \mu(\alpha S_u(x_0, t/2)). \tag{1.5}$$

Here $\alpha S_u(x_0, t)$ denotes the α-dilation of $S_u(x_0, t)$ with respect to its center of mass x^*:

$$\alpha S_u(x_0, t) = \{x^* + \alpha(x - x^*) : x \in S_u(x_0, t)\}.$$

Let $g_V(x, y)$ be the Green's function of L_u in V where $V \subset\subset \Omega$.

1.1. The main result

In this note, we obtain the sharp upper bounds for g_V in all dimensions when u satisfies (1.3) and (1.4). We also obtain the sharp lower bounds for g_V when μ satisfies a more general doubling condition (1.5). Our main result states:

Theorem 1.1. *Fix $x_0 \in V$. Suppose that $0 < t < 1/4$, $S_u(x_0, 2t) \subset\subset V$ if $n \geq 3$ and $S_u(x_0, t^{1/2}) \subset\subset V$ if $n = 2$.*

(i) Assume that (1.3) and (1.4) are satisfied. Then, for $x \in S_u(x_0, t)$, we have

$$g_V(x, x_0) \geq \begin{cases} c(n, \lambda, \Lambda)t^{-\frac{n-2}{2}} & \text{if } n \geq 3 \\ c(n, \lambda, \Lambda)|\log t| & \text{if } n = 2. \end{cases}$$

Moreover, for $x \in \partial S_u(x_0, t)$, we have

$$g_V(x, x_0) \leq \begin{cases} C(V, \Omega, n, \lambda, \Lambda)t^{-\frac{n-2}{2}} & \text{if } n \geq 3 \\ C(V, \Omega, n, \lambda, \Lambda)|\log t| & \text{if } n = 2. \end{cases}$$

(ii) Assume that (1.3) and (1.5) are satisfied. Then, for $x \in S_u(x_0, t)$, we have

$$g_V(x, x_0) \geq \begin{cases} c(n, \alpha, \beta)t\,(\mu(S_u(x_0, t)))^{-1} & \text{if } n \geq 3 \\ c(n, \alpha, \beta)\dfrac{|\log t|^2}{\int_t^{t^{1/2}} \frac{\mu(S_u(x_0,s))ds}{s^2}} & \text{if } n = 2. \end{cases}$$

(iii) Suppose that $n = 2$ and (1.3) and (1.4) are satisfied. Then there exists $p_*(n, \lambda, \Lambda) > 1$ such that for all $1 < p < p_*$ and all $S_u(x_0, r^{1/2}) \subset\subset V$, we have

$$\left(\int_{S_u(x_0,r)} |\nabla g_V(x, x_0)|^p dx\right)^{\frac{1}{p}} \leq C(V, \Omega, n, p, \lambda, \Lambda, r).$$

Our estimates in Theorem 1.1 depend only on the dimension, the upper and lower bound of the Hessian determinant. They do not depend on the bounds on eigenvalues of the Hessian matrix D^2u. Properties of the Green's function g_V have played an important role in establishing Sobolev inequality for the Monge–Ampère quasi-metric structure [8,11].

Remark 1.2. In Theorem 1.1 (iii), we can choose

$$p_* = 1 + \frac{\varepsilon}{2 + \varepsilon}$$

where $\varepsilon = \varepsilon(n, \lambda, \Lambda)$ comes from De Philippis–Figalli–Savin and Schmidt's $W^{2,1+\varepsilon}$ estimates [3,10] for the Monge–Ampère equation satisfying (1.3) and (1.4). Thus $p_* \to 2$ when $\varepsilon \to \infty$. Hence, by Caffarelli's $W^{2,p}$ estimates for the Monge–Ampère equations [1], we can take $p^* = 2$ when f is continuous.

Remark 1.3. In the case of Green's function of uniformly elliptic operators, Theorem 1.1 (iii) with all $p < 2$ is attributed to Stampacchia. In higher dimensions, Grüter and Widman [4] proved the L^p integrability of the gradient of the Green's function for all $p < \frac{n}{n-1}$. It would be interesting to prove the L^p integrability for some $p > 1$ for the gradient of the Green's function of the linearized Monge–Ampère operator in dimensions $n \geq 3$.

As a corollary, we use the sharp lower bound for the Green's function in Theorem 1.1 to prove a removable singularity result for the linearized Monge–Ampère equation.

Corollary 1.4. *Assume that* $V \subset\subset \Omega$ *and* $\lambda \leq \det D^2 u \leq \Lambda$ *in* Ω. *Suppose that a function* v *solves* $U^{ij} v_{ij} = 0$ *in* $S_u(0, R)\backslash\{0\} \subset V$ *and satisfies*

$$|v(x)| = \begin{cases} o(r^{\frac{2-n}{2}}) & \text{if } n \geq 3 \\ o(|\log r|) & \text{if } n = 2 \end{cases} \quad \text{on } \partial S_u(0, r) \text{ as } r \to 0.$$

Then v *has a removable singularity at* 0.

1.2. Previous results

Various properties of the Green's function of the linearized Monge–Ampère operator L_u under different conditions on μ have been studied by several authors, including Tian–Wang [11] and Maldonado [8]. Tian–Wang [11] proved a decay estimate for the distribution function of g_V under an (A_∞) weight condition on μ [called (**CG**) there] and certain conditions on the size of sections of u.

Proposition 1.5. ([11, Lemma 3.3]) *Assume that* μ *satisfies the structure condition:* **CG**. *For any given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any convex set* $S \subset \Omega$ *and any set* $E \subset S$, *if* $|E| \leq \delta |S|$, *then* $\mu(E) \leq \varepsilon \mu(S)$ *where* $|\cdot|$ *denotes the Lebesgue measure. Suppose that for any section* $S_u(x, h) \subset \Omega$ *of* u, *we have*

$$C_1 |S_u(x, h)|^{1+\theta} \leq \mu(S_u(x, h)) \leq C_2 |S_u(x, h)|^{\frac{1}{n-1}+\sigma},$$

where $\theta \geq 0, C_1, C_2, \sigma > 0$ *are constants. Then, for any* $y \in V$,

$$\mu\{x \in V : g_V(x, y) > t\} \leq K t^{-\frac{n(1+\theta)}{(n-1)(1+\theta)-1}}.$$

When μ satisfies (1.5) only, and $V = S_u(x, t)$, Maldonado [8] obtained a similar result on the decay estimate for the distribution function of g_V. His result can be stated as follows.

Proposition 1.6. ([8, Theorem 3]) *Suppose* $V = S_u(x, t) \subset\subset \Omega$. *There exists a constant* K_1 *depending only on* n, α, β *such that for all* $z \in S_u(x, t/2)$, *we have*

$$\mu(\{y \in V : g_V(y, z) > T\}) \leq K_1 (\mu(S_u(x, t)))^{-\frac{1}{n-1}} t^{\frac{n}{n-1}} T^{-\frac{n}{n-1}} \; \forall T > 0.$$

Remark 1.7. 1. If u satisfies (1.3) and (1.4), then in dimensions $n \geq 3$, Proposition 1.5 gives a sharp upper bound for g_V. In particular, for small t and $x \in \partial S_u(x_0, t)$, we have

$$g_V(x, x_0) \leq Ct^{-\frac{n-2}{2}}.$$

2. If u satisfies (1.3) and (1.5), then Proposition 1.6 gives a sharp upper bound for g_V in dimensions $n \geq 3$ when V is a section of u. When $V = S_u(x_0, t)$, we have

$$g_V(x, x_0) \leq K_1^{\frac{n-1}{n}} t[\mu(S_u(x_0, t)]^{-\frac{1}{n}}[\mu(S_u(x_0, s))]^{-\frac{n-1}{n}}$$
$$\forall\, x \in \partial S_u(x_0, s) \quad (0 < s < t).$$

In particular, by Lemma 2.4, we have

$$g_V(x, x_0) \leq C(K_1, \alpha, \beta)t[\mu(S_u(x_0, t)]^{-1}\ \forall\, x \in \partial S_u(x_0, t/2).$$

For reader's convenient, we will prove the estimates in this remark in Sect. 3.

The proof of (1.2) in [7] was based on potential theory employing capacity and the fundamental result of De Giorgi–Nash–Moser on Hölder continuity of solutions of uniformly elliptic equations in divergence form. Our proof of Theorem 1.1 (i) is based on the fundamental result of Caffarelli–Gutiérrez [2] on Hölder continuity of solutions of the linearized Monge–Ampère equation. We find a direct argument for Theorem 1.1 (i) without using capacity; see Sect. 3. We also provide an alternate proof for the lower bound of the Green's function in Theorem 1.1 using capacity; see Sect. 4. This potential theoretic approach works for general doubling Monge–Ampère measures, thus allowing us to prove Theorem 1.1 (ii); one of the key ingredients here is Maldonado's Harnack inequality [9] for linearized Monge–Amnpère equations under a doubling condition. The proof of Theorem 1.1 (iii) makes use of De Philippis–Figalli–Savin and Schmidt's $W^{2,1+\varepsilon}$ estimates [3,10] for the Monge–Ampère equation that are valid for all dimensions and the L^q integrability of the Green's function for all finite q in two dimensions.

2. Preliminaries

Throughout, we denote by c, C positive constants depending on λ, Λ, n, α, β, and their values may change from line to line whenever there is no possibility of confusion. We refer to such constants as *universal constants*.

2.1. Monge–Ampère measure bounded away from 0 *and* ∞

In this section, we assume that

$$\lambda \leq \det D^2u \leq \Lambda \text{ in } \Omega.$$

Throughout, we use the following volume growth for compactly supported sections:

Lemma 2.1. *If* $S_u(x,t) \subset\subset \Omega$ *then*

$$c_1(n,\lambda,\Lambda)t^{\frac{n}{2}} \leq |S_u(x,t)| \leq C_1(n,\lambda,\Lambda)t^{\frac{n}{2}}.$$

The Caffarelli–Gutiérrez's Harnack inequality for the linearized Monge–Ampère equation states:

Theorem 2.2. ([2]) *For each compactly supported section* $S_u(x,t) \subset\subset \Omega$*, and any nonnegative solution* v *of* $L_u v = 0$ *in* $S_u(x,t)$*, we have*

$$\sup_{S_u(x,\tau t)} v \leq C \inf_{S_u(x,\tau t)} v$$

for universal τ, C.

Since the linearized Monge–Ampère operator $L_u v$ can be written in both divergence form and non-divergence form, Caffarelli–Gutiérrez's theorem is the affine invariant analogue of De Giorgi–Nash–Moser's theorem and also Krylov–Safonov's theorem on Hölder continuity of solutions of uniformly elliptic equations in nondivergence form. Theorem 2.2 will play an important role in our proof of the main result.

We also need the following Vitali type covering lemma.

Lemma 2.3. (Vitali covering, [3]) *Let* D *be a compact set in* Ω *and assume that to each* $x \in D$ *we associate a corresponding section* $S_u(x,h) \subset\subset \Omega$*. Then we can find a finite number of these sections* $S_u(x_i,h_i)$*,* $i = 1,\ldots,m$*, such that*

$$D \subset \bigcup_{i=1}^{m} S_u(x_i,h_i), \text{ with } S_u(x_i,\delta h_i) \text{ disjoint},$$

where $\delta > 0$ *is a small constant that depends only on* λ*,* Λ *and* n*.*

2.2. Monge–Ampère measure satisfying a doubling condition

In this section, we assume that $\det D^2 u = \mu$ where μ satisfies (1.5). Then μ is *doubling with respect to the parameter* on the sections of u:

Lemma 2.4. ([5], Corollary 3.3.2) *If* $S_u(x,2t) \subset\subset \Omega$ *then there is a constant* β' *depending only on* n*,* β *and* α *such that*

$$\mu(S_u(x,2t)) \leq \beta' \mu(S_u(x,t)).$$

Maldonado [9], extending the work of Caffarelli–Gutiérrez, proved the following Harnack inequality for the linearized Monge–Ampère under minimal geometric condition, namely, the doubling condition (1.5).

Theorem 2.5. ([9]) *For each compactly supported section* $S_u(x,t) \subset\subset \Omega$*, and any nonnegative solution* v *of* $L_u v = 0$ *in* $S_u(x,t)$*, we have for*

$$\sup_{S_u(x,\tau t)} v \leq C \inf_{S_u(x,\tau t)} v$$

for universal τ, C *depending only on* n*,* β *and* α*.*

3. Bounding the Green's function

In this section, we prove Theorem 1.1 (i) and (iii) and Corollary 1.4. Assume throughout this section that (1.3) and (1.4) are satisfied.

The proof of Theorem 1.1 (i) relies on three Lemmas 3.1, 3.2 and 3.3. Lemma 3.1 gives the bounds for the Green's function $g_V(x, x_0)$ in the special case where V is itself a section of u centered at x_0. Lemma 3.2 estimates how the maximum of $g_V(x, x_0)$ on a section of u centered at x_0 changes when we pass to a concentric section with double height. Lemma 3.3 gives the upper bound for g_V near ∂V.

Lemma 3.1. *If* $V = S_u(x_0, t)$ *then*

$$g_V(x, x_0) \geq c(n, \lambda, \Lambda)t^{-\frac{n-2}{2}} \ \forall x \in S_u(x_0, t/2)$$

and

$$g_V(x, x_0) \leq C(n, \lambda, \Lambda)t^{-\frac{n-2}{2}} \ \forall x \in \partial S_u(x_0, t/2).$$

Lemma 3.2. *If* $S_u(x_0, 2t) \subset\subset V$, *then*

$$\max_{x \in \partial S_u(x_0,t)} g_V(x, x_0) \leq Ct^{-\frac{n-2}{2}} + \max_{z \in \partial S_u(x_0,2t)} g_V(z, x_0). \tag{3.1}$$

Lemma 3.3. *There exist constants* $r(V, \Omega, n, \lambda, \Lambda)$ *and* $C(V, \Omega, n, \lambda, \Lambda)$ *such that*

$$S_u(x_0, 2r) \subset\subset V \quad \text{and} \quad \max_{x \in \partial S_u(x_0,r)} g_V(x, x_0) \leq C(V, \Omega, n, \lambda, \Lambda). \tag{3.2}$$

Proof of Theorem 1.1. Part (i). We will prove the lower and upper bound for g_V.
Lower bound for g_V. Consider the following cases.
Case 1: $n \geq 3$ and $S_u(x_0, 2t) \subset\subset V$. In this case, the difference $w := g_V(x, x_0) - g_{S_u(x_0,2t)}(x, x_0)$ solves

$$U^{ij}w_{ij} = 0 \text{ in } S_u(x_0, 2t), \text{ with } w > 0 \text{ on } \partial S_u(x_0, 2t).$$

Thus, by the maximum principle, $w(x) \geq 0$ for $x \in S_u(x_0, t)$. It follows from Lemma 3.1 that

$$g_V(x, x_0) \geq g_{S_u(x_0,2t)}(x, x_0) \geq c(n, \lambda, \Lambda)t^{-\frac{n-2}{2}} \ \forall x \in S_u(x_0, t).$$

Case 2: $n = 2$ and $S_u(x_0, t^{1/2}) \subset\subset V$. Suppose that $S_u(x_0, 2h) \subset\subset V$. Then, the function

$$w(x) = g_V(x, x_0) - \inf_{y \in \partial S_u(x_0,2h)} g_V(y, x_0) - g_{S_u(x_0,2h)}(x, x_0)$$

satisfies

$$L_u w = 0 \text{ in } S_u(x_0, 2h) \text{ with } w \geq 0 \text{ on } \partial S_u(x_0, 2h).$$

By the maximum principle, we have $w \geq 0$ in $S_u(x_0, 2h)$. Thus, by Lemma 3.1, we find that

$$g_V(x, x_0) - \inf_{y \in \partial S_u(x_0, 2h)} g_V(y, x_0) \geq g_{S_u(x_0, 2h)}(x, x_0) \geq c \; \forall \, x \in S_u(x_0, h). \tag{3.3}$$

Choose an integer $k \geq 1$ such that $2^k \leq t^{-1/2} < 2^{k+1}$. Then

$$|\log \mathrm{t}| \leq Ck \quad \text{and} \quad 2^k t \leq t^{1/2}.$$

Applying (3.3) to $h = t, 2t, \ldots, 2^{k-1}t$, we get

$$\begin{aligned} \inf_{y \in \partial S_u(x_0, t)} g_V(y, x_0) \geq \inf_{y \in \partial S_u(x_0, 2t)} g_V(y, x_0) + c \geq \cdots \geq \inf_{y \in \partial S_u(x_0, 2^k t)} g_V(y, x_0) + kc \\ \geq kc \geq c |\log \mathrm{t}|. \end{aligned}$$

Upper bound for g_V. Our proof of the upper bound for g_V just follows from iterating the estimate in Lemma 3.2 and the upper bound for g_V near ∂V in Lemma 3.3.

Part (iii). Recall that in this part $n = 2$. Let $v(x) = g_V(x, x_0)$ and $S = S_u(x_0, r)$. Then the upper bound for v in Theorem 1.1 (i) implies that $v \in L^q(S)$ for all $q < \infty$ with the bound

$$\|v\|_{L^q(S)} \leq C(V, \Omega, \lambda, \Lambda, q, r). \tag{3.4}$$

By [9, Theorem 6.2], we have

$$\int_S U^{ij} v_i(x) v_j(x) \frac{1}{v(x)^2} dx \leq C(n, \lambda, \lambda) \frac{\mu(S)}{r} \leq C(n, \lambda, \lambda) \tag{3.5}$$

where we used the upper bound on volume of section in Lemma 2.1 in the last inequality. Next, we use the following inequality

$$U^{ij} v_i(x) v_j(x) \geq \frac{\det D^2 u |\nabla v|^2}{\Delta u}$$

whose simple proof can be found in [2, Lemma 2.1]. Thus, for all integrable function f we have

$$\begin{aligned} |\nabla v|^2 |f|^2 = (\Delta u |f|^2) \frac{|\nabla v|^2}{\Delta u} &\leq \frac{1}{\lambda} (\Delta u |f|^2) \frac{\det D^2 u |\nabla v|^2}{\Delta u} \\ &\leq \frac{1}{\lambda} (\Delta u |f|^2 v^2) U^{ij} v_i(x) v_j(x) \frac{1}{v(x)^2}. \end{aligned}$$

Integrating over S and using Cauchy-Schwartz inequality and (3.5), one finds

$$\begin{aligned} \int_S |\nabla v||f| &\leq \frac{1}{\sqrt{\lambda}} \left(\int_S U^{ij} v_i(x) v_j(x) \frac{1}{v(x)^2} \right)^{1/2} \left(\int_S \Delta u |f|^2 v^2 \right)^{1/2} \\ &\leq C(n, \lambda, \Lambda) \left(\int_S \Delta u |f|^2 v^2 \right)^{1/2}. \end{aligned} \tag{3.6}$$

By the De Philippis–Figalli–Savin and Schmidt's $W^{2,1+\varepsilon}$ estimates for the Monge–Ampère equation [3,10], there exists $\varepsilon = \varepsilon(n, \lambda, \Lambda) > 0$ such that $D^2u \in L^{1+\varepsilon}_{loc}(\Omega)$. Thus, by Hölder inequality,

$$\int_S \Delta u|f|^2 v^2 \le \left(\int_S (\Delta u)^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \left(\int_S |f|^{\frac{2(1+\varepsilon)}{\varepsilon}} v^{\frac{2(1+\varepsilon)}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}}$$
$$\le C(n, \lambda, \Lambda, r) \left(\int_S |f|^{\frac{2(1+\varepsilon)}{\varepsilon}} v^{\frac{2(1+\varepsilon)}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}}. \tag{3.7}$$

From (3.4), we find that $\left(\int_S |f|^{\frac{2(1+\varepsilon)}{\varepsilon}} v^{\frac{2(1+\varepsilon)}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}}$ is finite if $f \in L^{\frac{p}{p-1}}(S)$ where $\frac{p}{p-1} > \frac{2(1+\varepsilon)}{\varepsilon}$, or

$$p < 1 + \frac{\varepsilon}{2+\varepsilon} := p_*.$$

Combining (3.6) with (3.7) and (3.4), one finds that

$$\int_S |\nabla v||f| \le C(V, \Omega, n, p, \lambda, \Lambda, r)\|f\|_{L^{\frac{p}{p-1}}(S)}$$

for all $f \in L^{\frac{p}{p-1}}(S)$ where $1 < p < p_*$. Theorem 1.1 (iii) then follows from duality. □

Proof of Corollary 1.4. Let $\tilde{v}$ solves

$$\begin{cases} U^{ij}\tilde{v}_{ij} = 0 & \text{in } S_u(0, R), \\ \tilde{v} = v & \text{on } \partial S_u(0, R). \end{cases}$$

We will prove that $v = \tilde{v}$ in $S_u(0, R)\backslash\{0\}$. We only consider the case $n \ge 3$. The case $n = 2$ is similar. Let $w = \tilde{v} - v$ in $S_u(0, R)\backslash\{0\}$ and $M_r = \max_{\partial S_u(0,r)} |w|$. Let $\sigma(x) = g_{S_u(0,R)}(x, 0)$. By the lower bound for the Green's function in Theorem 1.1, it is obvious that

$$|w(x)| \le CM_r r^{\frac{n-2}{2}}\sigma(x) \quad \text{on } S_u(0, r).$$

Note that

$$U^{ij}(w - CM_r r^{\frac{n-2}{2}}\sigma(x))_{ij} = 0 \text{ in } S_u(0, R)\backslash S_u(0, r).$$

Thus, by the maximum principle in $S_u(0, R)\backslash S_u(0, r)$, we have

$$|w(x)| \le CM_r r^{\frac{n-2}{2}}\sigma(x) \text{ in } S_u(0, R)\backslash S_u(0, r).$$

Observe that

$$M_r = \max_{\partial S_u(0,r)} |v - \tilde{v}| \le M + \max_{\partial S_u(0,r)} |v|$$

where $M = \max_{\partial S_u(0,R)} |\tilde{v}|$. For each fixed $x \ne 0$, we can choose r small so that $x \notin S_u(0, r)$ and hence, by our hypothesis on the asymptotic behavior of v near 0,

$$|w(x)| \le CMr^{\frac{n-2}{2}}\sigma(x) + C\sigma(x)r^{\frac{n-2}{2}} \max_{\partial S_u(0,r)} |v| \to 0 \text{ as } r \to 0.$$

This proves $v = \tilde{v}$ in $S_u(0, R)\backslash\{0\}$. □

Proof of Lemma 3.1. By subtracting a linear function, we can assume that $u \geq 0$ and $u(x_0) = 0$. For simplicity, let us denote $\sigma(x) = g_V(x, x_0)$. Then on $V = S_u(x_0, t)$, σ satisfies

$$\begin{cases} L_u\sigma = \delta_{x_0} & \text{in } V, \\ \quad \sigma = 0 & \text{on } \partial V. \end{cases} \tag{3.8}$$

Multiplying both sides of (3.8) by $u(x) - t$ and integrating by parts twice, we get

$$\begin{aligned} -t = u(x_0) - t &= \int_V (L_u\sigma)(u - t) = \int_V -U^{ij}\sigma_{ij}(u - t) \\ &= \int_V -U^{ij}\sigma u_{ij} = \int_V -nf\sigma. \end{aligned}$$

The bounds on f then give the following bounds for the integral of σ:

$$\frac{t}{n\lambda} \geq \int_V \sigma \geq \frac{t}{n\Lambda}.$$

On the other hand, by the ABP estimate, for any $\varphi \in L^n(V)$, the solution ψ to

$$-U^{ij}\psi_{ij} = \varphi \text{ in } V, \ \ \psi = 0 \quad \text{on } \partial V,$$

satisfies

$$\begin{aligned} \left| \int_V \sigma(x)\varphi(x)dx \right| = |\psi(x_0)| &\leq C(n)|V|^{1/n} \left\| \frac{\varphi}{\det U} \right\|_{L^n(V)} \\ &\leq C(n, \lambda, \Lambda)|V|^{1/n} \|\varphi\|_{L^n(V)}. \end{aligned}$$

Here we used the identity $\det U = (\det D^2u)^{n-1}$. By duality, we obtain

$$\left(\int_V \sigma^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq C(n, \lambda, \Lambda)|V|^{1/n}.$$

This is essentially inequality (2.3) in [8]. Hence, by Lemma 2.1,

$$\|\sigma\|_{L^{\frac{n}{n-1}}(S_u(x_0,t))} \leq C(n, \lambda, \Lambda)t^{1/2}.$$

Let

$$K = (S_u(x_0, t) \setminus S_u(x_0, r_2t)) \cup S_u(x_0, r_1t)$$

where $0 < r_1 < 1/2 < r_2 < 1$. Then, by [5, Lemma 6. 5. 1] and Lemma 2.1, we can estimate

$$\begin{aligned} |K| &\leq n(1 - r_2)|S_u(x_0, t)| + |S_u(x_0, r_1t)| \\ &\leq C_1n(1 - r_2)t^{n/2} + C_1(r_1t)^{n/2} \leq \varepsilon^n t^{n/2} \end{aligned}$$

for

$$\varepsilon = \min \left\{ \frac{1}{2C_1(n, \lambda, \Lambda)n\Lambda}, \left(\frac{1}{2c_1} \right)^{1/n} \right\}$$

if $r_1, 1 - r_2$ are universally small. Then by Lemma 2.1,

$$\frac{c_1}{2} t^{n/2} \le |S_u(x_0, t) \backslash K| \le C_1 t^{n/2}. \tag{3.9}$$

On the other hand, by Holder inequality, we have

$$\int_K \sigma \le \|\sigma\|_{L^{\frac{n}{n-1}}(K)} |K|^{1/n} \le C(n, \lambda, \Lambda) t^{1/2} \varepsilon t^{1/2} = \frac{t}{2n\Lambda}.$$

It follows that

$$\frac{t}{n\lambda} \ge \int_{S_u(x_0,t)\backslash K} \sigma \ge \frac{t}{2n\Lambda}. \tag{3.10}$$

Given $0 < r_1 < r_2 < 1$ as above, we have

$$\sup_{S_u(x_0,t)\backslash K} \sigma \le C(n, \lambda, \Lambda) \inf_{S_u(x_0,t)\backslash K} \sigma. \tag{3.11}$$

Combining (3.9)–(3.11), we find that

$$C^{-1}(n, \lambda, \Lambda) t^{-\frac{n-2}{2}} \le \sigma(x) \le C(n, \lambda, \Lambda) t^{-\frac{n-2}{2}} \; \forall x \in S_u(x_0, t) \backslash K.$$

This line of argument is very similar to the proof of Lemma 5.1 in [6]. Since $r_2 > 1/2 > r_1$, we obtain the desired upper bound for $\sigma(x) = g_V(x, x_0)$ when $x \in \partial S_u(x_0, t/2)$ while from the maximum principle, we obtain the desired lower bound for $\sigma(x) = g_V(x, x_0)$ when $x \in S_u(x_0, t/2)$.

For completeness, we include the details of (3.11). By [5, Theorem 3.3.10], we can find a universal $\alpha \in (0, 1)$ such that for each $x \in S_u(x_0, t) \backslash K$, the section $S_u(x, \alpha t)$ satisfies

$$x_0 \notin S_u(x, \alpha t) \quad \text{and} \quad S_u(x, \alpha t) \subset S_u(x_0, t).$$

Using Lemma 2.3, we can find a collection of sections $S_u(x_i, \tau\alpha t)$ with $x_i \in S_u(x_0, t) \backslash K$ such that

$$S_u(x_0, t) \backslash K \subset \bigcup_{i \in I} S_u(x_i, \tau\alpha t)$$

and $S_u(x_i, \delta\tau\alpha t)$ are disjoint for some universal $\delta \in (0, 1)$. By using the volume estimates in Lemma 2.1, we find that $|I|$ is universally bounded. Now, we apply Theorem 2.2 to each $S_u(x_i, \alpha t)$ to obtain (3.11). □

Proof of Lemma 3.2. To prove (3.1), we consider

$$w(x) = g_V(x, x_0) - \inf_{y \in \partial S_u(x_0, 2t)} g_V(y, x_0) - g_{S_u(x_0, 2t)}(x, x_0).$$

It satisfies

$$L_u w = 0 \text{ in } S_u(x_0, 2t) \quad \text{with } w \ge 0 \text{ on } \partial S_u(x_0, 2t).$$

In $\overline{S_u(x_0, 2t)}$, w attains its maximum value on the boundary $\partial S_u(x_0, 2t)$. Thus, for $x \in \partial S_u(x_0, t)$, we have

$$g_V(x, x_0) - \inf_{y \in \partial S_u(x_0,2t)} g_V(y, x_0) - g_{S_u(x_0,2t)}(x, x_0) \leq \max_{z \in \partial S_u(x_0,2t)} w$$
$$= \max_{z \in \partial S_u(x_0,2t)} g_V(z, x_0) - \inf_{y \in \partial S_u(x_0,2t)} g_V(y, x_0)$$

since $g_{S_u(x_0,2t)}(x, x_0) = 0$ on $\partial S_u(x_0, 2t)$. This together with Lemma 3.1 gives

$$\max_{x \in \partial S_u(x_0,t)} g_V(x, x_0) \leq \max_{z \in \partial S_u(x_0,t)} g_{S_u(x_0,2t)}(z, x_0) + \max_{z \in \partial S_u(x_0,2t)} g_V(z, x_0)$$
$$\leq Ct^{-\frac{n-2}{2}} + \max_{z \in \partial S_u(x_0,2t)} g_V(z, x_0).$$

Therefore, (3.1) is proved. □

Proof of Lemma 3.3. The existence of $r(V, \Omega, n, \lambda, \Lambda)$ is easy to prove by the $C^{1,\alpha}$ estimate for u which implies in particular that $S_u(x_0, h) \subset B(x_0, Ch^{\alpha})$. We now prove

$$\max_{\partial S_u(x_0,r)} g_V(x, x_0) \leq C(V, \Omega, n, \lambda, \Lambda).$$

To do this, we first multiply $\sigma(x) := g_V(x, x_0)$ to $L_u \Phi$ for various choices of $\Phi = \Phi(x, u(x), Du(x))$ and then integrate by parts. Let ν be the unit outer-normal vector field on ∂V. Note that, on ∂V, we have $\nu = -\frac{D\sigma}{|D\sigma|}$. Integrating by parts, we get

$$\int_V (L_u \Phi)\sigma = \int_V -U^{ij}\sigma \Phi_{ij} = \int_V (U^{ij}\sigma)_i \Phi_j - \int_{\partial V} U^{ij}\sigma \Phi_j \nu_i$$
$$= \int_V -(U^{ij}\sigma)_{ij}\Phi + \int_{\partial V} U^{ij}\sigma_i \Phi \nu_j$$
$$= \Phi(x_0, u(x_0), Du(x_0)) - \int_V U^{ij}\Phi\sigma_i \frac{\sigma_j}{|D\sigma|}$$
$$= \Phi(x_0, u(x_0), Du(x_0)) - \int_V \Phi\rho dS. \quad (3.12)$$

Here, we denote

$$\rho = U^{ij}\sigma_i \frac{\sigma_j}{|D\sigma|}.$$

First, we choose $\Phi \equiv 1$. Then (3.12) gives

$$\int_{\partial V} \rho dS = 1. \quad (3.13)$$

Next, we choose $\Phi \equiv u$. Then, since

$$L_u u = -U^{ij}u_{ij} = -n \det D^2 u = -nf,$$

(3.12) gives

$$\int_V nf g_V(x, x_0)dx = \int_{\partial V} \rho u dS - u(x_0).$$

By Aleksandrov's maximum principle [5, Theorem 1.4.2], we have

$$|u(x_0)|, \max_{x\in\partial V} |u(x)| \leq C(V, \Omega, n, \lambda, \Lambda).$$

Combining these with (3.13), we get

$$\int_{S_u(x_0,2r)} g_V(x, x_0)dx \leq C(V, \Omega, n, \lambda, \Lambda).$$

Using the lower bound for volume of sections in Lemma 2.1 and Caffarelli–Gutiérrez's Harnack inequality in Theorem 2.2, we get the second inequality in (3.2). □

If we choose $\Phi \equiv |x|^2$ in (3.12) then, since $L_u\Phi = -2U^{ij}\delta_{ij} = -2\text{trace } U$, we get from (3.12) that

$$\int_V -2\text{trace } U\sigma = \int_V (L_u\Phi)\sigma = |x_0|^2 - \int_V |x|^2\rho dS.$$

Thus, by (3.13),

$$2\int_V \text{trace } U\sigma = \int_{\partial V} |x|^2\rho dS - |x_0|^2 \leq \max_{x\in\partial V} |x|^2 - |x_0|^2.$$

This combined with the lower bound of σ in Theorem 1.1 gives the following Corollary.

Corollary 3.4. *Assume that $V \subset\subset \Omega$ and $\lambda \leq \det D^2u \leq \Lambda$ in Ω. If $S_u(x_0, 2t) \subset\subset V$ or $S_u(x_0, t^{1/2}) \subset\subset V$ when $n = 2$ then, we have*

$$\int_{S_u(x_0,t)} \textit{trace } U \leq \begin{cases} C(n, \lambda, \Lambda)t^{\frac{n-2}{2}} \left(\max_{x\in\partial V} |x|^2 - |x_0|^2\right) & \textit{if } n \geq 3 \\ C(\lambda, \Lambda)|\log t|^{-1} \left(\max_{x\in\partial V} |x|^2 - |x_0|^2\right) & \textit{if } n = 2 \end{cases}. \tag{3.14}$$

We end this section with the proof of Remark 1.7.

Proof of Remark 1.7. **1**. In dimensions $n \geq 3$, we can establish the upper bound for g_V by using Proposition 1.5. When u satisfies (1.3) and (1.4), this proposition says that

$$|\{x \in V : g_V(x, x_0) > T\}| < K(V, \Omega, n, \lambda, \Lambda)T^{-\frac{n}{n-2}}.$$

We show that for small t and $x \in \partial S_u(x_0, t)$

$$g_V(x, x_0) \leq \left(\frac{K}{c_1}\right)^{\frac{n-2}{n}} t^{-\frac{n-2}{2}}$$

where c_1 is the constant in Lemma 2.1. Indeed, assume that for some $t > 0$, we have

$$T = \max_{x \in \partial S_u(x_0,t)} g_V(x, x_0) > \left(\frac{K}{c_1}\right)^{\frac{n-2}{n}} t^{-\frac{n-2}{2}}.$$

Then, by the maximum principle,

$$S_u(x_0, t) \subset \{x \in V : g_V(x, x_0) > T\}.$$

It follows from the lower bound on the volume of sections in Lemma 2.1 that

$$c_1 t^{\frac{n}{2}} \leq |S_u(x_0, t)| \leq |\{x \in V : g_V(x, x_0) > T\}| \leq K(V, n, \lambda, \Lambda) T^{-\frac{n}{n-2}}$$
$$< K\left(\left(\frac{K}{c_1}\right)^{\frac{n-2}{n}} t^{-\frac{n-2}{2}}\right)^{-\frac{n}{n-2}} = c_1 t^{\frac{n}{2}}.$$

This is a contradiction. Thus, we must have the desired upper bound.
2. The proof using Proposition 1.6 is similar to the above case and is thus omitted.
□

4. Capacity and lower bound for the Green's function

In this section, we bound the Green's function using capacity in potential theory and give the proof for the lower bound of the Green's function in Theorem 1.1 (ii).

Let u be convex with compact sections and satisfies the Monge–Ampère equation (1.3) with (1.5). Let V be a fixed, open, bounded set in $\mathbb{R}^n$ and let K be a closed subset of V. We define the capacity of K with respect to the linearized Monge–Ampère operator $L_u := -U^{ij}\partial_{ij}$ and the set V as the infimum of

$$Q_u(\Phi) = \int_V U^{ij} \Phi_i \Phi_j$$

among functions $\Phi \in H_0^1(V)$ satisfying $\Phi \geq 1$ on K. This infimum will be denoted by $cap_{L_u}(K, V)$. In what follows, our arguments do not depend on the lower and upper bounds of the eigenvalues of the matrix (U^{ij}). Thus, when necessary, we can assume that L_u is uniformly elliptic. In particular, we obtain as in [7] the following theorem:

Theorem 4.1. *Suppose that $S_u(x_0, 2t) \subset\subset V$. Let g_V be the Green's function for L_u in V. Then there is a constant $C(n, \alpha, \beta)$ such that for all $x \in \partial S_u(x_0, t)$*

$$C^{-1}\left[cap_{L_u}(\overline{S_u(x_0, t)}, V)\right]^{-1} \leq g_V(x, x_0) \leq C\left[cap_{L_u}(\overline{S_u(x_0, t)}, V)\right]^{-1}.$$

Proof of the lower bound of the Green's function in Theorem 1.1 (ii). In view of Theorem 4.1 and the maximum principle, the lower bound for the Green's function in Theorem 1.1 (ii) follows from the following capacity estimates:

$$cap_{L_u}(\overline{S_u(x_0, t)}, V) \leq \begin{cases} C(n, \alpha, \beta)\mu(S_u(x_0, t))t^{-1} & \text{if } n \geq 3 \\ \dfrac{8}{|\log \mathrm{t}|^2} \displaystyle\int_t^{t^{1/2}} \frac{\mu(S_u(x_0, s))ds}{s^2} & \text{if } n = 2. \end{cases}$$

We will prove these estimates in Lemmas 4.2 and 4.3 below. □

Lemma 4.2. *Assume $n \geq 3$. Suppose that $S_u(x_0, 2t) \subset\subset V$. Then*

$$cap_{L_u}(\overline{S_u(x_0,t)}, V) \leq C(n, \alpha, \beta)\mu(S_u(x_0,t))t^{-1}.$$

Lemma 4.3. *Assume $n = 2$. Suppose that $S_u(x_0, t^{1/2}) \subset\subset V$ and $0 < t < 1$. Then*

$$cap_{L_u}(\overline{S_u(x_0,t)}, V) \leq \frac{8}{|log\ t|^2}\int_t^{t^{1/2}} \frac{\mu(S_u(x_0,s))ds}{s^2}. \tag{4.1}$$

Remark 4.4. Lemma 4.2 can be deduced from the proof of [9, Theorem 7.2]. We present here a slightly different proof whose idea leads to the sharp bound for capacity in dimensions 2 in Lemma 4.3.

We now prove Lemmas 4.2 and 4.3. By subtracting a linear function, we can assume that $u \geq 0,\ u(x_0) = 0$. Then $u = s$ on $\partial S_u(x_0, s)$ for all $s > 0$. In the proofs of Lemmas 4.2 and 4.3, we use the following general fact:

Lemma 4.5. *We have*

$$\int_{\partial S_u(x_0,s)} U^{ij}\frac{u_i u_j}{|\nabla u|} = \int_{S_u(x_0,s)} n \det D^2 u.$$

Proof of Lemma 4.5. Let ϕ be any smooth function. Let $\nu = (\nu_1, \ldots, \nu_n)$ be the unit outer-normal to $\partial S_u(x_0, s)$. Then, integrating by parts twice, and noting that $\nu = \frac{\nabla u}{|\nabla u|}$ on $\partial S_u(x_0, s)$, we get

$$\begin{aligned}\int_{S_u(x_0,s)} (L_u\phi)u &= \int_{S_u(x_0,s)} -U^{ij}\phi_{ij}u = \int_{S_u(x_0,s)} U^{ij}\phi_i u_j - \int_{\partial S_u(x_0,s)} U^{ij}\phi_i u\nu_j \\ &= \int_{S_u(x_0,s)} -U^{ij}u_{ij}\phi + \int_{\partial S_u(x_0,s)} U^{ij}u_j\nu_i\phi - \int_{\partial S_u(x_0,s)} U^{ij}\phi_i u\nu_j \\ &= \int_{S_u(x_0,s)} -U^{ij}u_{ij}\phi + \int_{\partial S_u(x_0,s)} U^{ij}\frac{u_i u_j}{|\nabla u|}\phi - \int_{\partial S_u(x_0,s)} U^{ij}\frac{\phi_i u_j}{|\nabla u|}u.\end{aligned}$$

With $\phi \equiv 1$, using $U^{ij}u_{ij} = n \det D^2 u$, we obtain the equality claimed in the lemma. □

Proof of Lemma 4.2. Let us consider $h(x) = \gamma(u(x))$ where

$$\gamma(s) = \begin{cases} 1 & \text{if } s \leq t \\ \frac{t^{\frac{n-2}{2}}}{1-(1/2)^{\frac{n-2}{2}}}\left(\frac{1}{s^{\frac{n-2}{2}}} - \frac{1}{(2t)^{\frac{n-2}{2}}}\right) & \text{if } t \leq s \leq 2t \\ 0 & \text{if } s \geq 2t. \end{cases}$$

Then

$$h \in H_0^1(S_u(x_0, 2t)) \quad \text{and} \quad h \equiv 1 \text{ in } S_u(x_0, t).$$

We have

$$\nabla h(x) = \gamma'(u(x))\nabla u(x) = -\frac{n-2}{2}\frac{t^{\frac{n-2}{2}}}{1-(1/2)^{\frac{n-2}{2}}}u^{-\frac{n}{2}}\nabla u(x).$$

Therefore, by the coarea formula and Lemma 4.5, we get

$$\int_V U^{ij} h_i h_j = \left[\frac{n-2}{2} \frac{t^{\frac{n-2}{2}}}{1-(1/2)^{\frac{n-2}{2}}}\right]^2 \int_{S_u(x_0,2t)\setminus S_u(x_0,t)} U^{ij} \frac{u_i u_j}{u^n}$$
$$\leq C(n) t^{n-2} \int_t^{2t} \left(\int_{\partial S_u(x_0,s)} U^{ij} \frac{u_i u_j}{s^n} \frac{1}{|\nabla u|}\right) ds$$
$$= C(n) t^{n-2} \int_t^{2t} \frac{\mu(S_u(x_0,s))}{s^n} ds$$
$$\leq C(n,\alpha,\beta) \mu(S_u(x_0,t)) t^{-1}$$

where in the last inequality we used Lemma 2.4 which says that

$$\mu(S_u(x_0,s)) \leq C\mu(S_u(x_0,t)) \quad \text{for } t \leq s \leq 2t.$$

We now find from the definition of capacity that

$$cap_{L_u}(\overline{S_u(x_0,t)}, V) \leq \int_V U^{ij} h_i h_j \leq C(n,\alpha,\beta) \mu(S_u(x_0,t)) t^{-1}.$$

□

Proof of Lemma 4.3. Let us consider $h(x) = \gamma(u(x))$ where γ is the logarithmic cut off function

$$\gamma(s) = \chi_{(-\infty,t)}(s) + (2\log s/\log t - 1)\chi_{[t,t^{1/2}]}(s).$$

Then

$$h \in H_0^1(S_u(x_0,t^{1/2})) \quad \text{and} \quad h \equiv 1 \text{ in } S_u(x_0,t).$$

We have

$$\nabla h(x) = \gamma'(u(x))\nabla u(x) = \frac{2}{u \log \mathrm{t}} \nabla u(x).$$

Therefore, by the coarea formula and Lemma 4.5, we get

$$\int_V U^{ij} h_i h_j = \frac{4}{|\log \mathrm{t}|^2} \int_{S_u(x_0,t^{1/2})\setminus S_u(x_0,t)} U^{ij} \frac{u_i u_j}{u^2}$$
$$= \frac{4}{|\log t|^2} \int_t^{t^{1/2}} \left(\int_{\partial S_u(x_0,s)} U^{ij} \frac{u_i u_j}{s^2} \frac{1}{|\nabla u|}\right) ds$$
$$= \frac{4}{|\log \mathrm{t}|^2} \int_t^{t^{1/2}} \left(\frac{1}{s^2} \int_{S_u(x_0,s)} n \det D^2 u\right) ds$$
$$= \frac{8}{|\log \mathrm{t}|^2} \int_t^{t^{1/2}} \frac{\mu(S_u(x_0,s)) ds}{s^2}.$$

In the last equality, we used $n = 2$. By the definition of capacity, we obtain (4.1).

□

Sketch of proof of Theorem 4.1. We sketch here the proof of Theorem 4.1, following [7]. We can assume that $L_u := -U^{ij}\partial_{ij}$ is uniformly elliptic. The set of functions $\Phi \in H_0^1(V)$ satisfying $\Phi \geq 1$ on K is a closed convex set and $H_0^1(V)$ is a Hilbert space. It is then easy to see that there is a unique function $\Phi \in H_0^1(V)$ satisfying $\Phi \geq 1$ on K and

$$cap_{L_u}(K, V) = Q_u(\Phi).$$

This function Φ is called the *capacitary potential* of the set K with respect to the operator L_u and the set V. Moreover, by a simple truncation argument, we find that this Φ satisfies $\Phi \equiv 1$ on K.

The capacitary potential Φ of the compact set K with respect to the operator L_u and the set V has the following properties:

(i) $\Phi \equiv 1$ on K, $\Phi = 0$ on ∂V, $0 \leq \Phi \leq 1$ on $V \setminus K$.
(ii) $L_u\Phi = 0$ on $V \setminus K$.
(iii) For all $\varphi \in H_0^1(V)$ with $\varphi \geq 0$ on K, we have

$$\int_V U^{ij}\Phi_i\varphi_j \geq 0.$$

From (iii) and Schwartz's theorem on positive distributions, there is a nonnegative measure μ on K, called the *capacitary distribution* of K with respect to the operator L_u and the set V, such that

$$\int_V U^{ij}\Phi_i\varphi_j = \int_V \varphi d\mu \quad \text{for all } \varphi \in H_0^1(V) \text{ with } \varphi \geq 0 \text{ on } K. \tag{4.2}$$

Since $\Phi \equiv 1$ on K, the support of μ is on ∂K. Choosing $\varphi = \Phi$ in the above equation, we find that

$$\mu(K) = cap_{L_u}(K, V). \tag{4.3}$$

Moreover, we find from (4.2) that $L_u\Phi = \mu$ in V. Thus, we have the representation

$$\Phi(y) = \int_V g_V(x, y)d\mu(x)$$

where we recall that $g_V(x, y)$ is the Green's function of L_u in V.

Consider the set

$$J_a = \{x \in V : g_V(x, x_0) \geq a\}.$$

Let ν_a be the capacitary distribution of J_a with respect to the operator L_u and the set V. Then the capacitary potential of J_a with respect to the operator L_u and the set V is equal to 1 at x_0. Thus

$$1 = \int_V g_V(x, x_0)d\nu_a(x).$$

The support of ν_a is on ∂J_a where $g_V(x, x_0) = a$. Thus, (4.3) gives

$$cap_{L_u}(J_a, V) = \frac{1}{a}.$$

Let $a = \min_{x\in\partial S_u(x_0,t)} g_V(x, x_0)$. Then, by the maximum principle $\overline{S_u(x_0,t)} \subset J_a$. Therefore

$$cap_{L_u}(\overline{S_u(x_0,t)}, V) \le cap_{L_u}(\overline{J_a}, V) = \frac{1}{a} = \frac{1}{\min_{x\in\partial S_u(x_0,t)} g_V(x, x_0)}.$$

Similarly, if we let $b = \max_{x\in\partial S_u(x_0,t)} g_V(x, x_0)$. Then

$$cap_{L_u}(\overline{S_u(x_0,t)}, V) \le cap_{L_u}(\overline{J_b}, V) = \frac{1}{b} = \frac{1}{\max_{x\in\partial S_u(x_0,t)} g_V(x, x_0)}.$$

It follows that

$$\min_{x\in\partial S_u(x_0,t)} g_V(x, x_0) \le (cap_{L_u}(\overline{S_u(x_0,t)}, V))^{-1} \le \max_{x\in\partial S_u(x_0,t)} g_V(x, x_0). \quad (4.4)$$

Since $g_V(x, x_0)$ is a positive solution of $L_u g_V(\cdot, x_0)$ in $V\backslash\{x_0\}$, by Theorem 2.5, for each t where $S_u(x_0, 2t) \subset\subset V$, we have

$$\max_{x\in\partial S_u(x_0,t)} g_V(x, x_0) \le \beta \min_{x\in\partial S_u(x_0,t)} g(x, x_0).$$

This combined with (4.4) gives the desired conclusion. □

Acknowledgements. The research of the author was supported in part by the National Science Foundation under grant DMS-1500400 and an Indiana University Summer Faculty Fellowship.

References

[1] Caffarelli, L.A.: Interior $W^{2,p}$ estimates for solutions to the Monge–Ampère equation. Ann. Math. **131**(1), 135–150 (1990)
[2] Caffarelli, L.A., Gutiérrez, C.E.: Properties of the solutions of the linearized Monge–Ampère equation. Am. J. Math. **119**(2), 423–465 (1997)
[3] De Philippis, G., Figalli, A., Savin, O.: A note on interior $W^{2,1+\varepsilon}$ estimates for the Monge–Ampère equation. Math. Ann. **357**, 11–22 (2013)
[4] Grüter, M., Widman, K.O.: The Green function for uniformly elliptic equations. Manuscr. Math. **37**(3), 303–342 (1982)
[5] Gutiérrez, C.E.: The Monge–Ampère Equation. Birkhaüser, Boston (2001)
[6] Le, N.Q., Savin, O.: Some minimization problems in the class of convex functions with prescribed determinant. Anal. PDE. **6**(5), 1025–1050 (2013)
[7] Littman, W., Stampacchia, G., Weinberger, H.F.: Regular points for elliptic equations with discontinuous coefficients. Ann. Scuola Norm. Sup. Pisa (3) **17**, 43–77 (1963)
[8] Maldonado, D.: The Monge–Ampère quasi-metric structure admits a Sobolev inequality. Math. Res. Lett. **20**(3), 527–536 (2013)
[9] Maldonado, D.: On the $W^{2,1+\varepsilon}$ estimates for the Monge–Ampère equation and related real analysis. Calc. Var. Partial Differ. Equ. **50**(1-2), 94–114 (2014)
[10] Schmidt, T.: $W^{2,1+\varepsilon}$ estimates for the Monge–Ampère equation. Adv. Math. **240**, 672–689 (2013)
[11] Tian, G.J., Wang, X.J.: A class of Sobolev type inequalities. Methods Appl. Anal. **15**(2), 263–276 (2008)
[12] Trudinger, N.S., Wang, X.-J.: The Monge–Ampère equation and its geometric applications. Handbook of geometric analysis. No. 1, 467–524. Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA (2008)