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Rigidity of flat sections on non-negatively curved pullback submersions

Abstract. We give a geometric obstruction to the non-negativity of the sectional curvature in the total spaces of Riemannian submersions defined as pullbacks. Applications of this obstruction to several examples are given.

1. Introduction

The study of manifolds which admit a metric of positive or non-negative sectional curvature has lead to certain standard constructions and conditions that guarantee that these constructions furnish a metric with the desired properties. The classical example is the following: let M be a manifold of positive sectional curvature, G a compact Lie group and $G \cdots P \to M$ be a principal bundle-with-connection over a Riemannian manifold M. One can then endow P with the so-called *connection metric*, constructed by declaring the vertical and horizontal spaces orthogonal; inducing the metric on the horizontal space by the metric of M and on the vertical space by some canonical metric on G (typically biinvariant). This procedure makes the bundle a Riemannian submersion with totally geodesic fibers ([11], Proposition 2.7.1). Grey–O'Neill theory then says that the geometry of the total space is mostly controlled by the Grey–O'Neill tensor A and the metric on the base ([11,16]).

An immediate necessary condition for the positivity of curvature is that for X vertical and U horizontal, the Grey–O'Neill tensor $A_X^* U$ cannot be zero, since the vertizontal (unnormalized) sectional curvatures are given by $|A_X^* U|^2$ in this case. This condition, called *fatness* [21], depends only on the connection on the bundle, and imposes upon it strong topological restrictions [5,10]. Assuming fatness, in [4] conditions are given for the total spaces of these principal bundles to admit such metrics of positive curvature. These conditions take the form of differential inequalities relating the curvatures of M and the bundle connection.

In this paper we are interested in *non-negative* curvature. The case of non-negative curvature on compact Riemannian manifolds still has many unanswered questions: for example, do all 7-dimensional exotic spheres or any sphere of other dimension admit a metric of non-negative curvature? (it is known to be true for exotic 7-spheres which can be realized as

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sphere bundles over S^4, [12], and to be false for spheres which do not bound *spin* manifolds, see [14], for example). For non-negative curvature, the fatness condition on the Grey–O'Neill tensor can in principle be relaxed, but we shall see that there is a lot of rigidity along the degenerate directions.

Theorem 1. *Let* $\xi : E \to M$ *be a principal Riemannian submersion with totally geodesic fibers. Then, the set of vectors defined at each point of* $m \in M$ *by*

$$\mathcal{D}_m = \{X \in T_m M \mid A_{\hat{X}} = 0, \ \hat{X} = \text{horizontal lift of } X\} \tag{1.1}$$

defines a smooth involutive distribution $\mathcal{D}$ *in an open and dense subset of* M. *Furthermore, a necessary condition for the total space* E *to have non-negative curvature is that the integral manifolds of* $\mathcal{D}$ *are totally geodesic.*

We call the distribution $\mathcal{D}$ the A-flat distribution of the submersion. We distinguish the trivial case of the A-flat distribution being 0-dimensional:

Definition 1.1. A Riemannian submersion is *non-degenerate* if the map $X \mapsto A_X$ is a one-to-one mapping from the horizontal space into Hom(Horizontal, Vertical).

Like fatness, non-degeneracy only depends on the connection form of the bundle is a; however, it is a much weaker condition. Fatness means that for each horizontal $X \neq 0$ the map A_X : Horizontal $\to$ Vertical is *onto* and, which is a strong way of being non-zero, i.e., non-degenerate.

In the non-degenerate case, the leaves of the A-flat distribution are just the points of M and the theorem gives no information. However, of particular interest are *pullback* bundles; abstractly, because all principal bundles are topologically pullbacks of the universal bundles, and also in practice, we have found that many of the relevant examples in the literature of non-negative curvature are actually modeled as pullbacks where Theorem 3 applies non-trivially.

There are *two* natural metrics on the total space of a pullback bundle: one is the connection metric associated with the pullback connection and the other is the *pullback metric* of Definition 2.1. On both cases the curvature of the connection will be degenerate at least along the fibers of the pullback map, and, when we pull back a non-degenerate submersion, the leaves of the A-flat distribution are the fibers.

The next two theorems are respectively specializations of Theorem 1 to the case of pullback bundles for each of the two natural metrics of the previous paragraph:

Theorem 2. *Let* $\xi : G \cdots P \to B$ *be a non-degenerate principal Riemannian submersion with totally geodesic fibers,* M *a Riemannian manifold, and* $f : M \to B$ *a differentiable function. Endow the total space* E *of the pullback bundle* $f^*\xi$ *with the connection metric relative to the pullback connection. A necessary condition for* E *to have non-negative curvature in the connection metric is that for each regular value* $b \in B$, $f^{-1}(b)$ *is totally geodesic in* M.

When the bundle ξ is some canonical construction such as Hopf bundles, the easiest way to represent a particular connection metric is to consider the total space of $f^*\xi$ as a Riemannian submanifold of $M \times P$. This imposes a change on the metric on M in order to have a Riemannian submersion, but we will see (Lemma 2.4) that this is inconsequential.

Theorem 3. *Let* $\xi : G \cdots P \to B$ *be a non-degenerate principal Riemannian submersion with totally geodesic fibers,* M *a Riemannian manifold, and* $f : M \to B$ *a differentiable function. A necessary condition for the total space of the pullback* $f^*\xi$ *to have non-negative curvature with the pullback metric is that for each regular value* $b \in B$, $f^{-1}(b)$ *is totally geodesic in* M.

The Gromoll-Meyer construction of an exotic sphere as a quotient of $Sp(2)$ falls within the scope of these results, since $Sp(2)$ with its canonical biinvariant metric is the pullback of the Hopf bundle (Example 1). It has been recently realized that by studying other pullback maps many further geometric models of exotic spheres can be constructed (see [9,20] and Sect. 5).

The condition of each regular fiber of a map $f : M \rightarrow B$ being totally geodesic places strong restrictions on the metric of M; typically, there will be no metric on M for which the regular leaves are totally geodesic. A key element of this analysis is the study of how the map f behaves near the singular leaves. We offer here a simple stability result that is useful in concrete examples.

Proposition 1.2. *Let M be a complete Riemannian manifold and $f : M \rightarrow B$ be a map whose regular fibers are totally geodesic. If S is a submanifold contained in a (possibly singular) fiber of f, then fibers contained in a small enough neighbourhood of S immerse into S.*

Therefore, if one suspects that the total space of a pullback bundle admits non-negative curvature, the first test should be to look at the behavior of the fibers of the pullback map on the base space near singular points. We shall see in Sect. 5 how this test discards some examples (in which a direct computation of the curvature would be rather involved), and gives hope for others. In fact, Theorems 1–3 should not be seen as "structure theorems for manifolds of non-negative curvature"; instead, their value is in the construction of new examples of manifolds with non-negative curvature (which always use highly non-generic set-ups), as a quick way to check if there is any hope in a given metric.

The paper is organized as follows: in the next section we study the geometry of pullback bundles, in particular, the relationship between their two natural metrics we mentioned. Section 3 contains the proofs of Theorems 1, 2 and 3. In Sect. 4 we prove Proposition 1.2 and Sect. 5 is devoted to applications.

Notation: Given a Riemannian metric, ∇ denotes its Levi–Civita connection, R the curvature tensor $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$, and K the unnormalized sectional curvature associated to R. When necessary, left superscripts will be added to identify different spaces, e.g. $^M \nabla$.

In a Riemannian submersion, $P \xrightarrow{\pi} B$, given $Y_x \in T_b B$ we denote its horizontal lift at $p \in \pi^{-1}(y) \subset P$ by a hat, $\hat{Y}_p \in T_p P$. The Grey–O'Neill tensor will be denoted by the letter A. Anywhere we give a property of a vector $X \in T_b B$ in terms of a horizontal lift $\hat{X}$ in $T_p P$, it will be clear that the property is independent of the point $p \in \pi^{-1}(b)$.

2. Geometry of pullback bundles

Consider a principal bundle $G \cdots P \xrightarrow{\pi} B$. Let M be a Riemannian manifold, $f : M \rightarrow B$ a differentiable map and consider the pullback $G \cdots f^*P \rightarrow M$.

Definition 2.1. The *pullback metric* on f^*P is the metric induced as a submanifold by the definition of a pullback, i.e., $f^*P = \{(m, p) \in M \times P \mid f(m) = \pi(p)\}$, where we endow $M \times P$ with the product metric.

Example 1. The Lie group $Sp(2)$ can be realized as the total space of the pullback of the Hopf bundle by minus the Hopf map:

$$
\begin{array}{ccc}
S^3 & & S^3 \\
\vdots & & \vdots \\
S^3 \cdots Sp(2) & \overset{s}{\longrightarrow} & S^7 \\
f \downarrow & & \downarrow h \\
S^3 \cdots \ \ S^7 & \underset{a \circ h}{\longrightarrow} & S^4
\end{array}
$$

In the diagram, h denotes the Hopf map corresponding to the right Hopf action of S^3 on S^7, a the antipodal map of S^4, and f and s denote the first and second columns of a matrix in $Sp(2)$. When both S^7's in the diagram have the same constant curvature, then the pullback metric of $Sp(2)$ is biinvariant (see [19] and proposition 2.1 in [22]).

Some more sophisticated examples will be considered in Sect. 5.

Let us establish some geometrical properties of the pullback bundle. First, the submersion $G \cdots f^* P \to M$, where $f^* P$ has the pullback metric and M has its given metric, is *not* a Riemannian submersion. It is easy to see that in order to make it a Riemannian submersion, we must endow M with the *graph metric* induced from the embedding of M as a subset $\Gamma_f = \{(m, b) \in M \times B \mid f(m) = b\} \subset M \times B$, the later endowed with the product metric. Indeed, let $\pi : P \to B$ be a principal G-bundle endowed with the connection metric defined by a connection 1-form ω, an inner product β on the Lie algebra of G and a metric g_B on B.

Proposition 2.2. *Let $\pi : P \to (B, g_B)$ be as above, M a Riemannian manifold and $f : M \to B$ a smooth map. Then, the pullback metric on $f^* P$ is the connection metric defined by the pullback connection $f^* \omega$ and the graph metric g_{Γ_f}.*

Proof. Let $W = (X, Y)$ be a vector tangent to $f^* P$. So, by the definition of $f^* P$, $df(X) = d\pi(Y)$. Spelling out the induced metric on $f^* P$, we have

$$
\begin{aligned}
g_{f^* P}(W, W) = g_{M \times P}((X, Y), (X, Y)) &= g_M(X, X) + g_P(Y, Y) \\
&= g_M(X, X) + g_B(d\pi Y, d\pi Y) + \beta(\omega(Y), \omega(Y)) \\
&= g_M(X, X) + g_B(df X, df X) + \beta(\omega(Y), \omega(Y)) \\
&= g_{\Gamma_f}(X, X) + \beta(\omega(Y), \omega(Y))
\end{aligned}
$$

Now the proof follows by observing that Y is the image of W by the derivative of the induced pullback map. In particular, $\omega(Y)$ is the image of W by the pullback connection, as desired. $\square$

Remark 2.3. The fact that M has to change its original metric in order for $f^* P \to M$ be a Riemannian submersion is already present in the example from the introduction: with a biinvariant metric the submersion $Sp(2) \to S^7$ subduces a metric on S^7 that is not the round one, a fact used in [7].

The vertical space is given by vectors of the form $(0, U)$, $U \in TP$ vertical. Horizontal vectors have the form $(Z, \widehat{df(Z)})$, $Z \in TM$. In particular, if Z is tangent to a level set of f, then $(Z, 0)$ is horizontal.

The following property will be essential in Theorem 3:

Lemma 2.4. *Let M, B be Riemannian manifolds, $f : M \to B$, and $b \in B$ be a regular value of f. Then $f^{-1}(b)$ is totally geodesic in M if and only if $f^{-1}(b) \times \{b\}$ is totally geodesic in the graph Γ_f.*

Proof. A regular level set $L = f^{-1}(b) \subset M$ is totally geodesic on M if and only if $f^{-1}(b) \times \{b\}$ is totally geodesic in $M \times B$. For the last subset we have the chain of inclusions

$$f^{-1}(b) \times \{b\} \subset \Gamma_f \subset M \times B$$

If $f^{-1}(b) \times \{b\}$ is totally geodesic in $M \times B$, then it is totally geodesic in Γ_f. The converse statement follows from the fact that tangent vectors to $f^{-1}(b) \times \{b\}$ inside of $M \times B$ have the form $(Y, 0)$. □

3. Flat vertizontal sections

In this section, $G \cdots E \to M$ will be a Riemannian submersion with totally geodesic fibers; in the typical applications the total space E will be assumed to have non-negative curvature.

We first need the following elementary lemma on flat sections on non-negatively curved spaces:

Lemma 3.1. *Let (Q, g) be a non-negatively curved manifold and suppose the plane spanned by X and U has zero sectional curvature. Then $^Q R(X, U)X = 0$.*

Proof. This follows immediately by the well-known "quadratic trick" (see e.g. Proposition 1 in [17] or compare [4]): let Z be an arbitrary vector. Consider the sectional curvature of the plane spanned by X and $tU + Z$. Unnormalized, this is

$$g(R(X, tU + Z)tU + Z, X) = t^2 K(X, U) - 2tg(R(X, U)X, Z) + K(Z, U).$$

If $K(X, U) = 0$ then the quadratic term vanishes and the resulting expression is a non-negative linear function. Thus we get that $K \geq 0$ implies that the linear coefficient $g(R(X, U) X, Z) = 0$ for all $Z \in TQ$. □

Remark 3.2. The condition being symmetric in X and U, the conclusion also holds interchanging X and U. This means that what really happens is best explained in terms of the curvature operator $\mathcal{R} : \Lambda^2(TQ) \to \Lambda^2(TQ)$: given $\eta \in \Lambda^2(TQ)$ we define $\phi_\eta : \Lambda^2(TQ) \to \Lambda^4(TQ), \phi_\eta(\xi) = \eta \wedge \xi$. Then if $K(X, U) = 0$, then $\mathcal{R}(X \wedge U) \in \ker \phi_{X \wedge U}$. This gives some rigidity to zero curvature sections, an example of which is the main result of this paper (see also proposition 1 in [17]).

Remark 3.3. One may notice that an analogous proof works in the non-positive curvature case. In fact, being here where the non-negativity requirement enters, our main results work replacing non-negatively curved by non-positively curved.

Definition 3.4. A vector $X \in TM$ is said to be *A-flat* if $A_{\hat{X}} = 0$. An *A*-flat vector X is called regular if it can be locally extended to a vector field of *A*-flat vectors; we call such fields *A*-flat vector fields.

We abuse notation by also calling *A*-flats, the horizontal vectors $\hat{X}$ such that $A_{\hat{X}} = 0$. Grey–O'Neill equations imply that a vector is *A*-flat if and only if the curvature of all vertizontal planes containing $\hat{X}$ vanish.

Theorem 3.5. *Let $G \cdots E \to M$ be a Riemannian submersion with totally geodesic fibers such that E has non-negative curvature. If X is an A-flat vector field, then so is the covariant acceleration $\nabla_X X$.*

Proof. Let Z be another vector field on M, and U a vertical field on E.

Spelling out the curvature term in (3.1) and using the identities in [11], we have:

$$0 = g_P(R(\hat{X}, U)\hat{X}, \hat{Z}) = -\left\langle \nabla_{\hat{X}}(A_{\hat{X}}\hat{Z}), U \right\rangle + \left\langle A_{\nabla_{\hat{X}}\hat{X}}\hat{Z}, U \right\rangle + \left\langle A_{\hat{X}}\nabla_{\hat{X}}\hat{Z}, U \right\rangle$$

$$= \left\langle A_{\nabla_{\hat{X}}\hat{X}}\hat{Z}, U \right\rangle$$

and since the equality holds for all $\hat{Z}$ and U, $A_{\nabla_{\hat{X}}\hat{X}} = 0$. The theorem follows from the identity $\widehat{\nabla_X X} = \nabla_{\hat{X}}\hat{X}$. □

Definition 3.6. A submanifold S of M is said to be A-flat if vectors tangent to S are A-flat. It is maximal if for all $s \in S$ and $X \in T_s M$, X A-flat implies that $X \in T_s S$.

If X is an A-flat vector field tangent to a submanifold S, the proof of Theorem 3.5 uses only information of the vector field X along the submanifold. Then, it follows from Theorem 3.5 that

Theorem 3.7. *Let $G \cdots E \to M$ be a Riemannian submersion with totally geodesic fibers such that E has non-negative curvature. Then a maximal A-flat submanifold S is totally geodesic in M.*

Now we see that there are plenty of maximal A-flat submanifolds:

Lemma 3.8. *The set of all A-flat vector fields is in involution.*

Proof. The key issue is that the A-tensor only depends on the horizontal distribution; if $\hat{X}$, $\hat{Z}$ are horizontal vector fields then $A_{\hat{X}}\hat{Z}$ is the vertical part of their Lie bracket. Therefore, a vector field X on M is A-flat if and only if $g_E([\hat{X}, \hat{Z}], U) = 0$ for all vector fields Z on M and U vertical vector field on E. Thus, the lemma can be expressed in terms of the appropriate brackets: let X, Y, Z be vector fields on M where X and Y are A-flat then we want to prove that $g_E([\widehat{[X, Y]}, \hat{Z}], U) = 0$ for all vertical U.

Since $[\hat{W}_1, \hat{W}_2] = \widehat{[W_1, W_2]} + A_{W_1}W_2$ for all fields W_1, W_2 on M, it follows that $\widehat{[W_1, W_2]} = [\hat{W}_1, \hat{W}_2]$ if one of them is A-flat.

$$g_E([\widehat{[X, Y]}, \hat{Z}], U) = g_E([[\hat{X}, \hat{Y}], \hat{Z}], U)$$

$$= g_E([[\hat{X}, \hat{Z}], \hat{Y}], U) + g_E([[\hat{Z}, \hat{Y}], \hat{X}], U)$$

$$= g_E([\hat{Y}, \widehat{[Z, X]}], U) + g_E([\hat{X}, \widehat{[Y, Z]}], U)$$

$$= 0$$

where the second equality follows by the Jacobi identity and the last one by the hypothesis that X and Y are A-flat. □

Remark 3.9. Definition 3.4 establishes the A-flat, regular vectors as a *nullity distribution* as in [18], section 2.3. It is fairly common for the leaves of such distribution to have geometric properties (umbilic, totally geodesic, etc.); in our case, however, we need the simultaneous nullity of the A-tensor as in Definition 3.4 and of the curvature operator as in Remark 3.2.

To complete the proof of Theorem 1 we show that the A-flat vectors define a smooth involutive distribution in an open and dense subset of M.

Let $\mathcal{D}_m$ be as in Theorem 1, i.e., the set of all A-flat vectors, then M is partitioned according to $0 \leq \dim \mathcal{D}_m \leq \dim M$. It is easy to see that the set of points in M such that $\dim \mathcal{D}_m$ is locally *minimal* is open and dense. Indeed, $\mathcal{D}_m$ is the kernel of a continuous family of linear maps parametrized by $m \in M$, and, as such, the function $d : M \to \mathbb{N}$, $d(m) = \dim(\mathcal{D}_m)$ is lower semicontinuous. Thus the set of local minima, on which the rank of $\mathcal{D}_m$ is locally constant and $\mathcal{D}_m$ is a regular foliation, is open and dense (compare [13]).

Now let us deduce Theorems 2 and 3 from Theorem 1:

Recall that if $G \cdots E \to M$ is a principal Riemannian submersion, then, for horizontal fields X, Z and U vertical, the Grey–O'Neill tensor satisfies $g_P(A_X U, Z) = -\beta(\Omega(X, Z), \xi_U)$, where Ω is the curvature form of the bundle connection induced by the horizontal-vertical decomposition, β is a biinvariant metric on G and $\xi_U \in \mathfrak{g}$ is the infinitesimal generator of U. Thus, the condition of being A-flat can be completely written in terms of the curvature of the connection form of the bundle: it is in the kernel $\{X \in T_m M : \Omega(X, Z) = 0 \ \forall \ Z\}$

Consider now a pullback $G \cdots f^* P \to M$ of a non-degenerate principal Riemannian submersion $G \cdots P \to B$, with bundle curvature form Ω. Since the main theorem concerns regular values $b \in B$, by restricting to the open set of regular values, we might assume that f is a submersion.

The curvature form of the pullback bundle is just the pullback form $f^*\Omega$. Thus $X \in T_m M$ is A-flat if and only if $\Omega(df_m(X), df_m(Z)) = 0$ for all $Z \in T_m M$, and since f is a submersion then for all $Y \in T_{f(b)} B$ there exists $Z \in T_m M$ such that $df_m(Z) = Y$. This means that $\Omega(df_m(X), Y) = 0$ for all $Y \in T_{f(m)} B$, and since by hypothesis $G \cdots P \to B$ is non-degenerate, it follows that $df_m(X) = 0$. We have thus concluded that A-flat vectors are tangent to level sets of f, and regular fibers of f are maximal A-flat submanifolds. Then Theorem 3.7 implies that the fibers of f are totally geodesic on M.

For Theorem 3, put the graph metrics on the total space and on M. Then we have a connection metric and we can apply Theorem 2 *with respect to the graph metric of M*. However, Lemma 2.4 then finishes the proof.

4. Stability of regular totally geodesic fibers

In this section we prove Proposition 1.2.

Let $S \subset M$ be a compact submanifold, possibly with boundary, and ν its normal bundle, with δ-disk bundle ν_δ. For δ small enough, the exponential map $\exp : \nu_\delta \to M$ is a diffeomorphism onto a tubular neighborhood $U_\delta \subset M$. There is radial projection $p : U_\delta \to S$, given by $u \mapsto p(u)$, the (unique) closest point on S to U. With this context, we have

Proposition 4.1. *There exists $\delta > 0$ such that, if L is a complete totally geodesic submanifold contained in U_δ, then $p|_L : L \to S$ is an immersion.*

Note that the kernel of Dp is given by vectors tangent to the fibers of the radial projection. Since L is totally geodesic and complete, Proposition 4.1 is reduced to the following:

Lemma 4.2. *There exists $\delta > 0$ such that, if $m \in U_\delta$ and $v_m \in \ker Dp_m$, then $\exp_m tv \notin U_\delta$ for some t.*

Proof. The proof essentially reflects the convexity of the squared distance function to a submanifold in a sufficiently small tubular neighborhood. In the case S is a point, the method of proof is similar to lemma 4.1 of [6]; see also [1].

Let δ' be small enough so that $\exp : \nu_{\delta'} \to U_{\delta'}$ is an embedding and denote $\rho : U_{\delta'} \to \mathbb{R}$ as the square of the distance function to S. Given a vector v tangent to $U_{\delta'}$, we denote $v^{\perp}$ as the orthogonal projection onto $\ker dp_m$.

Then, at $m \in S$, we have:

(S1) The derivative $d\rho_m = 0$;
(S2) the Hessian $d^2\rho_m(v_m, v_m) = 2|v_m^{\perp}|^2$;
(S3) any geodesic such that $\gamma(0) = m \in S$ and $\dot{\gamma}(0) \in \ker dp_m$ satisfies that $\dot{\gamma}(t) \in \ker dp_{\gamma(t)}$ for all t.

Given these conditions at S we can find $\delta < \delta'$ satisfying for every $v_m \in T_m U_{\delta}$:

(U1) $|d\rho_m(v_m)| < \frac{1}{4}|v_m|$;
(U2) $|d^2\rho_m(v_m, v_m)| > |v_m^{\perp}|^2$;
(U3) for any geodesic $\gamma(t)$ such that $\dot{\gamma}(0) \in \ker dp_m$, $|\dot{\gamma}(t)^{\perp}| > \frac{1}{2}|\dot{\gamma}(t)|$ for $|t| \leq 2$,

Given an unit geodesic $\gamma(t)$, let $l(t) = \rho \circ \gamma(t)$; then $l'(t) = d\rho_{\gamma(t)}(\dot{\gamma}(t))$ and $l''(t) = d^2\rho_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))$.

Suppose that $\gamma(t)$ is an unit geodesic such that $\gamma(0) \in U_{\delta}$, $\dot{\gamma}(0) \in \ker dp_{\gamma(0)}$ and that $\gamma(t) \in U_{\delta}$ for all t. According to conditions (U1)–(U3), we have

$$|l'(t)| = \left| l'(0) + \int_0^t l''(s)ds \right| > \left| \frac{t}{2} - \frac{1}{4} \right|$$

which contradicts (U1) for $t \geq 1$. $\square$

Now Proposition 1.2 is just stating Proposition 4.1 in the situation we are interested in, that is, when S is a subset of a fiber of $f : M \to N$.

5. Applications

In this section we provide several applications; in each one the obstructions apply in different ways. Let us remark that, given Theorem 2, the obstruction apply not only for pullback metrics, but also for connection metrics associated to the pullback connection.

- The first example is the family of bundles studied by Wilhelm [22] (5.1). This family is a ladder of bundles, which extends $Sp(2)$ as presented in the Example 1; there the fibers are totally geodesic (just the Hopf fibers) and $Sp(2)$ with the pullback metric has non-negative curvature, whereas for the higher elements $Sp(2,m)$ one must carefully chose the metric in order to satisfy the hypothesis of Theorem 3.
- The bundles studied by Rigas [19] (5.2). For these bundles the main obstruction (Theorem 3) vanishes for the canonical metrics. However, the "secondary" obstruction furnished by Proposition 3.7 does not vanish for the canonical round metric on the base sphere; a deformation is necessary to make all obstructions vanish.
- The examples given in Sects. 5.3, (5.4) study the geometric presentations of exotic spheres constructed in [9] and [20]. In these cases, the obstruction is absolute: we show that for the pullback maps presented, for any given metric the fibers cannot be totally geodesic. These examples arise as pullbacks of the Gromoll–Meyer construction, which we think in the context of the pullback of the Hopf map $h : S^7 \to S^4$ as in Example 1; that is, we pullback the fibration $S^3 \cdots Sp(2) \to S^7$ by appropriate maps $f : X \to S^7$. However, since the canonical connection of $S^3 \cdots Sp(2) \to S^7$ is degenerate (being zero along the Hopf fibers), we go all the way to the Hopf bundle and pull back $S^3 \cdots S^7 \to S^4$ by maps $f : X \to S^4$ of the form $f = a \circ h \circ \phi$, where $\phi : X \to S^7$ and a is the

antipodal map of S^4. All actions in sight will be isometric with respect to the induced connections and metrics. In both examples, our results implies that one cannot put non-negative curvature in the total space of these bundles via the pullback construction, be it from using the pullback metric or the connection metric associated with the pullback connection.

- Finally in 5.5, we describe the pullback structure of Kervaire spheres. Here, in contrast with the previous examples, there is no *topological* obstruction to the presence of the totally geodesic foliation, however the authors have not been able to describe a metric with totally geodesic fibers in that case.

5.1. Wilhelm bundles

Consider the bundles constructed on [22]: let $Sp(2, m)$ be the subset of m copies of S^7 defined by the following condition

$$Sp(2, m) = \{(u_1, \ldots, u_m) \in (S^7)^m \mid h(u_1) = ah(u_2),\ \tilde{h}(u_i) = ah(u_{i+1})\ for\ i > 1\}$$

where a and h are as in Example 1 and $\tilde{h}$ is the dual Hopf map corresponding to the left S^3-action on S^7. Quotients of these bundles by free $S^3 \times \cdots \times S^3$-actions give models of 3-sphere bundles over S^4, and, in particular, exotic spheres.

As we observed earlier, the pull-back diagram of $Sp(2)$ endowed with its canonical metric does not have the obstruction given by Theorem 3. We also observe that $Sp(2, m+1)$ fits in the following diagram

$$
\begin{array}{ccc}
Sp(2, m+1) & \xrightarrow{pr_{m+1}} & S^7 \\
\downarrow & & \downarrow{ah} \\
Sp(2, m) & \xrightarrow{\tilde{h} \circ pr_m} & S^4
\end{array}
\tag{5.1}
$$

where $pr_m : Sp(2, m) \to S^7$ is the projection in the last coordinate and the unnamed down arrow is the projection in the first m coordinates. The right-hand side of the diagram is the Hopf fibration, which is non-degenerate, and thus the A-flat sections are given by the preimages of $\tilde{h} \circ pr_m$. Endowing $Sp(2, m)$ with the metric induced as a submanifold of $(S^7)^m$, the generic preimage is *not* totally geodesic.[1] This implies, according to Theorem 3, that $Sp(2, m)$ has sections of negative curvatures in an open and dense set of points. However, we can easily change the metric so that it satisfies the hypothesis of Theorem 3: the map $Sp(2, m) \to S^4$ is actually principal bundle projection with fiber $(S^3)^m$; then for any connection metric the fibers are totally geodesic. Alternatively, we can also begin at $m = 2$, where the map $Sp(2) \xrightarrow{\tilde{h} \circ pr_2} S^4$ is easily seen to be induced as the quotient map of the free $S^3 \times S^3$-action

$$
(q, r) \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \bar{r} & 0 \\ 0 & 1 \end{pmatrix};
$$

the r factors kills the first column whereas the q-factor induces Hopf in the second column. Now, as above, endow $Sp(2)$ with a connection metric, for which the fibers are totally

---

[1] We thank F. Wilhelm for pointing this out to us.

geodesic. Pullback diagrams like 5.1 preserve the "totally geodesic fibers" property: if we have

$$
\begin{array}{ccc}
P & \longrightarrow & E \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\
M & \xrightarrow{\ g\ } & B
\end{array}
$$

and the maps f and g have totally geodesic fibers, then so does the map $g \circ p$ if we endow P with the pullback metric as a submanifold of the product $M \times B$. Then we can proceed inductively. This presentation shows that the failure of the hypothesis of Theorem 3 for the canonical metric occurs at the $Sp(2)$-level, not at the maps $Sp(2, m+1) \to Sp(2, m)$.

5.2. Rigas bundles

Let $\phi_k : S^4 \to S^4$ be a degree k map, and $r_k = \phi_k \circ h : S^7 \to S^4$. Then the bundles $\tilde{P}_k$ constructed in [19] are defined as the pullback of Hopf by r_k (for example, if ϕ_{-1} is represented by the antipodal map of S^4, then $\tilde{P}_{-1}$ is $S^3 \cdots Sp(2) \to S^7$ as in Example 1).

$$
\begin{array}{ccccc}
\tilde{P}_k & \longrightarrow & P_k & \longrightarrow & S^7 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle h} \\
S^7 & \xrightarrow{\ h\ } & S^4 & \xrightarrow{\ \phi_k\ } & S^4
\end{array}
\qquad (5.2)
$$

We choose $\phi_k : S^4 \to S^4$ to be the suspension of p_k, the quaternion kth power map $S^3 \to S^3$ (note that, for $k = -1$, this is different from the antipodal map). We suspend this map in the simplest way,

$$
S^4 \supset \mathbb{R} \times \mathbb{H} \ni \begin{pmatrix} x \\ y \end{pmatrix} \overset{\phi_k}{\mapsto} \frac{1}{\sqrt{x^2 + |y|^{2k}}} \begin{pmatrix} x \\ y^k \end{pmatrix}.
$$

The critical values of maps ϕ_k, and *a fortiori* f_k, since the Hopf map is a submersion, are given by $(\pm 1, 0)^\top$ and the suspensions of the meridians $(x, \cos(\ell\pi/k) + \sin(\ell\pi/k)\hat{\alpha})$, α a purely imaginary quaternion and $1 < \ell < k$. Restricted to the inverse images of the complement of such points, ϕ_k is a local diffeomorphism. Therefore, the regular fibers of r_k will be given by sets of k disjoint standard Hopf fibers, and the obstruction given by Theorem 3 also vanishes in this case for the canonical round metric on S^7. However, if $|k| \geq 2$ the secondary obstruction given by Theorem 3.7 does *not* vanish for the round metric on S^7 which projects by the Hopf map to the round metric on S^4, since the pullback by Hopf of the suspensions of the meridians $(x, \cos(\ell\pi/k) + \sin(\ell\pi/k)\hat{\alpha})$ will not be totally geodesic. Note that, by continuity, the singular fibers detect negative curvature in the regular set that was invisible by just using Theorem 3.

Therefore, if one wants to construct a pullback metric of non-negative curvature on the Rigas bundles, the metric on S^4 must be changed so that this suspended meridians are totally geodesic, (by making S^3 to be a cylinder $S^2 \times I$, in a set that contains the meridians), and then the metric on S^7 is defined by the connection metric over the Hopf map.

5.3. Exotic 7-spheres

Consider the map $\phi_n : S^7 \to S^7$ given by the nth power of the Cayley octonions $\mathbb{O}$. If we write a unit octonion $q = \cos(t) + \alpha \sin(t)$ where α is purely imaginary, then $\phi_n(t) =$

$\cos(nt) + \alpha \sin(nt)$. Pulling back the bundle $S^3 \cdots Sp(2) \to S^7$ by ϕ_n we obtain principal S^3-bundles $S^3 \cdots E_n^{10} \to S^7$. Now writing octonion as pairs $(a, b)^\top$ of (column) quaternions, then the total space is given by

$$E_n \cong \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mid \phi_n\begin{pmatrix} a \\ b \end{pmatrix} \, \begin{matrix} c \\ d \end{matrix} \in Sp(2) \right\}.$$

With the projection onto S^7 given by the projection onto the first column $(a, b)^\top$. The unit quaternions act on E_n, by

$$q \star \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} qa\bar{q} & qc \\ qb\bar{q} & qd \end{pmatrix}.$$

The following is proven in [9]: the quotient of E_n^{10} by this action is diffeomorphic to Σ_n^7, n-times the Gromoll–Meyer sphere in the group of 7-dimensional homotopy spheres. Going all the way to the Hopf bundle, consider E_n^{10} as the pullback of the Hopf bundle by $f_n = a \circ h \circ \phi_n$, where a and h are as in Examples 1 and 5.1.

$$
\begin{array}{ccccc}
\tilde{E}_n & \longrightarrow & Sp(2) & \longrightarrow & S^7 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle h} \\
S^7 & \xrightarrow{\phi_n} & S^7 & \xrightarrow{a \circ h} & S^4
\end{array}
\qquad (5.3)
$$

Remark 5.1. It is instructive to compare these spaces to Rigas' bundles on the previous section; here the power map is on the Cayley numbers as the last map of the composition giving the pullback, and there the power map (of the quaternions) is the first step in the composition. Both cases are pullbacks of the Hopf bundle over S^4 by functions $S^7 \to S^4$, and the total spaces $\tilde{P}_k$ of and E_n are diffeomorphically related by the formula $E_n \cong \tilde{P}_k$, where $n = k(k+1)/2 \mod 12$ [2]. However, for the presentation given by the pullback by r_k the main obstruction vanishes, and by modifying the metric all obstructions can be eliminated, whereas we will presently see that this is impossible for the pullbacks by f_n: no metric on S^7 makes the regular fibers totally geodesic, and thus any pullback metric on $E_n = f_n^*\text{Hopf}$ has some negative curvature. In principle, the Grey–O'Neill tensor of the exotic actions could push the curvature in the base exotic sphere to be non-negative, although recent results suggests that the odds are not good [17]; this comment also applies to the exotic 8-spheres of next section.

We have

Lemma 5.2. *The only singular value of f is the south pole $(-1, 0) \in S^4 \subset \mathbb{R} \times \mathbb{H}$.*

Proof of lemma. Observe that the only singular values of ϕ_n are $\pm 1 \in S^7 \subset \mathbb{O}$, on the preimage of which the derivative of ϕ_n has a kernel of dimension at least 6 (ϕ being constant on the distance spheres $\cos(\frac{k\pi}{n}) + \alpha \sin(\frac{k\pi}{n})$). Thus the only singular value of $h \circ \phi_n$ is the class of the Hopf fiber through 1, which maps to $(1, 0) \in S^4 \subset \mathbb{R} \times \mathbb{H}$. The lemma now follows by composing with the antipodal map. $\square$

Restricted to the set $\phi_n^{-1}(\pm 1)$, the map ϕ_n is a (trivial) n-fold covering. Thus, the inverse image of regular values in S^4 consists the inverse image of Hopf fibers by ϕ_n, which are n disjoint 3-spheres, each one contained in a "belt" $\{\cos(t) + \alpha \sin(t) \mid \frac{k\pi}{n} < t < \frac{(k+1)\pi}{n}\}$.

The singular set $\phi^{-1}(-1, 0)$ is the union of the exponential 3-sphere $(x, 0)$, x a unit quaternion, with the distance spheres $\cos(\frac{k\pi}{n}) + \alpha \sin(\frac{k\pi}{n})$.

We have now

Proposition 5.3. *if $n > 1$, there is no metric on S^7 such that the inverse images of the regular values of f_n are totally geodesic.*

The remaining of this example is devoted to the proof of Proposition 5.3; the idea is to find a one parameter family of regular fibers of f_n whose convergence to the singular fiber contradicts Proposition 1.2.

Consider the one-dimensional manifold T of $S^4 \subset \mathbb{R} \times \mathbb{H}$ given by points having the second coordinate real, The inverse image of T by the Hopf map (and also by $a \circ h$, since T is invariant under the antipodal map) is the elements of $S^7 \subset \mathbb{H} \times \mathbb{H}$ which have both coordinates linearly dependent over the reals. This condition is invariant by the Cayley power map, and thus the inverse image of T by $f_n = a \circ h \circ \phi_n$ is also described by the real dependence of the first and second coordinates.

Consider now the curve $\sigma(\theta)$ in $T \subset S^4$ $\sigma(t) = (\cos(\theta), \sin(\theta))$, $\theta \in [\pi - \epsilon, \pi)$, the interval being chosen so that the trace of the curve curve is lies inside of the regular set, but approaches the critical value $(-1, 0)$ as $\theta \to \pi$.

The inverse image $(a \circ h \circ \phi_n)^{-1}(\sigma(\theta))$ is made of n disjoint 3-spheres, which we do not need to characterize precisely. In order to apply Proposition 1.2 we need to describe the evolution of these spheres as $\theta \to \pi$; we will also consider just the connected component of the preimage contained in the north polar cap $N = \{\cos(t) + \alpha \sin(t) \in S^7, \, |t \in [0, \pi/n]\}$.

Let then q be a unit Cayley number written as a pair of quaternions, $q = (a, b)^\top$, where $a = \cos(t) + \sin(t)p$ and $b = \sin(t)w$, $|p|^2 + |w|^2 = 1$, $t \in [0, \pi/n]$. We have

$$f_n(a, b)^\top = (2 \sin^2(nt)|w|^2 - 1, \star),$$

where the $\star$ in the second coordinate is determined by the first when we are close to $(-1, 0) \in S^4$ and both coordinates are real. Given $\theta \in [\pi - \epsilon, \pi)$, the formula for $f_n(a, b)^\top = \cos(\theta), \sin(\theta)$ implies that neither w,$\sin(nt)$ nor $\sin(t)$ are zero. Then

$$2 \sin^2(nt)|w|^2 - 1 = 2\frac{\sin^2(nt)}{\sin^2(t)}(\sin^2(t)|w|^2) - 1 = \eta(t)|b|^2 - 1,$$

where $\eta(t) = 2\frac{\sin^2(nt)}{\sin^2(t)}$ is bounded away from zero in the interval $[0, \pi/n]$. Then as $\theta \to \pi$, $|b| \to 0$. This means that the 3-spheres $f_n^{-1}(\cos(\theta), \sin(\theta)) \cap N$ converge to a set S contained in the meridian $(a, 0)$, $|a| = 1$, and being contained in the north polar cap we can also say that $S \subset \{(a, 0) \in S^7 \,|\, \Re(a) < \frac{3\pi}{2n}\}$. This set is diffeomorphic to an open ball in $\mathbb{R}^3$. Thus, Proposition 1.2 would furnish an embedding of a 3-sphere into a 3-ball in $\mathbb{R}^3$, which is impossible.   □

5.4. Exotic 8-sphere

Consider the map $\phi : S^8 \to S^7$ given by suspending the Hopf map $\eta : S^3 \to S^2$ in a smooth way. We write the Hopf map $S^3 \to S^2$ using the quaternions, $\eta(y) = yi\bar{y}$, where y is a unit quaternion and the image of η is contained in the unit purely imaginary quaternions. We also extend η to all quaternions in the obvious way.

We write S^8 as the unit sphere of $\mathbb{H} \times \mathbb{R} \times \mathbb{H}$, and let $\psi : S^8 \to \mathbb{R}^8 \cong \mathbb{H} \times \mathbb{H}$ be given by

$$\psi \begin{pmatrix} x \\ \lambda \\ y \end{pmatrix} = \begin{pmatrix} x \\ \lambda + \eta(y) \end{pmatrix}$$

The image does not fall in the Euclidean sphere S^7. However since $|\psi(x)| = |x|^2 + \lambda^2 + |y|^4 \neq 0$, we can normalize and $\phi = \frac{\eta}{|\eta|} : S^8 \to S^7$ clearly suspends the Hopf map.

Remark 5.4. There are other ways of building smooth suspensions of the Hopf map, which in principle would furnish different examples of the application of Theorem 3. We present the simplest one.

Then, the total space of the pull-back of $S^3 \cdots Sp(2) \to S^7$ by ϕ is readily identified with the set

$$E^{11} = \left\{ \begin{pmatrix} x & c \\ \lambda & d \\ y & \end{pmatrix} \in S^8 \times S^7 \;\middle|\; \left(\phi \begin{pmatrix} x \\ \lambda \\ y \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right) \in Sp(2), \right\}$$

The unit quaternions act on E^{11} as follows:

$$q \star \begin{pmatrix} qx\bar{q} & c \\ \lambda & d \\ qy & \end{pmatrix} = \begin{pmatrix} qx\bar{q} & qc \\ \lambda & qd \\ qy & \end{pmatrix} \in E^{11} \quad \text{if} \quad \begin{pmatrix} x & c \\ \lambda & d \\ y & \end{pmatrix} \in E^{11} \tag{5.4}$$

On one hand, the projection in the first column $pr_1 : E^{11} \to S^8$ define it as the pull-back bundle of $Sp(2) \to S^7$, on the other hand $\star$ in (5.4) defines a new free action on E^{11}. The quotient of this action is diffeomorphic to the only exotic sphere in dimension 8, [20]. Again, pulling back all the way from the Hopf map, we have that E^{11} is the pullback of the Hopf bundle $S^3 \cdots S^7 \to S^4$ by $f = a \circ h \circ \phi$.

We now have to study the inverse images by f of its regular values. We have:

Lemma 5.5. *The only singular value of f is the south pole $(-1, 0) \in S^4 \subset \mathbb{R} \times \mathbb{H}$.*

Proof of lemma. One can first note that the derivative of ϕ at $p = (1, 0, 0)^T$ is $D\phi_p(X, \Lambda, Y) = (X, \Lambda)$ which spans only one dimension transversal to the fiber $(z, 0) \subset S^7$ making the point $(-1, 0) \in S^4$ singular for $ah\phi$. On the other hand if $y \neq 0$, ϕ is a submersion and therefore so is f, and the last case in hand is $p = (x, \lambda, 0)$ which goes to a fiber different from $(z, 0)$. In this last case, the differential of ϕ is again $D\phi_p(X, \Lambda, Y) = (X, \Lambda)$ however, now the space spanned by this differential is completely transversal to the fiber of $\phi(p)$ since the tangent space to the last is of the form $(x\xi, \lambda\xi)$, for purely imaginary quaternions ξ, in particular, it has no real part in the second coordinate. $\qquad\square$

Then we see that the inverse image of the singular point is $S^3 \subset S^8$ given by points $(x, 0, 0) \in S^8$. The inverse images of regular points are 4-dimensional submanifolds of S^3. In this case Proposition 1.2 applies immediately, since there can be no embedding of a 4-dimensional manifold into a 3-dimensional one.

5.5. Kervaire spheres

Consider the principal bundle of special orthonormal frames over the round S^{2n+1}, I.e., consider the principal bundle $SO(2n+1) \cdots SO(2n+2) \overset{pr}{\to} S^{2n+1}$ given by projection on the first column. Consider also $\tau : S^{2n} \to SO(2n+1)$ where $\tau(x)$ is defined as the reflection by the hyperplane orthogonal to x. As a map from S^{4n+1} to S^{2n+1} we can consider $J\tau$, defined as

$$J\tau(x, y) = \exp\tau\left(\frac{y}{|y|}\right)x$$

where $(x, y) \in \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}$. This map extends continuously to $y = 0$ and has an appropriate equivariant smoothing with same fibers (this situation is quite common in this kind of construction, see e.g. section 2.3 of [8]). Its homotopy type is the image of τ by the Hopf-Whitehead J-homomorphism.

The bundle $P = (J\tau)^*\pi \to S^{4n+1}$ admits a free $SO(2n+1)$ action (which is isometric with respect to natural metrics) with quotient that can be identified with Σ^{4n+1}, the Kervaire sphere of dimension $4n + 1$ as presented in Chapter I, section 7 of [3,20]. These examples are known to be exotic [15] for infinitely many n's.

$$\begin{array}{ccc} P & \longrightarrow & SO(2n+2) \\ & \searrow & \searrow \\ \downarrow & \Sigma^{4n+1} \longrightarrow & S^{2n+1} \\ & & \downarrow \\ S^{4n+1} & \xrightarrow{\ J\tau\ } & S^{2n+1} \end{array} \qquad (5.5)$$

The fibers of the map $J\tau$ are the spheres $y = 0$ and

$$\{(\tau(y)^{-1}x, \lambda y) \mid y \in S^{2n}, \ \lambda \in [-1, 0)\}$$

The information given by the dimensions and topology of the fibers is not sufficient to perceive an obstruction to Theorems 3 and 3.7, making $(J\tau)^*\pi$ a candidate to induce non-negative curvature on Σ^{4n+1}.

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