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# Commuting involutions on surfaces of general type with $p_g = 0$ and $K^2 = 7$

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**Abstract.** The aim of this article is to classify the pairs $(S, G)$, where $S$ is a smooth minimal surface of general type with $p_g = 0$ and $K^2 = 7$, $G$ is a subgroup of the automorphism group of $S$ and $G$ is isomorphic to the group $\mathbb{Z}_2^2$. We show that there are only three possible cases for such pairs. Two of them correspond to known examples, but the existence of the third one remains an open problem. The Inoue surfaces with $K^2 = 7$, which are finite Galois $\mathbb{Z}_2^2$-covers of the 4-nodal cubic surface, are the first examples of such pairs. More recently, the author constructed a new family of such pairs. They are finite Galois $\mathbb{Z}_2^2$-covers of certain 6-nodal Del Pezzo surfaces of degree one. We prove that the base of the Kuranishi family of deformations of a surface in this family is smooth. We show that, in the Gieseker moduli space of canonical models of surfaces of general type, the subset corresponding to the surfaces in this family is an irreducible connected component, normal, unirational of dimension 3.

## 1. Introduction

The first examples of surfaces of general type with $p_g = 0$ were constructed in the 1930's (cf. [9,20]). Since then, these surfaces have been studied by many mathematicians, and more and more examples were constructed (cf. [1, Table 14,p. 304] and the references given there). Nowadays, there is a long list of examples (cf. [2, Table 1–3]). Minimal smooth surfaces of general type with $p_g = 0$ have invariants $1 \leq K^2 \leq 9$. Surprisingly, there are few examples of surfaces of general type with $p_g = 0$ and $K^2 = 7$. The first family of such surfaces was constructed by M. Inoue (cf. [21]).

The bicanonical map plays an important role in the classification of surfaces of general type with $p_g = 0$. It is shown in [25] and [26] that the bicanonical morphism of a smooth minimal surface of general type with $p_g = 0$ and $K^2 = 7$ has degree 1 or 2; and if the bicanonical morphism has degree 2, the surface has a genus 3 hyperelliptic fibration and the fibration has five double fibers and one reducible fiber. Another important way to classify surfaces with $p_g = 0$ is to study surfaces with certain automorphisms (for example, cf. [8,22]). Involutions on surfaces of general type with $p_g = 0$ and $K^2 = 7$ are investigated in [24] and [29]. All the

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possibilities of the quotient surfaces and the fixed loci of the involutions are listed in these articles.

Recently, the author constructed a new family of surfaces of general type with $p_g = 0$ and $K^2 = 7$ (cf. [16]). In this article, we illustrate the process which leads to the discovery of these surfaces. We explain the main idea. From the results of [24] and [29], we observe that, under the assumption of the existence of one involution, it is still hard to construct new examples of surfaces of general type with $p_g = 0$ and $K^2 = 7$ (cf. [16, Section 6]). So it is natural to try to classify such surfaces with two distinct involutions. We restrict our attention to the situation where the two involutions commute.

**Theorem 1.1.** *Let $S$ be a minimal smooth surface of general type with $p_g = 0$ and $K_S^2 = 7$. Assume that $\mathrm{Aut}(S)$ contains a subgroup $G = \{1, g_1, g_2, g_3\}$, which is isomorphic to $\mathbb{Z}_2^2$. Let $R_i$ be the divisorial part of the fixed locus of the involution $g_i$ for $i = 1, 2, 3$ and let $\pi : S \to \Sigma := S/G$ be the quotient map. Then the canonical divisor $K_S$ is ample and $R_i^2 = -1$ for $i = 1, 2, 3$. Moreover, one of the following cases holds:*

(a) $(K_S R_1, K_S R_2, K_S R_3) = (7, 5, 5)$*; in this case,* $(R_1 R_2, R_1 R_3, R_2 R_3) = (5, 9, 7)$ *or* $(9, 5, 7)$ *and $\pi : S \to \Sigma$ is a finite Galois $\mathbb{Z}_2^2$-cover of the 4-nodal cubic surface as described in Example 3.1;*
(b) $(K_S R_1, K_S R_2, K_S R_3) = (5, 5, 3)$*; in this case,* $(R_1 R_2, R_1 R_3, R_2 R_3) = (7, 5, 1)$ *or* $(7, 1, 5)$ *and $\pi : S \to \Sigma$ a finite Galois $\mathbb{Z}_2^2$-cover of a 6-nodal Del Pezzo surface of degree one as described in Example 4.1;*
(c) $(K_S R_1, K_S R_2, K_S R_3) = (5, 3, 1)$*; in this case,* $(R_1 R_2, R_1 R_3, R_2 R_3) = (1, 3, 1)$ *and the surface $\Sigma$ is a rational surface with $K_\Sigma^2 = -1$ and containing 8 nodes.*

We adopt the convention $K_S R_1 \geq K_S R_2 \geq K_S R_3$ in the theorem.

The surfaces in the case (a) are called Inoue surfaces. They were originally constructed by Inoue [21] and they were described as $\mathbb{Z}_2^2$-covers of the 4-nodal cubic surface in [25]. The bicanonical morphisms of Inoue surfaces have degree 2 (cf. [25, Example 4.1]). The surfaces in the case (b) were constructed by the author; they are different from the Inoue surfaces because they have birational bicanonical morphisms (cf. [16, Section 3–4]).

The proof of Theorem 1.1 is organized as follows. We describe the general properties of the pair $(S, G)$ in Sect. 2, showing in particular that $K_S$ is ample and that $\Sigma$ is rational in Proposition 2.7. Then we prove that there are only three possible cases for $(K_S R_1, K_S R_2, K_S R_3)$ in Theorem 2.9 and prove Theorem 1.1(c) in Proposition 2.10. In Sects. 3 and 4, we study the two remaining cases in detail and prove Theorem 1.1(a)–(b). In particular, Sect. 4.2 provides a detailed exposition of the classification process which leads to the discovery of the surfaces in the case (b). Here we explain the key strategy. Because the Picard number of the surface $S$ is 3, the four divisors $K_S$, $R_1$, $R_2$ and $R_3$ are linearly dependent in $\mathrm{Pic}(S)_{\mathbb{Q}}$. Combining this fact, the algebraic index theorem and the adjunction formula, we could easily analyze the configuration of the divisors $R_1$, $R_2$, $R_3$ and determine the number of nodes of the quotient surface $\Sigma$. When $\Sigma$ is a Del Pezzo surface (the cases (a) and (b)), we get a complete classification.

However, we have difficulties when dealing with the case (c) because $\Sigma$ is no longer a Del Pezzo surface. We do not know any example for this case. We can not exclude this case at the moment. Nevertheless, we remark that the minimal resolution of the quotient surface $S/g_3$ is a numerical Campedelli surface containing 5 disjoint nodal curves (cf. Proposition 2.10(c)). In particular, $S/g_3$ is of general type. Note that for surfaces in Examples 3.1 and 4.1, all the intermediate quotient surfaces have Kodaira dimensions at most 1 (cf. [24, Section 5], [16, Section 5–6]). So it is worth finding pairs $(S, G)$ in the case (c). We shall pursue this in the future.

The classification in Theorem 1.1 contributes to the study of the moduli of the surfaces in the case (b). The notation $\mathcal{M}^{\text{can}}_{1,7}$ in the following theorem stands for the Gieseker moduli space of canonical models of surfaces of general type with $\chi(\mathcal{O}) = 1$ and $K^2 = 7$ (cf. [19]).

**Theorem 1.2.** *Assume that $(S, G)$ is a pair in the case (b) of Theorem 1.1. Then the base of the Kuranishi family of deformations of $S$ is smooth.*

*In the Gieseker moduli space $\mathcal{M}^{\text{can}}_{1,7}$, the subset $\mathfrak{B}$ corresponding to the surfaces in the case (b) of Theorem 1.1 is an irreducible connected component, normal, unirational of dimension* 3.

We point out that similar statements for Inoue surfaces have been achieved in [3]: The base of the Kuranishi family of deformations of an Inoue surface is smooth; in the Gieseker moduli space, the Inoue surfaces form a 4-dimensional irreducible connected component, normal and unirational. Theorem 1.2 is proved in Sect. 5. The smoothness of the base of the Kuranishi family is obtained by calculating the dimensions of the cohomology groups of the tangent sheaf of the surface $S$. And the openness of $\mathfrak{B}$ in $\mathcal{M}^{\text{can}}_{1,7}$ follows from this. We emphasize that the closedness of $\mathfrak{B}$ in $\mathcal{M}^{\text{can}}_{1,7}$ follows from the classification in Theorem 1.1.

*Notation and conventions*

We adopt the convention that the indices $i \in \{1, 2, 3\}$ should be understood as residue classes modulo 3. Denote by $g_1, g_2, g_3$ the nontrivial elements of the group $G \cong \mathbb{Z}_2^2$. Denote by $G^* = \{1, \chi_1, \chi_2, \chi_3\}$ the group of characters of $G$, where $\chi_i(g_i) = 1$ and $\chi_i(g_{i+1}) = \chi_i(g_{i+2}) = -1$ for $i = 1, 2, 3$. If $G$ acts on a finite dimensional linear space $V$ over $\mathbb{C}$, we denote by $V^{\text{inv}}$ the $G$-invariant subspace of $V$ and by $V^{\chi_i}$ the eigenspace of $V$ corresponding to the character $\chi_i$.

Linear equivalence between divisors is denoted by $\equiv$. Numerical equivalence between divisors is denoted by $\overset{num}{\sim}$. A $-m$-curve ($m \geq 0$) on a smooth projective surface stands for a smooth rational curve with self intersection number $-m$. A $-2$-curve is also called a nodal curve. We denote by $c_1(\mathcal{L})$ (respectively $c_1(D)$) the first Chern class of an invertible sheaf $\mathcal{L}$ (respectively a Cartier divisor $D$). The rest of the notation is standard in algebraic geometry.

## 2. Commuting involutions on surfaces with $p_g = 0$

Let $S$ be a smooth irreducible projective surface over $\mathbb{C}$. A nontrivial automorphism $\alpha$ on $S$ is called an involution if $\alpha^2 = \text{Id}_S$. We refer to [8, Section 3] for the properties of an involution on a surface. We follow the ideas and the techniques there to study commuting involutions on a surface of general type with $p_g = 0$ and $K^2 = 7$.

*2.1. General case*

To study the general case, we assume that $S$ is a smooth minimal surface of general type with $p_g = 0$ and $\mathrm{Aut}(S)$ contains a subgroup $G = \{1, g_1, g_2, g_3\}$, which is isomorphic to $\mathbb{Z}_2^2$. Burniat surfaces are examples of such surfaces with $2 \le K^2 \le 6$ (cf. [7,28]).

Let $\pi : S \to \Sigma = S/G$ be the quotient map. We use Cartan's lemma (see [14]) to analyze the local properties of the ramification locus and branch locus of $\pi$. More precisely, for $i = 1, 2, 3$, let $R_i$ be the divisorial part of the fixed locus of the involution $g_i$ and let $B_i := \pi(R_i)$. Cartan's lemma implies that the divisors $R_1$, $R_2$ and $R_3$ satisfy the following properties:

(i) if $R_i$ is not 0, it is a disjoint union of irreducible smooth curves;
(ii) the divisor $R_1 + R_2 + R_3$ is normal crossing.

For each $i \in \{1, 2, 3\}$, if $R_{i+1} \cap R_{i+2} \neq \emptyset$, then the intersection points of $R_{i+1}$ and $R_{i+2}$ are isolated fixed points of the involution $g_i$. The image of these points under $\pi$ are smooth points of $\Sigma$. Besides the intersection points of $R_{i+1}$ and $R_{i+2}$, the involution $g_i$ has $l_i$ pairs of isolated fixed points $(p_{ij_i}, q_{ij_i})$ for $j_i = 1, \ldots, l_i$. The two points $p_{ij_i}$ and $q_{ij_i}$ of such a pair are permutated by $g_{i+1}$ and $g_{i+2}$. Their images $r_{ij_i} = \pi(p_{ij_i}) = \pi(q_{ij_i})$ are nodes of $\Sigma$. The nodes $r_{ij_i}$ ($j_i = 1, \ldots, l_i$ and $i = 1, 2, 3$) are the only singularities of $\Sigma$. In particular, $\Sigma$ is Gorenstein. We have the following formula

$$K_S = \pi^* K_\Sigma + R_1 + R_2 + R_3. \tag{2.1}$$

*Remark 2.1.* The discussion above also shows that the divisors $B_1$, $B_2$ and $B_3$ are contained in the smooth locus of $\Sigma$. Moreover, the statements (i) and (ii) still hold if we replace $R_i$ by $B_i$ (cf. [13, Theorem 2]).

**Proposition 2.2.** *Let $S$ be a minimal smooth surface of general type with $p_g = 0$. Assume that* $\mathrm{Aut}(S)$ *contains a subgroup $G = \{1, g_1, g_2, g_3\}$, which is isomorphic to $\mathbb{Z}_2^2$. Then:*

(a) *for $i = 1, 2, 3$, $2l_i + R_{i+1}R_{i+2} = K_S R_i + 4$, $0 \le K_S R_i \le K_S^2$ and the integers $K_S R_i$, $R_{i+1}R_{i+2}$ and $K_S^2$ are of the same parity;*
(b) $h^0(S, \mathcal{O}_S(2K_S))^{\mathrm{inv}} = \frac{1}{4}(K_S^2 + K_S R_1 + K_S R_2 + K_S R_3) + 1$ *and for $i = 1, 2, 3$,* $h^0(S, \mathcal{O}_S(2K_S))^{\chi_i} = \frac{1}{4}(K_S^2 + K_S R_i - K_S R_{i+1} - K_S R_{i+2})$*;*
(c) $K_\Sigma^2 = \frac{1}{4}(K_S^2 + R_1^2 + R_2^2 + R_3^2) + 6 - l_1 - l_2 - l_3$ *and the integer $K_S^2 + R_1^2 + R_2^2 + R_3^2$ is divisible by* 4.

*Proof.* The discussion above shows that the number of the isolated fixed points of the involution $g_i$ is $2l_i + R_{i+1}R_{i+2}$. By [8, Lemma 3.2andProposition 3.3(v)], $2l_i + R_{i+1}R_{i+2} = K_S R_i + 4 \le K_S^2 + 4$ and $2l_i + R_{i+1}R_{i+2}$ has the same parity of $K_S^2$. This implies (a).

Fix $i$. The invariant subspace of $H^0(S, \mathcal{O}_S(2K_S))$ for the action of $g_i$ has dimension $\frac{1}{2}(K_S^2 + K_S R_i) + 1$ (cf. [8, Proposition 3.3(iii)andCorollary 3.4(i)]). In our notation, we have

$$\begin{aligned} &\dim H^0(S, \mathcal{O}_S(2K_S))^{\mathrm{inv}} + \dim H^0(S, \mathcal{O}_S(2K_S))^{\chi_i} \\ &= \frac{1}{2}(K_S^2 + K_S R_i) + 1 \quad \text{for} \quad i = 1, 2, 3. \end{aligned}$$

Since $\dim H^0(S, \mathcal{O}_S(2K_S)) = K_S^2 + 1$, assertion (b) follows.

By (2.1), we have

$$K_\Sigma^2 = \frac{1}{4}\left(K_S^2 + R_1^2 + R_2^2 + R_3^2\right) - \frac{1}{2}K_S(R_1 + R_2 + R_3) + \frac{1}{2}(R_1R_2 + R_1R_3 + R_2R_3) \tag{2.2}$$

Because $\Sigma$ has only nodes, $K_\Sigma^2$ is an integer. Then assertion (c) follows by (a). $\square$

**Corollary 2.3.** (cf. Corollary 3.6 in [8]) *Fix $i$. The bicanonical map $\varphi\colon S \dashrightarrow \mathbb{P}^{K_S^2}$ is composed with the involution $g_i$ if and only if $K_S R_i = K_S^2$. In this case, $K_S R_{i+1} = K_S R_{i+2}$.*

Let $\eta\colon W \to \Sigma$ be the minimal resolution of $\Sigma$. For each $i = 1, 2, 3$, let $\overline{N}_i$ be the disjoint union of the $l_i$ nodal curves over the nodes $r_{ij_i}$ for $j_i = 1, \dots, l_i$ and let $\overline{B}_i := \eta^* B_i$. Let $\varepsilon\colon V \to S$ be the blowup at $p_{ij_i}$ and $q_{ij_i}$ for $i = 1, 2, 3$ and $j_i = 1, \dots, l_i$. Then the $\mathbb{Z}_2^2$-action on $S$ lifts to $V$ and $V/G \cong W$. There is a commutative diagram:

$$\begin{array}{ccccccc} V & \xrightarrow{\varepsilon} & S & & R_i & & \{(p_{ij_i}, q_{ij_i})\}_{j_i=1,\dots,l_i} \\ \downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ W & \xrightarrow[\eta]{} & \Sigma & \overline{B}_i \xrightarrow[\eta]{} & B_i & \overline{N}_i \xrightarrow[\eta]{} & \{r_{ij_i}\}_{j_i=1,\dots,l_i} \end{array} \tag{2.3}$$

The quotient map $\overline{\pi}\colon V \to W$ is a finite flat $\mathbb{Z}_2^2$-cover branched on the divisors $\Delta_1 := \overline{B}_1 + \overline{N}_1$, $\Delta_2 := \overline{B}_2 + \overline{N}_2$ and $\Delta_3 := \overline{B}_3 + \overline{N}_3$. There are three divisors $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ of $W$ such that

$$2\mathcal{L}_i \equiv \overline{B}_{i+1} + \overline{N}_{i+1} + \overline{B}_{i+2} + \overline{N}_{i+2}, \quad \mathcal{L}_i + \overline{B}_i + \overline{N}_i \equiv \mathcal{L}_{i+1} + \mathcal{L}_{i+2} \tag{2.4}$$

for $i = 1, 2, 3$ (cf. [13, Section 2]).

**Lemma 2.4.** *Let*

$$D := 2K_W + \overline{B}_1 + \overline{B}_2 + \overline{B}_3 \quad \textit{and} \quad M := K_W + D \tag{2.5}$$

*Then:*

(a) *the divisor $D$ is nef and big, $\overline{\pi}^* D = \varepsilon^*(2K_S)$ and $D^2 = K_S^2$;*
(b) *for each $i = 1, 2, 3$, $\overline{B}_i^2 = R_i^2$, $\overline{B}_i \overline{B}_{i+1} = R_i R_{i+1}$ and $D\overline{B}_i = K_S R_i$;*
(c) $h^0(W, \mathcal{O}_W(D)) = \frac{1}{4}(K_S^2 + K_S R_1 + K_S R_2 + K_S R_3) + 1$ *and* $\dim |D| \geq 1$*;*

(d) *any divisor in* $|D|$ *is* 1*-connected and* $p_a(D) \geq 1$*;*
(e) $h^0(W, \mathcal{O}_W(M)) = p_a(D) \geq 1$.

*Assume furthermore that* $K_S$ *is ample. Then:*

(f) *if* $DC = 0$ *for an irreducible curve* $C$*, then* $C$ *is one of the nodal curves in* $\overline{N}_1 \cup \overline{N}_2 \cup \overline{N}_3$*;*
(g) *the divisor* $M$ *is nef.*

*Proof.* Note that $\pi^* B_i = 2R_i$ for $i = 1, 2, 3$. By (2.1) and commutativity of (2.3), we have

$$\overline{\pi}^* D = \overline{\pi}^* \eta^* (2K_\Sigma + B_1 + B_2 + B_3) = \varepsilon^* \pi^* (2K_\Sigma + B_1 + B_2 + B_3) = \varepsilon^* (2K_S).$$

Then $D^2 = \frac{1}{4} 4K_S^2 = K_S^2$. The divisor $D$ is nef and big because so is $2K_S$.

Assertion (b) follows by $\overline{\pi}^* D = \varepsilon^*(2K_S)$ and $\overline{\pi}^* \overline{B}_i = \varepsilon^*(2R_i)$.

Note that

$$K_W + \overline{B}_1 + \overline{B}_2 + \overline{B}_3 \overset{num}{\sim} \frac{1}{2} D + \frac{1}{2}\left(\overline{B}_1 + \overline{B}_2 + \overline{B}_3\right).$$

The divisor $D$ is nef and big, whereas $\frac{1}{2}(\overline{B}_1 + \overline{B}_2 + \overline{B}_3)$ is effective with zero integral part, and its support has normal crossings (see Remark 2.1). Thus $H^k(W, \mathcal{O}_W(D)) = 0$ for $k = 1, 2$ by Kawamata–Vieweg vanishing theorem (cf. [18, Corollary 5.12(c)]). The Riemann–Roch theorem gives $h^0(W, \mathcal{O}_W(D)) = \frac{1}{2}(D^2 - DK_W) + 1$. Then (c) follows by (b) and (2.5).

By (2.5), (b) and Proposition 2.2(b), one has

$$\frac{1}{2}\left(D^2 + K_W D\right) = \frac{1}{4}\left(3D^2 - D\overline{B}_1 - D\overline{B}_2 - D\overline{B}_3\right) = \sum_{i=1}^{3} \dim H^0(S, \mathcal{O}_S(2K_S))^{\chi_i}.$$

So $\frac{1}{2}(D^2 + K_W D) \geq 0$, i.e., $p_a(D) \geq 1$. Because $D$ is nef and big, $D$ is 1-connected (cf. [27, Lemma 2.6]). This proves (d).

Assertion (e) follows from the long exact sequence of cohomology groups obtained from

$$0 \to \mathcal{O}_W(K_W) \to \mathcal{O}_W(K_W + D) \to \omega_D \to 0$$

and the fact $p_g(W) = q(W) = 0$.

Now we prove (f). Assertion(a) implies $\varepsilon^* K_S . \overline{\pi}^* C = 0$ and thus $K_S . \varepsilon_*(\overline{\pi}^* C) = 0$. Since $K_S$ is ample, $\varepsilon_*(\overline{\pi}^* C) = 0$ and Supp $\overline{\pi}^* C$ is contained in the exceptional divisors of $\varepsilon$. The nodal curves $\overline{N}_1 \cup \overline{N}_2 \cup \overline{N}_3$ are exactly the images of the exceptional divisors of $\varepsilon$ under $\overline{\pi}$. So $C$ is one of them.

For (g), assume by contradiction that $MC < 0$ for an irreducible curve $C$. Because $M$ is effective by (e), we have $C^2 < 0$. Since $K_W C = (M - D)C < -DC \leq 0$, $C$ is a $(-1)$-curve and $D.C = 0$. This contradicts (f). Hence $M$ is nef. □

*2.2. Surfaces with $p_g = 0$ and $K^2 = 7$*

In the remainder of the article, we always assume that $S$ is a smooth minimal surface of general type with $p_g = 0$ and $K_S^2 = 7$. We list some basic properties of $S$. The surface $S$ has irregularity $q(S) = 0$ and has Picard number $\rho(S) = 3$ by Noether's formula and Hodge decomposition. The expotential cohomology sequence gives $\mathrm{Pic}(S) \cong H^2(S, \mathbb{Z})$. Poincaré duality implies that the intersection form on $\mathrm{Num}(S) := \mathrm{Pic}(S)/\mathrm{Pic}(S)_{\mathrm{tor}}$ is unimodular. The the bicanonical map $\varphi\colon S \to \mathbb{P}^7$ has degree either 1 or 2 (cf. [25,26]). We also need the following lemmas.

**Lemma 2.5.** (cf. Theorem 1.4(1)(f) in [23]) *The surface $S$ contains at most one nodal curve.*

*Proof.* Assume by contradiction that $S$ contains two nodal curves $C_1$ and $C_2$. Then $C_1C_2 \leq 1$. The matrix of the intersection numbers of $K_S$, $C_1$ and $C_2$ has determinant 21 or 28, either of which is not a square integer. Since $\rho(S) = 3$, this contradicts the fact that the intersection form on $\mathrm{Num}(S)$ is unimodular. □

**Lemma 2.6.** (cf. Proposition 3.6 in [6]) *Let $\alpha$ be an involution on $S$ and let $R_\alpha$ be the divisorial part of the fixed locus of $\alpha$. Then $R_\alpha^2 = \pm 1$.*

*Proof.* Let $\mathrm{tr}(\alpha^*)$ be the trace of the induced linear map $\alpha^*\colon H^2(S, \mathbb{C}) \to H^2(S, \mathbb{C})$. Then $R_\alpha^2 = 2 - \mathrm{tr}(\alpha^*)$ by [17, Lemma 4.2]. The Chern classes $c_1(K_S)$ and $c_1(R_\alpha) \in H^2(S, \mathbb{C})$ are invariant under $\alpha^*$. So $\mathrm{tr}(\alpha^*) = -1, 1$ or 3. It suffices to exclude the case $\mathrm{tr}(\alpha^*) = -1$.

If $\mathrm{tr}(\alpha^*) = -1$, then $\alpha^*$ has eigenvalues $-1, -1$ and 1. This implies $R_\alpha \overset{num}{\sim} rK_S$ for some positive rational number $r$. Because $R_\alpha^2 = 2 - \mathrm{tr}(\alpha^*) = 3$ and $K_S^2 = 7$, this is impossible. □

We assume furthermore that $\mathrm{Aut}(S)$ contains a subgroup $G = \{1, g_1, g_2, g_3\} \cong \mathbb{Z}_2^2$. We keep the same notation introduced in the previous subsection and denote by

$$A = (R_i R_j)_{1 \leq i \leq j \leq 3} \tag{2.6}$$

the matrix of intersection numbers of the ramification divisors $R_1$, $R_2$ and $R_3$ of the quotient map $\pi: S \to \Sigma = S/G$.

Without loss of generality, we may assume $K_S R_1 \geq K_S R_2 \geq K_S R_3$. Since the bicanonical map $\varphi$ has degree at most 2, $\varphi$ is composed with at most one involution in $G$. Then by Proposition 2.2(a)–(b) and Corollary 2.3, one of the following cases occurs:

- if $\varphi$ is composed with exact one involution in $G$, then

$$(K_S R_1, K_S R_2, K_S R_3) \in \{(7, 1, 1), (7, 3, 3), (7, 5, 5)\}; \tag{2.7}$$

- if $\varphi$ is not composed with any involution in $G$, then

$$(K_S R_1, K_S R_2, K_S R_3) \in \{(3, 1, 1), (3, 3, 3), (5, 3, 1), (5, 5, 3)\}. \tag{2.8}$$

**Proposition 2.7.** *Let $S$ be a minimal smooth surface of general type with $p_g = 0$ and $K_S^2 = 7$. Assume that* $\mathrm{Aut}(S)$ *contains a subgroup $G = \{1, g_1, g_2, g_3\}$, which is isomorphic to $\mathbb{Z}_2^2$. Then:*

(a) *for $i = 1, 2, 3$, $R_i^2 = -1$;*
(b) *the quotient $\Sigma$ is a rational surface with $K_\Sigma^2 = 7 - l_1 - l_2 - l_3$ and $\rho(\Sigma) = 3$;*
(c) *the divisors $R_1$, $R_2$, $R_3$ generate a sublattice of* $\mathrm{Num}(S)$ *and* $\det A$ *is a positive square integer;*
(d) *the numbers $l_1$, $l_2$ and $l_3$ are even integers;*
(e) *the canonical divisor $K_S$ is ample.*

*Proof.* Recall that $R_i^2 = \pm 1$ for $i = 1, 2, 3$ by Lemma 2.6. We recall in the diagram (2.3) that $W$ is the minimal resolution of $\Sigma$. Note that $R_i R_{i+1} (= \overline{B}_i \overline{B}_{i+1})$ is a positive odd integer (see Proposition 2.2(a) and Lemma 2.4(b)).

We first exclude the case $(K_S R_1, K_S R_2, K_S R_3) = (3, 1, 1)$. Since $K_S^2 = 7$, the algebraic index theorem gives $(R_1 + R_2 + R_3)^2 \le \frac{5^2}{7}$. It follows that $R_i^2 = -1$ and $R_i R_{i+1} = 1$ for $i = 1, 2, 3$. Then $\overline{B}_i^2 = -1$, $\overline{B}_i \overline{B}_{i+1} = 1$ and $D\overline{B}_2 = 1$ by Lemma 2.4(b). So $K_W B_2 = \frac{1}{2}(D - \overline{B}_1 - \overline{B}_2 - \overline{B}_3)\overline{B}_2 = 0$ by (2.5). This contradicts the adjunction formula.

Now we show that $W$ (and thus $\Sigma$) is a rational surface and $K_W^2 \le 3$. By (2.5) and Lemma 2.4(b), we have

$$DK_W = \frac{1}{2}\left(D^2 - D\overline{B}_1 - D\overline{B}_2 - D\overline{B}_3\right) = \frac{1}{2}\left(K_S^2 - K_S R_1 - K_S R_2 - K_S R_3\right) \tag{2.9}$$

It follows that $DK_W \in \{-5, -3, -1\}$ for all the possibilities (2.7) and (2.8). Since $D$ is nef and big (see Lemma 2.4(a)), $W$ is a rational surface. The algebraic index theorem gives $K_W^2 \le 3$ since $D^2 = 7$.

Because $W$ is a rational surface, the Picard number $\rho(W) = 10 - K_W^2 \ge 7$. By Proposition 2.2(c), one of the following two cases holds:

- for $i = 1, 2, 3$, $R_i^2 = -1$ and $K_W^2 = 7 - l_1 - l_2 - l_3$;
- two of the three integers $R_i^2$ equal 1, the third one equals $-1$ and $K_W^2 = 8 - l_1 - l_2 - l_3$.

Assume that the latter holds. Without loss of generality, assume that $R_1^2 = R_2^2 = 1$ and $R_3^2 = -1$. Then $\rho(W) = l_1 + l_2 + l_3 + 2$ and $W$ contains $\rho(W) - 2$ disjoint nodal curves $\overline{N}_1 \cup \overline{N}_2 \cup \overline{N}_3$. By [17, Theorem 3.3], $l_1 + l_2 + l_3$ is an even integer. Also $\overline{B}_1$, $\overline{B}_3$ and these nodal curves generate a sublattice of $\mathrm{Num}(W)$. The matrix of intersection numbers of this sublattice has determinant $-2^{l_1+l_2+l_3}[1 + (\overline{B}_1\overline{B}_3)^2]$. Since the intersection form on $\mathrm{Num}(W)$ is unimodular, $2^{l_1+l_2+l_3}[1 + (\overline{B}_1\overline{B}_3)^2]$ is a positive square integer. Because $l_1 + l_2 + l_3$ is an even integer and $\overline{B}_1\overline{B}_3$ is a positive odd integer, this is impossible. Thus the former case holds. We have proved (a) and (b).

The matrix $A$ has determinant

$$\det A = (R_1R_2)^2 + (R_1R_3)^2 + (R_2R_3)^2 + 2(R_1R_2)(R_1R_3)(R_2R_3) - 1 \tag{2.10}$$

Because the numbers $R_i R_{i+1}$ are positive odd integer, det $A$ is a positive integer. Since $\rho(S) = 3$, $R_1$, $R_2$, $R_3$ generate a sublattice of $\mathrm{Num}(S)$ and therefore det $A$ is a square integer by Poincaré duality. This proves (c).

By Lemma 2.4(b), the matrix $A$ is also the intersection number matrix of the divisors $\overline{B}_1$, $\overline{B}_2$, $\overline{B}_3$. It follows that $\overline{B}_1$, $\overline{B}_2$ and $\overline{B}_3$ and the $l_1+l_2+l_3$ nodal curves $\overline{N}_1 \cup \overline{N}_2 \cup \overline{N}_3$ generate a sublattice of $\mathrm{Num}(W)$. Therefore $2^{l_1+l_2+l_3}$ det $A$ is a positive square integer by Poincaré duality. Hence $l_1 + l_2 + l_3$ is an even integer by (c).

Note that $R_i$ ($B_i$) is a disjoint union of smooth irreducible curves (see Remark 2.1). We apply the Hurwitz formula for the double cover $\pi|_{R_i} : R_i \to B_i$ induced by the action of $g_{i+1}$ ($g_{i+2}$):

$$K_S R_i + R_i^2 = 2(2p_a(B_i) - 2) + R_i(R_{i+1} + R_{i+2})$$

By Proposition 2.2(a), we have

$$2p_a(B_i) - 2 = \frac{1}{2}(K_S R_i + R_i^2 - K_S R_{i+1} - K_S R_{i+2}) + l_{i+1} + l_{i+2} - 4 \quad (2.11)$$

For all the possibilities (2.7) and (2.8), $K_S R_i + R_i^2 - K_S R_{i+1} - K_S R_{i+2}$ is divisible by 4 for each $i = 1, 2, 3$. So $l_{i+1} + l_{i+2}$ is an even integer for $i = 1, 2, 3$ by (2.11). We have seen that $l_1 + l_2 + l_3$ is an even integer. Therefore $l_1$, $l_2$ and $l_3$ are even integers. This proves (d).

Now we prove (e). Assume by contradiction that $K_S$ is not ample. Then $S$ contains exactly one nodal curve $C$ by Lemma 2.5. Thus $C$ is $G$-invariant. Let $C' = \pi(C) \subset \Sigma$ and let $\bar{C}'$ be the strict transform of $C'$ on $W$.

First assume that $C$ is contained in some $R_i$, i.e., $C'$ is contained in some $B_i$. By Remark 2.1, $C'$ is contained in the smooth locus of $\Sigma$ and $\pi^* C' = 2C$. Thus $C'^2 = -2$. It follows that $\bar{C}'$ is a nodal curve, which is disjoint from the nodal curves $\overline{N}_1 \cup \overline{N}_2 \cup \overline{N}_3$. Then $W$ contains $l_1 + l_2 + l_3 + 1$ pairwise disjoint nodal curves. By (d), this contradicts [17, Theorem 3.3].

Hence $C$ is not contained in $R_i$, i.e., $\overline{B}_i \not\geq \bar{C}'$ for $i = 1, 2, 3$. By Proposition 2.4(a), $D\bar{C}' = 0$ and $\bar{C}'^2 < 0$. It follows that $2K_W \bar{C}' = -(\overline{B}_1 + \overline{B}_2 + \overline{B}_3)\bar{C}' \leq 0$. Thus $\bar{C}'$ is either a $(-1)$-curve or a nodal curve.

If $\bar{C}'$ is a nodal curve, then $\overline{B}_i \bar{C}' = 0$ for $i = 1, 2, 3$, i.e., $R_i C = 0$ for $i = 1, 2, 3$. This contradicts (c). So $\bar{C}'$ is a $(-1)$-curve and $(\overline{B}_1 + \overline{B}_2 + \overline{B}_3)\bar{C}' = 2$, i.e., $C'(B_1 + B_2 + B_3) = 2$. We remark that for any Galois $\mathbb{Z}_2^2$-cover $\mathbb{P}^1 \to \mathbb{P}^1$, in the target space $\mathbb{P}^1$, the branch locus consists of three distinct points. Applying this remark to the cover $\pi|_C : C \to C'$, because $C'$ intersects the divisorial part $B_1 + B_2 + B_3$ of the branch locus of $\pi$ at two points, we conclude that $C'$ passes through exactly one node of $\Sigma$. Equivalently, $\bar{C}'$ intersects exactly one nodal curve $C_1$ in $\overline{N}_1 \cup \overline{N}_2 \cup \overline{N}_3$ and $\bar{C}' C_1 = 1$. Blowing down $\bar{C}'$ and then the image of $C_1$, we obtain a surface $W'$ containing $l_1 + l_2 + l_3 - 1$ disjoint nodal curves and $\rho(W') = l_1 + l_2 + l_3 + 1$. By (d), this again contradicts [17, Theorem 3.3].

Hence $K_S$ is ample. □

*Remark 2.8.* We see in the proof of Proposition 2.7(c)–(d) that the three curves $\overline{B}_1, \overline{B}_2, \overline{B}_3$ and the $l_1 + l_2 + l_3$ pairwise disjoint nodal curves $\overline{N}_1 \cup \overline{N}_2 \cup \overline{N}_3$ generate a sublattice of $\mathrm{Num}(W)$.

Also $\overline{B}_1+\overline{B}_2+\overline{B}_3$ does not contain any nodal curve. Otherwise, since $\overline{B}_1+\overline{B}_2+\overline{B}_3$ is disjoint from the nodal curves $\overline{N}_1\cup\overline{N}_2\cup\overline{N}_3$, $W$ contains at least $l_1+l_2+l_3+1$ pairwise disjoint nodal curves. But this contradicts Proposition 2.7(b), (d) and [17, Theorem 3.3].

The following theorem is the main result of this section. We keep the same notation introduced above and adopt the convention $K_S R_1 \geq K_S R_2 \geq K_S R_3$.

**Theorem 2.9.** *Let $S$ be a minimal smooth surface of general type with $p_g = 0$ and $K_S^2 = 7$. Assume that* $\mathrm{Aut}(S)$ *contains a subgroup $G = \{1, g_1, g_2, g_3\}$, which is isomorphic to $\mathbb{Z}_2^2$. Then there are only three possibilities for the the intersection numbers $(K_S R_1, K_S R_2, K_S R_3)$:* (a) $(7, 5, 5)$; (b) $(5, 5, 3)$; (c) $(5, 3, 1)$.

*Proof.* All the possibilities for $(K_S R_1, K_S R_2, K_S R_3)$ are listed in (2.7)–(2.8) and the case $(3, 1, 1)$ has been excluded in the proof of Proposition 2.7. Note that $R_i^2 = -1$ by Proposition 2.7(a). Fix one possibility for $(K_S R_1, K_S R_2, K_S R_3)$. We calculate $(R_1 R_2, R_1 R_3, R_2 R_3)$ as follows.

- Step 1: Bound $R_i R_{i+1}$ $(i = 1, 2, 3)$ and $R_1 R_2 + R_1 R_3 + R_2 R_3$ from above by using the algebraic index theorem: $2R_i R_{i+1} - 2 = (R_i + R_{i+1})^2 \leq \frac{1}{7}(K_S R_i + K_S R_{i+1})^2$ and $2(R_1 R_2 + R_1 R_3 + R_2 R_3) - 3 = (R_1 + R_2 + R_3)^2 \leq \frac{1}{7}(K_S R_1 + K_S R_2 + K_S R_3)^2$.
- Step 2: Propositions 2.2(a) and 2.7(d) show that $K_S R_i - R_{i+1} R_{i+2}$ is divisible by 4. This fact narrows down the possibilities for $R_i R_{i+1}$.
- Step 3: The fact that $\det A$ is a square integer also narrows down the possibilities (cf. Proposition 2.7(c)). Note that $\det A = (R_1 R_2)^2 + (R_1 R_3)^2 + (R_2 R_3)^2 + 2(R_1 R_2)(R_1 R_3)(R_2 R_3) - 1$.

Take the case $(K_S R_1, K_S R_2, K_S R_3) = (7, 3, 3)$ for example. Step 1 yields $R_1 R_2 \leq 8$, $R_1 R_3 \leq 8$, $R_2 R_3 \leq 3$ and $R_1 R_2 + R_1 R_3 + R_2 R_3 \leq 13$. Then Step 2 gives $R_1 R_2, R_1 R_3 \in \{3, 7\}$ and $R_2 R_3 = 3$. Since $\det A = 80$ or $192$, this case is excluded by Step 3. The same reasoning exclude the cases $(7, 1, 1)$ and $(3, 3, 3)$. □

We shall study the case (a) and the case (b) separately in the following two sections. We do not know any example for the case (c). But we have the following proposition.

**Proposition 2.10.** *Let $(S, G)$ be a pair as in 1.1(c). Then:*

(a) $(R_1 R_2, R_1 R_3, R_2 R_3) = (1, 3, 1)$ *and* $K_S \overset{num}{\sim} R_1 + 2R_3$;
(b) $\Sigma$ *is a rational surface with* $K_\Sigma^2 = -1$ *and containing* 8 *nodes.*

*Let $Y$ be the minimal resolution of the intermediate quotient surface $S/g_3$. Then*

(c) $Y$ *contains* 5 *disjoint nodal curves and* $K_Y$ *is nef with* $K_Y^2 = 2$.

*Proof.* To obtain $(R_1R_2, R_1R_3, R_2R_3) = (1, 3, 1)$, one proceeds as in the proof of Theorem 2.9. Since $(K_SR_1, K_SR_2, K_SR_3) = (5, 3, 1)$, Proposition 2.7(c) yields $K_S \overset{num}{\sim} R_1 + 2R_3$.

We also have $(l_1, l_2, l_3) = (4, 2, 2)$ by Proposition 2.2(a). Thus $\Sigma$ is a rational surface with $K_\Sigma^2 = -1$ and containing 8 nodes by Proposition 2.7(b).

Since $K_SR_3 = 1$, the involution $g_3$ has 5 isolated fixed points by [8, Lemma 3.2]. This is the case $k = 5$ in the list of [24, Introduction] and assertion (c) follows. □

## 3. Inoue surfaces

We treat the case (a) of Theorem 2.9 in this section and prove Theorem 1.1(a).

### *3.1. Examples*

We briefly describe the Inoue surfaces, which are the very first examples of surfaces of general type with $p_g = 0$ and $K^2 = 7$ (cf. [21]). The following description is from [25] (see [29] for an equivalent description; cf. [16, Section 6]).

*Example 3.1.* Let $p_1, p_2, p_3, p_1', p_2', p_3'$ be the six vertices of a complete quadrilateral on $\mathbb{P}^2$. Let $\sigma\colon W \to \mathbb{P}^2$ be the blowup of these points. Denote by $E_i$ (respectively $E_i'$) the exceptional curve of $W$ over $p_i$ (respectively $p_i'$) and by $L$ the pullback of a general line by $\sigma$ (Fig. 1).

The surface $W$ has four disjoint nodal curves, their divisor classes are

$$Z_i \equiv L - E_i - E_{i+1}' - E_{i+2}' \text{ for } i = 1, 2, 3 \text{ and } Z \equiv L - E_1 - E_2 - E_3.$$

Note that $Z_1, Z_2, Z_3$ and $Z$ are the proper transforms of the four sides of the quadrilateral. Let $\eta\colon W \to \Sigma$ be the morphism contracting there curves. The surface $\Sigma$ has four nodes, $-K_\Sigma$ is ample and $\Sigma$ is the 4-nodal cubic surface.

Let $\Gamma_1, \Gamma_2$ and $\Gamma_3$ be the proper transforms of the three diagonals of the quadrilateral, i.e., $\Gamma_i \equiv L - E_i - E_i'$ for $i = 1, 2, 3$. For each $i = 1, 2, 3$, $W$ has a pencil of rational curves $|F_i| := |2L - E_{i+1} - E_{i+2} - E_{i+1}' - E_{i+2}'|$. Note that $-K_W \equiv \Gamma_1 + \Gamma_2 + \Gamma_3 \equiv \Gamma_i + F_i$ for $i = 1, 2, 3$.

Now we define three effective divisors on $W$

$$\Delta_1 := \Gamma_1 + F_2 + Z_1 + Z_3, \quad \Delta_2 := \Gamma_2 + F_3, \quad \Delta_3 := \Gamma_3 + F_1 + F_1' + Z_2 + Z, \tag{3.1}$$

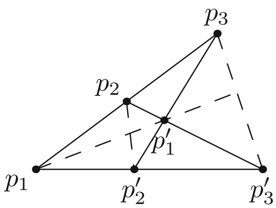



**Fig. 1.** Configurations of the points $p_1, \ldots, p_3'$

where $F_i$ $(i = 1, 2, 3)$ and $F_1'$ $(\in |F_1|)$ are smooth 0-curves such that the divisor $\Delta := \Delta_1 + \Delta_2 + \Delta_3$ has only nodes. There is a smooth finite $\mathbb{Z}_2^2$-cover $\overline{\pi}\colon V \to W$ branched on the divisors $\Delta_1$, $\Delta_2$ and $\Delta_3$ (cf. [13, Section 2]). The (set theoretic) inverse image $\overline{\pi}^{-1} Z_i$ or $\overline{\pi}^{-1} Z$ is a disjoint union of two $(-1)$-curves. Let $\varepsilon\colon V \to S$ be the blowdown of these eight $(-1)$-curves. Then $S$ is a smooth minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 7$.

From the construction, there is a finite $\mathbb{Z}_2^2$-cover $\pi : S \to \Sigma$ such that $\pi\varepsilon = \eta\overline{\pi}$. Recall that $Z_1$, $Z_2$, $Z_3$ and $Z$ are contracted by $\eta$. In the notation of Sect. 2 (see the diagram (2.3)), we have $\overline{N}_1 = Z_1 + Z_3$, $\overline{N}_2 = 0$ and $\overline{N}_3 = Z_2 + Z$ by comparing (3.1) with the first formula of (2.4). In the same notation, we have $\overline{B}_1 = \Gamma_1 + F_2$, $\overline{B}_2 = \Gamma_2 + F_3$ and $\overline{B}_3 = \Gamma_3 + F_1 + F_1'$ since $\overline{B}_i = \Delta_i - \overline{N}_i$. And then $D = 2K_W + \overline{B}_1 + B_2 + B_3 \equiv -K_W + F_1'$. It follows that $(D\overline{B}_1, D\overline{B}_2, D\overline{B}_3) = (7, 5, 5)$. So the Inoue surfaces satisfy Theorem 2.9(a) by Lemma 2.4(b).

*Remark 3.2.* We need the following remarks in the Proof of Theorem 1.1(a) in Sect. 3.2.

(1) Note that $W$ contains exactly nine $(-1)$-curves $E_i$, $E_i'$ and $\Gamma_i$ for $i = 1, 2, 3$. There are exactly three $(-1)$-curves $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, which are disjoint from the nodal curves.

(2) If we replace $Z_1 + Z_3$ in $\Delta_1$ by $Z_2 + Z$ and replace $Z_2 + Z$ in $\Delta_3$ by $Z_1 + Z_3$, these new divisors $\Delta_1$, $\Delta_2$ and $\Delta_3$ still define a smooth finite $\mathbb{Z}_2^2$-cover. In this way, we get another 4-dimensional family of surfaces of general type. However, this family is the same as the original one.
Indeed, let $\alpha$ be the involution on $\mathbb{P}^2$ such that $\alpha(p_k) = p_k'$ and $\alpha(p_k') = p_k$ for $k = 1, 2$. Then $\alpha(p_3) = p_3$ and $\alpha(p_3') = p_3'$. It induces an involution $\alpha'$ on $W$. The $(-1)$-curves $\Gamma_1, \Gamma_2, \Gamma_3$ are $\alpha'$-invariant and the divisors classes of $F_1, F_2, F_3$ are $\alpha'$-invariant. We also have $\alpha'(Z_1) = Z_2$, $\alpha'(Z_2) = Z_1$, $\alpha'(Z_3) = Z$ and $\alpha'(Z) = Z_3$.

(3) Observe that the two nodal curves in the same $\Delta_k$ $(k = 1, 3)$ are in the same singular member of the pencil $|F_2|$. Indeed, the singular members of the pencil $|F_2|$ are $\Gamma_1 + \Gamma_3$, $Z_1 + 2E_2' + Z_3$ and $Z_2 + 2E_2 + Z$.

### 3.2. *Classification*

The subsection is devoted to classify the pairs $(S, G)$ satisfying the property (a) of Theorem 2.9. We assume $(K_S R_1, K_S R_2, K_S R_3) = (7, 5, 5)$ throughout this subsection and stick to the same notation in Sect. 2.

Recall that $W$ is the minimal resolution of $\Sigma$. Also recall that $D = 2K_W + \overline{B}_1 + \overline{B}_2 + \overline{B}_3$ and $M = K_W + D$. We have $(D\overline{B}_1, D\overline{B}_2, D\overline{B}_3) = (7, 5, 5)$ by Lemma 2.4(b) and then we also have

$$DK_W = \frac{1}{2}D(D - \overline{B}_1 - \overline{B}_2 - \overline{B}_3) = -5, \quad DM = D(K_W + D) = 2,$$
$$K_W M = K_W(K_W + D) = K_W^2 - 5, \quad M^2 = M(K_W + D) = K_W^2 - 3 \quad (3.2)$$

The following lemma describes the surface $W$.

**Lemma 3.3.** (a) *The linear system $|M|$ is a base point free pencil of rational curves.*
(b) *The surface $W$ is a weak Del Pezzo surface of degree three, i.e., $-K_W$ is nef and $K_W^2 = 3$.*

*Proof.* Since $DK_W = -5$, the algebraic index theorem yields $K_W^2 \le 3$. Because $K_S$ is ample by Proposition 2.7(e), $M$ is nef by Lemma 2.4(f) and thus $M^2 \ge 0$. So $K_W^2 = 3$, $M^2 = 0$ and $K_W M = -2$ by (3.2). The adjunction formula gives $p_a(M) = 0$. Also Proposition 2.4(e) yields $\dim |M| = p_a(D) - 1 = 1$.

Assume $|M| = |\Phi| + \Psi$, where $|\Phi|$ is the moving part and $\Psi$ is the fixed part. Because both $M$ and $\Phi$ is nef, we have $0 = M^2 = M(\Phi + \Psi) \ge M\Phi = \Phi^2 + \Phi\Psi \ge \Phi^2 \ge 0$. It follows that $M\Phi = M\Psi = 0$ and $\Phi^2 = \Phi\Psi = \Psi^2 = 0$. In particular, $|\Phi|$ is base point free. Moreover, $D\Phi + D\Psi = DM = 2$ and $D\Phi = (M - K_W)\Phi = -K_W\Phi$. Since $D$ is nef and big, the adjunction formula implies $D\Phi = -K_W\Phi = 2$ and then $D\Psi = 0$. Then Lemma 2.4(f) yields $\Psi = 0$ since $\Psi^2 = 0$. Hence $|M| = |\Phi|$ is base point free.

It remains to show that $-K_W$ is nef. Assume by contradiction that $-K_W C < 0$ for an irreducible curve $C$. The Riemann–Roch theorem implies $h^0(W, \mathcal{O}_W(-K_W)) \ge 4$. So $C^2 < 0$ and $C$ is contained in the fixed part of $|-K_W|$. Because both $D$ and $M$ are nef, $1 \le K_W C + DC = MC \le M(-K_W) = 2$. Lemma 2.4(f) implies $DC \ge 1$. It follows that $K_W C = DC = 1$ and $MC = 2$. Since $M(-K_W - C) = 0$, the Zariski lemma yields $(-K_W - C)^2 = 5 + C^2 \le 0$. Then $p_a(C) = \frac{1}{2}(K_W C + C^2) + 1 < 0$, a contradiction. Hence $-K_W$ is nef. □

**Corollary 3.4.** *Either* $(R_1R_2, R_1R_3, R_2R_3) = (5, 9, 7)$ *or* $(R_1R_2, R_1R_3, R_2R_3) = (9, 5, 7)$.

*Proof.* Recall that $R_i^2 = -1$ for $i = 1, 2, 3$ and $K_S = \pi^* K_\Sigma + R_1 + R_2 + R_3$. According to the lemma above, one has $3 = K_\Sigma^2 = \frac{1}{4}(K_S - R_1 - R_2 - R_3)^2$, i.e., $R_1R_2 + R_1R_3 + R_2R_3 = 21$.

Step 1 and Step 2 in the proof of Theorem 2.9 show that $R_1R_2, R_1R_3 \in \{1, 5, 9\}$ and $R_2R_3 \in \{3, 7\}$. Therefore $(R_1R_2, R_1R_3, R_2R_3) \in \{(9, 9, 3), (5, 9, 7), (9, 5, 7)\}$. The case $(9, 9, 3)$ is excluded by Step 3 in the proof of Theorem 2.9 since $\det A = 656$, which is not a square integer. □

Since $K_S R_2 = K_S R_3 = 5$, we may assume $(R_1R_2, R_1R_3, R_2R_3) = (5, 9, 7)$ in the rest of this section. Then we have $(\overline{B}_1\overline{B}_2, \overline{B}_1\overline{B}_3, \overline{B}_2\overline{B}_3) = (5, 9, 7)$ by Lemma 2.4(b). Also $(l_1, l_2, l_3) = (2, 0, 2)$ by Lemma 2.2(a). This means that $\overline{N}_k$ consists of two disjoint nodal curves for $k = 1, 3$ and $\overline{N}_2 = 0$. The following lemma determines the branch divisors $\overline{B}_1$, $\overline{B}_2$ and $\overline{B}_3$.

**Lemma 3.5.** *The divisors $\overline{B}_1$, $\overline{B}_2$ and $\overline{B}_3$ are as follows:*

$$\overline{B}_1 = \Gamma_1 + F_2, \qquad \overline{B}_2 = \Gamma_2 + F_3, \qquad \overline{B}_3 = \Gamma_3 + F_1 + F_1', \tag{3.3}$$

*where the curves $\Gamma_i$ are $(-1)$-curves, $F_i$ and $F_1'$ are 0-curves such that $F_1, F_1' \in |M|$. Moreover, $\Gamma_i + F_i \equiv -K_W$ for $i = 1, 2, 3$ and $\Gamma_1 + \Gamma_2 + \Gamma_3 \equiv -K_W$.*

*Proof.* Recall that $(D\overline{B}_1, D\overline{B}_2, D\overline{B}_3) = (7,5,5)$, $(\overline{B}_1\overline{B}_2, \overline{B}_1\overline{B}_3, \overline{B}_2\overline{B}_3) = (5,9,7)$ and $\overline{B}_i^2 = -1$ for $i = 1,2,3$. Since $D = 2K_W + \overline{B}_1 + \overline{B}_2 + \overline{B}_3$, we have $(K_W\overline{B}_1, K_W\overline{B}_2, K_W\overline{B}_3) = (-3,-3,-5)$.

Also recall that $\overline{B}_i$ is a disjoint union of smooth irreducible curves (see Remark 2.1). Because $-K_W$ is nef and $\overline{B}_i$ does not contain any nodal curve (see Remark 2.8), $-K_W\overline{\Upsilon} > 0$ for any irreducible component $\overline{\Upsilon}$ of $\overline{B}_i$.

Since $-K_W\overline{B}_1 = -K_W\overline{B}_2 = -3$ and $\overline{B}_1^2 = \overline{B}_2^2 = -1$, the algebraic index theorem and the adjunction formula imply $\overline{B}_1 = \Gamma_1 + F_2$ and $\overline{B}_2 = \Gamma_2 + F_3$, where $\Gamma_1$ and $\Gamma_2$ are $(-1)$-curves, while $F_2$ and $F_3$ are 0-curves. Since $M\overline{B}_3 = (D + K_W)\overline{B}_3 = 0$, $\overline{B}_3$ is contained in some divisors of $|M|$. The Zariski lemma, the algebraic index theorem and the adjunction formula imply $\overline{B}_3 = \Gamma_3 + F_1 + F_1'$, where $\Gamma_3$ is $(-1)$-curve, $F_1$ and $F_1'$ are 0-curves in $|M|$.

We claim that $\Gamma_i\Gamma_{i+1} = 1$ for $i = 1,2,3$. Actually, since $-K_W(\Gamma_i+\Gamma_{i+1}) = 2$, the algebraic index theorem implies $\Gamma_i\Gamma_{i+1} \leq 1$. As an irreducible component of the curve $\overline{B}_i$, $\Gamma_i$ is disjoint from the nodal curves $\overline{N}_1 \cup \overline{N}_3$. Now fix $i$. If $\Gamma_i\Gamma_{i+1} = 0$, then blowing down $\Gamma_i$ and $\Gamma_{i+1}$, we obtain a rational surface $W'$ containing four disjoint nodal curves and $\rho(W') = 5$. This gives a contradiction to [17, Theorem 3.3]. The claim is proved and thus $(\Gamma_1 + \Gamma_2 + \Gamma_3)^2 = 3$. Since $-K_W(\Gamma_1 + \Gamma_2 + \Gamma_3) = 3$, the algebraic index theorem implies $\Gamma_1 + \Gamma_2 + \Gamma_3 \equiv -K_W$.

Since $F_1, F_1' \in |M|$ and $M \equiv K_W + D \equiv 3K_W + \overline{B}_1 + \overline{B}_2 + \overline{B}_3$, we have $F_1 + F_2 + F_3 \equiv -2K_W$ by (3.3). It follows that $F_i(F_1 + F_2 + F_3) = 4$ and $F_iF_{i+1} = 2$ for $i = 1,2,3$.

Since $\overline{B}_i$ is a disjoint union of smooth curves, we have $\Gamma_1F_2 = 0$, $\Gamma_2F_3 = 0$ and $\Gamma_3F_1 = 0$. Also $5 = \overline{B}_1\overline{B}_2 = (\Gamma_1 + F_2)(\Gamma_2 + F_3) = 3 + \Gamma_1F_3 + \Gamma_2F_2$, i.e., $\Gamma_1F_3+\Gamma_2F_2 = 2$. Similarly, $\overline{B}_1\overline{B}_3 = 9$ and $\overline{B}_2\overline{B}_3 = 7$ imply that $2\Gamma_1F_1+\Gamma_3F_2 = 4$ and $2\Gamma_2F_1 + \Gamma_3F_3 = 2$. Note that $(F_1 + F_2 + F_3)\Gamma_i = -2K_W\Gamma_i = 2$ for $i = 1,2,3$. It follows that $\Gamma_iF_i = 2$ and $F_i\Gamma_{i+1} = F_i\Gamma_{i+2} = 0$ for $i = 1,2,3$. Since $(\Gamma_i + F_i)^2 = 3$ and $-K_W(\Gamma_i + F_i) = 3$, the algebraic index theorem gives $\Gamma_i + F_i \equiv -K_W$. □

**Lemma 3.6.** *The surface $W$ is isomorphic to the minimal resolution of the* 4*-nodal cubic surface.*

*Proof.* It is well known that any weak Del Pezzo surface of degree three is isomorphic to the minimal resolution of a normal cubic surface in $\mathbb{P}^3$. We have seen that $W$ contains four nodal curves $\overline{N}_1 \cup \overline{N}_3$. It remains to show that $W$ contains exactly four nodal curves.

Assume that $C$ is a nodal curve on $W$. Because $\Gamma_1 + \Gamma_2 + \Gamma_3 \equiv -K_W$ and $\Gamma_i + F_i \equiv -2K_W$, $\Gamma_iC = F_iC = 0$ for $i = 1,2,3$. By (3.3), we $DC = 0$ since $D = 2K_W + \overline{B}_1 + \overline{B}_2 + \overline{B}_3$. Then Lemma 2.4(f) implies $C \in \overline{N}_1 \cup \overline{N}_3$. □

We have shown in diagram (2.3) that the cover $\overline{\pi}\colon V \to W$ is branched on $\overline{B}_1 + \overline{N}_1$, $\overline{B}_2$ and $\overline{B}_3 + \overline{N}_3$, where both $\overline{N}_1$ and $\overline{N}_3$ consist of two disjoint nodal curves, $\overline{B}_1$, $\overline{B}_2$ and $\overline{B}_3$ are described in Lemma 3.5. We conclude that $F_2 + \overline{N}_1$ and $F_2 + \overline{N}_3$ are divisible in $\mathrm{Pic}(W)$ by (2.4) and (3.3). It follows that the two nodal curves in $\overline{N}_k$ $(k = 1,3)$ are in the same singular member of the pencil $|F_2|$.

Comparing with (3.3) and Example 3.1, we conclude that $S$ is an Inoue surface by Remark 3.2.

## 4. Bidouble covers of Del Pezzo surfaces of degree one

We treat the case (b) of Theorem 2.9 and prove Theorem 1.1(b).

### *4.1. Examples*

We briefly describe the surfaces constructed in [16, Sections 2–3].

*Example 4.1.* Let $p_0, p_1, p_2, p_3$ be four points of $\mathbb{P}^2$ in general position and let $p'_j$ be the infinitely near point over $p_j$ corresponding to the line $\overline{p_0 p_j}$ for $j = 1, 2, 3$. Finally, let $p$ be the eighth point satisfying the following Zariski open conditions:

(I) $p \notin \cup_{i=1}^{3}\{\overline{p_0 p_i} : x_{i+1} = x_{i+2}\} \cup_{i=1}^{3} \{\overline{p_{i+1} p_{i+2}} : x_i = 0\}$;
(II) $p \notin c_1 \cup c_2 \cup c_3$, where $c_i$ is the unique conic passing through the five points $p_i, p_{i+1}, p'_{i+1}, p_{i+2}$ and $p'_{i+2}$ (Fig. 2).

Let $\sigma: W \to \mathbb{P}^2$ be the blowup of these eight points. Denote by $E_j$ (respectively $E'_j$, $E$) the total transform of the point $p_j$ (respectively $p'_j$, $p$), and by $L$ the pullback of a general line by $\sigma$. Then $\mathrm{Pic}(W) = \mathbb{Z}L \oplus \mathbb{Z}E_0 \oplus \oplus_{j=1}^{3}(\mathbb{Z}E_j \oplus \mathbb{Z}E'_j) \oplus \mathbb{Z}E$ and $-K_W \equiv 3L - E_0 - \sum_{j=1}^{3}(E_j + E'_j) - E$. We list some properties of the surface $W$.

(1) The surface $W$ is a weak Del Pezzo surface of degree one, i.e., $-K_W$ is nef and $K_W^2 = 1$.
(2) There are exactly six nodal curves on $W$. Their divisor classes are

$$C_j \equiv L - E_0 - E_j - E'_j,\ C'_j \equiv E_j - E'_j \quad \text{for} \quad j = 1, 2, 3. \tag{4.1}$$

Let $\eta: W \to \Sigma$ be the morphism contracting these curves. The surface $\Sigma$ has six nodes and $-K_\Sigma$ is ample.
(3) The pencil of lines on $\mathbb{P}^2$ passing through $p_0$ induces a fibration $g: W \to \mathbb{P}^1$. Denote by $F$ a general fiber of $g$. Then $F \equiv L - E_0$. The fibration $g$ has exactly four singular fibers: $C_j + 2E'_j + C'_j$ for $j = 1, 2, 3$ and $\Gamma + E$, where $\Gamma$ is the strict transform of the line $\overline{p_0 p}$ and $\Gamma \equiv L - E_0 - E$.

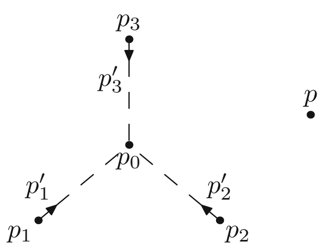



**Fig. 2.** Configurations of $p_0, p_1, \ldots, p'_3$ and $p$

(4) The linear system $|-2K_W-\Gamma|$ consists of a single $(-1)$-curve. Denote this curve by $\overline{B}_2$. The linear system $|-2K_W-E|$ consists of a single $(-1)$-curve. Denote this curve by $\overline{B}_3$. We have $\overline{B}_2\Gamma=\overline{B}_3E=3$ and $\overline{B}_2\overline{B}_3=\overline{B}_2E=\overline{B}_3\Gamma=1$. Moreover, the divisor $\Gamma+E+\overline{B}_2+\overline{B}_3$ has only nodes (cf. [16, Proposition 2.5]).

Now we define three effective divisors on $W$

$$\begin{aligned}\Delta_1 &:= F_b+\Gamma+(C_1+C_1'+C_2+C_2')\equiv 4L-4E_0-2E_1'-2E_2'-E,\\ \Delta_2 &:= \overline{B}_2+(C_3+C_3')\equiv -2K_W-2E_3'+E,\\ \Delta_3 &:= \overline{B}_3\equiv -2K_W-E.\end{aligned}\tag{4.2}$$

Here we require that the curve $F_b$ is a smooth fiber of $g$ and the divisor $\Delta:=\Delta_1+\Delta_2+\Delta_3$ has only nodes. We also define three divisors

$$\begin{aligned}\mathcal{L}_1&=-2K_W-E_3',\\ \mathcal{L}_2&=-K_W+(2L-2E_0-E_1'-E_2'-E),\\ \mathcal{L}_3&=-K_W+(2L-2E_0-E_1'-E_2'-E_3').\end{aligned}\tag{4.3}$$

It follows that $\Delta_i$ and $\mathcal{L}_i$ $(i=1,2,3)$ satisfy (2.4). These data define a smooth finite $\mathbb{Z}_2^2$-cover $\overline{\pi}\colon V\to W$ branched on the divisors $\Delta_1$, $\Delta_2$ and $\Delta_3$ (cf. [13, Section 2]). The (set theoretic) inverse image $\overline{\pi}^{-1}C_j$ or $\overline{\pi}^{-1}C_j'$ is a disjoint union of two $(-1)$-curves. Let $\varepsilon\colon V\to S$ be the blowdown of these twelve $(-1)$-curves. Then $S$ is a smooth minimal surface of general type with $p_g(S)=0$ and $K_S^2=7$.

From the construction, there is a finite $\mathbb{Z}_2^2$-cover $\pi: S\to\Sigma$ such that $\pi\varepsilon=\eta\overline{\pi}$. Recall that $C_1, C_1', C_2, C_2', C_3$ and $C_3'$ are contracted by $\eta$. In the notation of Sect. 2 (see the diagram (2.3)), we have $\overline{N}_1=C_1+C_1'+C_2+C_2'$, $\overline{N}_2=C_3+C_3'$ and $\overline{N}_3=0$ by comparing (4.2) with the first formula of (2.4). In the same notation, we have $\overline{B}_1=F_b+\Gamma$ since $\overline{B}_1=\Delta_1-\overline{N}_1$ and thus $D=2K_W+\overline{B}_1+\overline{B}_2+\overline{B}_3\equiv -2K_W+\Gamma$. It follows that $(D\overline{B}_1, D\overline{B}_2, D\overline{B}_3)=(5,5,3)$. So these surfaces satisfy the property (b) of Theorem 2.9 by Lemma 2.4(b).

We need the following lemma to study deformations of the surfaces in Example 4.1. See Sect. 5.

**Lemma 4.2.** *Let $W$ be a weak Del Pezzo surface of degree one as in Example 4.1. For $j=1,2,3$, the linear system $|-K_W+E_j'-E|$ consists of a single $(-1)$-curve. Denote this curve by $\Lambda_j$. Then $\Lambda_j$ intersects $F_b$ transversely with $\Lambda_jF_b=2$.*

*Proof.* Fix $j\in\{1,2,3\}$. Note that $(-K_W+E_j'-E)^2=-1$ and $K_W(-K_W+E_j'-E)=-1$. Also it is clear that $h^0(W,2K_W-E_j'+E)=0$. The Riemann–Roch theorem yields $h^0(W,\mathcal{O}_W(-K_W+E_j'-E))\geq 1$. From (4.1), we can easily show that any effective divisor $\Lambda_j$ in $|-K_W+E_j'-E|$ does not contain $C_k$ or $C_k'$ for $k=1,2,3$. Because the nodal curves $C_1,\ldots,C_3'$ are exactly all the nodal curves of $W$ and $-K_W\Lambda_j=1$, we conclude that $\Lambda_j$ is irreducible. Therefore $\Lambda_j$ is a $(-1)$-curve.

It is directly to show $C_j + 2\Lambda_j + C'_j \equiv -2K_W + L - E_0 - 2E \equiv \overline{B}_3 + \Gamma$ for $j = 1, 2, 3$. Since $\overline{B}_3$ and $\Gamma$ are $(-1)$-curves with $\overline{B}_3\Gamma = 1$, $|\overline{B}_3 + \Gamma|$ induces a fibration $g' \colon W \to \mathbb{P}^1$ and $C_j + 2\Lambda_j + C'_j$ is a singular fiber of $g'$ for $j = 1, 2, 3$.

Since $F_b(\overline{B}_3 + \Gamma) = 4$, the restriction $g'|_{F_b} \colon F_b \to \mathbb{P}^1$ has degree 4. The curve $F_b$ is disjoint from the nodal curves, so $F_b\Lambda_j = 2$ for $j = 1, 2, 3$. Because the multiplicity of $\Lambda_j$ in the singular fiber is 2, the ramification divisor of $g'|_{F_b}$ has degree at least $2 \times 3 = 6$. On the other hand, the Hurwitz formula shows that the ramification divisor of $g'|_{F_b}$ has degree exactly 6. This implies that $F_b$ intersects $\Lambda_j$ transversely for $j = 1, 2, 3$. □

### *4.2. Classification*

This subsection is devoted to classify the pairs $(S, G)$ satisfying the property (b) of Theorem 2.9. We assume $(K_SR_1, K_SR_2, K_SR_3) = (5, 5, 3)$ throughout this section and stick to the same notation in Sect. 2.

Recall that $W$ is the minimal resolution of $\Sigma$. Also recall that $D = 2K_W + \overline{B}_1 + \overline{B}_2 + \overline{B}_3$ and $M = K_W + D$. We have $(D\overline{B}_1, D\overline{B}_2, D\overline{B}_3) = (5, 5, 3)$ by Lemma 2.4(b) and then we have

$$DK_W = \frac{1}{2}D\left(D - \overline{B}_1 - \overline{B}_2 - \overline{B}_3\right) = -3, \quad DM = D(K_W + D) = 4,$$
$$K_WM = K_W(K_W + D) = K_W^2 - 3, \quad M^2 = M(K_W + D) = K_W^2 + 1 \qquad (4.4)$$

**Lemma 4.3.** *One of the following two cases holds:*

(b1) $K_W^2 = 1$, $M^2 = 2$ *and* $-K_W$ *is nef;*
(b2) $K_W^2 = -1$ *and* $M^2 = 0$.

*Proof.* Note that $K_W^2 = K_\Sigma^2$ is an odd integer by Proposition 2.7(b) and (d). Since $DK_W = -3$, the algebraic index theorem yields $K_W^2 \leq 1$. Because $K_S$ is ample by Proposition 2.7(e), $M$ is nef by Lemma 2.4(f) and thus $M^2 \geq 0$. So either $K_W^2 = 1$, $M^2 = 2$ or $K_W^2 = -1$, $M^2 = 0$ by (4.4). To prove the last statement of (b1), one proceeds as in Lemma 3.3(b). □

To prove Theorem 1.1(b), we shall show that the case (b1) corresponds to the surfaces in Example 4.1 and that the case (b2) never occurs.

*4.2.1. Existence result* Assume $K_W^2 = 1$ and $M^2 = 2$. We have seen that $W$ is a weak Del Pezzo surface of degree one by the lemma above.

**Corollary 4.4.** *Either* $(R_1R_2, R_1R_3, R_2R_3) = (7, 5, 1)$ *or* $(R_1R_2, R_1R_3, R_2R_3) = (7, 1, 5)$.

*Proof.* Recall that $R_i^2 = -1$ for $i = 1, 2, 3$ and $K_S = \pi^*K_\Sigma + R_1 + R_2 + R_3$. According to the lemma above, one has $1 = K_\Sigma^2 = \frac{1}{4}(K_S - R_1 - R_2 - R_3)^2$, i.e., $R_1R_2 + R_1R_3 + R_2R_3 = 13$. Then Step 1-Step 3 in the proof of Theorem 2.9 give $(R_1R_2, R_1R_3, R_2R_3) = (7, 5, 1)$ or $(7, 1, 5)$. □

Since $K_S R_1 = K_S R_2 = 5$, we may assume $(R_1 R_2, R_1 R_3, R_2 R_3) = (7, 5, 1)$. Then we have $(\overline{B}_1 \overline{B}_2, \overline{B}_1 \overline{B}_3, \overline{B}_2 \overline{B}_3) = (7, 5, 1)$ by Lemma 2.4(b). Also $(l_1, l_2, l_3) = (4, 2, 0)$ by Proposition 2.2(a). This means that $\overline{N}_1$ consists of four disjoint nodal curves, $\overline{N}_2$ consists of two disjoint nodal curves and $\overline{N}_3 = 0$. The following lemma determines the branch divisors $\overline{B}_1$, $\overline{B}_2$ and $\overline{B}_3$.

**Lemma 4.5.** *The divisors $\overline{B}_1, \overline{B}_2, \overline{B}_3$ satisfy the following properties.*

(a) *The curve $\overline{B}_1$ is reducible: $\overline{B}_1 = F_b + \Gamma$, where $\Gamma$ is a $(-1)$-curve and $\Gamma \equiv M + K_W$, while $F_b$ is a 0-curve and $DF_b = 4$.*
(b) *The divisors $\overline{B}_2$ and $\overline{B}_3$ are $(-1)$-curves. Moreover, $\overline{B}_2 \equiv -2K_W - \Gamma$ and $\overline{B}_3 \equiv -2K_W - F_b + \Gamma$.*

*Proof.* Recall that $(D\overline{B}_1, D\overline{B}_2, D\overline{B}_3) = (5, 5, 3)$, $(\overline{B}_1\overline{B}_2, \overline{B}_1\overline{B}_3, \overline{B}_2\overline{B}_3) = (7, 5, 1)$ and $\overline{B}_i^2 = -1$ for $i = 1, 2, 3$. Since $D = 2K_W + \overline{B}_1 + \overline{B}_2 + \overline{B}_3$, we have $(K_W\overline{B}_1, K_W\overline{B}_2, K_W\overline{B}_3) = (-3, -1, -1)$.

Also recall that $\overline{B}_i$ is a disjoint union of smooth irreducible curves (see Remark 2.1). Because $-K_W$ is nef and $\overline{B}_i$ does not contain any nodal curve (see Remark 2.8), $-K_W\overline{\Upsilon} > 0$ for any irreducible component $\overline{\Upsilon}$ of $\overline{B}_i$.

Since $-K_W\overline{B}_1 = 3$ and $\overline{B}_1^2 = -1$, the algebraic index theorem and the adjunction formula show that $\overline{B}_1$ is a disjoint union of a $(-1)$-curve $\Gamma$ and a 0-curve $F_b$. Now we prove that $\Gamma \equiv M + K_W$. Note that $\overline{B}_1(M + K_W) = \overline{B}_1(2K_W + D) = -1$. It suffices to show that the linear system $|M + K_W|$ consists of a $(-1)$-curve. Since $M$ is nef and big, it is 1-connected (cf. [27, Lemma 2.6]). The long exact sequence obtained from

$$0 \to \mathcal{O}_W(K_W) \to \mathcal{O}_W(M + K_W) \to \omega_M \to 0$$

gives $h^0(W, \mathcal{O}_W(M + K_W)) = p_a(M) = 1$. Since $D(M + K_W) = 1$, by Lemma 2.4(f), we may assume that $K_W + M \equiv \Phi + \Psi$, where $\Phi$ is an irreducible curve with $D\Phi = 1$ and $\mathrm{Supp}(\Psi) \in \overline{N}_1 \cup \overline{N}_2$. So $\Phi^2 \leq 0$ by the algebraic index theorem. Since $K_W\Phi = K_W(M + K_W) = -1$, $\Phi$ is a $(-1)$-curve. Then $\Psi^2 = (M + K_W - \Phi)^2 = (2K_W + D - \Phi)^2 = 0$ and thus $\Psi = 0$. Hence $|M + K_W|$ consists of a $(-1)$-curve and $\Gamma \equiv M + K_W$. It follows that $D\Gamma = DM + DK_W = 1$ and $DF_b = D(\overline{B}_1 - \Gamma) = 4$. This proves (a).

For (b), because $-K_W\overline{B}_2 = -K_W\overline{B}_3 = 1$, the curves $\overline{B}_2$ and $\overline{B}_3$ are irreducible. Since $\overline{B}_2^2 = \overline{B}_3^2 = -1$, $\overline{B}_2$ and $\overline{B}_3$ are $(-1)$-curves. Because $\Gamma \equiv M + K_W = 3K_W + \overline{B}_1 + \overline{B}_2 + \overline{B}_3$ and $\overline{B}_1 \equiv F_b + \Gamma$, we have $\overline{B}_2 + \overline{B}_3 \equiv -4K_W - F_b$. So it suffices to show $\overline{B}_2 \equiv -2K_W - \Gamma$.

Note that $\overline{B}_2 - (-2K_W - \Gamma) \equiv 3D - 2\overline{B}_1 - \overline{B}_2 - 2\overline{B}_3$. It follows that $[\overline{B}_2 - (-2K_W - \Gamma)]^2 = 0$ and $D[\overline{B}_2 - (-2K_W - \Gamma)] = 0$. The algebraic index theorem yields $\overline{B}_2 \equiv -2K_W - \Gamma$. □

The next lemma describes a rational fibration on $W$.

**Lemma 4.6.** *The linear system $|F_b|$ induces a genus 0 fibration $g\colon W \to \mathbb{P}^1$.*

(a) *The curve $\Gamma$ is contained in a singular fiber $F_0$ of $g$ and $F_0 = \Gamma + E$, where $E$ is a $(-1)$-curve and $E\Gamma = 1$.*

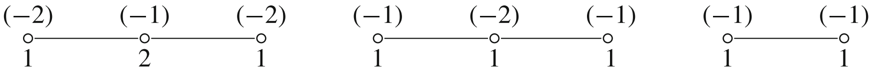
(b) *The fibration $g$ has exactly four singular fibers: $F_0$ and $C_j + 2E'_j + C'_j$ ($j = 1, 2, 3$), where $E'_j$ is a $(-1)$-curve, $C_j$ and $C'_j$ are the nodal curves contained in $\overline{N}_1 \cup \overline{N}_2$ and $C_j E'_j = C'_j E'_j = 1$.*

(c) *If $E_0$ is a smooth section of $g$ and $E_0^2 < 0$, then $E_0$ is a $(-1)$-curve.*

*Proof.* It is well known that a 0-curve on a smooth rational surface induces a genus 0 fibration. Because $F_b\Gamma = 0$, $\Gamma$ is contained in a fiber $F_0$ of $g$. Similarly, the nodal curves $\overline{N}_1 \cup \overline{N}_2$ are contained in the fibers of $g$. Denote by $F$ the general fiber of $g$.

Assume that $E$ is an irreducible component of $F_0$ such that $E\Gamma = 1$. Then $E$ is a smooth rational curve with $E^2 < 0$. Since $\overline{B}_2$ is a $(-1)$-curve and $\overline{B}_2 F = (-2K_W - \Gamma)F = 4$, $\overline{B}_2 \neq E$ and thus $\overline{B}_2 E = (-2K_W - \Gamma)E \geq 0$. It follows that $K_W E < 0$. The adjunction formula shows that $E$ is $(-1)$-curve. Then the Zariski lemma gives $F_0 = \Gamma + E$.

Assume that $F_s$ is a singular fiber different from $F_0$ and $A$ is an irreducible component of $F_s$. Then $A\Gamma = 0$ since $A$ and $\Gamma$ are in different fibers. Then $2K_W A = (\Gamma - D)A = -DA$. By Proposition 2.4(f), $A$ is either a nodal curve in $\overline{N}_1 \cup \overline{N}_2$ or a $(-1)$-curve with $DA = 2$. Because $DF = 4$, any singular fiber has one of the following types:




Here a vertex stands for an irreducible component of a singular fiber, and the number above the vertex represents the self-intersection number of the irreducible component, and the number below the vertex represents the multiplicity of the irreducible component in the fiber. Note that each fiber of the first two types contributes 2 to the Picard number $\rho(W)$. The surface $W$ contains six disjoint nodal curves and $\rho(W) = 10 - K_W^2 = 9$. We have seen that $g$ has one fiber $F_0 = \Gamma + E$ of the third type. So the other fibers are of the first type. This proves (a) and (b).

Assume that $E_0$ is a smooth section of $g$ with $E_0^2 < 0$. Lemma 4.3(b1) implies $K_W E_0 \leq 0$. If $K_W E_0 = 0$, then $E_0$ is a nodal curve. Also $E_0(\Gamma + E) = E_0 F = 1$. By Lemma 4.5(b), either $\overline{B}_2 E_0 = -1$ or $\overline{B}_3 E_0 = -1$. This is impossible because $\overline{B}_2$ and $\overline{B}_3$ are $(-1)$-curves. Hence $K_W E_0 < 0$ and $E_0$ is a $(-1)$-curve by the adjunction formula. □

**Lemma 4.7.** *The surface $W$ contains exactly six nodal curves $\overline{N}_1 \cup \overline{N}_2$.*

*Proof.* Assume that $C$ is a nodal curve different from the six nodal curves $\overline{N}_1 \cup \overline{N}_2$. According to Lemma 4.6(b), $C$ is not contained in the fibers of $g$. Lemma 4.6(c) implies $FC \geq 2$. Then $(\overline{B}_2 + \overline{B}_3)C = -FC < 0$ by Lemma 4.5(b). This contradicts that $\overline{B}_2$ and $\overline{B}_3$ are $(-1)$-curves. Thus $W$ contains exactly six nodal curves $\overline{N}_1 \cup \overline{N}_2$. □

We have shown in diagram (2.3) that the cover $\overline{\pi}\colon V \to W$ is branched on $\overline{B}_1 + \overline{N}_1$, $\overline{B}_2 + \overline{N}_2$ and $\overline{B}_3$, where

$$\overline{B}_1 = F_b + \Gamma, \quad \overline{B}_2 \equiv -2K_W - \Gamma, \quad \overline{B}_3 \equiv -2K_W - E \equiv -2K_W - F + \Gamma \tag{4.5}$$

Now we consider how the nodal curves $C_1, \dots, C'_3$ in the fibers of $g$ (see Lemma 4.6) distribute along two divisors $\overline{N}_1$ and $\overline{N}_2$. We conclude that $\overline{N}_1$ and $\overline{N}_2 + F$ are divisible by 2 in $\mathrm{Pic}(W)$ from (2.4) and (4.5). Since $C_j + C'_j \equiv F - 2E'_j$ for $j = 1, 2, 3$, we may assume

$$\overline{N}_1 = (C_1 + C'_1) + (C_2 + C'_2) \equiv 2F - 2E'_1 - 2E'_2,\ \overline{N}_2 = C_3 + C'_3 \equiv F - 2E'_3. \tag{4.6}$$

Finally, we shall show that $W$ arises as the blowup of $\mathbb{P}^2$ as described in Example 4.1. We first claim that there exists a smooth section $E_0$ of $g$, such that $E_0$ is a $(-1)$-curve, $E_0\Gamma = 1$ and $E_0 E = 0$. Let $p\colon W \to W'$ be the morphism blowing down $E$. Because $K^2_{W'} = 2$, the fibration $g'\colon W \to \mathbb{P}^1$ induced by $g$ is not relatively minimal. Thus $g'$ has a smooth section $E'_0$ such that $E'^2_0 < 0$. The strict transform $E_0$ of $E'_0$ is a smooth section of $g$ and thus it is a $(-1)$-curve by Lemma 4.6(c). It follows that $E_0\Gamma = 1$ and $E_0 E = 0$. The claim is proved.

Since $F \equiv C_j + 2E'_j + C'_j$, we have $E_0 E'_j = 0$. After possibly relabeling $C_j$ and $C'_j$ for each $j = 1, 2, 3$, we may assume that $E_0 C_j = 1$ and $E_0 C'_j = 0$.

Blowing down $E'_j (j = 1, 2, 3)$, then blowing down the image of $C'_j (j = 1, 2, 3)$ and finally blowing down $E_0$ and $E$, we obtain a birational morphism $\sigma\colon W \to \mathbb{P}^2$. Let $p_0 := \sigma(E_0)$, $p_1 := (E'_j \cup C'_j)$ $(j = 1, 2, 3)$ and $p := \sigma(E)$. Denote by $p'_j$ the infinitely near point over $p_j$ corresponding to the line $\overline{p_j p_0}$. Note that the exceptional divisor on $W$ corresponding to $p'_j$ is $E'_j$. The fibration $g$ corresponds to the pencil of lines passing through the point $p_0$. Then Lemma 4.6(b) implies that $p_0, p_i, p_{i+1}$ are not collinear and $p_0, p, p_i$ are not collinear for each $i = 1, 2, 3$. Lemma 4.7 implies that $p_1, p_2, p_3$ are not collinear and the point $p$ satisfies the conditions (I) and (II) in Example 4.1. Otherwise, for example, if $p \in c_1$, then the strict transform of $c_1$ on $W$ is a nodal curve, which is different from $C_1, \dots, C'_3$, a contradiction to Lemma 4.7.

Hence we have shown that $S$ is a surface in Example 4.1.

*4.2.2. Non-existence result* Our goal is to exclude the case (b2) of Lemma 4.3. Assume $K^2_W = -1$ and $M^2 = 0$. The following lemma describes a fibration on the surface $W$.

**Lemma 4.8.** (a) *The linear system $|M|$ is composed with a pencil $|F|$ and $M \equiv 2F$, where $|F|$ is a base point free pencil of rational curves and $DF = 2$.*

(b) *Let $g\colon W \to \mathbb{P}^1$ be the fibration defined by $|F|$. Then $g$ has exactly five singular fibers: $E_1 + E_2$ and $C_j + 2\Gamma_j + C'_j$ $(j = 1, 2, 3, 4)$, where $E_1$, $E_2$ and $\Gamma_j$ are $(-1)$-curves, $C_j$ and $C'_j$ are the nodal curves in $\overline{N}_1 \cup \overline{N}_2 \cup \overline{N}_3$ and $E_1E_2 = C_j\Gamma_j = C'_j\Gamma_j = 1$. Moreover, $DE_1 = DE_2 = 1$.*

*Proof.* Lemma 2.4(e) gives $h^0(W, \mathcal{O}_W(M)) = p_a(D) = 3$. Assume $|M| = |\Phi| + \Psi$, where $|\Phi|$ is the moving part and $\Psi$ is the fixed part. Because both $M$ and $\Phi$ is nef, we have $0 = M^2 = M(\Phi + \Psi) \geq M\Phi = \Phi^2 + \Phi\Psi \geq \Phi^2 \geq 0$. It follows that $M\Phi = M\Psi = 0$ and $\Phi^2 = \Phi\Psi = \Psi^2 = 0$. Then $|\Phi|$ is composed a pencil $|F|$ and $\Phi \equiv 2F$. It follows that $F^2 = 0$, $MF = 0$ and $DF = (M - K_W)F = -K_W F$.

Since $D$ is nef and big, $D\Phi \le DM = 4$ and then the adjunction formula shows that $DF = 2$ and $D\Phi = 4$. Then $D\Psi = 0$. Lemma 2.4(f) implies $\Psi = 0$ since $\Psi^2 = 0$. Thus $M \equiv \Phi \equiv 2F$, $DF = 2$ and $K_W F = -2$. This proves (a).

Since $F\overline{N}_i = \frac{1}{2}(D+K_W)\overline{N}_i = 0$ for $i = 1, 2, 3$, the nodal curves $\overline{N}_1 \cup \overline{N}_2 \cup \overline{N}_3$ are contained in the singular fibers of $g$. Assume that $A$ is an irreducible component of a singular fiber of $g$. Then $A$ is a smooth rational curve with $A^2 < 0$. Note that $0 = 2FA = MA = K_W A + DA \ge K_W A = -2 - A^2$. By Lemma 2.4(f), $A$ is either one of the nodal curves in $\overline{N}_1 \cup \overline{N}_2 \cup \overline{N}_3$ or a $(-1)$-curve with $DA = 1$. The rest of the proof is similar to that of Lemma 4.6(a)–(b). □

**Lemma 4.9.** (a) *The curves $\overline{B}_1$, $\overline{B}_2$ and $\overline{B}_3$ satisfy the following properties: $\overline{B}_1\overline{B}_2 = 7$, $\overline{B}_1\overline{B}_3 = \overline{B}_2\overline{B}_3 = 1$ and $\overline{B}_i F = 2$ for $i = 1, 2, 3$.*
(b) *The divisor $\overline{B}_3$ is an irreducible smooth elliptic curve and $\overline{B}_3 \equiv -K_W$.*

*Proof.* Recall that $\overline{B}_i\overline{B}_{i+1}$ is a positive odd integer and $\overline{B}_i^2 = -1$ for $i = 1, 2, 3$.

Note that $F \overset{num}{\sim} \frac{1}{2}(D + K_W) \overset{num}{\sim} \frac{1}{4}(3D - \overline{B}_1 - \overline{B}_2 - \overline{B}_3)$ and $F\overline{N}_i = 0$ for $i = 1, 2, 3$. It follows that

$$F\overline{B}_1 = 4 - \frac{1}{4}(\overline{B}_2 + \overline{B}_3)\overline{B}_1 \le 3, \;\; F\overline{B}_2 = 4 - \frac{1}{4}(\overline{B}_1 + \overline{B}_3)\overline{B}_2 \le 3,$$
$$F\overline{B}_3 = \frac{5}{2} - \frac{1}{4}(\overline{B}_1 + \overline{B}_2)\overline{B}_3 \le 2$$

Also $F\overline{B}_1$, $F\overline{B}_2$ and $F\overline{B}_3$ are of the same parity by (2.4). Since $F(\overline{B}_1 + \overline{B}_2 + \overline{B}_3) = \frac{1}{2}(D + K_W)(D - 2K_W) = 6$, we have $F\overline{B}_i = 2$ for $i = 1, 2, 3$ and then $\overline{B}_1\overline{B}_2 = 7$, $\overline{B}_1\overline{B}_3 = \overline{B}_2\overline{B}_3 = 1$.

By Remark 2.8, from the intersection numbers $D\overline{B}_i$ and $D\overline{N}_i = 0$, we can write $D$ as a $\mathbb{Q}$-linear combination of $\overline{B}_1$, $\overline{B}_2$ and $\overline{B}_3$. It turns out to be a $\mathbb{Z}$-linear combination: $D \equiv \overline{B}_1 + \overline{B}_2 - \overline{B}_3$. Since $\mathrm{Pic}(W)$ has no torsion, $\overline{B}_3 \equiv -K_W$ by (2.5). Assume that $\overline{B}_3 = \Phi_1 + \ldots + \Phi_t$, where the curves $\Phi_k$ are irreducible, smooth and pairwise disjoint. Then $\Phi_1^2 = \Phi_1\overline{B}_3 = -K_W\Phi_1$. It follows that $\Phi_1$ is a smooth elliptic curve. The long exact sequence obtained from

$$0 \to \mathcal{O}_W(K_W) \to \mathcal{O}_W(K_W + \Phi_1) \to \omega_{\Phi_1} \to 0$$

gives $h^0(W, \mathcal{O}_W(K_W + \Phi_1)) = 1$. Since $K_W + \Phi_1 \equiv -\Phi_2 - \ldots - \Phi_t$, $t = 1$ and $\overline{B}_3 = \Phi_1$. This is the desired conclusion. □

**Lemma 4.10.** *For $k = 1, 2$, the linear system $|\overline{B}_3 + E_k|$ induces an elliptic fibration $h_k \colon W \to \mathbb{P}^1$.*

*Proof.* Note that $(\overline{B}_1 + \overline{B}_2)E_k = (D - K_W)E_k = (M - 2K_W)E_k = 2$ for $k = 1, 2$. Since $E_k$ is disjoint from the nodal curves $\overline{N}_1 \cup \overline{N}_2 \cup \overline{N}_3$, by (2.4), $\overline{B}_1 E_k$, $\overline{B}_2 E_k$ and $\overline{B}_3 E_k$ are of the same parity. Since $\overline{B}_3 E_k = -K_W E_k = 1$, $\overline{B}_1 E_k$ and $\overline{B}_2 E_k$ are odd integers.

Fix $k \in \{1, 2\}$. Assume $\overline{B}_1 + \overline{B}_2 \not\ge E_k$. Then $\overline{B}_1 E_k = \overline{B}_2 E_k = 1$ and then the intersection number matrix of $\overline{B}_1$, $\overline{B}_2$, $\overline{B}_3$ and $E_k$ is non-degenerate. This contradicts Remark 2.8. Therefore $\overline{B}_1 + \overline{B}_2 \ge E_k$ for $k = 1, 2$.

Without loss of generality, we may assume $\overline{B}_1 \geq E_1$. Then $\overline{B}_1 E_1 = -1$ and $\overline{B}_2 E_1 = 3$. By Remark 2.8, we may write $E_1$ as the $\mathbb{Q}$-linear combination of $\overline{B}_1, \overline{B}_2, \overline{B}_3$. That is $E_1 \overset{num}{\sim} \frac{1}{2}(\overline{B}_1 - \overline{B}_3)$, i.e., $\overline{B}_1 \equiv 2E_1 + \overline{B}_3$. Then $\overline{B}_1 E_2 = (2E_1 + \overline{B}_3)E_2 = 3$ and $\overline{B}_2 E_2 = 2 - \overline{B}_1 E_2 = -1$. Thus $\overline{B}_2 \geq E_2$ and $\overline{B}_2 \equiv 2E_2 + \overline{B}_3$ again by Remark 2.8.

For $k = 1, 2$, the divisor $\overline{B}_k$ is a disjoint union of smooth irreducible curves, so is $\overline{B}_k - E_k$. Also $\overline{B}_k - E_k$ and $\overline{B}_3 + E_k$ have no common irreducible component (cf. Remark 2.1). Since $\overline{B}_k \equiv \overline{B}_3 + 2E_k$, $\overline{B}_k - E_k$ and $\overline{B}_3 + E_k$ generate a base point free pencil of elliptic curves. □

We are now in a position to deduce a contradiction. Let $P := E_1 + E_2 + \overline{B}_3 \equiv -K_W + F$. We have shown $\overline{B}_3 \equiv -K_W$ and $\overline{B}_1 + \overline{B}_2 \geq E_1 + E_2$. So $\overline{B}_3$, $E_1$ and $E_2$ have no common points by Remark 2.1. Because $|E_1 + E_2|$ $(= |F|)$, $|\overline{B}_3 + E_1|$ and $|\overline{B}_3 + E_2|$ are base point free pencils of curves, $|P|$ is base point free. We also have $E_1 E_2 = \overline{B}_3 E_1 = \overline{B}_3 E_2 = 1$ and $P\overline{B}_3 = PE_1 = PE_2 = 1$.

Note that $F$ is a rational curve and that $\overline{B}_3$ is an elliptic curve. The Riemann–Roch theorem yields $H^1(W, \mathcal{O}_W(F)) = 0$ and $\dim H^0(\overline{B}_3, \mathcal{O}_{\overline{B}_3}(P)) = 1$. Then the following exact sequence

$$0 \to \mathcal{O}_W(F) \to \mathcal{O}_W(P) \to \mathcal{O}_{\overline{B}_3}(P) \to 0$$

yields $\dim H^0(W, \mathcal{O}_W(P)) = 3$. So $|P|$ defines a morphism $f : W \to \mathbb{P}^2$. The argument above also shows that the trace of $|P|$ on $\overline{B}_3$ is complete and of 0-dimensional. Therefore $f(\overline{B}_3)$ is a point. Since $P = f^*\mathcal{O}_{\mathbb{P}^2}(1)$, this contradicts $P\overline{B}_3 = 1 > 0$. □

Hence we exclude the case (b2) of Lemma 4.3 and complete the proof of Theorem 1.1.

## 5. Deformations and the moduli space

We study the local deformations and the moduli of the surfaces in the case (b) of Theorem 1.1. These surfaces are exactly the ones in Example 4.1. Our goal is to prove Theorem 1.2. Throughout this section, we assume that $S$ is a smooth minimal surface as in Example 4.1. We denote by $\Theta_S$ the tangent sheaf of $S$. The following proposition estimates the dimension of the cohomology group $H^2(S, \Theta_S)$.

**Proposition 5.1.** *The dimensions of the eigenspaces of $H^2(S, \Theta_S)$ (for the $\mathbb{Z}_2^2$-action) satisfy the following properties:*

$$\dim H^2(S, \Theta_S)^{\text{inv}} = 0,\ \dim H^2(S, \Theta_S)^{\chi_1} \leq 2,\ \dim H^2(S, \Theta_S)^{\chi_2} \leq 2,\ \dim H^2(S, \Theta_S)^{\chi_3} \leq 3.$$

We use the methods in [3–5], and [15] to prove Proposition 5.1. The techniques involved depend on the exact sequences in [18, Properties 2.3(c), p. 13]. We also need the following lemma, which generalizes [5, Lemma 5.1].

**Lemma 5.2.** (cf. Lemma 4.4 [15]) *Let $X$ be a projective smooth surface. Let $Y_1, \dots, Y_{k-1}$ and $Y$ be $k$ irreducible smooth curves on $X$. Assume that $Y_1 + \dots + Y_{k-1} + Y$ has only nodes.*

(a) *There is an exact sequence*

$$0 \to \Omega^1_X(\log Y_1, \dots, \log Y_{k-1}, \log Y) \to \Omega^1_X(\log Y_1, \dots, \log Y_{k-1})(Y) \\ \to \Omega^1_Y(Y_1 + \dots + Y_{k-1} + Y) \to 0.$$

(b) *If $\mathcal{L}$ is a divisor of $X$ and $Y.(K_X + 2Y + Y_1 + \dots + Y_{k-1} + \mathcal{L}) < 0$, then*

$$\dim H^0(X, \Omega^1_X(\log Y_1, \dots, \log Y_{k-1})(Y + \mathcal{L})) \\ = \dim H^0(X, \Omega^1_X(\log Y_1, \dots, \log Y_{k-1}, \log Y)(\mathcal{L})).$$

*Proof of Proposition 5.1.* We recall in the Example 4.1 that $V$ is a blowup of the surface $S$ and $\overline{\pi}\colon V \to W$ is a finite Galois $\mathbb{Z}_2^2$-cover branched on $\Delta_1$, $\Delta_2$ and $\Delta_3$ [see the diagram (2.3) and (4.2)]. We often refer to Example 4.1 (2)–(4) for the intersection numbers of the curves and the classes of curves in the Picard group $\mathrm{Pic}(W) \cong H^2(W, \mathbb{Z})$.

Denote by $\Theta_V$ the tangent sheaf of the surface $V$. Because blowing down a $(-1)$-curve does not change the dimension of the second cohomology group of the tangent sheaf, we have $\dim H^2(S, \Theta_S)^{\mathrm{inv}} = \dim H^2(V, \Theta_V)^{\mathrm{inv}}$ and $\dim H^2(S, \Theta_S)^{\chi_i} = \dim H^2(V, \Theta_V)^{\chi_i}$ for $i = 1, 2, 3$. Serre duality implies $H^2(V, \Theta_V) \cong H^0(V, \Omega^1_V \otimes \Omega^2_V)^*$. The $\mathbb{Z}_2^2$-cover structure gives (cf. [12, Theorem 2.16])

$$\begin{aligned} &\overline{\pi}_*(\Omega^1_V \otimes \Omega^2_V) = \Omega^1_W(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_W) \\ &\oplus (\oplus_{i=1}^3 \Omega^1_W(\log \Delta_i)(K_W + \mathcal{L}_i)) \\ &H^0(V, \Omega^1_V \otimes \Omega^2_V) = H^0(W, \Omega^1_W(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_W)) \\ &\oplus (\oplus_{i=1}^3 H^0(W, \Omega^1_W(\log \Delta_i)(K_W + \mathcal{L}_i))), \end{aligned} \tag{5.1}$$

where the invertible sheaves $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are given by (4.3). It is sufficient to calculate the dimension of each summand.

We first show $H^0(W, \Omega^1_W(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_W)) = 0$. We have the following exact sequence from [12, Lemma 3.7] and [10, Lemma 3]

$$0 \to \Omega^1_W(K_W) \to \Omega^1_W(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_W) \to \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_W) \to 0 \tag{5.2}$$

Since $H^0(W, \Omega^1_W) = 0$ and $-K_W$ is effective, $H^0(W, \Omega^1_W(K_W)) = 0$. It suffices to show that the boundary map $\delta\colon H^0(W, \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_W)) \to H^1(W, \Omega^1_W(K_W))$ is injective. The linear system $|-K_W|$ has only one simple base point because $W$ is a weak Del Pezzo surface of degree one. So there is a morphism $\mathcal{O}_W(K_W) \to \mathcal{O}_W$, which is not identically zero on any component of the divisors $\Delta_i$. This morphism $\mathcal{O}_W(K_W) \to \mathcal{O}_W$ induces a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \Omega^1_W(K_W) & \to & \Omega^1_W(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_W) & \to & \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_W) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \Omega^1_W & \to & \Omega^1_W(\log \Delta_1, \log \Delta_2, \log \Delta_3) & \to & \oplus_{i=1}^3 \mathcal{O}_{\Delta_i} & \to 0 \end{array}$$

It gives a commutative diagram of cohomology groups

$$\begin{array}{ccc} H^0(W, \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_W)) & \xrightarrow{\delta} & H^1(W, \Omega_W^1(K_W)) \\ \psi_2 \downarrow & \searrow^{\psi} & \downarrow \\ H^0(W, \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}) & \xrightarrow{\psi_1} & H^1(W, \Omega_W^1) \end{array}$$

Recall that $C_i$ and $C_i'$ are nodal-curves and that other components of $\Delta_i$ are $(-1)$-curves or 0-curves (see (4.2)). So we have $H^0(W, \oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_W)) \cong \oplus_{i=1}^3 H^0(W, \mathcal{O}_{C_i}) \oplus H^0(W, \mathcal{O}_{C_i'})$. By [12, Lemma 3.7], the image of the function identically equal to 1 on $C_i$ (respectively $C_i'$) maps under $\psi_1$ to the first Chern class of $C_i$ (respectively $C_i'$). Because the curves $C_i$ and $C_i'$ are disjoint nodal curves, their Chern classes are linearly independent in $H^1(W, \Omega_W^1)$. Thus the composite map $\psi = \psi_1\psi_2$ is injective. It follows that $\delta$ is also injective. We thus get

$$\dim H^2(S, \Theta_S)^{\text{inv}} = \dim H^0(W, \Omega_W^1(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_W)) = 0.$$

We now calculate $\dim H^0(W, \Omega_W^1(\log \Delta_1)(K_W + \mathcal{L}_1))$. Recall that $\Delta_1 = F_b + \Gamma + C_1 + C_1' + C_2 + C_2'$ and $K_W + \mathcal{L}_1 = -K_W - E_3'$ (see (4.2) and (4.3)). Applying Lemma 5.2(b) to $\Gamma$, $C_3$ and $C_3'$, we have

$$\begin{aligned} &\dim H^0(W, \Omega_W^1(\log \Delta_1)(K_W + \mathcal{L}_1)) = \qquad (5.3)\\ &\dim H^0(W, \Omega_W^1(\log F_b, \log C_1, \log C_1', \log C_2, \log C_2', \log C_3, \log C_3')(K_W \\ &\quad + \mathcal{L}_1 - C_3 - C_3' + \Gamma)) \end{aligned}$$

Note that $K_W + \mathcal{L}_1 - C_3 - C_3' + \Gamma \equiv (-K_W - E_3') - (L - E_0 - 2E_3') + (L - E_0 - E) = -K_W + E_3' - E$ (see (4.1) and (4.3)). By Lemma 4.2, $|-K_W + E_3' - E|$ consists of a $(-1)$-curve $\Lambda$ and $\Lambda$ meets $F_b$ transversely. Now $\Lambda(K_W + 2\Lambda + F_b + C_1 + \ldots + C_3') = 1$, Lemma 5.2(a) gives the following exact sequence

$$\begin{aligned} 0 \to\ & \Omega_W^1(\log F_b, \log C_1, \log C_1', \log C_2, \log C_2', \log C_3, \log C_3', \log \Lambda) \\ & \to \Omega_W^1(\log F_b, \log C_1, \log C_1', \log C_2, \log C_2', \log C_3, \log C_3')(\Lambda) \to \mathcal{O}_\Lambda(1) \to 0. \end{aligned}$$

The first Chern classes of $F_b, C_1, \ldots, C_3'$ and $\Lambda$ are linearly independent in $H^2(W, \mathbb{C})$. By [12, Lemma 3.7], it follows that

$$H^0(W, \Omega_W^1(\log F_b, \log C_1, \log C_1', \log C_2, \log C_2', \log C_3, \log C_3', \log \Lambda)) = 0.$$

From the long exact sequence of cohomology we conclude that

$$\dim H^0(W, \Omega_W^1(\log F_b, \log C_1, \log C_1', \log C_2, \log C_2', \log C_3, \log C_3')(\Lambda)) \leq 2 \qquad (5.4)$$

We thus get $\dim H^2(S, \Theta_S)^{\chi_1} = \dim H^0(W, \Omega_W^1(\log \Delta_1)(K_W + \mathcal{L}_1)) \leq 2$ by (5.3) and (5.4).

We proceed to calculate $\dim H^0(W, \Omega_W^1(\log \Delta_2)(K_W + \mathcal{L}_2))$. Recall that $\Delta_2 = \overline{B}_2 + C_3 + C_3'$ and $K_W + \mathcal{L}_2 = 2\mathcal{L} - 2E_0 - E_1' - E_2' - E$. Note that $\Gamma$ intersects

$\overline{B}_2$ transversely (see Example 4.1(4)) and $\Gamma(K_W+2\Gamma+\overline{B}_2+C_3+C_3'+(K_W+\mathcal{L}_2-\Gamma))=0$. Lemma 5.2(a) yields an exact sequence

$$0\to\Omega^1_W(\log\overline{B}_2,\log C_3,\log C_3',\log\Gamma)(K_W+\mathcal{L}_2-\Gamma)\\ \to\Omega^1_W(\log\overline{B}_2,\log C_3,\log C_3')(K_W+\mathcal{L}_2)\to\mathcal{O}_\Gamma\to 0.$$

It follows that

$$\dim H^0(W,\Omega^1_W(\log\overline{B}_2,\log C_3,\log C_3')(K_W+\mathcal{L}_2))\\ \leq\dim H^0(W,\Omega^1_W(\log\overline{B}_2,\log C_3,\log C_3',\log\Gamma)(K_W+\mathcal{L}_2-\Gamma))+1 \quad (5.5)$$

Note that $\overline{B}_2(K_W+\mathcal{L}_2-\Gamma)=0$. Tensoring the exact sequence (cf. [18, Properties 2.3(b)])

$$0\to\Omega^1_W(\log C_3,\log C_3',\log\Gamma)\to\Omega^1_W(\log\overline{B}_2,\log C_3,\log C_3',\\ \log\Gamma)\to\mathcal{O}_{\overline{B}_2}\to 0$$

with $\mathcal{O}_W(K_W+\mathcal{L}_2-\Gamma)$, we have

$$\dim H^0(W,\Omega^1_W(\log\overline{B}_2,\log C_3,\log C_3',\log\Gamma)(K_W+\mathcal{L}_2-\Gamma))\\ \leq\dim H^0(W,\Omega^1_W(\log C_3,\log C_3',\log\Gamma)(K_W+\mathcal{L}_2-\Gamma))+1 \quad (5.6)$$

Note that $K_W+\mathcal{L}_2-\Gamma\equiv L-E_0-E_1'-E_2'\equiv C_1+E_1'+C_1'-E_2'$. Applying Lemma 5.2(b) to $C_1$, $E_1'$ and $C_1'$, we have

$$\dim H^0(W,\Omega^1_W(\log C_3,\log C_3',\log\Gamma)(K_W+\mathcal{L}_2-\Gamma))\\ =\dim H^0(W,\Omega^1_W(\log C_3,\log C_3',\log\Gamma,\log C_1,\log E_1',\log C_1')(-E_2')) \quad (5.7)$$

The first Chern classes of $\Gamma, C_3, C_3', C_1, E_1'$ and $C_1'$ are linearly independent in $H^2(S,\mathbb{C})$. By [12, Lemma 3.7], it follows that

$$\dim H^0(W,\Omega^1_W(\log C_3,\log C_3',\log\Gamma,\log C_1,\log E_1',\log C_1')(-E_2'))=0 \quad (5.8)$$

We thus obtain $\dim H^2(S,\Theta_S)^{\chi_2}=\dim H^0(W,\Omega^1_W(\log\Delta_2)(K_W+\mathcal{L}_2))\leq 2$ from (5.5)–(5.8).

It remains to calculate $\dim H^0(W,\Omega^1_W(\log\Delta_3)(K_W+\mathcal{L}_3))$. Note that $\Delta_3=\overline{B}_3$ and $K_W+\mathcal{L}_3=2\mathcal{L}-2E_0-E_1'-E_2'-E_3'$. Since $\Gamma(K_W+2\Gamma+\overline{B}_3+(K_W+\mathcal{L}_3-\Gamma))=-1$, Lemma 5.2(b) yields

$$\dim H^0(W,\Omega^1_W(\log\overline{B}_3)(K_W+\mathcal{L}_3))\\ =\dim H^0(W,\Omega^1_W(\log\overline{B}_3,\log\Gamma)(K_W+\mathcal{L}_3-\Gamma)) \quad (5.9)$$

Note that $\overline{B}_3(K_W+\mathcal{L}_3-\Gamma)=1$. Tensoring the exact sequence (cf. [18, Properties 2.3(b)])

$$0\to\Omega^1_W(\log\Gamma)\to\Omega^1_W(\log\overline{B}_3,\log\Gamma)\to\mathcal{O}_{\overline{B}_3}\to 0$$

with $\mathcal{O}_W(K_W + \mathcal{L}_3 - \Gamma)$, we have

$$\dim H^0(W, \Omega^1_W(\log \overline{B}_3, \log \Gamma)(K_W + \mathcal{L}_3 - \Gamma)) \\ \leq \dim H^0(W, \Omega^1_W(\log \Gamma)(K_W + \mathcal{L}_3 - \Gamma)) + 2 \tag{5.10}$$

Note that $K_W + \mathcal{L}_3 - \Gamma = L - E_0 + E - E'_1 - E'_2 - E'_3 \equiv E + C_1 + E'_1 + C'_1 - E'_2 - E'_3$. Applying Lemma 5.2(b) to $E$, $C_1$, $E'_1$ and $C'_1$, we have

$$\dim H^0(W, \Omega^1_W(\log \Gamma)(K_W + \mathcal{L}_3 - \Gamma)) = \\ \dim H^0(W, \Omega^1_W(\log \Gamma, \log E, \log C_1, \log E'_1, \log C'_1)(-E'_2 - E'_3)) \tag{5.11}$$

Since $E + \Gamma \equiv F \equiv C_1 + 2E'_1 + C'_1$, applying [12, Lemma 3.7] to the following exact sequence

$$0 \to \Omega^1_W \to \Omega^1_W(\log \Gamma, \log E, \log C_1, \log E'_1, \log C'_1) \to \mathcal{O}_\Gamma \oplus \mathcal{O}_E \oplus \mathcal{O}_{C_1} \\ \oplus \mathcal{O}_{E'_1} \oplus \mathcal{O}_{C'_1} \to 0,$$

we have

$$H^0(W, \Omega^1_W(\log \Gamma, \log E, \log C_1, \log E'_1, \log C'_1)) = 1 \tag{5.12}$$

We obtain $\dim H^2(S, \Theta_S)^{\chi_3} \leq 3$ from (5.9)–(5.12) and complete the proof. □

*The proof of Theorem 1.2.* From the construction of the surfaces in Example 4.1, it is clear that $\mathfrak{B}$ is irreducible, unirational and of dimension 3 (cf. [16, thelastparagraphofSection 3]).

Let $S$ be a surface in Example 4.1. Since $-\dim H^1(S, \Theta_S) + \dim H^2(S, \Theta_S) = 2K_S^2 - 10\chi(S) = 4$, by Proposition 5.1, we have $\dim H^2(S, \Theta_S) \leq 7$ and $\dim H^1(S, \Theta_S) \leq 3$. Because $S$ is smooth and $K_S$ is ample, the minimal model and the canonical model of $S$ coincide. We denote by $\mathrm{B}(S)$ the base of the Kuranishi family of deformations of $S$ and by $[S]$ the corresponding point of the moduli space $\mathcal{M}^{\mathrm{can}}_{1,7}$. Recall the fact that the germ of the complex space $(\mathcal{M}^{\mathrm{can}}_{1,7}, [S])$ is analytically isomorphic to the quotient $\mathrm{B}(S)/\mathrm{Aut}(S)$.

We have the following inequalities

$$3 \geq \dim H^1(S, \Theta_S) \geq \text{the dimension of } \mathrm{B}(S) \geq \text{the dimension of } (\mathcal{M}^{\mathrm{can}}_{1,7}, [S]) \\ \geq \text{the local dimension of } \mathfrak{B} \text{ at the point } [S] = 3.$$

Consequently all the equalities hold. The second equality shows that $\mathrm{B}(S)$ is smooth. Therefore $(\mathcal{M}^{\mathrm{can}}_{1,7}, [S])$ is normal. Since $\mathfrak{B}$ is irreducible, the last equality shows that $\mathfrak{B}$ coincides with $\mathcal{M}^{\mathrm{can}}_{1,7}$ locally at $[S]$ for any $S$ in Example 4.1. Hence $\mathfrak{B}$ is an open subset of $\mathcal{M}^{\mathrm{can}}_{1,7}$ and $\mathfrak{B}$ is normal.

It remains to prove that $\mathfrak{B}$ is a closed subset of $\mathcal{M}^{\mathrm{can}}_{1,7}$. It suffices to prove the following statement. *Let $T$ be a smooth affine curve and $o \in T$. Let $\mathcal{F}: \mathcal{S} \to T$ be a flat family of canonical models of surfaces of general type with $K^2 = 7$ and $p_g = 0$. Set $\mathcal{S}_t := \mathcal{F}^{-1}(t)$ for $t \in T$. Assume that $\mathcal{S}_t$ is a surface in Example 4.1 for $t \neq o$. Then so is $\mathcal{S}_o$.*

In fact, by construction, we have a $G$ ($\cong \mathbb{Z}_2^2$)-action on $\mathcal{S} \setminus \mathcal{S}_o$. The $G$-action extends to $\mathcal{S}$ by [11, Theorem 1.8]. In particular, $\mathcal{S}_o$ admits a $G$-action. This action lifts to the minimal model $\mathcal{S}'_o$ of $\mathcal{S}_o$. By Proposition 2.7(e), the canonical divisor of $\mathcal{S}'_o$ is ample and thus $\mathcal{S}_o = \mathcal{S}'_o$. It suffices to show that the pair $(\mathcal{S}_o, G)$ satisfies property (b) of Theorem 1.1.

For $t \in T$, we denote by $\mathcal{R}_t$ be the union of the divisorial parts of the fixed loci of the three involutions $g_1$, $g_2$ and $g_3$. Since $\mathcal{F}\colon \mathcal{S} \to T$ is a flat family with the $G$-action on each fiber, $K_{\mathcal{S}_o}\mathcal{R}_o \geq K_{\mathcal{S}_t}\mathcal{R}_t$ for $t \neq o$. From the construction in Example 4.1, we have $K_{\mathcal{S}_t}\mathcal{R}_t = 5 + 5 + 3 = 13$ for $t \neq o$ and thus $K_{\mathcal{S}_o}\mathcal{R}_o \geq 13$. We see that $(\mathcal{S}_o, G)$ satisfies the property (a) or (b) of Theorem 1.1.

Set $\mathcal{Y} := \mathcal{S}/G$. Then $\mathcal{Y} \to T$ is a flat family of surfaces with only nodes. For $t \neq o$, by construction, $\mathcal{Y}_t := \mathcal{S}_t/G$ is a 6-nodal singular Del Pezzo surface. It follows that $\mathcal{Y}_o := \mathcal{S}_o/G$ has at least 6 nodes. Since the quotient of an Inoue surface is the 4-nodal cubic surface, by Theorem 1.1, $(\mathcal{S}_o, G)$ satisfies property (b) and $\mathcal{S}_o$ is indeed a surface in Example 4.1.

Hence $\mathfrak{B}$ is a closed subset of $\mathcal{M}^{\mathrm{can}}_{1,7}$ and we complete the proof of Theorem 1.2. □

*Remark 5.3.* From the proof of Theorem 1.2, we see that all the inequalities in Proposition 5.1 hold, $\dim H^1(S, \Theta_S) = 3$ and $\dim H^2(S, \Theta_S) = 7$. Here we show $H^1(S, \Theta_S)$ is $\mathbb{Z}_2^2$-invariant.

It suffices to show $\dim H^1(S, \Theta_S)^{\mathrm{inv}} = \dim H^1(W, \Omega^1_W(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_W)) = 3$ by (5.1). We have seen $\dim H^0(W, \Omega^1_W(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_W)) = 0$. According to (5.1), $H^2(W, \Omega^1_W(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_W))$ is a direct sum of $H^0(V, \Theta_V)^*$. Since $V$ is of general type, we have $H^0(V, \Theta_V) = 0$. It remains to show $\chi(\Omega^1_W(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_W)) = -3$. The exact sequence (5.2) yields

$$\chi(\Omega^1_W(\log \Delta_1, \log \Delta_2, \log \Delta_3)(K_W) = \chi(\Omega^1_W(K_W)) + \chi(\oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_W)).$$

The splitting principle and the Riemann–Roch theorem imply $\chi(\Omega^1_W(K_W)) = -8$. Note that each component of $\Delta_1$, $\Delta_2$ and $\Delta_3$ is a smooth rational curve (see (4.2)). We easily obtain $\chi(\oplus_{i=1}^3 \mathcal{O}_{\Delta_i}(K_W)) = 5$ by Riemann–Roch Theorem and complete the proof.

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