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Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term

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Abstract. The orbital stability of standing waves of nonlinear Schrödinger equations with a general nonlinear term is investigated in this paper. We study the corresponding minimizing problem with L^2-constraint:

$$E_\alpha = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(|u|) dx ; u \in H^1(\mathbb{R}^N), \|u\|^2_{L^2(\mathbb{R}^N)} = \alpha \right\}.$$

We discuss when a minimizing sequence with respect to E_α is precompact. We prove that there exists $\alpha_0 \geq 0$ such that there exists a global minimizer if $\alpha > \alpha_0$ and there exists no global minimizer if $\alpha < \alpha_0$. Moreover, some almost critical conditions which determine $\alpha_0 = 0$ or $\alpha_0 > 0$ are established, and the existence results with respect to E_{α_0} under some conditions are obtained.

1. Introduction and main results

In this paper, we study stability results regarding standing waves of nonlinear Schrödinger equations with general nonlinearity:

$$iu_t + \Delta u + f(u) = 0 \quad \text{if } (t, x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1}$$

where $N \geq 1$. We are interested in existence and orbital stability for standing waves for (1). That is, solutions of (1) of the special form $u(t, x) = e^{i\mu t} v(x)$, where $\mu \in \mathbb{R}$ and $v \in H^1(\mathbb{R}^N)$. For the nonlinear term, we assume the following conditions throughout this paper:

(F1) $f \in C(\mathbb{C}, \mathbb{C}),\ f(0) = 0$.
(F2) $f(r) \in \mathbb{R}$ for $r \in \mathbb{R},\ f(e^{i\theta} z) = e^{i\theta} f(z)$ for $\theta \in \mathbb{R},\ z \in \mathbb{C}$.
(F3) $\lim_{z \to 0} f(z)/|z| = 0$.
(F4) $\lim_{|z| \to \infty} f(z)/|z|^{l-1} = 0$, where $l = 2 + 4/N$.
(F5) There exists $s_0 > 0$ such that $F(s_0) > 0$, where $F(s) = \int_0^s f(\tau) d\tau$ for $s \in \mathbb{R}$.

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Under these conditions, for a solution u of (1), it has been established that the following conservations laws:

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^N)} = \|u(0,\cdot)\|_{L^2(\mathbb{R}^N)}, \quad I[u(t,\cdot)] = I[u(0,\cdot)] \quad \text{for any } t \in \mathbb{R},$$

where I is the energy functional associated with (1) defined by

$$I[u] = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(|u|)dx$$

for any $u \in H^1(\mathbb{R}^N)$. Moreover, we consider the following conditions:

(F6) There exist $K > 0$ and $p \in (2, 2^*)$ such that $|f(z_1) - f(z_2)| \leq K(1 + |z_1| + |z_2|)^{p-2}|z_1 - z_2|$ for $z_1, z_2 \in \mathbb{C}$, where $2^* = 2N/(N-2)_+$.

(F7) There exist $L > 0$ and $q \in (2, l)$ such that $F(|z|) \leq L(|z|^2 + |z|^q)$ for $z \in \mathbb{C}$.

It is recognized that the global well-posedness in $H^1(\mathbb{R}^N)$ about (1) holds under assumptions (F1), (F2), (F6), and (F7). Regarding global well-posedness, see, for example, [5].

If u is a standing wave, i.e., $u(t,x) = e^{i\mu t}v(x)$, then $v \in H^1(\mathbb{R}^N)$ and $\mu \in \mathbb{R}$ satisfy the following equation:

$$\Delta v + f(v) = \mu v \quad \text{if } x \in \mathbb{R}^N. \tag{2}$$

In this paper, we look for solutions (v, μ) with a priori prescribed L^2-norm. More precisely, we consider a constrained variational problem as follows. For a given $\alpha > 0$, we put

$$M_\alpha = \left\{u \in H^1(\mathbb{R}^N);\ \|u\|^2_{L^2(\mathbb{R}^N)} = \alpha\right\}.$$

If v is a critical point of I on M_α, then v is a solution of (2) where μ is determined as the Lagrange multiplier. Since I is bounded below by the assumption $(F4)$, the energy

$$E_\alpha = \inf_{u \in M_\alpha} I[u] \tag{3}$$

is well-defined and the existence of a global minimizer of (3) is expected. We define S_α by the set of all global minimizers, i.e.,

$$S_\alpha = \{u \in M_\alpha;\ I[u] = E_\alpha\}.$$

In this paper, we study the existence and the non-existence of global minimizers of E_α.

This type problem was first studied in the works of Stuart [11,12]. Subsequently, in [4], orbital stability of the set of minimizers, which suppose to establish the compactness of any minimizing sequence, was obtained using the concentration compactness principle [10]. In [4], it is assumed that $E_\alpha < 0$ for all $\alpha > 0$ and that the strict subadditivity condition

$$E_{\alpha+\beta} < E_\alpha + E_\beta \tag{4}$$

holds. This strict inequality was established in the special case of $f(u) = |u|^{p-2}u$ $(2 < p < l)$ in [4]. However, for a general f, it is not clear if (4) hold. Another difficulty is that $E_\alpha < 0$ for all $\alpha > 0$ may not be satisfied. Actually, under the assumptions (F1)–(F5), we show that there exists a $\alpha_0 \geq 0$ uniquely determined by f and N such that

$$E_\alpha = 0 \quad \text{if } 0 \leq \alpha \leq \alpha_0, \quad E_\alpha < 0 \quad \text{if } \alpha > \alpha_0. \tag{5}$$

These last years have seen a renew interest for L^2-constraint minimizing problems, or more generally for constrained minimization problem, see, e.g., [1,2,6–9]. Our work uses in particular some arguments of [7] where a new scaling is introduced to exclude the dichotomy of minimizing sequences on a related problem. Also note that a study of existence and non-existence of minimizers, in the same spirit as the present work, is made in [8] on the Schrödinger-Poisson equation.

Our primarily goal is the following theorem:

Theorem 1.1. *Suppose (F1)–(F5) and that a constant $\alpha_0 \geq 0$ which satisfies (5) is uniquely determined. If $\alpha > \alpha_0$,*

(i) *There exists a global minimizer with respect to E_α, i.e., $S_\alpha \neq \emptyset$.*
(ii) *Under the assumptions (F6)–(F7), S_α is orbitally stable, i.e., for any $\epsilon > 0$, there exists $\delta > 0$ such that for any solution u of (1) with $\mathrm{dist}(u(0,\cdot), S_\alpha) < \delta$, it holds that*

$$\mathrm{dist}(u(t,\cdot), S_\alpha) < \epsilon \quad \textit{for any } t \in \mathbb{R},$$

where $\mathrm{dist}(\phi, S_\alpha) = \inf_{\psi \in S_\alpha} \|\phi - \psi\|_{H^1(\mathbb{R}^N)}$.

If $0 < \alpha < \alpha_0$, there is no global minimizer with respect to E_α.

Theorem 1.1 is proved by the following Theorem 1.2 and the argument presented by Cazenave and Lions [4].

Theorem 1.2. *Suppose (F1)–(F5) and that $\alpha > 0$. If $\{u_n\}_{n\in\mathbb{N}} \subset M_\alpha$ is a minimizing sequence with respect to E_α, then one of the following holds:*

(i)

$$\overline{\lim_{n\to\infty}} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n|^2 dx = 0. \tag{6}$$

(ii) *Taking a subsequence if necessary, there exist $u \in M_\alpha$ and a family $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $u_n(\cdot - y_n) \to u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Specifically, u is a global minimizer.*

It is a natural question that "When $\alpha_0 > 0$ holds". To answer the question, the behavior of f near 0 is important. We can show that the following results:

Theorem 1.3. *Suppose (F1)–(F5).*

(i) *If $\underline{\lim}_{s\to 0} F(s)/s^l = \infty$ holds, then $\alpha_0 = 0$ holds.*
(ii) *If $\overline{\lim}_{s\to 0} F(s)/s^l < \infty$ holds, then $\alpha_0 > 0$ holds.*

In the case $\alpha_0 > 0$, existence of a global minimizer with respect to E_{α_0} is still unknown. Under some conditions, we can obtain existence results as follows.

Theorem 1.4. *Suppose (F1)–(F5).*

(i) *If* $\overline{\lim}_{s\to 0} F(s)/s^l = 0$ *holds, then there exists a global minimizer with respect to* E_{α_0}.
(ii) *There exist positive constants* C *and* s_1 *such that* $F(s) = C|s|^l$ *if* $|s| \le s_1$. *Then, there exists a global minimizer with respect to* E_{α_0}.

Finally, we can state the strict subadditivity condition.

Theorem 1.5. *Suppose (F1)–(F5). Then the strict subadditivity condition holds, i.e., for* $\alpha, \beta > 0$ *with* $\alpha + \beta > \alpha_0$,

$$E_{\alpha+\beta} < E_\alpha + E_\beta$$

holds.

We remark that the condition $\alpha + \beta > \alpha_0$ is necessary because $E_\alpha = E_\beta = E_{\alpha+\beta} = 0$ if $\alpha + \beta \le \alpha_0$.

In Sect. 2, we introduce the framework and provide the appropriate setting for the proof of our main theorem. In Sect. 3, we give the proof of Theorem 1.2. In Sect. 4, we give the proof of Theorem 1.1. In Sect. 5, we give the proof of the remaining theorems. In Sect. 6 (Appendix), we give the proof of some lemmas stated in Sect. 2.

2. Preliminaries

In this paper, $L^p(\Omega)$ $(p \ge 1)$ is the usual Lebesgue space and $H^1(\Omega)$ is the usual Sobolev space on a domain $\Omega \subset \mathbb{R}^N$. We denote the norm of $L^p(\Omega)$ and $H^1(\Omega)$ by $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$, respectively. We consider $H^1(\Omega)$ as a real Hilbert space with the inner product

$$(u, v)_{H^1(\Omega)} = \Re \int_\Omega \nabla u \cdot \overline{\nabla v} + u\overline{v}dx.$$

Under the assumptions (F1)–(F4), it is known that I is continuously differentiable in $H^1(\mathbb{R}^N)$ as follows.

Lemma 2.1. *The energy functional* I *is continuously differentiable in* $H^1(\mathbb{R}^N)$. *Moreover, for* $u, v \in H^1(\mathbb{R}^N)$,

$$I'[u]v = \Re \int_{\mathbb{R}^N} \nabla u \cdot \overline{\nabla v} - f(u)\overline{v}dx$$

holds. If u *is a global minimizer with respect to* E_α, *the following holds:*

$$\Re \int_{\mathbb{R}^N} \nabla u \cdot \overline{\nabla \phi} - f(u)\overline{\phi} + \mu u\overline{\phi}dx = 0 \quad \text{for } \phi \in H^1(\mathbb{R}^N),$$

where μ is a Lagrange multiplier determined by

$$\mu = \frac{-1}{\alpha} \int_{\mathbb{R}^N} |\nabla u|^2 - f(|u|)|u| dx.$$

In particular, u is a solution of (2).

Lemma 2.1 is clear, so we omit the proof.

Lemma 2.2. (i) *Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R}^N)$. If either $\lim_{n\to\infty} \|u_n\|_{L^2(\mathbb{R}^N)} = 0$ or $\lim_{n\to\infty} \|u_n\|_{L^l(\mathbb{R}^N)} = 0$ holds, then it is true that $\lim_{n\to\infty} \int_{\mathbb{R}^N} F(|u_n|)dx = 0$.*

(ii) *There exists a positive constant $C(f, N, \alpha)$ depending f, N and α such that*

$$I[u] \geq \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u|^2 dx - C(f, N, \alpha) \tag{7}$$

holds for any $u \in M_\alpha$. Specifically, $E_\alpha \geq -C(f, N, \alpha) > -\infty$.

Proof. (i): By the assumptions (F1)–(F4), for any $\epsilon > 0$, there exists a positive constant $C(f, \epsilon)$ which depends on ϵ and f such that

$$|F(|u|)| \leq C(f, \epsilon)|u|^2 + \epsilon|u|^l, \quad |F(|u|)| \leq \epsilon|u|^2 + C(f, \epsilon)|u|^l,$$

where $l = 2 + 4/N$. For $u \in H^1(\mathbb{R}^N)$, we have

$$\left| \int_{\mathbb{R}^N} F(|u|)dx \right| \leq C(f, \epsilon)\|u\|^2_{L^2(\mathbb{R}^N)} + \epsilon\|u\|^l_{L^l(\mathbb{R}^N)}, \tag{8}$$

$$\left| \int_{\mathbb{R}^N} F(|u|)dx \right| \leq \epsilon\|u\|^2_{L^2(\mathbb{R}^N)} + C(f, \epsilon)\|u\|^l_{L^l(\mathbb{R}^N)}. \tag{9}$$

The Gagliardo–Nirenberg inequality implies that

$$\|u\|^l_{L^l(\mathbb{R}^N)} \leq C(N)\|\nabla u\|^2_{L^2(\mathbb{R}^N)}\|u\|^{4/N}_{L^2(\mathbb{R}^N)},$$

where $C(N)$ is a positive constant which depends on N. Thus, we obtain

$$\left| \int_{\mathbb{R}^N} F(|u|)dx \right| \leq C(f, \epsilon)\|u\|^2_{L^2(\mathbb{R}^N)} + \epsilon C(N)\|\nabla u\|^2_{L^2(\mathbb{R}^N)}\|u\|^{4/N}_{L^2(\mathbb{R}^N)}. \tag{10}$$

We take the case where $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R}^N)$ satisfying $\lim_{n\to\infty} \|u_n\|_{L^2(\mathbb{R}^N)} = 0$. By (10), we have $\lim_{n\to\infty} \int_{\mathbb{R}^N} F(|u_n|)dx = 0$. Alternatively, we can take the case where $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R}^N)$ satisfying $\lim_{n\to\infty} \|u_n\|_{L^l(\mathbb{R}^N)} = 0$. By (9), we have

$$\varlimsup_{n\to\infty} \left| \int_{\mathbb{R}^N} F(|u_n|)dx \right| \leq \epsilon\|u_n\|^2_{L^2(\mathbb{R}^N)}.$$

Since we can choose $\epsilon > 0$ arbitrary, we obtain $\lim_{n\to\infty} \int_{\mathbb{R}^N} F(|u_n|)dx = 0$.

(ii): In (10), we choose $\epsilon > 0$ satisfying $C(N)\alpha^{2/N}\epsilon = 1/4$. Then, for $u \in M_\alpha$, we have

$$\int_{\mathbb{R}^N} F(|u|)dx \le C(f, N, \alpha) + \frac{1}{4}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} = \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u|^2 dx + C(f, N, \alpha),$$

where $C(f, N, \alpha)$ is a positive constant which depends on f, N and α. This implies (7). □

In relation to the energy E_α, the following lemma holds.

Lemma 2.3. (i) $E_\alpha \le 0$ *for* $\alpha > 0$.
(ii) $E_{\alpha+\beta} \le E_\alpha + E_\beta$ *for* $\alpha, \beta > 0$.
(iii) $\alpha \mapsto E_\alpha$ *is nonincreasing.*
(iv) *For sufficiently large* α, $E_\alpha < 0$ *holds.*
(v) $\alpha \mapsto E_\alpha$ *is continuous.*

We define α_0 by

$$\alpha_0 = \inf\{\alpha > 0;\ E_\alpha < 0\}.$$

By Lemma 2.3, α_0 is well-defined and (5) holds. We state the proof of Lemma 2.3 in the Appendix.

Lemma 2.4. *Let* $\{u_n\}_{n\in\mathbb{N}}$ *be a bounded sequence in* $H^1(\mathbb{R}^N)$ *satisfying* $\lim_{n\to\infty} \|u_n\|^2_{L^2(\mathbb{R}^N)} = \alpha > 0$. *Let* $a_n = \sqrt{\alpha}/\|u_n\|_{L^2(\mathbb{R}^N)}$ *and* $\tilde{u}_n = a_n u_n$. *Then the following holds:*

$$\tilde{u}_n \in M_\alpha, \quad \lim_{n\to\infty} a_n = 1, \quad \lim_{n\to\infty} |I[\tilde{u}_n] - I[u_n]| = 0.$$

Proof. Clearly, $\tilde{u}_n \in M_\alpha$ and $\lim_{n\to\infty} a_n = 1$ hold. We can compute

$$\begin{aligned} I[\tilde{u}_n] - I[u_n] &= \frac{(a_n^2-1)}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \int_{\mathbb{R}^N} F(|a_n u_n|) - F(|u_n|)dx \\ &= \frac{(a_n^2-1)}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \int_{\mathbb{R}^N}\left(\int_0^1 f(|u_n| + (|a_n|-1)\theta|u_n|)(|a_n|-1)|u_n| d\theta\right) dx \\ &= \frac{(a_n^2-1)}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx - (|a_n|-1)\int_{\mathbb{R}^N}\left(\int_0^1 f(|u_n| + (|a_n|-1)\theta|u_n|)|u_n| d\theta\right) dx. \end{aligned}$$

We have $0 \le |u_n| + (|a_n| - 1)\theta|u_n| \le (|a_n| + 2)|u_n|$. Under the assumptions (F1)–(F4), we have $|f(s)| \le |s| + C(f)|s|^{l-1}$. Hence, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} \int_0^1 f(|u_n| + (|a_n| - 1)\theta|u_n|)|u_n| d\theta dx \right| \\
&\le \int_{\mathbb{R}^N} \left(\int_0^1 (|a_n| + 2)|u_n|^2 + C(f)(|a_n| + 2)^{l-1}|u_n|^l d\theta \right) dx \\
&= \int_{\mathbb{R}^N} (|a_n| + 2)|u_n|^2 + C(f)(|a_n| + 2)^{l-1}|u_n|^l dx.
\end{aligned}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, we achieve our conclusion. □

3. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. For our purposes, we will use the concentration-compactness argument. The following Lemma 3.1 is necessary for this argument.

Lemma 3.1. (Lions [10, Lemma I.1]) *Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ which satisfies*

$$\sup_{z\in\mathbb{R}^n} \int_{B(z,1)} |u_n|^2 dx \to 0 \quad \text{as } n \to \infty.$$

Then, for $p \in (2, 2^*)$,

$$\|u_n\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty$$

holds, where $2^* = 2N/(N-2)_+$ is the critical Sobolev exponent.

To prove Theorem 1.2, the new scaling argument introduced in [7] plays important role. By using the scaling argument in [7], similar to [8, Lemma 3.2], we can obtain the following useful lemma.

Lemma 3.2. (i) *Assume that there exists a global minimizer $u \in M_a$ with respect to E_a for some $a > 0$. Then $E_b < E_a$ for any $b > a$. In particular, we have $E_b < 0$ for any $b > a$.*

(ii) *Assume that there exist global minimizers $u \in M_a$ and $v \in M_b$ with respect to E_a and E_b respectively for some $a, b > 0$. Then $E_{a+b} < E_a + E_b$.*

Proof. (i): By Lemma 2.3, we have $I[u] \le 0$. Now setting $\lambda = b/a > 1$ and $\tilde{u}(x) = u(\lambda^{-1/N}x)$, by the assumption, we have $\|\tilde{u}\|^2_{L^2(\mathbb{R}^N)} = b$ and

$$I[\tilde{u}] = \lambda \left(\frac{\lambda^{-2/N}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(u) dx \right) < \lambda I[u] = \lambda E_a.$$

Hence, we obtain $E_b \le I[\tilde{u}] < \lambda E_a \le E_a$.

(ii): By the assumption and the argument as above, we have

$$E_{\lambda a} < \lambda E_a \quad \text{for any } \lambda > 1,$$
$$E_{\tau b} \le \tau E_b \quad \text{for any } \tau \ge 1.$$

Noting that we can assume $0 < b \le a$ without loss of generality, taking $\lambda = (a+b)/a$ and $\tau = a/b$, we obtain

$$E_{a+b} < \frac{a+b}{a} E_a = E_a + \frac{b}{a} E_a \le E_a + E_b.$$

It completes the lemma. □

Proof of Theorem 1.2. Suppose that $\{u_n\}_{n\in\mathbb{N}} \subset M_\alpha$ is a minimizing sequence which does not satisfy (6). It is sufficient to show that (ii) holds. Since (6) does not hold and $\{u_n\}_{n\in\mathbb{N}} \subset M_\alpha$, we have

$$0 < \varlimsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n|^2 dx \le \alpha < \infty.$$

Taking a subsequence if necessary, there exists a family $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$0 < \lim_{n\to\infty} \int_{B(0,1)} |u_n(x-y_n)|^2 dx < \infty. \tag{11}$$

Since $\{u_n\}_{n\in\mathbb{N}} \subset M_\alpha$ is a minimizing sequence, Lemma 2.2 (ii) asserts that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R}^N)$. Hence $\{u_n(\cdot - y_n)\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R}^N)$. Using the weak compactness of a Hilbert space and the Rellich compactness, for some subsequence, there exists $u \in H^1(\mathbb{R}^N)$ such that

$$u_n(\cdot - y_n) \rightharpoonup u \quad \text{weakly in } H^1(\mathbb{R}^N), \tag{12}$$
$$u_n(\cdot - y_n) \to u \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N), \tag{13}$$
$$u_n(\cdot - y_n) \to u \quad \text{a.e. in } \mathbb{R}^N. \tag{14}$$

Equations (11) and (13) assert that $\|u\|_{L^2(\mathbb{R}^N)} > 0$. We put $v_n = u_n(\cdot - y_n) - u$. By (12), $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$. Thus, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u + \nabla v_n|^2 dx &= \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v_n|^2 dx + 2\Re \int_{\mathbb{R}^N} \nabla u \cdot \overline{\nabla v_n} \\ &= \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v_n|^2 dx + o(1) \quad \text{as } n \to \infty,\end{aligned} \tag{15}$$

$$\begin{aligned}\int_{\mathbb{R}^N} |u + v_n|^2 dx &= \int_{\mathbb{R}^N} |u|^2 + |v_n|^2 dx + 2\Re \int_{\mathbb{R}^N} u\overline{v_n} \\ &= \int_{\mathbb{R}^N} |u|^2 + |v_n|^2 dx + o(1) \quad \text{as } n \to \infty.\end{aligned} \tag{16}$$

Using (14), the Brezis-Lieb theorem (see [3]) implies that

$$\int_{\mathbb{R}^N} F(|u+v_n|)dx = \int_{\mathbb{R}^N} F(|u|) + F(|v_n|)dx + o(1) \quad \text{as } n \to \infty.$$

Since $I[u_n] = I[u_n(\cdot - y_n)] = I[u + v_n]$, we can obtain

$$\begin{aligned} &I[u_n] = I[u] + I[v_n] + o(1), \\ &\|u_n\|^2_{L^2(\mathbb{R}^N)} = \|u\|^2_{L^2(\mathbb{R}^N)} + \|v_n\|^2_{L^2(\mathbb{R}^N)} + o(1) \quad \text{as } n \to \infty. \end{aligned} \tag{17}$$

We will show the following claim.

Claim.

$$\overline{\lim_{n\to\infty}} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |v_n|^2 dx = 0. \tag{18}$$

Suppose that (18) does not hold. Since $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, similarly as above, for some subsequence, there exist a family $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $v \in H^1(\mathbb{R}^N)$ satisfying $\|v\|_{L^2(\mathbb{R}^N)} > 0$ such that

$$\begin{aligned} &v_n(\cdot - z_n) \rightharpoonup v \quad \text{weakly in } H^1(\mathbb{R}^N), \\ &v_n(\cdot - z_n) \to v \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N), \\ &v_n(\cdot - z_n) \to v \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

We put $w_n = v_n(\cdot - z_n) - v$. Then, similarly as above, we can obtain

$$\begin{aligned} &I[v_n] = I[v + w_n] = I[v] + I[w_n] + o(1), \\ &\|v_n\|^2_{L^2(\mathbb{R}^N)} = \|v\|^2_{L^2(\mathbb{R}^N)} + \|w_n\|^2_{L^2(\mathbb{R}^N)} + o(1) \quad \text{as } n \to \infty. \end{aligned}$$

Consequently, we have

$$I[u_n] = I[u] + I[v] + I[w_n] + o(1) \quad \text{as } n \to \infty, \tag{19}$$

$$\|u_n\|^2_{L^2(\mathbb{R}^N)} = \|u\|^2_{L^2(\mathbb{R}^N)} + \|v\|^2_{L^2(\mathbb{R}^N)} + \|w_n\|^2_{L^2(\mathbb{R}^N)} + o(1) \quad \text{as } n \to \infty. \tag{20}$$

Here, we set $\beta = \|u\|^2_{L^2(\mathbb{R}^N)}$, $\gamma = \|v\|^2_{L^2(\mathbb{R}^N)}$ and $\delta = \alpha - \beta - \gamma$. Then, we have $\lim_{n\to\infty} \|w_n\|^2_{L^2(\mathbb{R}^N)} = \delta \geq 0$. We will consider cases $\delta > 0$ and $\delta = 0$.

In the case $\delta > 0$, we set $\tilde{w}_n = a_n w_n$ and $a_n = \sqrt{\delta}/\|w_n\|_{L^2(\mathbb{R}^N)}$. By Lemma 2.4, we have $\tilde{w}_n \in M_\delta$ and $I[w_n] = I[\tilde{w}_n] + o(1)$. Thus, by (19) and the definition of E_δ, we have

$$\begin{aligned} I[u_n] &= I[u] + I[v] + I[w_n] + o(1) \\ &= I[u] + I[v] + I[\tilde{w}_n] + o(1) \\ &\geq I[u] + I[v] + E_\delta + o(1) \quad \text{as } n \to \infty. \end{aligned}$$

As $n \to \infty$, Lemma 2.3 implies that

$$E_\alpha \geq I[u] + I[v] + E_\delta \geq E_\beta + E_\gamma + E_\delta \geq E_{\beta+\gamma+\delta} = E_\alpha. \tag{21}$$

Hence u and v are global minimizers with respect to E_β and E_γ respectively. Here, we can apply Lemma 3.2 (ii) to obtain

$$E_{\beta+\gamma} < E_\beta + E_\gamma.$$

It contradicts to (21).

In the case $\delta = 0$, the equations $\alpha = \beta + \gamma$ and $\lim_{n\to\infty} \|w_n\|_{L^2(\mathbb{R}^N)} = 0$ hold. By Lemma 2.2 (i), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(|w_n|)dx = 0.$$

Thus, we obtain

$$\varliminf_{n\to\infty} I[w_n] \geq 0.$$

As $n \to \infty$ in (19), we have

$$E_\alpha \geq I[u] + I[v] \geq E_\beta + E_\gamma \geq E_\alpha.$$

Hence u and v are global minimizers with respect to E_β and E_γ respectively. Similarly as above, by Lemma 3.2 (ii), we can obtain

$$E_\alpha = E_{\beta+\gamma} < E_\beta + E_\gamma,$$

which is a contradiction. It completes the proof of the claim.

By (18) and Lemma 3.1, we have $\lim_{n\to\infty} \|v_n\|_{L^l(\mathbb{R}^N)} = 0$. Lemma 2.2 (i) asserts that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(|v_n|)dx = 0. \tag{22}$$

Next, we estimate the L^2 norm of v_n.

Claim. $\lim_{n\to\infty} \|v_n\|^2_{L^2(\mathbb{R}^N)} = 0$. *In particular,* $\|u\|^2_{L^2(\mathbb{R}^N)} = \alpha$.

By (16) and $\beta = \|u\|^2_{L^2}$, it is sufficient to show that $\beta = \alpha$. Otherwise, $\beta < \alpha$ holds because $\beta \leq \alpha$. By (22), we have

$$\varliminf_{n\to\infty} I[v_n] \geq \varliminf_{n\to\infty} -\int_{\mathbb{R}^N} F(|v_n|)dx = 0.$$

Taking the limit in (17), we obtain $E_\alpha \geq I[u]$. Using Lemma 2.3 (iii) along with $u \in M_\beta$, we have

$$E_\alpha \geq I[u] \geq E_\beta \geq E_\alpha. \tag{23}$$

This requires $E_\beta = E_\alpha$. Moreover, u is a global minimizer with respect to E_β. By Lemma 3.2 (i), we obtain $E_\beta > E_\alpha$ because $\beta < \alpha$. It contradicts to (23).

Finally, we estimate the H^1-norm of v_n. Using the above claim, $u \in M_\alpha$. This gives $I[u] \geq E_\alpha$. Therefore, we have

$$I[u_n] = I[u] + I[v_n] + o(1) \geq E_\alpha + I[v_n] + o(1) \quad \text{as } n \to \infty.$$

As $n \to \infty$, we obtain

$$\overline{\lim_{n\to\infty}} I[v_n] \leq 0,$$

while (22) asserts that

$$\overline{\lim_{n\to\infty}} \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \leq \overline{\lim_{n\to\infty}} I[v_n] + \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} F(|v_n|) dx \leq 0.$$

Since $\lim_{n\to\infty} \|v_n\|^2_{L^2(\mathbb{R}^N)} = 0$, we have $\lim_{n\to\infty} \|v_n\|^2_{H^1(\mathbb{R}^N)} = 0$. Hence $\lim_{n\to\infty} u_n(\cdot - y_n) = u$ in $H^1(\mathbb{R}^N)$. □

4. Proof of Theorem 1.1

In this section, we will give a proof of Theorem 1.1. To our purpose, we show the following proposition.

Proposition 4.1. *Suppose that $\alpha > \alpha_0$. If $\{u_n\}_{n\in\mathbb{N}} \subset M_\alpha$ is a minimizing sequence with respect to E_α, i.e., $\lim_{n\to\infty} I[u_n] = E_\alpha$. Then, taking a subsequence if necessary, there exist a family $\{y_n\} \subset \mathbb{R}^N$ and $u \in M_\alpha$ such that $\lim_{n\to\infty} u_n(\cdot - y_n) = u$ strongly in $H^1(\mathbb{R}^N)$. In particular, u is a global minimizer, i.e., $u \in S_\alpha$.*

Proof. By the assumption of the proposition and (5), we have $E_\alpha < 0$. Let $\{u_n\}_{n\in\mathbb{N}} \subset M_\alpha$ be a minimizing sequence with respect to E_α. It is sufficient to show that $\{u_n\}_{n\in N}$ satisfies (ii) in Theorem 1.2. Otherwise, by Theorem 1.2, $\{u_n\}_{n\in N}$ satisfies (6). By Lemma 2.2 (ii), $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, so (6) and Lemma 3.1 imply that $u_n \to 0$ in $L^l(\mathbb{R}^N)$. By Lemma 2.2 (i), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(|u_n|) dx = 0.$$

Since $I[u_n] \geq -\int_{\mathbb{R}^N} F(|u_n|)$, we can obtain

$$E_\alpha = \lim_{n\to\infty} I[u_n] \geq \underline{\lim_{n\to\infty}} - \int_{\mathbb{R}^N} F(|u_n|) dx = 0,$$

contradicting to $E_\alpha < 0$. □

Now, we can show Theorem 1.1.

Proof of Theorem 1.1. First, we consider the case $0 < \alpha < \alpha_0$ and suppose by contradiction that there exists a global minimizer with respect to E_α. By the assumption, we have $E_\alpha = 0$. Here, Lemma 3.2 (i) asserts that

$$0 = E_\alpha > E_{\alpha_0}.$$

It contradicts to the definition of α_0 and Lemma 2.3 (v).

Next, we consider the case $\alpha > \alpha_0$. Proposition 4.1 asserts Theorem 1.1 (i). Moreover, Theorem 1.1 (ii) follows from Proposition 4.1 according to [4]. So we omit the proof. □

5. Proof of Theorems 1.3, 1.4 and 1.5

In this section, we give the proofs of the remaining theorems.

Proof of Theorem 1.3. (i): We fix $\alpha > 0$ and take some function $u \in M_\alpha \cap C_0^\infty(\mathbb{R}^N) \setminus \{0\}$. For $\lambda > 0$, let $u_\lambda(x) = \lambda^{N/2}u(\lambda x)$. Then, we see that $u_\lambda \in M_\alpha$. By the assumption of (i), there exist a positive constant δ such that

$$F(s) \geq C|s|^l \quad \text{if } |s| < \delta,$$

where C is a constant determined by

$$C = \int_{\mathbb{R}^N} |\nabla u|^2 dx \Big/ \int_{\mathbb{R}^N} |u|^l dx.$$

Hence $F(|u_\lambda|) \geq C|u_\lambda|^l$ holds for a sufficiently small λ. Thus we have

$$I[u_\lambda] \leq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 dx - C\int_{\mathbb{R}^N} |u_\lambda|^l dx = -\frac{\lambda^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

It concludes that $E_\alpha \leq I[u_\lambda] < 0$ for any $\alpha > 0$.

(ii): By the assumption of (ii), there exists a positive constant $C(f)$ depending on f such that $F(s) \leq C(f)|s|^l$ holds for any $s \geq 0$. For $u \in M_\alpha$, using the Gagliardo-Nirenberg inequality, we have

$$\int_{\mathbb{R}^N} F(|u|)dx \leq C(f)\|u\|^l_{L^l(\mathbb{R}^N)} \leq C(f)C(N)\|\nabla u\|^2_{L^2(\mathbb{R}^N)}\alpha^{2/N}.$$

For a sufficiently small $\alpha > 0$, it can be shown that $C(f)C(N)\alpha^{2/N} \leq 1/2$ holds. After choosing an appropriately small α, we have

$$I[u] \geq \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} = 0.$$

This means $E_\alpha \geq 0$ for a small $\alpha > 0$. Hence, we obtain $\alpha_0 > 0$. □

Proof of Theorem 1.4. (i): By Theorem 1.3 (ii), we have $\alpha_0 > 0$. Let $\alpha_n = \alpha_0 + 1/n$. By (5), $E_{\alpha_n} < 0$ holds. Moreover, Theorem 1.1 asserts that there exists a global minimizer $u_n \in M_{\alpha_n}$ with respect to E_{α_n}. Using the symmetric rearrangement, we can assume that u_n is radially symmetric with respect to the origin and and that it is nonincreasing. Since $I[u_n]$ and $\|u_n\|_{L^2(\mathbb{R}^N)}$ are bounded, u_n is bounded in $H^1(\mathbb{R}^N)$. By the definition of u_n, we have $\lim_{n\to\infty} \|u_n\|^2_{L^2(\mathbb{R}^N)} = \alpha_0$. Let $v_n = \sqrt{\alpha_0}u_n/\|u_n\|_{L^2(\mathbb{R}^N)}$. Then, by Lemma 2.4, we can obtain $v_n \in M_{\alpha_0}$ and

$$\overline{\lim_{n\to\infty}} I[v_n] = \overline{\lim_{n\to\infty}} I[u_n] = \overline{\lim_{n\to\infty}} E_{\alpha_0+1/n} \leq 0.$$

On the other hand, by (5), $E_{\alpha_0} = 0$. Thus, $\{v_n\}_{n\in\mathbb{N}}$ is a minimizing sequence with respect to E_{α_0}. We now show the following claim.

Claim. *For some subsequence, there exist a family* $\{y_n\}_{n\in\mathbb{N}}$ *and* $v \in H^1(\mathbb{R}^N)$ *such that* $\lim_{n\to\infty} v_n(\cdot - y_n) = v$ *in* $H^1(\mathbb{R}^N)$.

If the claim holds, then v is a global minimizer. Therefore, suppose instead that the claim does not hold. By Theorem 1.2, $\{v_n\}_{n\in\mathbb{N}}$ satisfies (6). By the definition of v_n, we see that $\{u_n\}_{n\in\mathbb{N}}$ satisfies (6). Here, u_n is a solution of

$$-\Delta u_n + \mu_n u_n = f(u_n) \quad \text{in } \mathbb{R}^N.$$

As $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, we find that μ_n is bounded in $\mathbb{R}$. Using the elliptic regularity theory, we see that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $C^1(\overline{B(0,1)})$. Thus, by (6), we have

$$u_n(0) = \|u_n\|_{L^\infty(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

On the other hand, by the assumption $\overline{\lim}_{s\to 0} F(s)/s^l = 0$, for any $\epsilon > 0$, there exists $s_3 > 0$ such that $F(s) \leq \epsilon|s|^l$ if $|s| < s_3$. Thus, for a sufficiently large n, we have

$$\int_{\mathbb{R}^N} F(|u_n|)dx \leq \epsilon\|u_n\|^l_{L^l(\mathbb{R}^N)} \leq \epsilon C(N)\|\nabla u_n\|^2_{L^2(\mathbb{R}^N)} \left(\alpha_0 + \frac{1}{n}\right)^{2/N}.$$

We choose $\epsilon > 0$ satisfying $\epsilon C(N)(\alpha_0+1)^{2/N} \leq 1/2$. We can then obtain $I[u_n] \geq 0$, contradicting the definition of u_n.

(ii): By Theorem 1.3 (ii), we have $\alpha_0 > 0$. Let u_n be a global minimizer with respect to $E_{\alpha_0+1/n}$. We can assume u_n is radially symmetric with respect to the origin and nonincreasing. Similar to the proof of (i), if there is no global minimizer with respect to E_{α_0}, then $\{u_n\}_{n\in\mathbb{N}}$ satisfies (6). Moreover, we have $\lim_{n\to\infty} \|u_n\|_{L^\infty(\mathbb{R}^N)} = 0$. Thus, we can choose a sufficiently large n such that $\|u_n\|_{L^\infty(\mathbb{R}^N)} \leq s_1/2$.

Let $v_\lambda(x) = \lambda^{N/2} u_n(\lambda x)$, so that $v_\lambda \in M_{\alpha_0+1/n}$. By the assumption of (ii), in the case $\lambda^{N/2} \leq 2$, we see that $F(|v_\lambda|) = C|v_\lambda|^l$ holds. Hence, we have

$$\begin{aligned} I[v_\lambda] =& \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 dx - C \int_{\mathbb{R}^N} |v_\lambda|^l dx \\ =& \frac{\lambda^2}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \lambda^2 C \int_{\mathbb{R}^N} |u_n|^l dx = \lambda^2 I[u_n]. \end{aligned}$$

Choose an λ which satisfies $\lambda > 1$ and $\lambda^{N/2} \leq 2$. As $I[u_n] < 0$, we can obtain

$$E_{\alpha_0+1/n} \leq I[v_\lambda] = \lambda^2 I[u_n] < I[u_n] = E_{\alpha_0+1/n},$$

which is a contradiction.

Proof of Theorem 1.5. We can assume that $0 < \alpha \leq \beta$ without loss of generality. It is then sufficient to consider the following cases:

Case 1: $E_\alpha = E_\beta = 0$.
Case 2: $E_\alpha = 0,\ E_\beta < 0$.
Case 3: $E_\alpha,\ E_\beta < 0$.

In Case 1, it is clear that $E_\alpha + E_\beta = 0 > E_{\alpha+\beta}$ because $\alpha + \beta > \alpha_0$.

In Case 2, there exists a minimizer with respect to E_β by Theorem 1.1. Thus, we can apply Lemma 3.2 (i) to obtain

$$E_{\alpha+\beta} < E_\beta.$$

In Case 3, there exist a minimizers with respect to E_α and E_β by Theorem 1.1. Therefore Lemma 3.2 (ii) asserts conclusion. □

6. Appendix

In this appendix, we will give the proofs of Lemmas 2.3.

Proof of Lemma 2.3. (i): Let $u \in M_\alpha$. For $\lambda > 0$, we set $u_\lambda(x) = \lambda^{N/2} u(\lambda x)$, giving $u_\lambda \in M_\alpha$. Moreover, $\|u_\lambda\|^l_{L^l(\mathbb{R}^N)} = \lambda^2 \|u\|^l_{L^l(\mathbb{R}^N)} \to 0$ as $\lambda \to 0$. By Lemma 2.2 (i), we have

$$\lim_{\lambda \to 0} \int_{\mathbb{R}^N} F(|u_\lambda|) dx = 0.$$

As $\|\nabla u_\lambda\|^2_{L^2(\mathbb{R}^N)} = \lambda^2 \|\nabla u\|^2_{L^2(\mathbb{R}^N)}$, we see that $\lim_{\lambda \to 0} I[u_\lambda] = 0$ holds. By the definition of E_α, we have $E_\alpha \leq I[u_\lambda]$. Thus, we obtain $E_\alpha \leq 0$.

(ii): We fix $\epsilon > 0$. By the definition of E_α and E_β, there exist $u \in M_\alpha \cap C_0^\infty(\mathbb{R}^N)$ and $v \in M_\beta \cap C_0^\infty(\mathbb{R}^N)$ such that

$$I[u] \leq E_\alpha + \epsilon, \quad I[v] \leq E_\alpha + \epsilon.$$

Since u and v have compact support, by using parallel translation, we can assume $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$. Therefore, we have $u + v \in M_{\alpha+\beta}$. Thus, we find

$$E_{\alpha+\beta} \le I[u+v] = I[u] + I[v] \le E_\alpha + E_\beta + 2\epsilon.$$

As $\epsilon \to 0$, we have $E_{\alpha+\beta} \le E_\alpha + E_\beta$.

(iii): By (i) and (ii), we have

$$E_{\alpha+\beta} \le E_\alpha + E_\beta \le E_\alpha$$

for any $\alpha, \beta > 0$. This gives (iii).

(iv): For $R > 0$, set $u_R \in H^1(\mathbb{R}^N)$ by according to the following:

$$u_R(x) = \begin{cases} s_0 & \text{if } |x| \le R, \\ s_0(R+1-|x|) & \text{if } R < |x| \le R+1, \\ 0 & \text{if } |x| > R+1, \end{cases}$$

where s_0 is a constant determined in (F5). We write $|S^{N-1}|$ for the surface area of the unit sphere. If $N = 1$, set $|S^0| = 2$. We estimate $I[u_R]$ as follows:

$$\begin{aligned} I[u_R] &= \int_{B(0,R)} \frac{1}{2}|\nabla u_R|^2 - F(|u_R|)dx + \int_{B(0,R+1)\setminus B(0,R)} \frac{1}{2}|\nabla u_R|^2 - F(|u_R|)dx \\ &\le \int_{B(0,R)} -F(s_0)dx + \int_{B(0,R+1)\setminus B(0,R)} \frac{1}{2}s_0^2 + \sup_{0\le s\le s_0} |F(s)|dx \\ &= -F(s_0)|S^{N-1}| \int_0^R r^{N-1}dr + \left(\frac{1}{2}s_0^2 + \sup_{0\le s\le s_0} |F(s)|\right)|S^{N-1}| \int_R^{R+1} r^{N-1}dr \\ &= \frac{\left((R+1)^N - R^N\right)|S^{N-1}|}{N}\left(\frac{1}{2}s_0^2 + \sup_{0\le s\le s_0}|F(s)| - \frac{R^N}{(R+1)^N - R^N}F(s_0)\right). \end{aligned}$$

Since

$$\frac{R^N}{(R+1)^N - R^N} = \frac{1}{(1+1/R)^N - 1} \to \infty \quad \text{as } R \to \infty,$$

for a sufficiently large R, we have $I[u_R] < 0$. By choosing such an R and setting $\alpha = \|u_R\|^2_{L^2(\mathbb{R}^N)}$, we obtain $E_\alpha \le I[u_R] < 0$. By (iii), we have $E_\beta \le E_\alpha < 0$ if $\beta \ge \alpha$.

(v): We fix $\alpha > 0$. By (iii), $E_{\alpha-h}$ and $E_{\alpha+h}$ are monotonic and bounded as $h \to 0+0$, so therefore they has limits. Moreover, $E_{\alpha-h} \ge E_\alpha \ge E_{\alpha+h}$ holds due to (iii). Thus, we obtain

$$\lim_{h\to 0+0} E_{\alpha-h} \ge E_\alpha \ge \lim_{h\to 0+0} E_{\alpha+h}.$$

Claim. $\lim_{h\to 0+0} E_{\alpha-h} \le E_\alpha$.

This is clear if $E_\alpha = 0$, so we consider the case $E_\alpha < 0$. Take $u \in M_\alpha$ and let $u_h(x) = \sqrt{1-h/\alpha}\, u(x)$ for $h > 0$. Since $\|u_h\|^2_{L^2(\mathbb{R}^N)} = (1-h/\alpha)\alpha = \alpha - h$, we have $u_h \in M_{\alpha-h}$. On the other hand, we have $\|u_h - u\|_{H^1(\mathbb{R}^N)} = (1 - \sqrt{1-h/\alpha})\|u\|_{H^1(\mathbb{R}^N)} \to 0$ as $h \to 0{+}0$. Thus, we obtain $\lim_{h\to 0+0} I[u_h] = I[u]$. By $E_{\alpha-h} \le I[u_h]$, we have

$$\lim_{h\to 0+0} E_{\alpha-h} \le \lim_{h\to 0+0} I[u_h] = I[u].$$

As we choose $u \in M_\alpha$ arbitrarily, for a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset M_\alpha$ with respect to E_α, we can obtain

$$\lim_{h\to 0+0} E_{\alpha-h} \le I[u_n] \quad \text{for any } n \in \mathbb{N}.$$

As $n \to \infty$, the claim holds.

Claim. $\lim_{h\to 0+0} E_{\alpha+h} \ge E_\alpha$.

Since the left hand side converges, it is sufficient to consider the case $h = 1/n$, where $n \in \mathbb{N}$. Choose a $\{u_n \in M_{\alpha+1/n}$ which satisfies $I[u_n] \le E_{\alpha+1/n} + 1/n$ for each $n \in \mathbb{N}$. By (i), $I[u_n] \le 1/n$. Lemma 2.2 (ii) asserts that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R}^N)$. By the definition of u_n, we have

$$\lim_{n\to\infty} I[u_n] = \lim_{h\to 0+0} E_{\alpha+h}. \tag{24}$$

Let $v_n = u_n/\sqrt{1+1/(\alpha n)}$ for $n \in \mathbb{N}$. Then, $\{v_n\}_{n\in\mathbb{N}}$ is also a bounded sequence in $H^1(\mathbb{R}^N)$. Moreover, we have

$$\|v_n\|^2_{L^2(\mathbb{R}^N)} = \frac{\|u_n\|^2_{L^2(\mathbb{R}^N)}}{1+1/(\alpha n)} = \frac{\alpha + 1/n}{1+1/(\alpha n)} = \alpha.$$

Hence, $v_n \in M_\alpha$ holds. Since Lemma 2.4 is independent of Lemma 2.3, we can use Lemma 2.4 to obtain

$$E_\alpha \le I[v_n] = I[u_n] + o(1) \quad \text{as } n \to \infty.$$

By (24), the claim holds. □

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