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Cohomology of locally closed semi-algebraic subsets

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Abstract. Let k be a non-Archimedean field, let ℓ be a prime number distinct from the characteristic of the residue field of k . If \mathcal{X} is a separated k -scheme of finite type, Berkovich's theory of germs allows to define étale ℓ -adic cohomology groups with compact support of locally closed semi-algebraic subsets of \mathcal{X}^{an} . We prove that these vector spaces are finite dimensional continuous representations of the Galois group of k^{sep}/k , and satisfy the usual long exact sequence and Künneth formula. This has been recently used by E. Hrushovski and F. Loeser in a paper about the monodromy of the Milnor fibration. In this statement, the main difficulty is the finiteness result, whose proof relies on a cohomological finiteness result for affinoid spaces, recently proved by V. Berkovich.

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1. Introduction

Let k be a non-Archimedean¹ field and \mathcal{X} a separated k -scheme of finite type. One can associate to it a k -analytic space \mathcal{X}^{an} [1]. Using [2] one can define ℓ -adic cohomology groups $H_c^i(\mathcal{X}^{an}, \mathbb{Q}_\ell)$ which have good properties if ℓ is different

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¹ This means that k is equipped with an ultrametric norm $|\cdot| : k \rightarrow \mathbb{R}_+$ for which it is complete.

from $\text{char}(\tilde{k})$ (in particular, they are finite dimensional vector spaces when k is algebraically closed).

If the k -scheme $\mathcal{X} = \text{Spec}(A)$ is affine, a subset U of \mathcal{X}^{an} is called *semi-algebraic* if it is a finite Boolean combination of subsets of the form $\{x \in \mathcal{X}^{an} \mid |f(x)| \leq \lambda|g(x)|\}$ where f and g belong to A and λ is a positive real number. This definition of semi-algebraic subsets extends to general k -varieties (see Definition 2.8). If U is a semi-algebraic subset of \mathcal{X}^{an} , using the theory of k -germs developed in [2], it is possible to define cohomology groups of the k -germ (\mathcal{X}^{an}, U) , that we will denote by $H_c^i(U, \mathbb{Q}_\ell)$. We want to point out that in general, U is not equipped with a structure of k -analytic space.

In this paper, we generalize the finiteness property mentioned above to locally closed semi-algebraic subsets of \mathcal{X}^{an} . More precisely, let $\widehat{k^a}$ be the completion of the algebraic closure of k and let us set $\overline{\mathcal{X}} := \mathcal{X}^{an} \widehat{\otimes}_k \widehat{k^a}$ and $\pi : \overline{\mathcal{X}} \rightarrow \mathcal{X}^{an}$ the natural morphism. If U is a subset of \mathcal{X}^{an} , we set $\overline{U} := \pi^{-1}(U)$. Our main result is then:

Theorem. 5.14 *Let \mathcal{X} be a separated k -scheme of finite type of dimension d , U a locally closed semi-algebraic subset of \mathcal{X}^{an} , and $\ell \neq \text{char}(k)$ be a prime number.*

1. *The groups $H_c^i(\overline{U}, \mathbb{Q}_\ell)$ are finite dimensional \mathbb{Q}_ℓ -vector spaces, on which $\text{Gal}(k^{sep}/k)$ acts continuously, and $H_c^i(\overline{U}, \mathbb{Q}_\ell) = 0$ for $i > 2d$.*
2. *If $V \subset U$ is a semi-algebraic subset which is open in U and $F = U \setminus V$, then there is a long exact sequence of Galois modules*

$$\cdots \rightarrow H_c^i(\overline{V}, \mathbb{Q}_\ell) \longrightarrow H_c^i(\overline{U}, \mathbb{Q}_\ell) \longrightarrow H_c^i(\overline{F}, \mathbb{Q}_\ell) \longrightarrow H_c^{i+1}(\overline{V}, \mathbb{Q}_\ell) \cdots \rightarrow$$

3. *For all integer n there are canonical isomorphisms of Galois modules:*

$$\bigoplus_{i+j=n} H_c^i(\overline{U}, \mathbb{Q}_\ell) \otimes H_c^j(\overline{V}, \mathbb{Q}_\ell) \simeq H_c^n(\overline{U \times V}, \mathbb{Q}_\ell).$$

We prove more generally this result when \mathcal{X} is a separated \mathcal{A} -scheme of finite type where \mathcal{A} is a k -affinoid algebra. The above result corresponds to the case $\mathcal{A} = k$.

This question was raised by F. Loeser and used in [8] where they study the Milnor fibration associated to a morphism $f : X \rightarrow \mathbb{A}_{\mathbb{C}}^1$ where X is a smooth complex algebraic variety. The non-Archimedean field is then $k = \mathbb{C}((t))$.

In fact, the main point in the above theorem is to prove that when k is algebraically closed, the groups $H_c^i(\overline{U}, \mathbb{Z}/\ell^n \mathbb{Z})$ are finite. This is obtained as a consequence of another analogous result which does not involve algebraic objects. If X is a k -affinoid space whose affinoid algebra is \mathcal{A} , we say that a subset S of X is *semianalytic* if it is a finite Boolean combination of subsets of the form $\{x \in X \mid |f(x)| \leq \lambda|g(x)|\}$ where f and g belong to \mathcal{A} and $\lambda > 0$. Now if X is a compact k -analytic space, we say that a subset $S \subset X$ is *G -semianalytic* if there exists a finite covering $\{X_i\}$ of X by some affinoid domains such that for each i , $S \cap X_i$ is semianalytic in X_i . We then prove:

Proposition. 4.1 *Let us assume that k is algebraically closed, and let X be a compact k -analytic space. Then for any locally closed G -semianalytic subset S of X and Λ a finite group whose cardinal is prime to $\text{char}(\tilde{k})$, the groups $H_c^q((X, S), \Lambda)$ are finite.*

We want to point out that this result relies deeply on the cohomological finiteness of affinoid spaces which has been recently² proved by V. Berkovich in [4].

We also want to mention that in the author’s thesis (Sect. 2.4) it is proved that our finiteness result Proposition 3.3 is also true if we assume that S is an overconvergent subanalytic subset³ of X .

Finally in Sect. 6, we explain some counterparts of these finiteness results for Huber’s adic spaces.

Notations. In what follows, k will be a complete non-Archimedean field, X a Hausdorff k -analytic space, and $S \subset X$ will always be a locally closed subset of X .

The étale site of a germ (see [2, 3.4]) If S is a subset of X , (X, S) is called a k -germ. If (Y, T) is another k -germ, a morphism of k -germs $f : (Y, T) \rightarrow (X, S)$ is a morphism of k -analytic spaces $f : Y \rightarrow X$ such that $f(T) \subset S$. This defines the category of k -germs.

Then the category of k -Germs is defined as the localization of the category of k -germs by the morphisms of k -germs $\varphi : (Y, T) \rightarrow (X, S)$ which induce an isomorphism of Y with some open neighbourhood of S in X (this implies that φ induces a homeomorphism between T and S). It is important to remark that the functor $k\text{-An} \rightarrow k\text{-Germs}$ defined by $X \rightarrow (X, |X|)$ is fully faithful.

A morphism of k -Germs f is called étale if it has a representative $\varphi : (Y, T) \rightarrow (X, S)$ such that $\varphi : Y \rightarrow X$ is étale and $T = \varphi^{-1}(S)$. Berkovich defines the small étale site $(X, S)_{\text{ét}}$ of the k -germ (X, S) , as the category of étale morphisms above (X, S) , a family $(Y_i, T_i) \xrightarrow{f_i} (Y, T)$ being a covering if $\cup_i f_i(T_i) = T$. The category of abelian sheaves on $(X, S)_{\text{ét}}$ is denoted by $\mathcal{S}(X, S)_{\text{ét}}$.

Cohomology groups with compact support (see [2, 5.1]) If S is a topological space, a family of supports Φ is a family of closed subsets of S which is stable under finite unions and such that if F is a closed subset of S and $F \subset T$ for some $T \in \Phi$ then $F \in \Phi$. If Φ is a family of supports of S , and $A \subset S$ we set

$$\Phi_A = \{F \in \Phi \mid F \subset A\} \tag{1}$$

which is a family of supports of A .

A family of supports Φ is said to be paracompactifying if for all $A \in \Phi$, A is paracompact, and if for all $A \in \Phi$, there exists $B \in \Phi$ which is a neighbourhood of A .

If Φ is a family of supports, the following functor

$$\begin{aligned} \Gamma_\Phi : \mathcal{S}(X, S)_{\text{ét}} &\rightarrow \text{Ab} \\ F &\mapsto \{s \in F(X, S) \mid \text{supp}(s) \in \Phi\} \end{aligned} \tag{2}$$

² In a previous version of this work (arXiv:1210.4521, [v1]), the results of [4] were not available. Instead, we used [3, Corollary 5.5] which is a cohomological finiteness result for affinoid spaces which are algebraizable in some sense. As a consequence, in this previous version, we obtained less general results. In particular, we could not prove Proposition 3.3.

³ If X is a k -affinoid space, the class of overconvergent subanalytic subsets of X , introduced by H. Schoutens in [14], properly contains the class of semianalytic subsets of X .

is left exact. Its right derived functors are denoted by $H_{\Phi}^q((X, S), F)$.

Let us assume that S is locally closed in X , that T is an open subset of S , and that $R := S \setminus T$. If Φ is a paracompactifying family of supports of S , and F is an abelian sheaf on (X, S) , there is a long exact sequence [2, 5.2.6 (ii)]:

$$\begin{aligned} \dots \rightarrow H_{\Phi_R}^{q-1}((X, R), F_{(X,R)}) &\rightarrow H_{\Phi_T}^q((X, T), F_{(X,T)}) \\ &\rightarrow H_{\Phi}^q((X, S), F) \rightarrow H_{\Phi_R}^q((X, R), F_{(X,R)}) \rightarrow \dots \end{aligned} \tag{3}$$

Now, we denote by C_S the family of compact subsets of S . If S is Hausdorff, this is a family of supports, and if S is locally compact, C_S is paracompactifying. Remind that if S is a locally closed subset of a locally compact topological set X , S is also locally compact. Since we will always consider locally closed subsets S of some Hausdorff k -analytic space X , the family of supports C_S will then be paracompactifying. If $F \in \mathcal{S}(X, S)_{\acute{e}t}$, we will denote by

$$H_c^q((X, S), F) := H_{C_S}^q((X, S), F)$$

the associated right derived functors.

Let $T \subset S$ be an open subset of S and $R := S \setminus T$ the complementary closed subset of S . Then, $(C_S)_T = C_T$ because being compact in T or in S is equivalent; likewise $(C_S)_R = C_R$. In this context, the long exact sequence (3) can be written:

$$\begin{aligned} \dots \rightarrow H_c^{q-1}((X, R), F_{(X,R)}) &\rightarrow H_c^q((X, T), F_{(X,T)}) \\ &\rightarrow H_c^q((X, S), F) \rightarrow H_c^q((X, R), F_{(X,R)}) \rightarrow \dots \end{aligned} \tag{4}$$

What we will look at Let Λ be a finite abelian group whose cardinal is prime to the characteristic of \tilde{k} . We set

$$H_c^n(S, \Lambda) := H_c^n((X, S), \underline{\Lambda}) \tag{5}$$

where $\underline{\Lambda}$ is the constant sheaf of value Λ on $(X, S)_{\acute{e}t}$. This notation is abusive since the cohomology of S itself is meaningless, only the cohomology of the k -germ (X, S) can be defined. Nonetheless, we will use the notation (5) to simplify the exposition.

If we still denote by $\underline{\Lambda}$ the constant sheaf of value Λ on $(X, S)_{\acute{e}t}$, then if $U \subset S$, $\underline{\Lambda}_{(X,U)}$ is isomorphic to the constant sheaf of value Λ on $(X, U)_{\acute{e}t}$. Hence, if T is an open subset of S and $R := S \setminus T$, the long exact sequence (4) becomes

$$\dots \rightarrow H_c^{q-1}(R, \Lambda) \rightarrow H_c^q(T, \Lambda) \rightarrow H_c^q(S, \Lambda) \rightarrow H_c^q(R, \Lambda) \rightarrow \dots \tag{6}$$

Quasi-immersions (see [2, 4.3]) A morphism of k -germs $\varphi : (Y, T) \rightarrow (X, S)$ is called a quasi-immersion if φ induces a homeomorphism of T with its image $\varphi(T)$ and for all $y \in T$, if we set $x := \varphi(y)$, the maximal purely inseparable extension of $\mathcal{H}(x)$ in $\mathcal{H}(y)$ is everywhere dense in $\mathcal{H}(y)$.

Here are two examples of quasi-immersions that we will use frequently. If U is an analytic domain of X , and $\varphi : U \rightarrow X$ is the natural inclusion morphism,

then $\varphi : (U, U) \rightarrow (X, U)$ is a quasi-immersion. If $\varphi : Z \rightarrow X$ is a closed immersion, then $(Z, Z) \rightarrow (X, Z)$ is a quasi-immersion. Moreover, quasi-immersions are stable under composition and base change.

Quasi-immersions will be very important for us through the following result:

Proposition. [2, 4.3.4 (i)] *If $\varphi : (Y, T) \rightarrow (X, S)$ is a quasi-immersion of k -germs, it induces an equivalence of categories*

$$\mathbf{S}(Y, T)_{\acute{e}t} \simeq \mathbf{S}(X, \varphi(T))_{\acute{e}t}.$$

In particular if U is an analytic domain of X , there are isomorphisms

$$H_c^q(U, \Lambda) \simeq H_c^q((U, U), \Lambda) \simeq H_c^q((X, U), \Lambda). \tag{7}$$

Similarly, if Z is a closed k -analytic subset of X ,

$$H_c^q(Z, \Lambda) \simeq H_c^q((Z, Z), \Lambda) \simeq H_c^q((X, Z), \Lambda). \tag{8}$$

We want to stress out that (7) and (8) partly justify the abuse of the notation made in (5).

2. Semianalytic and semi-algebraic sets

Definition 2.1. Let \mathcal{A} be a k -affinoid algebra, and let $X = \mathcal{M}(\mathcal{A})$ be the associated k -affinoid space. A subset $S \subset X$ is called *semianalytic* if it is a finite Boolean combination⁴ of sets of the form

$$\{x \in X \mid |f(x)| \leq \lambda |g(x)|\} \tag{9}$$

where $f, g \in \mathcal{A}$, and λ is a positive real number.

If one takes $g = 0$ in (9), one sees that if $f \in \mathcal{A}$, the hypersurface $V(f) = \{x \in X \mid f(x) = 0\}$ is semianalytic. More generally, any Zariski-closed subset of X is semianalytic. Using the Gerritzen–Grauert theorem, one can also check that any affinoid domain of X is semianalytic.

Definition 2.2. Let X be a compact k -analytic space. A subset $S \subseteq X$ is called *G-semianalytic* if there exists a finite cover $X = \cup_{i=1}^n U_i$ by affinoid domains such that $S \cap U_i$ is semianalytic in U_i for all i .

Remark 2.3. Let X be a k -affinoid space. If S is a semianalytic subset of X , then it is also a G -semianalytic subset of X (just consider the trivial cover with $n = 1$ and $X_1 = X$). The converse is false: if $S \subset X$ is G -semianalytic, it is not necessarily semianalytic in X .

We could have said that S is locally semianalytic if for each point $x \in X$, there is some affinoid neighbourhood V of x such that $S \cap V$ is semianalytic in V . With

⁴ Since we authorize finite Boolean combinations, we could have also authorized some $<$ and $=$ in this definition.

this definition, for a subset S of a k -affinoid space X , the following implications hold

$$S \text{ is semianalytic} \Rightarrow S \text{ is locally semianalytic} \Rightarrow S \text{ is G-semianalytic.}$$

However, these three classes of subsets are pairwise distinct. For more details on this, we refer to [11] and [12].

We remind the following definition of [6, 2.1]:

Definition 2.4. Let \mathcal{A} be a k -affinoid algebra, \mathcal{B} an \mathcal{A} -algebra of finite type, and $\mathcal{X} = \text{Spec}(\mathcal{B})$. A subset $S \subset \mathcal{X}^{an}$ is called *semi-algebraic* if it is a Boolean combination of subsets

$$\{x \in \mathcal{X}^{an} \mid |f(x)| \leq \lambda |g(x)|\}$$

where $f, g \in \mathcal{B}$ and $\lambda \in \mathbb{R}_+^*$.

Remark 2.5. We want to point out that this definition depends on the algebraic datum $\mathcal{X} = \text{Spec}(\mathcal{B})$ and not only on the k -analytic space \mathcal{X}^{an} .

For instance, let us consider the case $\mathcal{A} = k$ and $\mathcal{X} = \text{Spec}(k[T_1, T_2])$, so that

$$\mathcal{X}^{an} \simeq \mathbb{A}_k^{2,an}. \tag{10}$$

Let us consider $S = \{x \in \mathbb{A}_k^{2,an} \mid T_2(x) = 0\}$. Then S is semi-algebraic in $\mathbb{A}_k^{2,an}$ with respect to the presentation (10). Let then $f = \sum_{n \geq 0} a_n T_1^n$ be a series with $a_n \in k$ whose radius of convergence is infinite, and such that f is not a polynomial. Let us then consider the automorphism of $\mathbb{A}_k^{2,an}$ defined by

$$\begin{aligned} \varphi : \mathbb{A}_k^{2,an} &\rightarrow \mathbb{A}_k^{2,an} \\ (T_1, T_2) &\mapsto (T_1, T_2 + f(T_1)) \end{aligned}$$

It is easy to check that $\varphi(S) \subset \mathbb{A}_k^{2,an}$ is not semi-algebraic any more with respect to the presentation (10).

So, in order to be precise, when one talks about a semi-algebraic subset $S \subset \mathcal{X}^{an}$, one should always specify the algebraic datum \mathcal{X} . However, for simplicity, we will not do it when the algebraic presentation will be clear from the context.

Lemma 2.6. *Let \mathcal{A} be a k -affinoid algebra, \mathcal{X} an affine \mathcal{A} -scheme of finite type, and let $\mathcal{X} = \mathcal{U}_1 \cup \dots \cup \mathcal{U}_n$ be an affine covering of \mathcal{X} . If $S \subset \mathcal{X}^{an}$, then S is semi-algebraic in \mathcal{X}^{an} if and only if $S \cap \mathcal{U}_i^{an}$ is semi-algebraic in \mathcal{U}_i^{an} for all i .*

Proof. Let $S \subset \mathcal{X}^{an}$.

In one hand it follows from Definition 2.4 that if S is semi-algebraic in \mathcal{X}^{an} , then $S \cap \mathcal{U}_i^{an}$ is also semi-algebraic in \mathcal{U}_i^{an} , using the restriction morphisms $\mathcal{O}_{\mathcal{X}}(\mathcal{X}) \rightarrow \mathcal{O}_{\mathcal{X}}(\mathcal{U}_i)$.

Conversely, if $S \cap \mathcal{U}_i^{an}$ is semi-algebraic in \mathcal{U}_i^{an} for all i , according to [6, 2.6], $S \cap \mathcal{U}_i^{an}$ is also semi-algebraic in \mathcal{X}^{an} . Since $S = \cup_{i=1}^n (S \cap \mathcal{U}_i^{an})$, it follows that S is semi-algebraic in \mathcal{X}^{an} . □

Lemma 2.7. *Let \mathcal{X} be a separated \mathcal{A} -scheme of finite type, and $S \subset \mathcal{X}^{an}$. Let $\mathcal{X} = \mathcal{U}_1 \cup \dots \cup \mathcal{U}_n$ be some affine covering of \mathcal{X} . The following statements are equivalent:*

1. *For $i = 1 \dots n$, $S \cap \mathcal{U}_i^{an}$ is semi-algebraic in \mathcal{U}_i^{an} .*
2. *For every open affine subscheme $\mathcal{V} \subset \mathcal{X}$, $S \cap \mathcal{V}^{an}$ is semi-algebraic in \mathcal{V}^{an} .*

Proof. Let us assume that the condition (1) is fulfilled. Let \mathcal{V} be some open affine subscheme of \mathcal{X} . Then $\mathcal{U}_i \cap \mathcal{V}$ is an open affine subscheme of \mathcal{U}_i (because \mathcal{X} is separated), and since $S \cap \mathcal{U}_i^{an}$ is semi-algebraic in \mathcal{U}_i^{an} , $S \cap (\mathcal{U}_i \cap \mathcal{V})^{an}$ is semi-algebraic in $(\mathcal{U}_i \cap \mathcal{V})^{an}$. Since the family $\{\mathcal{U}_i \cap \mathcal{V}\}$ is a finite affine covering of \mathcal{V} and since $S \cap \mathcal{V}^{an} = \cup_{i=1}^n S \cap (\mathcal{U}_i \cap \mathcal{V})^{an}$, thanks to the previous lemma, $S \cap \mathcal{V}^{an}$ is semi-algebraic in \mathcal{V}^{an} , that is to say (2) is true.

For the converse statement one just has to take $\mathcal{V} = \mathcal{U}_i$. □

Definition 2.8. Let \mathcal{A} be a k -affinoid algebra, and let \mathcal{X} be a separated \mathcal{A} -scheme of finite type. A subset $S \subset \mathcal{X}^{an}$ is called semi-algebraic if it satisfies one of the two equivalent conditions of Lemma 2.7.

Remark 2.9. Thanks to Lemma 2.7, we can check whether $S \subset \mathcal{X}^{an}$ is semi-algebraic, with an affine covering of \mathcal{X} , and this does not depend on the covering. Moreover, if \mathcal{X} is affine, according to Lemma 2.6, the two Definitions 2.4 and 2.8 of a semi-algebraic set are equivalent.

We have already used the following result, which is proved as a consequence of the quantifier elimination in ACVF:

Proposition. [6, 2.5] *Let \mathcal{A} be a k -affinoid algebra, \mathcal{X} and \mathcal{Y} some affine \mathcal{A} -schemes of finite type, $f : \mathcal{X} \rightarrow \mathcal{Y}$ an \mathcal{A} -morphism of finite type, and S a semi-algebraic subset of \mathcal{X}^{an} . Then $f^{an}(S)$ is a semialgebraic subset of \mathcal{Y}^{an} .*

Thanks to Lemmas 2.6 and 2.7, it has the immediate generalization:

Proposition 2.10. *Let \mathcal{A} be a k -affinoid algebra, \mathcal{X} and \mathcal{Y} some separated \mathcal{A} -schemes of finite type, $f : \mathcal{X} \rightarrow \mathcal{Y}$ an \mathcal{A} -morphism of finite type, and S a semi-algebraic subset of \mathcal{X}^{an} . Then $f^{an}(S)$ is a semialgebraic subset of \mathcal{Y}^{an} .*

Remark 2.11. We want to point out that Definition 2.8 both generalizes the definition of a semianalytic set of an affinoid space, and the more classical definition of a semi-algebraic set of some \mathcal{X}^{an} where \mathcal{X} is an affine k -scheme of finite type.

Remark 2.12. Let \mathcal{A} be a k -affinoid algebra. Let \mathcal{X} be an affine \mathcal{A} -scheme of finite type, V be an affinoid domain of \mathcal{X}^{an} , and let $S \subset \mathcal{X}^{an}$ be a semi-algebraic set. Then $S \cap V$ is semianalytic in V . This is a straightforward consequence of the above definitions.

Lemma 2.13. *Let \mathcal{X} be a separated \mathcal{A} -scheme of finite type, $S \subset \mathcal{X}^{an}$ a semi-algebraic subset of \mathcal{X}^{an} and V an affinoid domain of \mathcal{X}^{an} . Then $S \cap V$ is G -semianalytic in V .*

Proof. It is possible to find a finite covering of V by affinoid domains $V = \cup_{i=1}^n V_i$, and some affine open subschemes \mathcal{U}_i of \mathcal{X} such that $V_i \subset \mathcal{U}_i^{an}$ for all i . Then $S \cap V_i = (S \cap \mathcal{U}_i^{an}) \cap V_i$, and since $S \cap \mathcal{U}_i^{an}$ is semi-algebraic in \mathcal{U}_i^{an} and $V_i \subset \mathcal{U}_i^{an}$, $S \cap V_i$ is semianalytic in V_i according to Remark 2.12. \square

In fact, in the above lemma, one can check that $S \cap V$ is even locally semianalytic.

3. A finiteness result in the affinoid case

In this section k will be a (complete) non-Archimedean algebraically closed field. We consider a k -affinoid algebra \mathcal{A} , and we set $X = \mathcal{M}(\mathcal{A})$. We remind that Λ is a finite abelian group whose order is prime to the characteristic of \bar{k} . The goal of this section is to prove Proposition 3.3.

Lemma 3.1. *Let $n \in \mathbb{N}$, and for $i = 1 \dots n$, let $f_i, g_i \in \mathcal{A}$, $\diamond_i \in \{<, \leq\}$ and $\lambda_i > 0$ be a positive real number. Let us consider*

$$S = \bigcap_{i=1}^n \{x \in X \mid |f_i(x)| \diamond_i \lambda_i |g_i(x)|\}.$$

Then, the groups $H_c^q(S, \Lambda)$ are finite for all $q \in \mathbb{N}$.

Proof. We prove the lemma by induction on n .

If $n = 0$, then $S = X$ and the result is a consequence of the finiteness result [4, Theorem 1.1.1].

Let $n \geq 0$ and assume that the result is true for n . Let $f, g \in \mathcal{A}$, $\lambda > 0$, and let

$$\begin{aligned} S &= \bigcap_{i=1}^n \left\{ x \in X \mid |f_i(x)| \diamond_i \lambda_i |g_i(x)|, i = 1 \dots n \right\} \\ T &= \{x \in S \mid |g(x)|\lambda < |f(x)|\} \\ R &= \{x \in S \mid |f(x)| \leq \lambda |g(x)|\} = S \setminus T. \end{aligned}$$

Let us show that the groups $H_c^q(T, \Lambda)$ and $H_c^q(R, \Lambda)$ are finite. This will achieve our induction step.

By its definition, S is a locally closed subset of X , T is an open subset of S , and $R = S \setminus T$ is the complementary closed subset of S . So we can apply the long exact sequence (6) to S , R and T . By induction hypothesis, the groups $H_c^q(S, \Lambda)$ are finite, so if we show that the groups $H_c^q(R, \Lambda)$ are finite, this will also prove the finiteness of the groups $H_c^q(T, \Lambda)$. Let us then show that the groups $H_c^q(R, \Lambda)$ are finite.

Let $Y = \mathcal{M}(\mathcal{A}\{\lambda^{-1}U\}/(f - Ug))$ and let $\varphi : Y \rightarrow X$ be the morphism of affinoid spaces induced by the natural map $\mathcal{A} \rightarrow \mathcal{A}\{\lambda^{-1}U\}/(f - Ug)$. The morphism φ induces an isomorphism between the analytic domain of Y :

$$A = \{y \in Y \mid g(y) \neq 0\}$$

and the analytic domain of X :

$$B = \{x \in X \mid |f(x)| \leq \lambda|g(x)| \text{ and } g(x) \neq 0\}.$$

As a consequence, φ induces a quasi-immersion $\varphi : (Y, A) \rightarrow (X, B)$, and also a quasi-immersion

$$(Y, A \cap \varphi^{-1}(S)) \rightarrow (X, B \cap S). \tag{11}$$

But

$$\varphi^{-1}(S) = \bigcap_{i=1}^n \{y \in Y \mid |f_i(y)| \leq \lambda_i |g_i(y)|\}.$$

Here we have written f_i (resp. g_i) whereas we should rather have written $\varphi^*(f_i)$ (resp. $\varphi^*(g_i)$). Hence by induction hypothesis, the groups $H_c^q((Y, \varphi^{-1}(S)), \Lambda)$ are finite.

Now $A \cap \varphi^{-1}(S)$ is an open subset of $\varphi^{-1}(S)$, whose complement in $\varphi^{-1}(S)$ is $\varphi^{-1}(S) \cap \{y \in Y \mid g(y) = 0\}$. Let Z be the Zariski closed subset of Y defined by

$$Z = \{y \in Y \mid g(y) = 0\}$$

and $\psi : Z \rightarrow Y$ the associated closed immersion. We then obtain a quasi-immersion:

$$(Z, \psi^{-1}(\varphi^{-1}(S))) \rightarrow (Y, Z \cap \varphi^{-1}(S)).$$

By the induction hypothesis the groups $H_c^q(\psi^{-1}(\varphi^{-1}(S)), \Lambda)$ are finite, therefore it is also true for the groups $H_c^q(Z \cap \varphi^{-1}(S), \Lambda)$. Thus in the long exact sequence

$$\dots \rightarrow H_c^q(A \cap \varphi^{-1}(S), \Lambda) \rightarrow H_c^q(\varphi^{-1}(S), \Lambda) \rightarrow H_c^q(Z \cap \varphi^{-1}(S), \Lambda) \rightarrow \dots$$

the written groups in the middle and in the right are finite and we conclude from this that the groups $H_c^q(A \cap \varphi^{-1}(S), \Lambda)$ are finite. We have already noticed that $(Y, A \cap \varphi^{-1}(S)) \rightarrow (X, B \cap S)$ is a quasi-immersion, hence

$$H_c^q(A \cap \varphi^{-1}(S), \Lambda) \simeq H_c^q(B \cap S, \Lambda).$$

From this, we conclude that the groups $H_c^q(B \cap S, \Lambda)$ are also finite.

If we go back to our starting point

$$B \cap S = \{x \in S \mid |f(x)| \leq \lambda|g(x)| \text{ and } g(x) \neq 0\}$$

is an open subset of

$$R = \{x \in S \mid |f(x)| \leq \lambda|g(x)|\}.$$

The complementary subset of $B \cap S$ in R is

$$D = \{x \in S \mid |f(x)| \leq \lambda|g(x)| \text{ and } g(x) = 0\} = \{x \in S \mid f(x) = g(x) = 0\}.$$

We denote by Z' the Zariski closed subset of X :

$$Z' = \{x \in X \mid f(x) = g(x) = 0\}$$

hence $D = Z' \cap S$, and using the same kind of arguments as above we can conclude that the groups $H_c^q(D, \Lambda)$ are finite.

We use for the last time the long exact sequence

$$\dots \rightarrow H_c^q(B \cap S, \Lambda) \rightarrow H_c^q(R, \Lambda) \rightarrow H_c^q(D, \Lambda) \rightarrow \dots$$

We have shown that the groups on the left, and on the right are finite, thus the groups $H_c^q(R, \Lambda)$ are also finite. \square

Next, we want to extend this result to an arbitrary locally closed semianalytic subset of X . In order to do so, we introduce the following notation.

Let $f_1, \dots, f_r, g_1, \dots, g_r \in \mathcal{A}$, and $\lambda_1, \dots, \lambda_r > 0$. For a subset $I \subseteq \{1 \dots r\}$ we set

$$C_I = \left(\bigcap_{i \in I} \{x \in X \mid |f_i(x)| \leq \lambda_i |g_i(x)|\} \right) \cap \left(\bigcap_{j \notin I} \{x \in X \mid |f_j(x)| > \lambda_j |g_j(x)|\} \right).$$

The subsets C_I induce a partition of X , and each C_I is a semianalytic set of X . If $A \subseteq \mathcal{P}(\{1 \dots r\})$, let us set

$$C_A = \coprod_{I \in A} C_I.$$

This is a semianalytic subset of X , and in fact every semianalytic subset of X is of this form.⁵ This follows from the fact that if S is a semianalytic subset of X , one can find some $f_1, \dots, f_r, g_1, \dots, g_r \in \mathcal{A}$ such that S is a finite union of subsets of the form

$$\{x \in \mid |f_{i_1}(x)| \diamond_{i_1} |g_{i_1}(x)| \text{ and } \dots \text{ and } |f_{i_m}(x)| \diamond_{i_m} |g_{i_m}(x)|\}$$

where $1 \leq i_1 < \dots < i_m \leq r$, and $\diamond_j \in \{\leq, >\}$.

For instance, if $S = \{|f_1| \leq |g_1|\} \cup \{|f_2| > |g_2|\}$, $A = \{\{1, 2\}, \{1\}, \emptyset\}$ is suitable:

$$S = \{|f_1| \leq |g_1| \text{ and } |f_2| \leq |g_2|\} \cup \{|f_1| \leq |g_1| \text{ and } |f_2| > |g_2|\} \\ \cup \{|f_1| > |g_1| \text{ and } |f_2| > |g_2|\}.$$

Lemma 3.2. *Let r and n be two integers, $f_1, \dots, f_r, g_1, \dots, g_r, F_1, \dots, F_n, G_1, \dots, G_n \in \mathcal{A}$, $A \subseteq \mathcal{P}(\{1 \dots r\})$, $\diamond_i \in \{<, \leq\}$ for $i = 1 \dots n$ and $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_n$ be some positive real numbers. Let us suppose that the semianalytic set of X*

$$C = C_A \cap \left(\bigcap_{i=1}^n \{x \in X \mid |F_i(x)| \diamond_i \mu_i |G_i(x)|\} \right)$$

is locally closed. Then the groups $H_c^q(C, \Lambda)$ are finite.

⁵ This is some kind of disjunctive normal form.

Proof. We prove this by induction on r .

If $r = 0$ this is precisely the preceding Lemma 3.1.

Let $r \geq 0$ and let us assume that we are given $f_1, \dots, f_{r+1}, g_1, \dots, g_{r+1} \in \mathcal{A}, A \subset \mathcal{P}(\{1 \dots r + 1\})$, and

$$C = C_A \cap \left(\bigcap_{i=1}^n \{x \in X \mid |F_i(x)| \diamond_i \mu_i |G_i(x)| \} \right)$$

a subset of X , assumed to be locally closed. Then we must show that the groups $H_c^q(C, \Lambda)$ are finite. The idea is to decompose C as

$$C = \{x \in C \mid |f_{r+1}(x)| \leq \lambda_{r+1} |g_{r+1}(x)|\} \coprod \{x \in C \mid |f_{r+1}(x)| > \lambda_{r+1} |g_{r+1}(x)|\}$$

and to use our induction hypothesis to this partition of C .

To formalize this, we set

$$\begin{aligned} A_1 &= \{P \in A \mid r + 1 \in P\} \\ A_2 &= \{P \in A \mid r + 1 \notin P\} = A \setminus A_1. \end{aligned}$$

Finally we set

$$B_1 = \{P \setminus \{r + 1\} \mid P \in A_1\}$$

and $B_2 = A_2$. In addition, we see B_1 and B_2 as subsets of $\mathcal{P}(\{1 \dots r\})$.

We now consider the subsets of X, C_{B_1} and C_{B_2} , associated with $f_1, \dots, f_r, g_1, \dots, g_r$ and $\lambda_1, \dots, \lambda_r$. Then, by definition of B_1 and B_2 ,

$$\begin{aligned} C_A &= (\{x \in X \mid |f_{r+1}(x)| \leq \lambda_{r+1} |g_{r+1}(x)|\} \cap C_{B_1}) \coprod (\{x \in X \mid |f_{r+1}(x)| \\ &> \lambda_{r+1} |g_{r+1}(x)|\} \cap C_{B_2}). \end{aligned}$$

Said more simply, we have partitioned the set $C_A = \coprod_{I \in A} C_I$ in two parts: on the left side, we have kept the C'_I 's where the inequality $|f_{r+1}(x)| \leq \lambda_{r+1} |g_{r+1}(x)|$ appears, and on the right side, we have kept the C'_I 's where the inequality $|f_{r+1}(x)| > \lambda_{r+1} |g_{r+1}(x)|$ appears, which allows us to restrict to subsets of $\{1 \dots r\}$. And now we set:

$$\begin{aligned} C_1 &= C_{B_1} \cap \left(\{x \in X \mid |f_{r+1}(x)| \leq \lambda_{r+1} |g_{r+1}(x)|\} \cap \bigcap_{i=1}^n \{x \in X \mid |F_i(x)| \diamond_i \mu_i |G_i(x)| \} \right) \\ C_2 &= C_{B_2} \cap \left(\{x \in X \mid |f_{r+1}(x)| > \lambda_{r+1} |g_{r+1}(x)|\} \cap \bigcap_{i=1}^n \{x \in X \mid |F_i(x)| \diamond_i \mu_i |G_i(x)| \} \right). \end{aligned}$$

The following holds:

$$\begin{aligned} C_1 &= \{x \in C \mid |f_{r+1}(x)| \leq \lambda_{r+1} |g_{r+1}(x)|\} \\ C_2 &= \{x \in C \mid |f_{r+1}(x)| > \lambda_{r+1} |g_{r+1}(x)|\}. \end{aligned}$$

So $C = C_1 \coprod C_2$, C_2 is an open subset of C , and C_1 is the closed complementary subset attached to it, in particular, C_1 and C_2 are locally closed in X . But we can now apply our induction hypothesis to C_1 and C_2 : the groups $H_c^q(C_i, \Lambda)$ are finite for $i = 1, 2$. Finally, according to long exact sequence (6) applied to $C_2 \subset C \supset C_1$, the groups $H_c^q(C, \Lambda)$ are finite. \square

The previous lemma, with $n = 0$ becomes:

Proposition 3.3. *Let S be a locally closed semianalytic subset of X . The groups $H_c^q(S, \Lambda)$ are finite.*

4. Global results

In this section, we will still assume that k is algebraically closed.

4.1. The compact case

Proposition 4.1. *Let X be a compact k -analytic space, and let S be a locally closed G -semianalytic subset of X . The groups $H_c^q(S, \Lambda)$ are finite.*

Proof. We prove by induction on m that if X is a Hausdorff k -analytic space which is covered by m affinoid domains: $X = \cup_{i=1}^m V_i$, and $S \subset X$ is a locally closed subset of X such that $S \cap V_i$ is semianalytic in V_i for all i (in particular S is G -semianalytic in X), then the groups $H_c^q(S, \Lambda)$ are finite.

For $m = 1$ this is Proposition 3.3.

Let then $m \geq 1$ and let us assume that X is covered by the affinoid domains V_i , $i = 1 \dots m + 1$, and that $S \subset X$ such that for all i , $S \cap V_i$ is semianalytic in V_i . We set

$$\begin{aligned} R &= S \cap (V_1 \cup \dots \cup V_m) \\ T &= S \setminus R \\ &= (V_{m+1} \cap S) \setminus (V_{m+1} \cap (V_1 \cup \dots \cup V_m)). \end{aligned} \tag{12}$$

If $X' := V_1 \cup \dots \cup V_m$, then X' is a compact analytic domain of X (not necessarily good), and R is a locally closed G -semianalytic subset of X' such that $R \cap V_i$ is semianalytic in V_i for $i = 1 \dots m$. Thus by induction hypothesis, the groups $H_c^q((X', R), \Lambda)$ are finite. In addition, since $(X', R) \rightarrow (X, R)$ is a quasi-immersion (because X' is an analytic domain of X), $H_c^q((X', R), \Lambda) \simeq H_c^q((X, R), \Lambda)$, hence these are finite groups.

Next, we claim that T is a locally closed semianalytic set of V_{m+1} . Indeed, for each i , $V_{m+1} \cap V_i$ is an affinoid domain of V_{m+1} (because X is separated), hence closed and semianalytic in V_{m+1} according to the Gerritzen–Grauert theorem. Hence $V_{m+1} \cap (V_1 \cup \dots \cup V_n)$ is a closed semianalytic set of V_{m+1} . Since $S \cap V_{m+1}$ is a locally closed semianalytic subset of V_{m+1} , according to (12), T is a locally closed semianalytic subset of V_{m+1} . Hence according to Proposition 3.3,

the groups $H_c^q((V_{m+1}, T), \Lambda)$ are finite, and since $(V_{m+1}, T) \rightarrow (X, T)$ is a quasi-immersion, $H_c^q((V_{m+1}, T), \Lambda) \simeq H_c^q((X, T), \Lambda)$, thus the groups $H_c^q((X, T), \Lambda)$ are finite.

Finally, since R is a closed subset of S and $T = S \setminus R$, the long exact sequence (6) allows to conclude that the groups $H_c^q((X, S), \Lambda)$ are finite. \square

4.2. The semi-algebraic case

Proposition 4.2. *Let \mathcal{A} be a k -affinoid algebra, \mathcal{X} a separated \mathcal{A} -scheme of finite type, and S a locally closed semi-algebraic subset of \mathcal{X}^{an} . Then the groups $H_c^q((\mathcal{X}^{an}, S), \Lambda)$ (that we abusively denote by $H_c^q(S, \Lambda)$) are finite.*

Proof. According to Nagata’s compactification theorem (see [5] for a modern proof), we can embed \mathcal{X} as an open subscheme of a proper \mathcal{A} -scheme $\overline{\mathcal{X}}$. Since $(\mathcal{X}^{an}, \mathcal{X}^{an}) \rightarrow (\overline{\mathcal{X}}^{an}, \mathcal{X}^{an})$ is a quasi-immersion, and since quasi-immersions are stable under base change, for all $q \geq 0$ we have an isomorphism of groups:

$$H_c^q((\mathcal{X}^{an}, S), \Lambda) \simeq H_c^q((\overline{\mathcal{X}}^{an}, S), \Lambda).$$

In addition, according to Proposition 2.10, S is still semi-algebraic in $\overline{\mathcal{X}}^{an}$. Moreover, S is still locally closed in $\overline{\mathcal{X}}^{an}$ because \mathcal{X}^{an} is open in $\overline{\mathcal{X}}^{an}$. So we can assume that \mathcal{X} is proper.

In that case, \mathcal{X}^{an} is compact, and according to Lemma 2.13, S is G -semianalytic in $\overline{\mathcal{X}}^{an}$, and the result follows from Proposition 4.1. \square

5. From torsion to ℓ -adic coefficients

5.1. Continuous Galois action

From now on, we do not assume any more that k is algebraically closed. We still consider X a Hausdorff k -analytic space. Let S be a locally closed subset of X and let us set $\overline{X} = X \widehat{\otimes}_k \widehat{k^a}$, $\pi : \overline{X} \rightarrow X$, the projection, and $\overline{S} = \pi^{-1}(S)$. This is a locally closed subset of \overline{X} . There is an action of $\text{Gal}(k^{sep}/k)$ on \overline{X} which stabilizes \overline{S} . Hence $\text{Gal}(k^{sep}/k)$ acts on the k -germ $(\overline{X}, \overline{S})$. If $\mathcal{F} \in \mathcal{S}(X, S)$, we set $\overline{\mathcal{F}} = \pi^*(\mathcal{F})$. The action of $\text{Gal}(k^{sep}/k)$ on $(\overline{X}, \overline{S})$ induces an action on $H_c^i(\overline{S}, \overline{\mathcal{F}})$. Indeed for $\sigma \in \text{Gal}(k^{sep}/k)$ we have the commutative diagram

$$\begin{array}{ccc} (\overline{X}, \overline{S}) & \xrightarrow{\sigma} & (\overline{X}, \overline{S}) \\ & \searrow \pi & \swarrow \pi \\ & (X, S) & \end{array}$$

Then the action of σ on the cohomology is given by :

$$\sigma^* : H_c^i((\overline{X}, \overline{S}), \overline{\mathcal{F}}) \simeq H_c^i((\overline{X}, \overline{S}), \sigma^*\overline{\mathcal{F}}) \simeq H_c^i((\overline{X}, \overline{S}), \overline{\mathcal{F}}),$$

the last isomorphism being a consequence of the isomorphism $\sigma^* \circ \pi^*(\mathcal{F}) \simeq \pi^*(\mathcal{F})$. If (X, S) is a k -germ, and K is a complete extension of k , we consider $\pi_K : X_K = X \hat{\otimes}_k K \rightarrow X$ and we set $S_K = \pi_K^{-1}(S)$, so that we can consider the K -germ (X_K, S_K) .

Proposition 5.1. *If X is a Hausdorff k -analytic space, F a locally closed subset of X , $\mathcal{F} \in \mathcal{S}_{\text{ét}}(X)$, there is an isomorphism of Galois modules:*

$$\varinjlim_{K/k} H_c^q((X_K, F_K), \mathcal{F}_K) \simeq H_c^q((\overline{X}, \overline{F}), \overline{\mathcal{F}})$$

where the limit is taken over all finite separable extensions K of k contained in k^{sep} .

Proof. We will use that if Y is a Hausdorff k -analytic space and $\mathcal{G} \in \mathcal{S}_{\text{ét}}(Y)$, the following is true [2, 5.3.5]:

$$\varinjlim_{K/k} H_c^q(Y_K, \mathcal{G}_K) \simeq H_c^q(\overline{Y}, \overline{\mathcal{G}}),$$

and it is an isomorphism of Galois-modules.

Since F is locally closed, it can be written $F = U \cap F'$ where U is open in X and F' is closed in X , and since $(U, F) \rightarrow (X, F)$ is a quasi-immersion, for all $q \geq 0$, $H_c^q((U, F), \Lambda) \simeq H_c^q((X, F), \Lambda)$. Now, F is closed in U , so we can replace X by U and assume that F is closed.

In this situation, let $U = X \setminus F$ be the complementary open subset of X . For K a finite separable extension of k , F_K is a closed subset of X_K whose complementary open subset is U_K . Hence we get a commutative diagram:

$$\begin{array}{ccccccc} \cdots \rightarrow & \varinjlim_{K/k} H_c^q(U_K, \mathcal{F}_K) & \longrightarrow & \varinjlim_{K/k} H_c^q(X_K, \mathcal{F}_K) & \longrightarrow & \varinjlim_{K/k} H_c^q((X_K, F_K), \mathcal{F}_K) & \cdots \rightarrow \\ & \parallel & & \parallel & & \downarrow & \\ \cdots \rightarrow & H_c^q(\overline{U}, \overline{\mathcal{F}}) & \longrightarrow & H_c^q(\overline{X}, \overline{\mathcal{F}}) & \longrightarrow & H_c^q((\overline{X}, \overline{F}), \overline{\mathcal{F}}) & \cdots \rightarrow \end{array}$$

Thanks to the long exact sequence (4), the first row is exact because \varinjlim is an exact functor (we consider a filtered inductive limit), and the second row is exact. We can then conclude thanks to the five lemma. \square

In particular, if \mathcal{A} is a k -affinoid algebra, \mathcal{X} is a separated \mathcal{A} -scheme of finite type, and if S is a locally closed subset of \mathcal{X}^{an} ,

$$H_c^q((\overline{\mathcal{X}}, \overline{S}), \Lambda)$$

is a continuous Galois module. Moreover, if T is an open subset of S and $R = S \setminus T$, the long exact sequence

$$\cdots \rightarrow H_c^q((\overline{\mathcal{X}}, \overline{T}), \Lambda) \rightarrow H_c^q((\overline{\mathcal{X}}, \overline{S}), \Lambda) \rightarrow H_c^q((\overline{\mathcal{X}}, \overline{R}), \Lambda) \rightarrow \cdots$$

is Galois equivariant.

5.2. About the dimension

Let X be a Hausdorff k -analytic space. We denote by d the dimension of X (cf. [1, p. 34] and [2, p. 23]).

Proposition 5.2. [2, Cor 5.3.8] *Let Y be a Hausdorff k -analytic space of dimension d , \mathcal{F} a torsion abelian sheaf on Y , then for all $i > 2d$, $H_c^i(Y, \mathcal{F}) = 0$.*

We can generalize this result in the following way:

Proposition 5.3. *Let X be a Hausdorff k -analytic space of dimension d . Let S be a locally closed subset of X . For $q > 2d$, and $F \in \mathcal{S}(X)$ an abelian torsion sheaf on X , $H_c^q((X, S), F) = 0$.*

Proof. Write $S = U \cap Z$ with U an open subset of X and Z a closed subset. Set $V = U \setminus S$ which is an open subset of U and X . Then $H_c^q((U, S), F) \simeq H_c^q((X, S), F)$ and $H_c^q(V, F) \simeq H_c^q((V, V), F) \simeq H_c^q((U, V), F)$, hence in the long exact sequence (6)

$$\dots \rightarrow H_c^q((U, V), F) \rightarrow H_c^q(U, F) \rightarrow H_c^q((U, S), F) \rightarrow \dots$$

according to the previous proposition, the groups are 0 on the left and in the middle for $q > 2d$, so this must also occur for the groups on the right. \square

In our situation, this result can be refined. If \mathcal{A} is a k -affinoid algebra, \mathcal{X} a separated \mathcal{A} -scheme of finite type, and $S \subset \mathcal{X}^{an}$ a semi-algebraic set, we set $Z := \overline{S}^{Zar}$. Then since $(Z, S) \rightarrow (X, S)$ is a quasi-immersion, $H_c^q((Z, S), F) \simeq H_c^q((X, S), F)$. Hence if we set $\dim(S) := \dim(Z)$, with the above notations, $H_c^q((X, S), F) = 0$ for all $q > 2 \dim(S)$.

5.3. Finiteness of the ℓ -adic cohomology

In this subsection, we assume again that k is algebraically closed. We fix \mathcal{A} a k -affinoid algebra, \mathcal{X} a separated \mathcal{A} -scheme of finite type, S a locally closed semi-algebraic subset of \mathcal{X}^{an} , and ℓ a prime number different from the characteristic of \tilde{k} .

In this situation, we have seen in Proposition 4.2 that for $n \geq 0$, the groups $H_c^q(S, \mathbb{Z}/\ell^n\mathbb{Z})$ are finite (we remind that the notation $H_c^q(S, \mathbb{Z}/\ell^n\mathbb{Z})$ is a shorthand for $H_c^q((X, S), \underline{\mathbb{Z}/\ell^n\mathbb{Z}})$).

We then set

$$H_c^q(S, \mathbb{Z}_\ell) = \varprojlim_{n>0} H_c^q(S, \mathbb{Z}/\ell^n\mathbb{Z}) \tag{13}$$

and

$$H_c^q(S, \mathbb{Q}_\ell) = H_c^q(S, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell. \tag{14}$$

It is a classical fact that Proposition 4.2 implies that the groups $H_c^q(S, \mathbb{Z}_\ell)$ are finitely generated \mathbb{Z}_ℓ -modules, and as a consequence, that $H_c^q(S, \mathbb{Q}_\ell)$ are finite-dimensional \mathbb{Q}_ℓ -vector spaces. For completeness, we give here a proof as simple as possible.

Definition 5.4. A \mathbb{Z}_ℓ -module M is called complete and separated (with respect to the ℓ -adic topology) if the canonical map

$$\pi : M \rightarrow \widehat{M} := \varprojlim_{k \geq 1} M/\ell^k M$$

is an isomorphism.

Proposition 5.5. *Let us consider a projective system of abelian groups*

$$M_1 \xleftarrow{d_1} M_2 \leftarrow \dots \xleftarrow{d_{n-1}} M_n \xleftarrow{d_n} \dots$$

where each M_n is a finite $\mathbb{Z}/\ell^n\mathbb{Z}$ -module. Then

$$M := \varprojlim_{n \geq 1} M_n$$

is a complete and separated \mathbb{Z}_ℓ -module.

Proof. We must show that

$$\pi : M \rightarrow \widehat{M} = \varprojlim_{k \geq 1} M/\ell^k M$$

is an isomorphism.

We first prove that π is injective. If $x = (x_n) \in M$, and $\pi(x) = 0$, this means that $x \in \ell^k M$ for all k . Since for each n , M_n is a $\mathbb{Z}/\ell^n\mathbb{Z}$ -module, taking $k = n$, this implies that $x_n \in \ell^n M_n$, so $x_n = 0$ for all n , hence $x = 0$.

Let us now prove that π is surjective. For this, we consider a Cauchy sequence $(y^{(k)})_{k \geq 1}$ in M , such that for all $j \geq k$, $y^{(j)} \equiv y^{(k)} \pmod{\ell^k M}$. In particular, if $j \geq k$, this implies that for all n ,

$$y_n^{(j)} \equiv y_n^{(k)} \pmod{\ell^k M_n}. \quad (15)$$

Now all we have to do is to find some $x \in M$ such that for all $k \geq 1$, $x \equiv y^{(k)} \pmod{\ell^k M}$.

First, we define $x = (x_n)$ by

$$x_n = (y_n^{(n)})_{n \geq 1}.$$

Thus we obtain:

$$d_n(x_{n+1}) = d_n \left(y_{n+1}^{(n+1)} \right) = y_n^{(n+1)} \equiv y_n^{(n)} \pmod{\ell^n M_n},$$

the last congruence being a consequence of (15). But since M_n is a $\mathbb{Z}/\ell^n\mathbb{Z}$ -module, $\ell^n M_n = \{0\}$, thus $d_n(x_{n+1}) = y_n^{(n)} = x_n$. Hence $(x_n) \in M$.

It is now sufficient to show that $x \equiv y^{(k)} \pmod{\ell^k M}$ for all $k \geq 1$. For this, let us consider some $n \in \mathbb{N}^*$. Then

$$x_n - y_n^{(k)} = y_n^{(n)} - y_n^{(k)}.$$

If $n < k$, then according to (15), $y_n^{(n)} - y_n^{(k)} \in \ell^n M_n$ and since $\ell^n M_n = \{0\}$, $y_n^{(n)} = y_n^{(k)}$, so in particular $y_n^{(n)} \equiv y_n^{(k)} \pmod{\ell^k M_k}$. If $n \geq k$, still according to (15), $y_n^{(n)} - y_n^{(k)} \in \ell^k M_k$. So in any case $(x - y^{(k)})_n \in \ell^k M_n$. But since the groups M_n are all finite, according to the Mittag-Leffler condition,

$$\ell^k \varprojlim_{n \geq 1} M_n \simeq \varprojlim_{n \geq 1} \ell^k M_n.$$

Hence $x - y^{(k)} \in \ell^k M$ which concludes the proof. □

Lemma 5.6. *Let M be a complete and separated \mathbb{Z}_ℓ -module. Then M is finitely generated if and only if $M/\ell M$ is finite.*

Proof. First, if M is a finitely generated \mathbb{Z}_ℓ -module, $M/\ell M$ is a finitely generated $\mathbb{Z}/\ell\mathbb{Z}$ -module, hence is finite.

Conversely, if $M/\ell M$ is generated by some elements m_1, \dots, m_N from M , we show by induction on n that for each $n \geq 0$, $\ell^n M/(\ell^{n+1} M)$ is generated by $\ell^n(m_1, \dots, m_N)$. Indeed, this is true by hypothesis for $n = 0$. Now, if $n > 0$, and $x \in \ell^n M$, say $x = \sum_{i=1}^N \ell^n x_i$, then $x = \ell \sum_{i=1}^N \ell^{n-1} x_i$ and by induction hypothesis, there exists $y \in \ell^{n-1}(m_1 \dots m_N)$ such that $\sum_{i=1}^N \ell^{n-1} x_i \equiv y \pmod{\ell^n M}$. Hence $\ell y \in \ell^n(m_1 \dots m_N)$ and $\ell y \equiv \sum_{i=1}^N \ell^n x_i \pmod{\ell^{n+1} M}$.

Hence if $x \in M$, one can inductively define a sequence $(x_n)_{n \geq 0}$ such that $x_n \in (m_1 \dots m_N)$, $x_n = x \pmod{\ell^n M}$ and $x_{n+1} - x_n \in \ell^n(m_1 \dots m_N)$. Hence in $\mathbb{Z}_\ell(m_1 \dots m_N)$, (x_n) has a limit which is x . □

Proposition 5.7. *The groups $H_c^q(S, \mathbb{Z}_\ell)$ are finitely generated \mathbb{Z}_ℓ -modules. Hence, $H_c^q(S, \mathbb{Q}_\ell)$ is a finitely generated vector space for all q , and $H_c^q(S, \mathbb{Q}_\ell) = \{0\}$ for $q > 2d$, where d is the dimension of \mathcal{X}^{an} .*

Proof. According to Propositions 4.2 and 5.5, $H_c^q(S, \mathbb{Z}_\ell)$ is a complete \mathbb{Z}_ℓ -module. So according to Lemma 5.6, it only remains to prove that $H_c^q(S, \mathbb{Z}_\ell)/\ell H_c^q(S, \mathbb{Z}_\ell)$ is finite. Let us prove this.

For each $n \geq 0$ we have the exact sequence of groups

$$0 \rightarrow \mathbb{Z}/\ell^n \mathbb{Z} \xrightarrow{\mu_n} \mathbb{Z}/\ell^{n+1} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/\ell \mathbb{Z} \rightarrow 0$$

where

$$\begin{aligned} \mu_n : \mathbb{Z}/\ell^n \mathbb{Z} &\rightarrow \mathbb{Z}/\ell^{n+1} \mathbb{Z} \\ x \pmod{\ell^n} &\mapsto \ell x \pmod{\ell^{n+1}} \end{aligned}$$

and π is the reduction map. If we take the long exact sequence in cohomology associated to this, we get the long exact sequence of projective systems:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & H_c^i(S, \mathbb{Z}/\ell^n \mathbb{Z}) & \xrightarrow{\mu_n} & H_c^i(S, \mathbb{Z}/\ell^{n+1} \mathbb{Z}) & \longrightarrow & H_c^i(S, \mathbb{Z}/\ell \mathbb{Z}) & \longrightarrow & H_c^{i+1}(S, \mathbb{Z}/\ell^n \mathbb{Z}) & \longrightarrow & \cdots \\
 & & \uparrow \alpha & & \uparrow \beta & & \parallel & & \uparrow & & \\
 \cdots & \longrightarrow & H_c^i(S, \mathbb{Z}/\ell^{n+1} \mathbb{Z}) & \xrightarrow{\mu_{n+1}} & H_c^i(S, \mathbb{Z}/\ell^{n+2} \mathbb{Z}) & \longrightarrow & H_c^i(S, \mathbb{Z}/\ell \mathbb{Z}) & \longrightarrow & H_c^{i+1}(S, \mathbb{Z}/\ell^{n+1} \mathbb{Z}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & &
 \end{array}$$

where the first two vertical arrows α and β correspond to the natural projections. In addition, one checks that the composite $\mu_n \circ \alpha$ is just multiplication by ℓ . By the previous section, the groups are all finite, so the functor $\varprojlim_{n \geq 0}$ (this is a particular case of the Mittag-Leffler condition). So, once we apply the functor $\varprojlim_{n \geq 0}$, we obtain the long exact sequence:

$$\cdots \longrightarrow H_c^i(S, \mathbb{Z}_\ell) \xrightarrow{\times \ell} H_c^i(S, \mathbb{Z}_\ell) \longrightarrow H_c^i(S, \mathbb{Z}/\ell \mathbb{Z}) \longrightarrow H_c^{i+1}(S, \mathbb{Z}_\ell) \longrightarrow \cdots$$

In particular, this implies that we have an injection:

$$0 \rightarrow H_c^q(S, \mathbb{Z}_\ell) / (\ell \cdot H_c^q(S, \mathbb{Z}_\ell)) \rightarrow H_c^q(S, \mathbb{Z}/\ell \mathbb{Z})$$

and since $H_c^q(S, \mathbb{Z}/\ell \mathbb{Z})$ is finite, we conclude that $H_c^q(S, \mathbb{Z}_\ell) / (\ell \cdot H_c^q(S, \mathbb{Z}_\ell))$ is finite. □

Using again the exactness of \varprojlim when all the groups are finite, and the long exact sequence (6), when S is a locally closed semi-algebraic subset, $V \subseteq S$ is a semi-algebraic subset which is open in S and $F = S \setminus V$, then we get a long exact sequence

$$\cdots \rightarrow H_c^i(V, \mathbb{Q}_\ell) \rightarrow H_c^i(S, \mathbb{Q}_\ell) \rightarrow H_c^i(F, \mathbb{Q}_\ell) \rightarrow \cdots$$

5.4. Künneth formula

Definition 5.8. Let Λ be a ring. A complex M^\bullet of Λ -modules is called *strictly perfect* if it is bounded, and for all n , M^n is a finitely generated projective Λ -module.

Proposition 5.9. *Let*

$$\begin{array}{ccc}
 (X \times_S Y, R \times_S T) = (X, R) \times_S (Y, T) & & \\
 \swarrow g' & \downarrow h & \searrow f' \\
 (X, R) & & (Y, T) \\
 \searrow f & & \swarrow g \\
 & S &
 \end{array}$$

be a cartesian square of k -germs, where R (resp. T) is locally closed in X (resp. in Y), X, Y and S being some Hausdorff k -analytic spaces. Let $\mathcal{F} \in \mathcal{D}^-(X, \mathbb{Z}/\ell^n\mathbb{Z})$ and $\mathcal{G} \in \mathcal{D}^-(Y, \mathbb{Z}/\ell^n\mathbb{Z})$. Then there is a canonical isomorphism:

$$\mathbf{R}f_! \mathcal{F} \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}}^L \mathbf{R}g_! \mathcal{G} \simeq \mathbf{R}h_! \left((g'^* \mathcal{F}) \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}}^L (f'^* \mathcal{G}) \right)$$

Proof. Since R is locally closed, $R = U \cap F$ where U is an open subset of X , and F a closed subset, so R is closed in U , and since the inclusion $(U, R) \rightarrow (X, R)$ is a quasi-immersion, replacing X by U , we can assume that R is closed in X . Let us set $U := X \setminus R$ the complementary open subset.

In a first step, let us assume that $T = Y$, that is to say that $(Y, T) = (Y, Y) \simeq Y$. Remark that (X, U) and U are isomorphic as k - \mathcal{G} erms. We then consider the following three cartesian diagrams:

$$\begin{array}{ccc}
 (X, R) \times_S Y & & \\
 \swarrow g' & \downarrow h & \searrow f' \\
 (X, R) & & Y \\
 \searrow f & & \swarrow g \\
 & S &
 \end{array}$$

$$\begin{array}{ccc}
 U \times_S Y & & \\
 \swarrow g'_U & \downarrow h_U & \searrow f'_U \\
 U & & Y \\
 \searrow f_U & & \swarrow g_U \\
 & S &
 \end{array}$$

$$\begin{array}{ccc}
 X \times_S Y & & \\
 \swarrow g'_X & \downarrow h_X & \searrow f'_X \\
 X & & Y \\
 \searrow f_X & & \swarrow g_X \\
 & S &
 \end{array}$$

We then obtain a commutative diagram of distinguished triangles:

$$\begin{array}{ccc}
 (\mathbf{R}f_{U!}(\mathcal{F}_U)) \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}}^L (\mathbf{R}g_!\mathcal{G}) & \xrightarrow{1} & \mathbf{R}h_{U!} \left(g_U'^*(\mathcal{F}_U) \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}}^L (f_U'^*(\mathcal{G})) \right) \\
 \downarrow & & \downarrow \\
 (\mathbf{R}f_{X!}(\mathcal{F})) \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}}^L (\mathbf{R}g_!\mathcal{G}) & \xrightarrow{2} & \mathbf{R}h_{X!} \left(g_X'^*\mathcal{F} \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}}^L (f_X'^*(\mathcal{G})) \right) \\
 \downarrow & & \downarrow \\
 (\mathbf{R}f_!(\mathcal{F}_{|(X,R)})) \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}}^L (\mathbf{R}g_!\mathcal{G}) & \xrightarrow{3} & \mathbf{R}h_! \left(g'^*(\mathcal{F}_{|(X,R)}) \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}}^L (f'^*(\mathcal{G})) \right) \\
 \downarrow [-1] & & \downarrow [-1]
 \end{array}$$

According to [2, 7.7.3] the arrows 1 and 2 are isomorphisms. So 3 (which is constructed in the same way as in *loc.cit.*) is also an isomorphism.

Next, if (Y, T) is a locally closed k -germ, as above we can assume that T is closed in Y , so that if we set $V := Y \setminus T$, V is an open subset of Y and $V \rightarrow (Y, V)$ is a quasi-immersion. Hence according to the first step, the proposition holds for the k -germs (Y, V) and Y , so using again the distinguished triangle associated to (Y, V) and (Y, T) , we can conclude. \square

Exactly in the same way, we can generalize [2, 5.3.10] to k -germs:

Proposition 5.10. *Let $\varphi : Y \rightarrow X$ be a Hausdorff morphism of finite dimension, $\mathcal{G} \in D^b(Y, \mathbb{Z}/\ell^n\mathbb{Z})$ of finite Tor-dimension, and $\mathcal{F} \in D(X, \mathbb{Z}/\ell^n\mathbb{Z})$. Let T be a locally closed subspace of Y , and let us set $f = \varphi|_{(Y,T)}$. Then $\mathbf{R}f_!(\mathcal{G}_{|(Y,T)})$ is also of finite Tor-dimension, and there is a canonical isomorphism*

$$\mathcal{F} \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}}^L \mathbf{R}f_!(\mathcal{G}_{|(Y,T)}) \simeq \mathbf{R}f_!(f^*(\mathcal{F}) \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}}^L \mathcal{G}_{|(Y,T)}).$$

We now apply Proposition 5.9 to the following situation: we assume that $S = \mathcal{M}(k)$ and we consider the constant sheaves

$$F = \underline{\mathbb{Z}/\ell^n\mathbb{Z}}.$$

In that case $\mathbf{R}f_! = \mathbf{R}\Gamma_c$, and we have the following isomorphism in $\mathcal{D}^-(\mathbb{Z}/\ell^n\mathbb{Z} - \text{Mod})$:

$$\mathbf{R}\Gamma_c((X, R), \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}}^L \mathbf{R}\Gamma_c((Y, T), \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) \simeq \mathbf{R}\Gamma_c((X \times Y, R \times T), \underline{\mathbb{Z}/\ell^n\mathbb{Z}}). \tag{16}$$

Our goal is now to pass from $\mathbb{Z}/\ell^n\mathbb{Z}$ coefficients to \mathbb{Q}_ℓ coefficients which is achieved in Proposition 5.13. The following arguments are a rewriting of the exposition of the ℓ -adic Künneth formula for étale cohomology of schemes made in [13, VI 8].

Using Proposition 5.10 with $\mathcal{F} = \mathbb{Z}/\ell^{n-1}\mathbb{Z}$ and $\mathcal{G} = \mathbb{Z}/\ell^n\mathbb{Z}$ yields the following isomorphism in $D^-(\mathbb{Z}/\ell^{n-1}\mathbb{Z} - \text{Mod})$:

$$\mathbf{R}\Gamma_c((X, R), \mathbb{Z}/\ell^n\mathbb{Z}) \underset{\mathbb{Z}/\ell^n\mathbb{Z}}{\overset{L}{\otimes}} \mathbb{Z}/\ell^{n-1}\mathbb{Z} \simeq \mathbf{R}\Gamma_c((X, R), \mathbb{Z}/\ell^{n-1}\mathbb{Z}) \quad (17)$$

In what follows, we will work with complexes M^\bullet of \mathbb{Z}_ℓ (resp. $\mathbb{Z}/\ell^n\mathbb{Z}$) modules. According to the context, we will either see M^\bullet as a complex of modules, or as its image in the derived category $\mathcal{D}(\mathbb{Z}_\ell - \text{Mod})$ (resp. $\mathcal{D}(\mathbb{Z}/\ell^n\mathbb{Z} - \text{Mod})$). For instance when we will consider projective limits $\varprojlim_n M_n^\bullet$, this will always mean that the M_n^\bullet 's are complexes of $\mathbb{Z}/\ell^n\mathbb{Z}$ -modules. In the same way, if M^\bullet and N^\bullet are complexes of \mathbb{Z}_ℓ -modules, $M^\bullet \otimes_{\mathbb{Z}_\ell} N^\bullet$ will denote the total tensor product of complexes of \mathbb{Z}_ℓ -modules, whereas $M^\bullet \overset{L}{\otimes}_{\mathbb{Z}_\ell} N^\bullet$ will denote the total tensor product of M^\bullet and N^\bullet seen as objects of the derived category.

Now we need the following lemma:

Lemma 5.11. *For each $n \geq 1$, let A_n^\bullet and B_n^\bullet be strictly perfect complexes of $\mathbb{Z}/\ell^n\mathbb{Z}$ -modules, and for each $n \geq 2$ let $\varphi_n : A_n^\bullet \rightarrow A_{n-1}^\bullet$ (resp. $\psi_n : B_n^\bullet \rightarrow B_{n-1}^\bullet$) be a morphism of complex of $\mathbb{Z}/\ell^n\mathbb{Z}$ -modules, such that the canonical morphism $A_n^\bullet \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}} \mathbb{Z}/\ell^{n-1}\mathbb{Z} \rightarrow A_{n-1}^\bullet$ (resp. $B_n^\bullet \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}} \mathbb{Z}/\ell^{n-1}\mathbb{Z} \rightarrow B_{n-1}^\bullet$) is a quasi-isomorphism. Then there is a canonical isomorphism in $\mathcal{D}(\mathbb{Z}_\ell - \text{Mod})$:*

$$\varprojlim_{n \geq 1} \left(A_n^\bullet \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}} B_n^\bullet \right) \simeq \left(\varprojlim_{n \geq 1} A_n^\bullet \right) \overset{L}{\otimes}_{\mathbb{Z}_\ell} \left(\varprojlim_{n \geq 1} B_n^\bullet \right)$$

Proof. According to [7, I 12.5], there exists a strictly perfect complex A^\bullet of \mathbb{Z}_ℓ -modules and for each n a quasi-isomorphism $\alpha_n : A^\bullet/(\ell^n A^\bullet) \rightarrow A_n^\bullet$ such that for all n the following diagram commutes up to homotopy:

$$\begin{CD} A^\bullet/(\ell^n A^\bullet) @>\alpha_n>> A_n^\bullet \\ @VredVV @VV\varphi_nV \\ A^\bullet/(\ell^{n-1} A^\bullet) @>\alpha_{n-1}>> A_{n-1}^\bullet \end{CD} \quad (18)$$

and likewise there exists a strictly perfect complex of \mathbb{Z}_ℓ -modules B^\bullet and some quasi-isomorphisms $\beta_n : B^\bullet/(\ell^n B^\bullet) \rightarrow B_n^\bullet$ such that the following diagram commutes up to homotopy:

$$\begin{CD} B^\bullet/(\ell^n B^\bullet) @>\beta_n>> B_n^\bullet \\ @VredVV @VV\psi_nV \\ B^\bullet/(\ell^{n-1} B^\bullet) @>\beta_{n-1}>> B_{n-1}^\bullet \end{CD} \quad (19)$$

Remind that if M is a \mathbb{Z}_ℓ -module of finite type, there is a functorial isomorphism:

$$M \xrightarrow{\sim} \varprojlim_{n \geq 1} M/(\ell^n M) \quad (20)$$

We then obtain the following quasi-isomorphisms:

$$A^\bullet \xrightarrow{\sim} \varprojlim_n (A^\bullet / (\ell^n A^\bullet)) \xrightarrow{\sim} \varprojlim_n (A_n^\bullet), \tag{21}$$

where the first arrow is an isomorphism of complexes according to (20) and the second arrow is a quasi-isomorphism according to Mittag–Leffler condition and the fact that all the modules involved are of finite type. The similar results holds for B^\bullet .

We then obtain the following sequence of isomorphisms in $\mathcal{D}(\mathbb{Z}_\ell - \text{Mod})$:

$$\left(\varprojlim_n A_n^\bullet\right) \otimes_{\mathbb{Z}_\ell}^L \left(\varprojlim_n B_n^\bullet\right) \xleftarrow{\sim} A^\bullet \otimes_{\mathbb{Z}_\ell}^L B^\bullet \tag{22}$$

$$\simeq A^\bullet \otimes_{\mathbb{Z}_\ell} B^\bullet \tag{23}$$

$$\xrightarrow{\sim} \varprojlim_n \left((A^\bullet \otimes_{\mathbb{Z}_\ell} B^\bullet) / (\ell^n (A^\bullet \otimes_{\mathbb{Z}_\ell} B^\bullet)) \right) \tag{24}$$

$$\xrightarrow{\sim} \varprojlim_n \left((A^\bullet / (\ell^n A^\bullet)) \otimes_{\mathbb{Z}/\ell^n \mathbb{Z}} (B^\bullet / (\ell^n B^\bullet)) \right) \tag{25}$$

$$\xrightarrow{\sim} \varprojlim_n \left(A_n^\bullet \otimes_{\mathbb{Z}/\ell^n \mathbb{Z}} (B^\bullet / (\ell^n B^\bullet)) \right) \tag{26}$$

$$\xleftarrow{\sim} \varprojlim_n (A_n^\bullet \otimes_{\mathbb{Z}/\ell^n \mathbb{Z}} B_n^\bullet). \tag{27}$$

The isomorphism (22) holds thanks to (21), (23) holds because A^\bullet and B^\bullet are flat, (24) is remark (20), (25) is base change for tensor product.

Finally to obtain (26) we take the tensor product of the first (resp. second) line of diagram (18) with $B^\bullet / (\ell^n B^\bullet)$ (resp. $B^\bullet / (\ell^{n-1} B^\bullet)$). The resulting diagram still commutes up to homotopy and since $B^\bullet / (\ell^n B^\bullet)$ is a projective complex, the horizontal lines are still quasi-isomorphisms. Hence (thanks to Mittag–Leffler condition), we obtain (26).

Similarly, for (27), we take the tensor product of the first (resp. second) line of diagram (19) with A_n^\bullet (resp. A_{n-1}^\bullet). Since A_n^\bullet is a projective complex, the horizontal lines remain quasi-isomorphisms and we can conclude with the same argument. □

Remark 5.12. Note that we have implicitly used the following result: if M_1^\bullet, M_2^\bullet and M_3^\bullet are bounded above complexes of Λ -modules such that M_3^\bullet is projective, and $f : M_1^\bullet \rightarrow M_2^\bullet$ is a quasi-isomorphism, then $f \otimes id : M_1^\bullet \otimes M_3^\bullet \rightarrow M_2^\bullet \otimes M_3^\bullet$ is a quasi-isomorphism [15, 10.6.2]

Proposition 5.13. *Let $(X, R), (Y, T)$ be k -germs such that for all $(n, \mathbf{R}\Gamma_c((X, R), \mathbb{Z}/\ell^n \mathbb{Z})$ and $\mathbf{R}\Gamma_c((Y, T), \mathbb{Z}/\ell^n \mathbb{Z})$ have finite cohomology groups. Then the cohomology groups of $\mathbf{R}\Gamma_c((X, R) \times (Y, T), \mathbb{Z}/\ell^n \mathbb{Z})$ are also finite and for all $r \geq 0$ we have a canonical isomorphism:*

$$H_c^r((X \times Y, R \times T), \mathbb{Q}_\ell) \simeq \bigoplus_{p+q=r} H_c^p((X, R), \mathbb{Q}_\ell) \otimes H_c^q((Y, T), \mathbb{Q}_\ell).$$

Proof. The complexes $\mathbf{R}\Gamma_c((X, R), \mathbb{Z}/\ell^n\mathbb{Z})$ and $\mathbf{R}\Gamma_c((Y, T), \mathbb{Z}/\ell^n\mathbb{Z})$ have bounded cohomology groups, are of finite type by hypothesis, and according to Proposition 5.10 are of finite Tor-dimension, so we can choose some resolutions by some strictly perfect complexes of the projective systems:

$$\begin{aligned} K_n^\bullet &\rightarrow \mathbf{R}\Gamma_c((X, R), \mathbb{Z}/\ell^n\mathbb{Z}) \\ P_n^\bullet &\rightarrow \mathbf{R}\Gamma_c((Y, T), \mathbb{Z}/\ell^n\mathbb{Z}) \\ Q_n^\bullet &\rightarrow \mathbf{R}\Gamma_c((X, R) \times (Y, T), \mathbb{Z}/\ell^n\mathbb{Z}). \end{aligned}$$

In addition, according to (16) we can find up to homotopy a quasi-isomorphism, of projective systems:

$$K_n^\bullet \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}} P_n^\bullet \simeq Q_n^\bullet. \tag{28}$$

Moreover according to (17), K_n^\bullet and P_n^\bullet fulfil the hypothesis of Lemma 5.11. We then denote by

$$\begin{aligned} K^\bullet &= \varprojlim_n (K_n^\bullet) \\ P^\bullet &= \varprojlim_n (P_n^\bullet) \\ Q^\bullet &= \varprojlim_n (Q_n^\bullet). \end{aligned}$$

Remark that (thanks to Mittag–Leffler property again)

$$H^p(K^\bullet) \simeq H_c^p((X, R), \mathbb{Z}_\ell) \tag{29}$$

$$H^p(P^\bullet) \simeq H_c^p((Y, T), \mathbb{Z}_\ell) \tag{30}$$

$$H^p(Q^\bullet) \simeq H_c^p((X \times Y, R \times T), \mathbb{Z}_\ell). \tag{31}$$

In $\mathcal{D}(\mathbb{Z}_\ell - \text{Mod})$ we consider the following sequence of isomorphisms:

$$K^\bullet \otimes_{\mathbb{Z}_\ell}^L P^\bullet \simeq \varprojlim_n (K_n^\bullet) \otimes_{\mathbb{Z}_\ell}^L \varprojlim_n (P_n^\bullet) \tag{32}$$

$$\simeq \varprojlim_n (K_n^\bullet \otimes_{\mathbb{Z}/\ell^n\mathbb{Z}} P_n^\bullet) \tag{33}$$

$$\simeq \varprojlim_n (Q_n^\bullet) \tag{34}$$

$$\simeq Q^\bullet. \tag{35}$$

The isomorphism (32) holds by definition of K^\bullet and P^\bullet , (33) holds thanks to Lemma 5.11, (34) is just a consequence of (28), and (35) holds by definition of Q^\bullet .

We then obtain the following isomorphisms in $\mathcal{D}(\mathbb{Q}_\ell - \text{Mod})$:

$$\left(K^\bullet \otimes_{\mathbb{Z}_\ell}^L \mathbb{Q}_\ell \right) \otimes_{\mathbb{Q}_\ell}^L \left(P^\bullet \otimes_{\mathbb{Z}_\ell}^L \mathbb{Q}_\ell \right) \simeq \left(K^\bullet \otimes_{\mathbb{Z}_\ell}^L P^\bullet \right) \otimes_{\mathbb{Z}_\ell}^L \mathbb{Q}_\ell \simeq Q^\bullet \otimes_{\mathbb{Z}_\ell}^L \mathbb{Q}_\ell$$

But since \mathbb{Q}_ℓ is flat over \mathbb{Z}_ℓ , we can replace all the $\overset{L}{\otimes}_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ by some $\otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Finally, since \mathbb{Q}_ℓ is a field,

$$H^r \left((K^\bullet \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} (P^\bullet \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \right) \simeq \bigoplus_{p+q=r} H^p(K^\bullet \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} H^q(P^\bullet \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell).$$

The result then follows from the isomorphisms (29)–(31). □

We must mention that the Proposition 5.9 is functorial in S . So let $(\mathcal{X}, R), (\mathcal{Y}, T)$ be k -germs, with \mathcal{X} (resp. \mathcal{Y}) a separated \mathcal{A} -scheme (resp. \mathcal{B} -scheme) of finite type, \mathcal{A} (resp. \mathcal{B}) being some k -affinoid algebra and R (resp. T) being some locally closed semi-algebraic subset of \mathcal{X}^{an} (resp. \mathcal{Y}^{an}). Then, there are isomorphisms of Galois modules:

$$\bigoplus_{p+q=r} H_c^p \left((\overline{\mathcal{X}^{an}}, \overline{R}), \mathbb{Q}_\ell \right) \otimes H_c^q \left((\overline{\mathcal{Y}^{an}}, \overline{T}), \mathbb{Q}_\ell \right) \simeq H_c^r \left((\overline{(\mathcal{X} \times \mathcal{Y})^{an}}, \overline{R \times T}), \mathbb{Q}_\ell \right).$$

5.5. Statement of the main theorem

We sum up all results of this section:

Theorem 5.14. *Let k be a non-Archimedean complete valued field, \mathcal{A} a k -affinoid algebra, \mathcal{X} a separated \mathcal{A} -scheme of finite type of dimension d , U a locally closed semi-algebraic subset of \mathcal{X}^{an} , and $\ell \neq \text{char}(k)$ be a prime number. We denote by $\pi : \mathcal{X}^{an} \rightarrow \mathcal{X}^{an}$ the morphism defined in 5.1 and we set $\overline{U} = \pi^{-1}(U)$.*

1. *The groups $H_c^i(\overline{U}, \mathbb{Q}_\ell)$ are finite dimensional \mathbb{Q}_ℓ -vector spaces, endowed with a continuous $\text{Gal}(k^{sep}/k)$ -action, and $H_c^i(\overline{U}, \mathbb{Q}_\ell) = 0$ for $i > 2d$.*
2. *Let $V \subset U$ be a semi-algebraic subset which is open in U , and let $F = U \setminus V$. Then there is a $\text{Gal}(k^{sep}/k)$ -equivariant long exact sequence*

$$\cdots \rightarrow H_c^i(\overline{V}, \mathbb{Q}_\ell) \longrightarrow H_c^i(\overline{U}, \mathbb{Q}_\ell) \longrightarrow H_c^i(\overline{F}, \mathbb{Q}_\ell) \longrightarrow H_c^{i+1}(\overline{V}, \mathbb{Q}_\ell) \cdots \rightarrow$$

3. *For all integer n there are canonical $\text{Gal}(k^{sep}/k)$ -equivariant isomorphisms:*

$$\bigoplus_{i+j=n} H_c^i(\overline{U}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} H_c^j(\overline{V}, \mathbb{Q}_\ell) \simeq H_c^n(\overline{U \times V}, \mathbb{Q}_\ell).$$

6. Analogous statements for adic spaces

In this section, k will be a non-Archimedean algebraically closed non-trivially valued field, Λ a finite group prime to the characteristic of k and \mathcal{A} a strictly k -affinoid algebra.

We want to stress out that in the previous sections, instead of working with the étale cohomology developed by Berkovich in [2], we could also have used the theory of adic spaces and its étale cohomology theory, developed by Huber in [9].

This might be interesting because this will define different groups (cf. Remark 6.1) and will apply for different semianalytic (resp. semi-algebraic) subsets (cf. Remark 6.7). To avoid confusion, we will denote by $H_{c,ad}^q(X, \Lambda)$ the groups defined by the cohomology with compact support of adic spaces.

In this framework, the analogue of a k -germ (X, S) is the notion of a pseudo-adic space (X, S) over $\text{Spa}(k, k^\circ)$ [ibid. 1.10.3]. The quasi-immersions will be replaced by locally closed embeddings [ibid. 1.10.8 (ii)], and the analogue of [2, 4.3.4] which states that cohomology is invariant by quasi-immersion is [9, 2.3.8] which states the same thing for locally closed embeddings. In Huber’s theory though, compactly supported cohomology isn’t defined as a derived functor, but with some compactification, like in the étale cohomology of schemes. Nonetheless, one can check that if $i : (X, S) \rightarrow (Y, T)$ is a locally closed embedding with $i(S) = T$, then $H_{c,ad}^q((X, S), i^*(\mathcal{F})) \simeq H_{c,ad}^q((Y, T), \mathcal{F})$. Indeed, in this case, $i_! = i_*$ is an exact functor (it induces an equivalence of categories), so $R^+i_! = i_! = i_*$ [ibid. 5.4.1]. So $R^+i_!(i^*\mathcal{F}) \simeq \mathcal{F}$, from what it follows that $H_{c,ad}^q((X, S), i^*(\mathcal{F})) \simeq H_{c,ad}^q((Y, T), \mathcal{F})$.

Remark 6.1. One has to keep in mind that compactly supported cohomology does not give the same groups in both theories, for instance if X is the closed disc of radius one:

i	0	1	2
$H_{c,Ber}^i(X, \Lambda)$	Λ	0	0
$H_{c,ad}^i(X, \Lambda)$	0	0	Λ

In Sect. 3, we systematically used the long exact sequence

$$\dots H_c^{q-1}(R, \Lambda) \rightarrow H_c^q(T, \Lambda) \rightarrow H_c^q(S, \Lambda) \rightarrow H_c^q(R, \Lambda) \rightarrow \dots$$

where

$$T = \{x \in S \mid |f(x)| < |g(x)|\} \tag{36}$$

and $R = S \setminus T$. Although the closed-open long exact sequence is still valid for pseudo-adic spaces [9, 5.5.11 (iv)], T as defined in (36) is not an open subset of S any more, so we cannot apply this long exact sequence. In fact the typical example of an open subset of an adic space is

$$T = \{x \in S \mid |f(x)| \leq |g(x)| \neq 0\}. \tag{37}$$

It will be then possible in that case to apply this long exact sequence (which includes the case $\{f \neq 0\} = \{0 \leq |f| \neq 0\}$).

Remark 6.2. A subset S is a finite Boolean combination of subsets of the form $\{|f| \leq |g| \neq 0\}$ if and only if it is a finite Boolean combination of subsets of the form $\{|f| \leq |g|\}$.

For instance, $\{|f| \leq |g|\} = \{|f| \leq |g| \neq 0\} \cup (\{g \neq 0\} \cup \{f \neq 0\})^c$.

Let \mathcal{A} be a (strictly) k -affinoid algebra. We will say that a subset $S \subset \text{Spa}(\mathcal{A}, \mathcal{A}^\circ)$ is semianalytic if it is a Boolean combination of subsets of the form

$$\{x \in \text{Spa}(\mathcal{A}, \mathcal{A}^\circ) \mid |f(x)| \leq |g(x)|\}$$

where $f, g \in \mathcal{A}$. This definition slightly differs from the one given for Berkovich spaces: here we do not allow real constants in the inequalities.

Lemma 6.3. *Let $X = \text{Spa}(\mathcal{A}, \mathcal{A}^\circ)$ be the affinoid adic space associated to \mathcal{A} , $S = \bigcap_{i=1}^n S_i$ where for each i , S_i is of the form $S_i = \{x \in X \mid |f_i(x)| \leq |g_i(x)| \neq 0\}$ or $S_i = \{x \in X \mid |f_i(x)| > |g_i(x)| \text{ or } g_i(x) = 0\}$, with $f_i, g_i \in \mathcal{A}$. Then the groups $H_{c,ad}^q(S, \Lambda)$ are finite.*

Proof. Mimic the proof of Lemma 3.1 using that

$$\{x \in X \mid |f_i(x)| \leq |g_i(x)| \neq 0\}^c = \{x \in X \mid |f_i(x)| > |g_i(x)| \text{ or } g_i(x) = 0\}.$$

The key point here (that makes possible the base case of the induction) is that for an affinoid adic space Y , the groups $H_{c,ad}^q(Y, \Lambda)$ are finite [10, 5.1]. \square

Proposition 6.4. *Let T be a locally closed, semianalytic subset of $X = \text{Spa}(\mathcal{A}, \mathcal{A}^\circ)$. Then the groups $H_{c,ad}^q(T, \Lambda)$ are finite.*

Proof. According to Remark 6.2, we can assume that T is a finite union of subsets S as in Lemma 6.3. Hence we can adapt the proof of Lemma 3.2. \square

In this context, if X is a quasi-separated adic space of finite type over k , we will say that S is *locally semianalytic* if there exists a finite affinoid covering $\{U_i\}$ of X such that $S \cap U_i$ is semianalytic in U_i for all i . Adapting the proofs of Proposition 4.1, we obtain:

Proposition 6.5. *Let X be a quasi-separated adic space of finite type over k , and S a locally closed, locally semianalytic subset of X . Then the groups $H_{c,ad}^q(S, \Lambda)$ are finite.*

We can define similarly semi-algebraic subsets $S \subset \mathcal{X}^{ad}$ where \mathcal{X} is an \mathcal{A} -scheme of finite type, like in Definition 2.4, but without the real constants λ . We then obtain:

Proposition 6.6. *Let \mathcal{X} be a separated \mathcal{A} -scheme of finite type, S a locally closed semi-algebraic subset of \mathcal{X}^{ad} . Then the groups $H_{c,ad}^q(S, \Lambda)$ are finite.*

Remark 6.7. As indicated above, if X is a k -analytic (resp. adic) affinoid space, the class of locally closed subspaces will be different according to the theories. To illustrate this we want to give two examples. Let us consider X the closed bidisc of radius 1: $X = \mathcal{M}(k\{x, y\})$ or $\text{Spa}(k\{x, y\}, k^\circ\{x, y\})$ according to theory we are using. Remind that a subset U is locally closed if and only if U is open in \bar{U} .

A subset which is locally closed for the topology of adic spaces but not for the Berkovich topology. Let $V = \{p \in X \mid |x(p)| > |y(p)|\} \cup \{p_0\}$. Here p_0 is the rigid

point corresponding to the origin. Then V is closed in the adic topology. Indeed its complement is

$$V^c = \{p \in X \setminus \{p_0\} \mid |x(p)| \leq |y(p)|\} = \{p \in X \mid |x(p)| \leq |y(p)| \neq 0\}$$

which is open by definition of the topology of adic spaces. But we claim that V is not locally closed for the Berkovich topology. To show this, for $r, s \leq 1$ let $\eta_{r,s} \in X$ be defined by

$$\eta_{r,s} \left(\sum_{i,j \in \mathbb{N}} a_{i,j} x^i y^j \right) = \max_{i,j \in \mathbb{N}} |a_{i,j}| r^i s^j.$$

Then for $r > s$, $\eta_{r,s} \in V$, and for $0 < r \leq 1$, $\eta_{r,r} \in \overline{V} \setminus V$. Now, if V was open in \overline{V} , since it contains p_0 , it should contain $\eta_{r,r}$ for r small enough which is a contradiction.

A subset which is locally closed for the Berkovich topology but not for the topology of adic spaces. Let us consider the set $U = \{p \in X \mid |x(p)| \leq |y(p)|\}$. Then U is closed for the Berkovich topology but not locally-closed for the topology of adic spaces. Indeed, if p_0 is the rigid point corresponding to the origin $(0, 0)$, $p_0 \in U$, but U is not a neighbourhood of p_0 in \overline{U} with respect to the topology of adic spaces. Otherwise for some $\varepsilon > 0$, U would contain a subset $B = \{p \in \overline{U} \mid |x(p)| \leq \varepsilon \text{ and } |y(p)| \leq \varepsilon\}$. But then for $0 < \alpha < \varepsilon$ with $\alpha \in |k^\times|$, we can define $\eta_\alpha \in \overline{U}$ a valuation of rank 2 such that $\eta_\alpha(x) = \alpha$ and $\eta_\alpha(y) = \alpha_-$ where $\alpha_- < \alpha$ but is infinitesimally closed. Now, $\eta_\alpha \in \overline{U}$ because $\eta_\alpha \in \overline{\{\eta_{\alpha,\alpha}\}}$ (cf. definition above). So by definition of B , $\eta_\alpha \in B$. So we should have $\eta_\alpha \in U$, which is false. So U is not locally closed for the adic topology.

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