manuscripta math. 148, 471–484 (2015) 

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# Harnack estimates for nonlinear heat equations with potentials in geometric flows

Received: 17 August 2014 / Accepted 27 April 2015
Published online: 21 May 2015

**Abstract.** Let $M$ be a closed Riemannian manifold with a family of Riemannian metrics $g_{ij}(t)$ evolving by geometric flow $\partial_t g_{ij} = -2S_{ij}$, where $S_{ij}(t)$ is a family of smooth symmetric two-tensors on $M$. In this paper we derive differential Harnack estimates for positive solutions to the nonlinear heat equation with potential:

$$\frac{\partial f}{\partial t} = \Delta f + \gamma(t) f \log f + aSf,$$

where $\gamma(t)$ is a continuous function on $t$, $a$ is a constant and $S = g^{ij}S_{ij}$ is the trace of $S_{ij}$. Our Harnack estimates include many known results as special cases, and moreover lead to new Harnack inequalities for a variety geometric flows.

## 1. Introduction

Let $M$ be a closed Riemannian $n$-manifold with a one parameter family of Riemannian metrics $g(t)$ evolving by the geometric flow

$$\frac{\partial}{\partial t} g_{ij} = -2S_{ij}, \tag{1}$$

where $S_{ij}(t)$ is a one parameter family of smooth symmetric two-tensors on $M$ and $t \in [0, T)$.

In a recent article [7], the present authors studied Harnack inequalities for all positive solutions to

$$\frac{\partial f}{\partial t} = -\Delta f + \gamma f \log f + aSf$$

where $\gamma$ and $a$ are constants. In the case where $S_{ij} = R_{ij}$, $\gamma = 0$ and $a = 1$, the above equation is Perelman's conjugate heat equation, and Harnack estimates for all positive solutions have been studied by Cao [2] and Kuang-Zhang [12].

The purpose of the current article is to study the forward nonlinear equations with potential terms under (1):

$$\frac{\partial f}{\partial t} = \Delta f + \gamma(t) f \log f + aSf, \tag{2}$$

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*Mathematics Subject Classification*: 53C44 · 53C21

DOI: 10.1007/s00229-015-0757-3

where $\gamma(t)$ is a funtion on $t$ and $a$ is a constant. In the Ricci flow case, the consideration of this equation is motivated by expanding gradient Ricci solitons. See [4] for more details. In the Ricci flow, Cao–Hamilton [3] proved various Harnack inequalities of (2) for $\gamma(t) \equiv 0$. For general geometric flows, many people have studied Harnack inequality for the time-dependant heat equation, see for instance [1,5,16]. For a positive solution $f$ of (2), let $u = -\log f$ and a direct computation tells us that $u$ satisfies

$$\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^2 + \gamma(t)u - aS. \tag{3}$$

Note that (2) and (3) are equivalent equations.

To state the main results, we introduce two quantities defined by Müller [17].

**Definition 1.** Suppose that $g(t)$ evolves by the geometric flow (1) and let $X = X^i \frac{\partial}{\partial x^i}$ be a vector field on $M$. One defines

$$\mathcal{H}(S_{ij}, X) = \frac{\partial S}{\partial t} + \frac{S}{t} - 2\nabla_i S X^i + 2S^{ij}X_iX_j,$$
$$\mathcal{D}(S_{ij}, X) = \frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2 + \left(4\nabla^i S_{i\ell} - 2\nabla_\ell S\right) X^\ell + 2\left(R^{ij} - S^{ij}\right) X_iX_j$$

where the upper indices are lifted by the metric, for instance $S^{ij} = g^{ik}g^{lj}S_{kl}$.

We notice that $\mathcal{H}$ and $\mathcal{D}$ were firstly introduced by Müller [17] to prove the monotonicity of Perelman type reduced volume under (1). Later on they were used to prove entropy monotonicity and Harnack inequalities in [6–8]. We also notice that when $M$ is static, namely when $S_{ij} = 0$ one has

$$\mathcal{H}(0, X) = 0, \quad \mathcal{D}(0, X) = R^{ij}X_iX_j.$$

In the Ricci flow, namely when $S_{ij} = R_{ij}$ one has

$$\mathcal{H}(R_{ij}, X) = \frac{\partial R}{\partial t} + \frac{R}{t} - 2\nabla_i R X^i + 2R^{ij}X_iX_j, \quad \mathcal{D}(R_{ij}, X) = 0$$

and in this case $\mathcal{H}$ is nothing but Hamilton's trace Harnack quantity.

For the Eq. (3) in the case where $a = 1$, we prove

**Theorem A.** Let $g(t)$ be a solution to the geometric flow (1) on a closed oriented smooth $n$-manifold $M$. Assume for all $X$ and $t \in [0, T)$, it holds

$$2\mathcal{H}(S_{ij}, X) + \mathcal{D}(S_{ij}, X) \geq 0, \;\; S \geq 0 \tag{4}$$

Let $u$ be a solution to

$$\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^2 + \gamma(t)u - S$$

with

$$-\frac{2}{t} \leq \gamma(t) \leq 0 \tag{5}$$

for all time $t \in (0, T)$. Then for all $t \in (0, T)$,

$$Q_S = 2\Delta u - |\nabla u|^2 - 3S - 2\frac{n}{t} \leq 0. \tag{6}$$

Notice that in [6], (6) was proved for $\gamma(t) \equiv 0$ under a slight different assumption. On the other hand, notice that (5) is not satisfied for all time $t \in (0, T)$ if $\gamma(t)$ is a nonzero constant. However, in the case where $\gamma(t) \equiv -1$, we are able to prove a similar result as follows:

**Theorem B.** Let $g(t)$ be a solution to the geometric flow (1) on a closed oriented smooth $n$-manifold $M$. Assume that (4) holds, namely $2\mathcal{H}(S_{ij}, X) + \mathcal{D}(S_{ij}, X) \geq 0$ and $S \geq 0$. Let $u$ be a solution to

$$\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^2 - u - S$$

then for all time $t \in (0, T)$, the following holds:

$$Q_S = 2\Delta u - |\nabla u|^2 - 3S - 2\frac{n}{t} \leq \frac{n}{4}. \tag{7}$$

For the Eq. (2) in the case where $a = 0$, we shall prove

**Theorem C.** Suppose that $g(t)$, $t \in [0, T)$, evolves by the geometric flow (1) on a closed oriented smooth $n$-manifold $M$ with

$$\mathcal{I}(S_{ij}, X) := \left(R^{ij} - S^{ij}\right) X_i X_j \geq 0 \tag{8}$$

for all $X$ and all time $t \in [0, T)$. Let $0 < f < 1$ be a positive solution to

$$\frac{\partial f}{\partial t} = \Delta f + \gamma(t) f \log f,$$

and $u = -\log f$. If $\gamma(t) \leq 0$ for all time $t \in [0, T)$, then

$$|\nabla u|^2 - \frac{u}{t} \leq 0 \tag{9}$$

holds for all time $t \in (0, T)$.

We notice that the above theorems in the case where $S_{ij} = R_{ij}$ imply the results proved in [4] as special cases. In Theorems A and B, the assumptions are the same as stated by (4). In Theorem C, the assumption is (8). In the following section we will discuss the assumptions in various geometric flows, and replace them by natural geometric assumptions in the corresponding flow. The rest of the article is devoted to proving the main theorems.

## 2. Examples

**(1) Static Riemannian manifold.** In this case $S_{ij} = 0$, $\mathcal{H} = 0$ and $\mathcal{D} = R^{ij} X_i X_j$. Thus the assumptions in Theorems A, B and C can be replaced by $R_{ij} \geq 0$.
**(2) The Ricci flow.** In this case $S_{ij} = R_{ij}$. Therefore, (4) is equivalent to

$$\mathcal{H}(S_{ij}, X) = \frac{\partial R}{\partial t} + \frac{R}{t} - 2\nabla_i R X^i + 2R^{ij} X_i X_j \geq 0, \;\; R \geq 0.$$

It is known [10] that these conditions are satisfied if the initial metric $g(0)$ has weakly positive curvature operator. Hence, the assumptions in Theorems A and B hold if $g(0)$ has weakly positive curvature operator. Moreover, the assumption (8) in Theorem C is automatically satisfied. Notice that Theorem A in the case where $\gamma(t) = -2/(t + 2)$ is Theorem 1.2 in [4]. On the other hand, our Theorem B is Theorem 1.1 in [4]. Theorem C in the case where $\gamma(t) = -1$ is nothing but Theorem 4.1 in [22].

**(3) List's extended Ricci flow.** In this case, $S_{ij} = R_{ij} - 2\nabla_i\psi\nabla_j\psi$ we have

$$\mathcal{D}(S_{ij}, X) = 4|\Delta\psi - \nabla_X\psi|^2$$

In particular, (4) is particularly satisfied if

$$\mathcal{H}(S_{ij}, X) \geq 0,\ \ R(0) \geq 2|\nabla\psi|^2_{t=0}.$$

To the best our knowledge, it is still unknown whether $\mathcal{H}(S_{ij}, X) \geq 0$ is preserved by the Bernhard List's flow under a suitable assumption. The Ricci flow case is due to Hamilton [10] as we already mentioned. On the other hand, (8) holds automatically.

**(4) Müller's Ricci flow coupled with harmonic map flow.** In this case, $S_{ij} = R_{ij} - \alpha(t)\nabla_i\phi\nabla_j\phi$ and moreover $\mathcal{D}(S_{ij}, X) = 2\alpha(t)\left|\tau_g\phi - \nabla_X\phi\right|^2 - (\frac{\partial\alpha(t)}{\partial t})|\nabla\phi|^2$. Therefore $\mathcal{D}(S_{ij}, X) \geq 0$ holds if $\alpha(t) \geq 0$ and $\frac{\partial\alpha(t)}{\partial t} \leq 0$. In this case, (4) is particularly satisfied if

$$\mathcal{H}(S_{ij}, X) \geq 0,\ \ R(0) \geq \alpha(0)|\nabla\phi|^2_{t=0}.$$

As in the case of List's flow, it is unknown whether $\mathcal{H}(S_{ij}, X) \geq 0$ is preserved by Müller's flow under a suitable assumption. As in other examples (8) holds automatically.

## 3. General evolution equations

In this section, we shall prove general evolution equations of Harnack quantities under the geometric flow, which are useful to prove the main results. See Theorems 1 and 2 stated below. In the Ricci flow case, these general evolution equations are firstly proved by Cao and Hamilton [3]. Theorems 1 and 2 can be seen as generalizations of Lemma 2.1 and Lemma 3.1 in [3] respectively.

### *3.1. Case of $u = -\log f$*

Let $M$ be a closed Riemannian manifold with a Riemannian metric $g_{ij}(t)$ evolving by a geometric flow $\partial_t g_{ij} = -2S_{ij}$. Let $f$ be a positive solution of the following equation:

$$\frac{\partial f}{\partial t} = \Delta f + \gamma(t) f \log f - cSf, \tag{10}$$

where $c$ is a constant and $\gamma(t)$ is a function depneds on $t$. Let $u = -\log f$. A direct computation tells us that $u$ satisfies

$$\frac{\partial u}{\partial t} = \Delta u - |\nabla u|^2 + cS + \gamma(t)u. \tag{11}$$

We introduce

**Definition 2.** Suppose that $g(t)$ evolves by (1) and let $S$ be the trace of $S_{ij}$. Let $X = X^i\frac{\partial}{\partial x^i}$ be a vector field on $M$. And $a, \alpha$ and $\beta$ are constants. Then, one defines

$$\mathcal{D}_{(a,\alpha,\beta)}(S_{ij}, X) = a\left(\frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2\right) - \alpha\left(2\nabla^i S_{i\ell} - \nabla_\ell S\right)X^\ell + 2\beta(R^{ij} - S^{ij})X_i X_j.$$

Then we prove

**Proposition 1.** *Let $g(t)$ be a solution to the geometric flow* (1) *and $u$ satisfies* (11). *Let*

$$Q_S = \alpha\Delta u - \beta|\nabla u|^2 + aS - b\frac{u}{t} - d\frac{n}{t},$$

*where $\alpha$, $\beta$, $a$, $b$ and $d$ are constants. Then $Q_S$ satisfies*

$$\begin{aligned}\frac{\partial Q_S}{\partial t} &= \Delta Q_S - 2\nabla^i Q_S\nabla_i u + 2(a-\beta c)\nabla^i S\nabla_i u - 2(\alpha-\beta)|\nabla\nabla u|^2\\ &\quad - 2\alpha R^{ij}\nabla_i u\nabla_j u + 2\alpha S^{ij}\nabla_i\nabla_j u + \alpha c\Delta S - \frac{b}{t}|\nabla u|^2 - \frac{b}{t}cS + \frac{b}{t^2}u + d\frac{n}{t^2}\\ &\quad + 2a|S_{ij}|^2 + \mathcal{D}_{(a,\alpha,\beta)}(S_{ij}, -\nabla u) + \alpha\gamma(t)\Delta u - 2\beta\gamma(t)|\nabla u|^2 - b\frac{\gamma(t)}{t}u.\end{aligned}$$

*Proof.* First of all, notice that we have the following evolution equations, which follow from standard computations:

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta u) &= 2S^{ij}\nabla_i\nabla_j u + \Delta\left(\frac{\partial u}{\partial t}\right) - g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)\nabla_k u,\\ \frac{\partial}{\partial t}(|\nabla u|^2) &= 2S^{ij}\nabla_i u\nabla_j u + 2\nabla^i\left(\frac{\partial u}{\partial t}\right)\nabla_i u.\end{aligned}$$

On the other hand, we also get the following by standard computations:

$$g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right) = -g^{k\ell}\left(2\nabla^i S_{i\ell} - \nabla_\ell S\right).$$

By using these formulas and (11), we obtain the following:

$$\begin{aligned}\frac{\partial Q_S}{\partial t} &= \alpha\frac{\partial}{\partial t}(\Delta u) - \beta\frac{\partial}{\partial t}(|\nabla u|^2) + a\frac{\partial S}{\partial t} - \frac{b}{t}\frac{\partial u}{\partial t} + \frac{b}{t^2}u + d\frac{n}{t^2}\\ &= \alpha\left(2S^{ij}\nabla_i\nabla_j u + \Delta(\frac{\partial u}{\partial t}) - g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)\nabla_k u\right)\\ &\quad - \beta\left(2S^{ij}\nabla_i u\nabla_j u + 2\nabla^i\left(\frac{\partial u}{\partial t}\right)\nabla_i u\right) + a\frac{\partial S}{\partial t} - \frac{b}{t}\frac{\partial u}{\partial t} + \frac{b}{t^2}u + d\frac{n}{t^2}\\ &= \alpha\left(2S^{ij}\nabla_i\nabla_j u + \Delta(\Delta u - |\nabla u|^2 + cS + \gamma(t)u) + g^{k\ell}\left(2\nabla^i S_{i\ell} - \nabla_\ell S\right)\nabla_k u\right)\\ &\quad - \beta\left(2S^{ij}\nabla_i u\nabla_j u + 2\nabla^i(\Delta u - |\nabla u|^2 + cS + \gamma(t)u)\nabla_i u\right) + a\frac{\partial S}{\partial t}\\ &\quad - \frac{b}{t}\left(\Delta u - |\nabla u|^2 + cS + \gamma(t)u\right) + \frac{b}{t^2}u + d\frac{n}{t^2}\\ &= 2\alpha S^{ij}\nabla_i\nabla_j u + \alpha\Delta(\Delta u) - \alpha\Delta(|\nabla u|^2) + \alpha c\Delta S + \alpha\left(2\nabla^i S_{i\ell} - \nabla_\ell S\right)\nabla^\ell u\\ &\quad - 2\beta S^{ij}\nabla_i u\nabla_j u - 2\beta\nabla^i(\Delta u)\nabla_i u + 2\beta\nabla^i(|\nabla u|^2)\nabla_i u - 2\beta c\nabla^i S\nabla_i u\\ &\quad + \frac{b}{t}|\nabla u|^2 - \frac{b}{t}cS + \frac{b}{t^2}u + d\frac{n}{t^2} + a\frac{\partial S}{\partial t} - \frac{b}{t}\Delta u + \alpha\gamma(t)\Delta u\\ &\quad - 2\beta\gamma(t)|\nabla u|^2 - b\frac{\gamma(t)}{t}u.\end{aligned}$$

On the other hand, we also have the following by the definition of $Q_S$:

$$\begin{aligned}\Delta Q_S &= \alpha\Delta(\Delta u) - \beta\Delta(|\nabla u|^2) + a\Delta S - \frac{b}{t}\Delta u.\\ \nabla^i Q_S &= \alpha\nabla^i(\Delta u) - \beta\nabla^i(|\nabla u|^2) + a\nabla^i S - \frac{b}{t}\nabla^i u\end{aligned}$$

Therefore we get

$$\Delta Q_S - 2\nabla^i Q_S \nabla_i u = \alpha\Delta(\Delta u) - \beta\Delta(|\nabla u|^2) + a\Delta S - \frac{b}{t}\Delta u \\ - 2\alpha\nabla^i(\Delta u)\nabla_i u + 2\beta\nabla^i(|\nabla u|^2)\nabla_i u - 2a\nabla^i S\nabla_i u + \frac{2b}{t}|\nabla u|^2.$$

By using this, we are able to obtain

$$\frac{\partial Q_S}{\partial t} = \Delta Q_S - 2\nabla^i Q\nabla_i u + 2\alpha S^{ij}\nabla_i\nabla_j u - 2\beta S^{ij}\nabla_i u\nabla_j u - (\alpha-\beta)\Delta(|\nabla u|^2) \\ + (\alpha c - a)\Delta S + \alpha(2\nabla^i S_{i\ell} - \nabla_\ell S)\nabla^\ell u + 2(\alpha-\beta)\nabla^i(\Delta u)\nabla_i u \\ + 2(a-\beta c)\nabla^i S\nabla_i u - \frac{b}{t}|\nabla u|^2 + a\frac{\partial S}{\partial t} - \frac{b}{t}cS + \frac{b}{t^2}u + d\frac{n}{t^2} \\ + \alpha\gamma(t)\Delta u - 2\beta\gamma(t)|\nabla u|^2 - b\frac{\gamma(t)}{t}u.$$

On the other hand, we also have the following Bochner-Weitzenbock type formula:

$$\Delta(|\nabla u|^2) = 2|\nabla\nabla u|^2 + 2\nabla^i(\Delta u)\nabla_i u + 2R^{ij}\nabla_i u\nabla_j u.$$

By using this formula, we get

$$\frac{\partial Q_S}{\partial t} = \Delta Q_S - 2\nabla^i Q_S\nabla_i u + 2(a-\beta c)\nabla^i S\nabla_i u - 2(\alpha-\beta)|\nabla\nabla u|^2 \\ - 2\alpha R^{ij}\nabla_i u\nabla_j u + 2\alpha S^{ij}\nabla_i\nabla_j u + \alpha c\Delta S - \frac{b}{t}|\nabla u|^2 - \frac{b}{t}cS + \frac{b}{t^2}u + d\frac{n}{t^2} \\ + 2a|S_{ij}|^2 + a\left(\frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2\right) + \alpha(2\nabla^i S_{i\ell} - \nabla_\ell S)\nabla^\ell u \\ + 2\beta(R^{ij} - S^{ij})\nabla_i u\nabla_j u + \alpha\gamma(t)\Delta u - 2\beta\gamma(t)|\nabla u|^2 - b\frac{\gamma(t)}{t}u \\ = \Delta Q_S - 2\nabla^i Q_S\nabla_i u + 2(a-\beta c)\nabla^i S\nabla_i u - 2(\alpha-\beta)|\nabla\nabla u|^2 \\ - 2\alpha R^{ij}\nabla_i u\nabla_j u + 2\alpha S^{ij}\nabla_i\nabla_j u + \alpha c\Delta S - \frac{b}{t}|\nabla u|^2 - \frac{b}{t}cS + \frac{b}{t^2}u + d\frac{n}{t^2} \\ + 2a|S_{ij}|^2 + \mathcal{D}_{(a,\alpha,\beta)}(S_{ij}, -\nabla u) + \alpha\gamma(t)\Delta u - 2\beta\gamma(t)|\nabla u|^2 - b\frac{\gamma(t)}{t}u,$$

where we used Definition 2. □

**Theorem 1.** *Suppose that $\alpha \neq 0$ and $\alpha \neq \beta$. Then, the evolution equation in Proposition 1 can be rewritten as follows:*

$$\frac{\partial Q_S}{\partial t} = \Delta Q_S - 2\nabla^i Q_S\nabla_i u - 2(\alpha-\beta)\left|\nabla_i\nabla_j u - \frac{\alpha}{2(\alpha-\beta)}S_{ij} - \frac{\lambda}{2t}g_{ij}\right|^2 \\ + 2(a-\beta c)\nabla^i u\nabla_i S - \frac{2(\alpha-\beta)}{\alpha}\frac{\lambda}{t}Q_S + \frac{(\alpha-\beta)n\lambda^2}{2t^2} - \left(b + \frac{2(\alpha-\beta)\lambda\beta}{\alpha}\right)\frac{|\nabla u|^2}{t} \\ + \left(2a + \frac{\alpha^2}{2(\alpha-\beta)}\right)|S_{ij}|^2 + \left(\alpha\lambda - bc + \frac{2(\alpha-\beta)\lambda a}{\alpha}\right)\frac{S}{t} + \left(1 - \frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{b}{t^2}u \\ + \left(1 - \frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{d}{t^2}n + \alpha c\Delta S - 2\alpha R^{ij}\nabla_i u\nabla_j u + \mathcal{D}_{(a,\alpha,\beta)}(S_{ij}, -\nabla u) \\ + \alpha\gamma(t)\Delta u - 2\beta\gamma(t)|\nabla u|^2 - b\frac{\gamma(t)}{t}u,$$

*where $\lambda$ is a constant.*

*Proof.* First of all, notice that a direct computation implies

$$
\begin{aligned}
&-2(\alpha-\beta)\left|\nabla_i\nabla_j u-\frac{\alpha}{2(\alpha-\beta)}S_{ij}-\frac{\lambda}{2t}g_{ij}\right|^2=-2(\alpha-\beta)|\nabla\nabla u|^2+2\alpha S^{ij}\nabla_i\nabla_j u\\
&\quad+2(\alpha-\beta)\frac{\lambda}{t}\Delta u-\frac{\lambda\alpha}{t}S-\frac{\alpha^2}{2(\alpha-\beta)}|S_{ij}|^2-\frac{(\alpha-\beta)\lambda^2 n}{2t^2}.
\end{aligned}
$$

Therefore we get the following:

$$
\begin{aligned}
&-2(\alpha-\beta)|\nabla\nabla u|^2+2\alpha S^{ij}\nabla_i\nabla_j u+2a|S_{ij}|^2\\
&=-2(\alpha-\beta)\left|\nabla_i\nabla_j u-\frac{\alpha}{2(\alpha-\beta)}S_{ij}-\frac{\lambda}{2t}g_{ij}\right|^2-2(\alpha-\beta)\frac{\lambda}{t}\left(\Delta u-\frac{\alpha S}{2(\alpha-\beta)}\right)\\
&\quad+\frac{(\alpha-\beta)\lambda^2 n}{2t^2}+\left(2a+\frac{\alpha^2}{2(\alpha-\beta)}\right)|S_{ij}|^2.
\end{aligned}
$$

By this and Lemma 1, we obtain

$$
\begin{aligned}
\frac{\partial Q_S}{\partial t}&=\Delta Q_S-2\nabla^i Q_S\nabla_i u-2(\alpha-\beta)\left|\nabla_i\nabla_j u-\frac{\alpha}{2(\alpha-\beta)}S_{ij}-\frac{\lambda}{2t}g_{ij}\right|^2\\
&\quad+2(a-\beta c)\nabla^i S\nabla_i u-2(\alpha-\beta)\frac{\lambda}{t}\left(\Delta u-\frac{\alpha S}{2(\alpha-\beta)}\right)+\frac{(\alpha-\beta)\lambda^2 n}{2t^2}\\
&\quad+\left(2a+\frac{\alpha^2}{2(\alpha-\beta)}\right)|S_{ij}|^2+\alpha c\Delta S-2\alpha R^{ij}\nabla_i u\nabla_j u-\frac{b}{t}|\nabla u|^2-\frac{b}{t}cS\\
&\quad+\frac{b}{t^2}u+d\frac{n}{t^2}+\mathcal{D}_{(a,\alpha,\beta)}(S_{ij},-\nabla u)+\alpha\gamma(t)\Delta u-2\beta\gamma(t)|\nabla u|^2-b\frac{\gamma(t)}{t}u.
\end{aligned}
$$

On the other hand, we also get the following by using the definition of $Q_S$:

$$
\begin{aligned}
&-2(\alpha-\beta)\frac{\lambda}{t}\left(\Delta u-\frac{\alpha S}{2(\alpha-\beta)}\right)-\frac{b}{t}|\nabla u|^2-\frac{b}{t}cS+\frac{b}{t^2}u+d\frac{n}{t^2}\\
&=-\frac{2(\alpha-\beta)}{\alpha}\frac{\lambda}{t}Q_S-\left(b+\frac{2(\alpha-\beta)\lambda\beta}{\alpha}\right)\frac{|\nabla u|^2}{t}+\left(1-\frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{b}{t^2}u\\
&\quad+\left(\alpha\lambda-bc+\frac{2(\alpha-\beta)\lambda a}{\alpha}\right)\frac{S}{t}+\left(1-\frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{d}{t^2}n.
\end{aligned}
$$

Using this equation, we get the claim. □

As a special case, we obtain the following result:

**Corollary 1.** *Let $g(t)$ be a solution to the geometric flow (1) and $u$ satisfies*

$$
\frac{\partial u}{\partial t}=\Delta u-|\nabla u|^2-(a+4)S+\gamma(t)u.
$$

*Let*

$$
Q_S=2\Delta u-|\nabla u|^2+aS-d\frac{n}{t},
$$

*where a and d are constants. Then $Q_S$ satisfies*

$$\begin{aligned}\frac{\partial Q_S}{\partial t} &= \Delta Q_S - 2\nabla^i Q_S \nabla_i u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}\right|^2 - \left(\frac{2}{t} - \gamma(t)\right) Q_S \\ &\quad + \left(-\frac{2}{t} - \gamma(t)\right)|\nabla u|^2 - a\gamma(t)S + 2(a+2)\mathcal{H}(S_{ij}, -\nabla u) + \frac{n}{t^2}(2-d) + d\gamma(t)\frac{n}{t} \\ &\quad - \left((a+4)\frac{\partial S}{\partial t} - 2|S_{ij}|^2 + (3a+8)\Delta S\right) + 2\left(2\nabla^i S_{i\ell} - \nabla_\ell S\right)\nabla^\ell u \\ &\quad - 2\left(R^{ij} + (2a+5)S^{ij}\right)\nabla_i u \nabla_j u\end{aligned}$$

*Proof.* By Theorem 1 in the case where $\alpha = 2, \beta = 1, b = 0, c = -a-4, \lambda = 2$, we obtain

$$\begin{aligned}\frac{\partial Q_S}{\partial t} &= \Delta Q_S - 2\nabla^i Q_S \nabla_i u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}\right|^2 - \frac{2}{t}Q_S - \frac{2}{t}|\nabla u|^2 - 2(a+4)\Delta S \\ &\quad + 2(1+a)|S_{ij}|^2 + 2(a+2)\left(\frac{\partial S}{\partial t} + \frac{S}{t} + 2\nabla^i S \nabla_i u + 2S^{ij}\nabla_i u \nabla_j u\right) \\ &\quad - 2(a+2)\frac{\partial S}{\partial t} - 4(a+2)S^{ij}\nabla_i u\nabla_j u - 4R^{ij}\nabla_i u \nabla_j u + \frac{n}{t^2}(2-d) \\ &\quad + \mathcal{D}_{(a,2,1)}(S_{ij}, -\nabla u) \\ &\quad + 2\gamma(t)(\Delta u - |\nabla u|^2) \\ &= \Delta Q_S - 2\nabla^i Q_S \nabla_i u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}\right|^2 - \frac{2}{t}Q_S - \frac{2}{t}|\nabla u|^2 - 2(a+4)\Delta S \\ &\quad + 2(1+a)|S_{ij}|^2 + 2(a+2)\mathcal{H}(S_{ij}, -\nabla u) - 2(a+2)\frac{\partial S}{\partial t} - 4(a+2)S^{ij}\nabla_i u \nabla_j u \\ &\quad - 4R^{ij}\nabla_i u\nabla_j u + \frac{n}{t^2}(2-d) + \mathcal{D}_{(a,2,1)}(S_{ij}, -\nabla u) + 2\gamma(t)(\Delta u - |\nabla u|^2).\end{aligned}$$

Since we have

$$\begin{aligned}\mathcal{D}_{(a,2,1)}(S_{ij}, -\nabla u) = a\left(\frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2\right) + 2\left(2\nabla^i S_{i\ell} - \nabla_\ell S\right)\nabla^\ell u \\ + 2(R^{ij} - S^{ij})\nabla_i u \nabla_j u,\end{aligned}$$

we get

$$\begin{aligned}\frac{\partial Q_S}{\partial t} &= \Delta Q_S - 2\nabla^i Q_S \nabla_i u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}\right|^2 - \frac{2}{t}Q_S - \frac{2}{t}|\nabla u|^2 \\ &\quad + 2(a+2)\mathcal{H}(S_{ij}, -\nabla u) - 2\left(R^{ij} + (2a+5)S^{ij}\right)\nabla_i u\nabla_j u + 2\left(2\nabla^i S_{i\ell} - \nabla_\ell S\right)\nabla^\ell u \\ &\quad - \left((a+4)\frac{\partial S}{\partial t} - 2|S_{ij}|^2 + (3a+8)\Delta S\right) + \frac{n}{t^2}(2-d) + 2\gamma(t)(\Delta u - |\nabla u|^2).\end{aligned}$$

On the other hand, we also get the following by a direct computation:

$$2\gamma(t)(\Delta u - |\nabla u|^2) = \gamma(t)Q_S - \gamma(t)|\nabla u|^2 - a\gamma(t)S + d\gamma(t)\frac{n}{t}.$$

Using this, we obtain the desired result. □

### 3.2. *Case of* $v = -\log f - \frac{n}{2}\log(4\pi t)$

As in Sect. 3.1, let $f$ be a positive solution of (10). Let $v = -\log f - \frac{n}{2}\log(4\pi t)$. A direct computation tells us that $v$ satsifies

$$\frac{\partial v}{\partial t} = \Delta v - |\nabla v|^2 + cS - \frac{n}{2t} + \gamma(t)\left(v + \frac{n}{2}\log(4\pi t)\right). \tag{12}$$

**Theorem 2.** *Let $g(t)$ be a solution to the geometric flow (1) and $u$ satisfies (12). Let*

$$R_S = \alpha\Delta v - \beta|\nabla v|^2 + aS - b\frac{v}{t} - d\frac{n}{t},$$

*where $\alpha, \beta, a, b$ and $d$ are constants. Assume that $\alpha \neq 0$ and $\alpha \neq \beta$. Then $R_S$ satisfies*

$$\begin{aligned}
\frac{\partial R_S}{\partial t} &= \Delta R_S - 2\nabla^i R_S \nabla_i u - 2(\alpha-\beta)\left|\nabla_i\nabla_j u - \frac{\alpha}{2(\alpha-\beta)}S_{ij} - \frac{\lambda}{2t}g_{ij}\right|^2 \\
&\quad + 2(a-\beta c)\nabla^i v\nabla_i S - \frac{2(\alpha-\beta)}{\alpha}\frac{\lambda}{t}R_S + \frac{(\alpha-\beta)n\lambda^2}{2t^2} - \left(b + \frac{2(\alpha-\beta)\lambda\beta}{\alpha}\right)\frac{|\nabla v|^2}{t} \\
&\quad + \left(2a + \frac{\alpha^2}{2(\alpha-\beta)}\right)|S_{ij}|^2 + \left(\alpha\lambda - bc + \frac{2(\alpha-\beta)\lambda a}{\alpha}\right)\frac{S}{t} + \left(1 - \frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{b}{t^2}v \\
&\quad + \left(1 - \frac{2(\alpha-\beta)\lambda}{\alpha}\right)\frac{d}{t^2}n + \alpha c\Delta S - 2\alpha R^{ij}\nabla_i v\nabla_j v + \mathcal{D}_{(a,\alpha,\beta)}(S_{ij}, -\nabla v) \\
&\quad + \alpha\gamma(t)\Delta v - 2\beta\gamma(t)|\nabla v|^2 - b\frac{\gamma(t)}{t}\left(v + \frac{n}{2}\log(4\pi t)\right) + \frac{bn}{2t^2},
\end{aligned}$$

*where $\lambda$ is a constant.*

*Proof.* The idea of the proof is similar to that of Theorem 1. In fact, notice that we have $v = u - \frac{n}{2}\log(4\pi t)$. Hence $\nabla u = \nabla v$ and $\Delta u = \Delta v$ hold. Moreover,

$$R_S = Q_S + \frac{bn}{2t}\log(4\pi t).$$

Then Theorem 1 and direct computations imply the desired result. □

As a special case, we get

**Corollary 2.** *Let $g(t)$ be a solution to the geometric flow (1) and $v$ satisfies*

$$\frac{\partial v}{\partial t} = \Delta v - |\nabla v|^2 - (a+4)S - \frac{n}{2t} + \gamma(t)\left(v + \frac{n}{2}\log(4\pi t)\right).$$

*Let*

$$R_S = 2\Delta v - |\nabla v|^2 + aS - d\frac{n}{t},$$

*where $a$ and $d$ are constants. Then $R_S$ satisfies*

$$\begin{aligned}
\frac{\partial R_S}{\partial t} &= \Delta R_S - 2\nabla^i R_S\nabla_i v - 2\left|\nabla_i\nabla_j v - S_{ij} - \frac{1}{t}g_{ij}\right|^2 - \left(\frac{2}{t} - \gamma(t)\right)R_S \\
&\quad + \left(-\frac{2}{t} - \gamma(t)\right)|\nabla v|^2 - a\gamma(t)S + 2(a+2)\mathcal{H}(S_{ij}, -\nabla v) + \frac{n}{t^2}(2-d) + d\gamma(t)\frac{n}{t} \\
&\quad - \left((a+4)\frac{\partial S}{\partial t} - 2|S_{ij}|^2 + (3a+8)\Delta S\right) + 2\left(2\nabla^i S_{i\ell} - \nabla_\ell S\right)\nabla^\ell v \\
&\quad - 2\left(R^{ij} + (2a+5)S^{ij}\right)\nabla_i v\nabla_j v
\end{aligned}$$

*Proof.* The idea of proof is similar to that of Corollary 1. Use Theorem 2 in the case where $\alpha = 2, \beta = 1, b = 0, c = -a - 4, \lambda = 2$. □

## 4. Proof of Theorem A

By Corollary 1 in the case where $a = -3$, we obtain

$$\begin{aligned}\frac{\partial Q_S}{\partial t} &= \Delta Q_S - 2\nabla^i Q_S \nabla_i u - 2\left|\nabla_i \nabla_j u - S_{ij} - \frac{1}{t} g_{ij}\right|^2 - \left(\frac{2}{t} - \gamma(t)\right) Q_S \\ &\quad + \left(-\frac{2}{t} - \gamma(t)\right)|\nabla u|^2 + 3\gamma(t)S + \frac{n}{t^2}(2-d) + d\gamma(t)\frac{n}{t} \\ &\quad - \left(2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u)\right) \\ &\leq \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \left(\frac{2}{t} - \gamma(t)\right) Q_S + \left(-\frac{2}{t} - \gamma(t)\right)|\nabla u|^2 + 3\gamma(t)S \\ &\quad + \frac{n}{t^2}(2-d) + d\gamma(t)\frac{n}{t} - \left(2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u)\right).\end{aligned}$$

Now we assume that $d \geq 2$ holds. Moreover, by the assumption of Theorem A, we also get

$$2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u) \geq 0,\ S \geq 0,\ -\frac{2}{t} \leq \gamma(t) \leq 0.$$

Therefore we are able to obtain

$$\frac{\partial Q_S}{\partial t} \leq \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \left(\frac{2}{t} - \gamma(t)\right) Q_S.$$

Notice that

$$Q_S < 0$$

holds for $t$ small enough which depends on $d$. By using the maximum principle, we get the desired result.

Similarly, we get the following by using Corollary 2:

**Theorem 3.** *Let $g(t)$ be a solution to the geometric flow (1) on a closed oriented smooth $n$-manifold $M$satisfying*

$$2\mathcal{H}(S_{ij}, X) + \mathcal{D}(S_{ij}, X) \geq 0,\ S \geq 0.$$

*hold for all vector fields $X$ and all time $t \in [0, T)$ for which the flow exists. Let $v$ satisfies*

$$\frac{\partial v}{\partial t} = \Delta v - |\nabla v|^2 + cS - \frac{n}{2t} + \gamma(t)\left(v + \frac{n}{2}\log(4\pi t)\right).$$

*and assume that $\gamma(t)$ satsisfies*

$$-\frac{2}{t} \leq \gamma(t) \leq 0$$

*for for all time $t \in (0, T)$. Let*

$$R_S = 2\Delta v - |\nabla v|^2 - 3S - d\frac{n}{t},$$

*where $d \geq 2$ is a constant. Then for all time $t \in (0, T)$,*

$$R_S \leq 0$$

*holds.*

## 5. Proof of Theorem B

By Corollary 1 in the case where $a = -3$ and $\gamma(t) \equiv \gamma$, where $\gamma$ is a constant, we obtain

$$\begin{aligned}
\frac{\partial Q_S}{\partial t} &= \Delta Q_S - 2\nabla^i Q_S \nabla_i u - 2\left|\nabla_i\nabla_j u - S_{ij} - \frac{1}{t}g_{ij}\right|^2 - \left(\frac{2}{t} - \gamma\right) Q_S \\
&\quad + \left(-\frac{2}{t} - \gamma\right)|\nabla u|^2 + 3\gamma S + \frac{n}{t^2}(2-d) + d\gamma\frac{n}{t} - \left(2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u)\right) \\
&\le \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \frac{2}{n}\left(\Delta u - S - \frac{n}{t}\right)^2 - \left(\frac{2}{t} - \gamma\right) Q_S - \frac{2}{t}|\nabla u|^2 \\
&\quad - \gamma|\nabla u|^2 + 3\gamma S + \frac{n}{t^2}(2-d) + d\gamma\frac{n}{t} - \left(2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u)\right) \\
&\le \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \frac{2}{n}\left(\Delta u - S - \frac{n}{t}\right)^2 - \left(\frac{2}{t} - \gamma\right) Q_S \\
&\quad - \gamma|\nabla u|^2 + 3\gamma S + \frac{n}{t^2}(2-d) + d\gamma\frac{n}{t} - \left(2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u)\right)
\end{aligned}$$

On the other hand, we have

$$|\nabla u|^2 = 2\left(\Delta u - S - \frac{n}{t}\right) - Q_S - S - \frac{n}{t}(d-2).$$

Therefore, we obtain the following:

$$\begin{aligned}
\frac{\partial Q_S}{\partial t} &\le \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \frac{2}{n}\left(\Delta u - S - \frac{n}{t}\right)^2 - \left(\frac{2}{t} - \gamma\right) Q_S \\
&\quad - \gamma\left(2\left(\Delta u - S - \frac{n}{t}\right) - Q_S - S - \frac{n}{t}(d-2)\right) + 3\gamma S + \frac{n}{t^2}(2-d) + d\gamma\frac{n}{t} \\
&\quad - \left(2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u)\right) \\
&= \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \frac{2}{n}\left(\Delta u - S - \frac{n}{t}\right)^2 - \left(\frac{2}{t} - \gamma\right) Q_S \\
&\quad - 2\gamma\left(\Delta u - S - \frac{n}{t}\right) + \gamma Q_S + 4\gamma S + \frac{n}{t}\gamma(d-2) + \frac{n}{t^2}(2-d) + d\gamma\frac{n}{t} \\
&\quad - \left(2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u)\right) \\
&= \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \frac{2}{n}\left(\Delta u - S - \frac{n}{t}\right)^2 - \left(\frac{2}{t} - 2\gamma\right) Q_S - 2\gamma\left(\Delta u - S - \frac{n}{t}\right) \\
&\quad + 4\gamma S + \frac{2n}{t}\gamma(d-1) + \frac{n}{t^2}(2-d) - \left(2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u)\right).
\end{aligned}$$

Since we also have the following by a direct computation:

$$-2\gamma\left(\Delta u - S - \frac{n}{t}\right) = \frac{2}{n}\gamma\left(\Delta u - S - \frac{n}{t} - \frac{n}{2}\right)^2 - \frac{2}{n}\gamma\left(\Delta u - S - \frac{n}{t}\right)^2 - \frac{n}{2}\gamma.$$

Hence, we obtain

$$\begin{aligned}
\frac{\partial Q_S}{\partial t} &\le \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \frac{2}{n}(1+\gamma)\left(\Delta u - S - \frac{n}{t}\right)^2 - \left(\frac{2}{t} - 2\gamma\right) Q_S \\
&\quad + \frac{2}{n}\gamma\left(\Delta u - S - \frac{n}{t} - \frac{n}{2}\right)^2 + 4\gamma S + \frac{2n}{t}\gamma(d-1) + \frac{n}{t^2}(2-d) - \frac{n}{2}\gamma \\
&\quad - \left(2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u)\right).
\end{aligned}$$

Now suppose that $-1 \leq \gamma \leq 0$. Then the above implies

$$\frac{\partial Q_S}{\partial t} \leq \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \left(\frac{2}{t} - 2\gamma\right) Q_S + 4\gamma S + \frac{2n}{t}\gamma(d-1) + \frac{n}{t^2}(2-d) - \frac{n}{2}\gamma - \left(2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u)\right).$$

Assume that $d \geq 2$ holds. Moreover, by the assumption of Theorem B, we also get the following:

$$2\mathcal{H}(S_{ij}, -\nabla u) + \mathcal{D}(S_{ij}, -\nabla u) \geq 0, \ S \geq 0.$$

Then we have

$$\frac{\partial Q_S}{\partial t} \leq \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \left(\frac{2}{t} - 2\gamma\right) Q_S + \frac{2n}{t}\gamma(d-1) - \frac{n}{2}\gamma.$$

Since we also have

$$-\left(\frac{2}{t} - 2\gamma\right) Q_S = -\left(\frac{2}{t} - 2\gamma\right)\left(Q_S + \frac{n}{4}\gamma\right) + \frac{n}{2t}\gamma - \frac{n}{2}\gamma^2,$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(Q_S + \frac{n}{4}\gamma\right) &\leq \Delta\left(Q_S + \frac{n}{4}\gamma\right) - 2\nabla^i\left(Q_S + \frac{n}{4}\gamma\right)\nabla_i u - \left(\frac{2}{t} - 2\gamma\right)\left(Q_S + \frac{n}{4}\gamma\right)\\ &\quad + \frac{n}{t}\gamma\left(2d - \frac{3}{2}\right) - \frac{n}{2}\left(\gamma^2 + \gamma\right)\\ &\leq \Delta\left(Q_S + \frac{n}{4}\gamma\right) - 2\nabla^i\left(Q_S + \frac{n}{4}\gamma\right)\nabla_i u - \left(\frac{2}{t} - 2\gamma\right)\left(Q_S + \frac{n}{4}\gamma\right)\\ &\quad - \frac{n}{2}\left(\gamma^2 + \gamma\right),\end{aligned}$$

where notice that

$$\frac{n}{t}\gamma\left(2d - \frac{3}{2}\right) \leq 0$$

under $-1 \leq \gamma \leq 0$ and $d \geq 2$. Finally, we get the following by taking $\gamma = -1$:

$$\frac{\partial}{\partial t}\left(Q_S - \frac{n}{4}\right) \leq \Delta\left(Q_S - \frac{n}{4}\right) - 2\nabla^i\left(Q_S - \frac{n}{4}\right)\nabla_i u - \left(\frac{2}{t} + 2\right)\left(Q_S - \frac{n}{4}\right).$$

Notice that

$$Q_S < \frac{n}{4}$$

holds for $t$ small enough which depends on $d$. By using the maximum principle, we obtain the desired result.

Similarly, we get the following by using Corollary 2:

**Theorem 4.** *Let $g(t)$ be a solution to the geometric flow (1) on a closed oriented smooth $n$-manifold $M$satisfying*

$$2\mathcal{H}(S_{ij}, X) + \mathcal{D}(S_{ij}, X) \geq 0, \ S \geq 0.$$

*hold for all vector fields $X$ and all time $t \in [0, T)$ for which the flow exists. Let $v$ satisfies*

$$\frac{\partial v}{\partial t} = \Delta v - |\nabla v|^2 - S + \frac{n}{2t} - \left(v + \frac{n}{2}\log(4\pi t)\right).$$

*Let*

$$R_S = 2\Delta v - |\nabla v|^2 - 3S - d\frac{n}{t},$$

*where $d \geq 2$ is any fixed constant. Then for all time $t \in (0, T)$,*

$$R_S \leq \frac{n}{4}$$

*holds.*

## 6. Proof of Theorem C

By Proposition 1 in the case where $\alpha = 0$, $\beta = -1$, $a = c = 0$, $b = 1$, $d = 0$, we obtain

$$Q_S = |\nabla u|^2 - \frac{u}{t}$$

and

$$\begin{aligned}\frac{\partial Q_S}{\partial t} &= \Delta Q_S - 2\nabla^i Q_S \nabla_i u - 2|\nabla\nabla u|^2 - \frac{1}{t}|\nabla u|^2 + \frac{1}{t^2}u + \mathcal{D}_{(0,0,-1)}(S_{ij}, -\nabla u) \\ &\quad + 2\gamma(t)|\nabla u|^2 - \frac{\gamma(t)}{t}u \\ &= \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \frac{1}{t}Q_S + 2\gamma(t)|\nabla u|^2 - \frac{\gamma(t)}{t}u + \mathcal{D}_{(0,0,-1)}(S_{ij}, -\nabla u) \\ &= \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \left(\frac{1}{t} - \gamma(t)\right) Q_S + \gamma(t)|\nabla u|^2 - 2\mathcal{I}(S_{ij}, -\nabla u),\end{aligned}$$

where notice that $\mathcal{D}_{(0,0,-1)}(S_{ij}, -\nabla u) = -2\mathcal{I}(S_{ij}, -\nabla u)$. Since we assumed $\gamma(t) \leq 0$ and $\mathcal{I}(S_{ij}, -\nabla u) \geq 0$, the above implies

$$\frac{\partial Q_S}{\partial t} \leq \Delta Q_S - 2\nabla^i Q_S \nabla_i u - \left(\frac{1}{t} - \gamma(t)\right) Q_S.$$

Since

$$Q_S < 0$$

holds for $t$ small enough, the maximum principle tells us that the desired result holds.

*Acknowledgments* The first author was supported by NSFC (Grant Nos. 11001203 and 11171143) and Zhejiang Provincial Natural Science Foundation of China (Project No. LY13A010009). The second author was partially supported by the Grant-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science, No. 25400074.

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