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# Nonlinear degenerate elliptic problems with $W_0^{1,1}(\Omega)$ solutions

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**Abstract.** We study a nonlinear equation with an elliptic operator having degenerate coercivity. We prove the existence of a unique $W_0^{1,1}(\Omega)$ distributional solution under suitable summability assumptions on the source in Lebesgue spaces. Moreover, we prove that our problem has no solution if the source is a Radon measure concentrated on a set of zero harmonic capacity.

## 1. Introduction and statement of the results

In this paper we are going to study the nonlinear elliptic equation

$$\begin{cases} -\operatorname{div}\left(\dfrac{a(x,\nabla u)}{(1+|u|)^\gamma}\right) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

under the following assumptions. The set $\Omega$ is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, $\gamma > 0$, $f$ belongs to some Lebesgue space, and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e., $a(\cdot, \xi)$ is measurable on $\Omega$ for every $\xi$ in $\mathbb{R}^N$, and $a(x, \cdot)$ is continuous on $\mathbb{R}^N$ for almost every $x$ in $\Omega$) such that

$$a(x,\xi)\cdot\xi \ge \alpha|\xi|^2, \tag{1.2}$$

$$|a(x,\xi)| \le \beta|\xi|, \tag{1.3}$$

$$[a(x,\xi) - a(x,\eta)]\cdot(\xi-\eta) > 0, \tag{1.4}$$

for almost every $x$ in $\Omega$ and for every $\xi$ and $\eta$ in $\mathbb{R}^N$, $\xi \neq \eta$, where $\alpha$ and $\beta$ are positive constants. We are going to prove that, under suitable assumptions on $\gamma$ and $f$, problem (1.1) has a unique distributional solution $u$ obtained by approximation, with $u$ belonging to the (nonreflexive) Sobolev space $W_0^{1,1}(\Omega)$. Furthermore, we are

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going to prove that problem (1.1) does not have a solution if $\gamma > 1$ and the datum $f$ is a bounded Radon measure concentrated on a set of zero harmonic capacity.

Problems like (1.1) have been extensively studied in the past. In [7] (see also [15,16,19]), existence and regularity results were proved, under the assumptions that $a(x,\xi) = A(x)\xi$, with $A$ a uniformly elliptic bounded matrix, and $0 < \gamma \le 1$, for the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{A(x)\nabla u}{(1+|u|)^\gamma}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $f$ belongs to $L^m(\Omega)$ for some $m \ge 1$.

The main difficulty in dealing with problem (1.5) (or (1.1)) is that the differential operator, even if well defined between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$, is not coercive on $H_0^1(\Omega)$ due to the fact that if $u$ is large, $\frac{1}{(1+|u|)^\gamma}$ tends to zero (see [19] for an explicit example).

This lack of coercivity implies that the classical methods used in order to prove the existence of a solution for elliptic equations (see [18]) cannot be applied even if the datum $f$ is regular. However, in [7], a whole range of existence results was proved, yielding solutions belonging to some Sobolev space $W_0^{1,q}(\Omega)$, with $q = q(\gamma, m) \le 2$, if $f$ is regular enough. Under weaker summability assumptions on $f$, the gradient of $u$ (and even $u$ itself) may not be in $L^1(\Omega)$: in this case, it is possible to give a meaning to solutions of problem (1.5), using the concept of *entropy solutions* which has been introduced in [3].

If $\gamma > 1$, a non existence result for problem (1.5) was proved in [1] (where the principal part is nonlinear with respect to the gradient), even for $L^\infty(\Omega)$ data $f$. Therefore, if the operator becomes "too degenerate", existence may be lost even for data expected to give bounded solutions. However, as proved in [5], existence of solutions can be recovered by adding a lower order term of order zero. Indeed, if we consider the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{A(x)\nabla u}{(1+|u|)^\gamma}\right) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with $f$ in $L^m(\Omega)$, then the following results can be proved in the case $\gamma > 1$ (see [5,11]):

(i) if $m > \gamma\frac{N}{2}$, then there exists a weak solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$;
(ii) if $m \ge \gamma + 2$, then there exists a weak solution in $H_0^1(\Omega) \cap L^m(\Omega)$;
(iii) if $\frac{\gamma+2}{2} < m < \gamma + 2$, then there exists a distributional solution in $W_0^{1,\frac{2m}{\gamma+2}}(\Omega) \cap L^m(\Omega)$;
(iv) if $1 \le m \le \frac{\gamma+2}{2}$, then there exists an entropy solution in $L^m(\Omega)$ whose gradient belongs to the Marcinkiewicz space $M^{\frac{2m}{\gamma+2}}(\Omega)$.

Note that if $\gamma + 2 \le m < \gamma \frac{N}{2}$ and $m$ tends to $\gamma \frac{N}{2}$, the summability result of (ii) is not "continuous" with the boundedness result of (i), according to the following example (see also Example 3.3 of [5]).

*Example 1.1.* If $\frac{2}{\gamma} < \sigma < N-2$, then $u(x) = \frac{1}{|x|^\sigma} - 1$ is a distributional solution of (1.6) with $A(x) \equiv I$, and $f(x) = \frac{\sigma(N-2+\sigma(\gamma-1))}{|x|^{2-\sigma(\gamma-1)}} + \frac{1}{|x|^\sigma} - 1$. Due to the assumptions on $\sigma$, both $f$ and $u$ belong to $L^m(\Omega)$, with $m < \gamma \frac{N}{2}$. If $m$ tends to $\gamma \frac{N}{2}$, i.e., if $\sigma$ tends to $\frac{2}{\gamma}$, the solution $u$ does not become bounded.

As stated before, this paper is concerned with two borderline cases connected with point (iv) above:

A. if $m = \frac{\gamma+2}{2}$, we will prove in Sect. 2 the existence of $W_0^{1,1}(\Omega)$ distributional solutions, and in Sect. 3 their uniqueness;
B. if $f$ is a bounded Radon measure concentrated on a set $E$ of zero harmonic capacity and $\gamma > 1$, we will prove in Sect. 4 non existence of solutions.

In the linear case, i.e., for the boundary value problem (1.6), a simple proof of the existence result is given in [6].

*Remark 1.2.* Let $a(x,\xi) = A(x)\xi$, with $A$ a bounded and measurable uniformly elliptic matrix, and let $u \ge 0$ be a solution of

$$-\text{div}\left(\frac{A(x)\nabla u}{(1+u)^\gamma}\right) + u = f,$$

with $\gamma > 1$ and $f \ge 0$. If we define

$$z = \frac{1}{\gamma - 1}\left(1 - \frac{1}{(1+u)^{\gamma-1}}\right),$$

then $z$ is a solution of

$$-\text{div}(A(x)\nabla z) + \left(\frac{1}{(1-(\gamma-1)z)^{\frac{1}{\gamma-1}}} - 1\right) = f,$$

which is an equation whose lower order term becomes singular as $z$ tends to the value $\frac{1}{\gamma-1}$. For a study of these problems, see [4,14].

*Remark 1.3.* We explicitely state that our existence results can be generalized to equations with differential operators defined on $W_0^{1,p}(\Omega)$, with $p > 1$: if $\gamma \ge \frac{(p-2)_+}{p-1}$ and if $m = \frac{\gamma(p-1)+2}{p}$, then it is possible to prove the existence of a distributional solution $u$ in $W_0^{1,1}(\Omega) \cap L^m(\Omega)$ of the boundary value problem

$$\begin{cases} -\text{div}\left(\dfrac{a(x,\nabla u)}{(1+|u|)^{\gamma(p-1)}}\right) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $a(x,\xi)$ satisfies (1.2), (1.3) and (1.4) with $p$ instead of 2 (in (1.3), $a$ grows as $|\xi|^{p-1}$).

## 2. Existence of a $W_0^{1,1}(\Omega)$ solution

In this section we prove the existence of a $W_0^{1,1}(\Omega)$ solution to problem (1.1). Our result is the following.

**Theorem 2.1.** *Let $\gamma > 0$, and let $f$ be a function in $L^{\frac{\gamma+2}{2}}(\Omega)$. Then there exists a distributional solution $u$ in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma+2}{2}}(\Omega)$ of (1.1), that is,*

$$\int_\Omega \frac{a(x, \nabla u) \cdot \nabla \varphi}{(1 + |u|)^\gamma} + \int_\Omega u\, \varphi = \int_\Omega f\, \varphi, \quad \forall \varphi \in W_0^{1,\infty}(\Omega). \tag{2.1}$$

*Remark 2.2.* The previous result gives existence of a solution $u$ in $W_0^{1,1}(\Omega)$ to (1.6) for every $\gamma > 0$. If $0 < \gamma \le 1$ existence results for (1.1) can also be proved by the same techniques of [7]. More precisely, if $f$ belongs to $L^m(\Omega)$ with $m > \frac{N}{N(1-\gamma)+1+\gamma}$ then (1.1) has a solution in $W_0^{1,q}(\Omega)$, with $q = \frac{Nm(1-\gamma)}{N-m(1+\gamma)}$. Note that when $m$ tends to $\frac{N}{N(1-\gamma)+1+\gamma}$, then $q$ tends to 1. We have now two cases: if $\frac{\gamma+2}{2} > \frac{N}{N(1-\gamma)+1+\gamma}$, that is, if $0 < \gamma < \frac{2}{N-1}$, our result is weaker than the one in [7]. On the other hand, if $\frac{2}{N-1} \le \gamma \le 1$, then our result, which strongly uses the lower order term of order zero, is better.

*Remark 2.3.* The same existence result, with the same proof, holds for the following boundary value problem

$$\begin{cases} -\mathrm{div}(b(x, u, \nabla u)) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ a Carathéodory function such that

$$\frac{\alpha|\xi|^2}{(1 + |s|)^\gamma} \le b(x, s, \xi) \cdot \xi \le \beta|\xi|^2,$$

where $\alpha$, $\beta$, $\gamma$ are positive constants.

To prove Theorem 2.1 we will work by approximation. First of all, let $g$ be a function in $L^\infty(\Omega)$. Then, by the results of [5], there exists a solution $v$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of

$$\begin{cases} -\mathrm{div}\left(\dfrac{a(x, \nabla v)}{(1 + |v|)^\gamma}\right) + v = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

In order for this paper to be self contained, we give here the easy proof of this fact. Let $M = \|g\|_{L^\infty(\Omega)} + 1$, and consider the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{a(x, \nabla v)}{(1 + |T_M(v)|)^\gamma}\right) + v = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Here and in the following we define $T_k(s) = \max(-k, \min(s, k))$ for $k \geq 0$ and $s$ in $\mathbb{R}$. Since the differential operator is pseudomonotone and coercive thanks to the assumptions on $a$ and to the truncature, by the results of [18] there exists a weak solution $v$ in $H_0^1(\Omega)$ of (2.3). Choosing $(|v| - \|g\|_{L^\infty(\Omega)})_+ \operatorname{sgn}(v)$ as a test function we obtain, dropping the nonnegative first term,

$$\int_\Omega |v|\,(|v| - \|g\|_{L^\infty(\Omega)})_+ \leq \int_\Omega \|g\|_{L^\infty(\Omega)}\,(|v| - \|g\|_{L^\infty(\Omega)})_+.$$

Thus,

$$\int_\Omega (|v| - \|g\|_{L^\infty(\Omega)})\,(|v| - \|g\|_{L^\infty(\Omega)})_+ \leq 0,$$

so that $|v| \leq \|g\|_{L^\infty(\Omega)} < M$. Therefore, $T_M(v) = v$, and $v$ is a bounded weak solution of (2.2).

Let now $f_n$ be a sequence of $L^\infty(\Omega)$ functions which converges to $f$ in $L^{\frac{\gamma+2}{2}}(\Omega)$, and such that $|f_n| \leq |f|$ almost everywhere in $\Omega$, and consider the approximating problems

$$\begin{cases} -\operatorname{div}\left(\dfrac{a(x, \nabla u_n)}{(1 + |u_n|)^\gamma}\right) + u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

A solution $u_n$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$ exists choosing $g = f_n$ in (2.2). We begin with some *a priori* estimates on the sequence $\{u_n\}$.

**Lemma 2.4.** *If $u_n$ is a solution to problem (2.4), then, for every $k \geq 0$,*

$$\int_{\{|u_n| \geq k\}} |u_n|^{\frac{\gamma+2}{2}} \leq \int_{\{|u_n| \geq k\}} |f|^{\frac{\gamma+2}{2}}; \tag{2.5}$$

$$\int_{\{|u_n| \geq k\}} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{\frac{\gamma+2}{2}}} \leq C \left[\int_{\{|u_n| \geq k\}} |f|^{\frac{\gamma+2}{2}}\right]^{\frac{2}{\gamma+2}}; \tag{2.6}$$

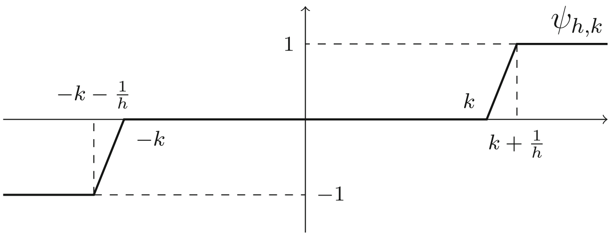
$$\int_{\{|u_n| \geq k\}} |\nabla u_n| \leq C \left[\int_{\{|u_n| \geq k\}} |f|^{\frac{\gamma+2}{2}}\right]^{\frac{1}{\gamma+2}}; \tag{2.7}$$

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 \leq k\,(1 + k)^\gamma \int_\Omega |f|. \tag{2.8}$$

*Here, and in the following, $C$ denotes a positive constant depending on $\alpha, \gamma, \operatorname{meas}(\Omega)$, and the norm of $f$ in $L^{\frac{\gamma+2}{2}}(\Omega)$.*

*Proof.* Let $k \geq 0$, $h > 0$, and let $\psi_{h,k}(s)$ be the function defined by

$$\psi_{h,k}(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq k, \\ h\,(s-k) & \text{if } k < s \leq k + \frac{1}{h}, \\ 1 & \text{if } s > k + \frac{1}{h}, \\ \psi_{h,k}(s) = -\psi_{h,k}(-s) & \text{if } s < 0. \end{cases}$$




Note that

$$\lim_{h \to +\infty} \psi_{h,k}(s) = \begin{cases} 1 & \text{if } s > k, \\ 0 & \text{if } |s| \leq k, \\ -1 & \text{if } s < -k. \end{cases}$$

Let $\varepsilon > 0$, and choose $(\varepsilon + |u_n|)^{\frac{\gamma}{2}}\,\psi_{h,k}(u_n)$ as a test function in (2.4); such a test function is admissible since $u_n$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and $\psi_{h,k}(0) = 0$. We obtain

$$\begin{aligned} &\frac{\gamma}{2} \int_\Omega a(x, \nabla u_n) \cdot \nabla u_n \frac{(\varepsilon + |u_n|)^{\frac{\gamma}{2}-1}}{(1+|u_n|)^\gamma}\,|\psi_{h,k}(u_n)| \\ &\quad + \int_\Omega \frac{a(x, \nabla u_n) \cdot \nabla u_n}{(1+|u_n|)^\gamma}\,\psi'_{h,k}(u_n)\,(\varepsilon + |u_n|)^{\frac{\gamma}{2}} \\ &\quad + \int_\Omega u_n\,(\varepsilon + |u_n|)^{\frac{\gamma}{2}}\,\psi_{h,k}(u_n) \\ &= \int_\Omega f_n\,(\varepsilon + |u_n|)^{\frac{\gamma}{2}}\,\psi_{h,k}(u_n). \end{aligned} \tag{2.9}$$

By (1.2), and since $\psi'_{h,k}(s) \geq 0$, the first two terms are nonnegative, so that we obtain, recalling that $|f_n| \leq |f|$,

$$\int_\Omega u_n(\varepsilon + |u_n|)^{\frac{\gamma}{2}}\,\psi_{h,k}(u_n) \leq \int_\Omega |f|(\varepsilon + |u_n|)^{\frac{\gamma}{2}}\,|\psi_{h,k}(u_n)|.$$

Letting $\varepsilon$ tend to zero and $h$ tend to infinity, we obtain, by Fatou's lemma (on the left hand side) and by Lebesgue's theorem (on the right hand side, recall that $u_n$

belongs to $L^\infty(\Omega)$),

$$\int_{\{|u_n|\geq k\}} |u_n|^{\frac{\gamma+2}{2}} \leq \int_{\{|u_n|\geq k\}} |f|\,|u_n|^{\frac{\gamma}{2}}.$$

Using Hölder's inequality on the right hand side we obtain

$$\int_{\{|u_n|\geq k\}} |u_n|^{\frac{\gamma+2}{2}} \leq \left[\int_{\{|u_n|\geq k\}} |f|^{\frac{\gamma+2}{2}}\right]^{\frac{2}{\gamma+2}} \left[\int_{\{|u_n|\geq k\}} |u_n|^{\frac{\gamma+2}{2}}\right]^{\frac{\gamma}{\gamma+2}}.$$

Simplifying equal terms we thus have

$$\int_{\{|u_n|\geq k\}} |u_n|^{\frac{\gamma+2}{2}} \leq \int_{\{|u_n|\geq k\}} |f|^{\frac{\gamma+2}{2}},$$

which is (2.5). Note that from (2.5), written for $k = 0$, it follows

$$\int_\Omega |u_n|^{\frac{\gamma+2}{2}} \leq \int_\Omega |f|^{\frac{\gamma+2}{2}} = \|f\|^{\frac{\gamma+2}{2}}_{L^{\frac{\gamma+2}{2}}(\Omega)}. \tag{2.10}$$

Now we consider (2.9) written for $\varepsilon = 1$. Dropping the nonnegative second and third terms, and using that $|f_n| \leq |f|$, we have

$$\frac{\gamma}{2}\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+|u_n|)^{\frac{\gamma+2}{2}}}|\psi_{h,k}(u_n)| \leq \int_\Omega |f|(1+|u_n|)^{\frac{\gamma+2}{2}}|\psi_{h,k}(u_n)|.$$

Using (1.2), and letting $h$ tend to infinity, we get (using again Fatou's lemma and Lebesgue's theorem)

$$\alpha\frac{\gamma}{2}\int_{\{|u_n|\geq k\}} \frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}} \leq \int_{\{|u_n|\geq k\}} |f|(1+|u_n|)^{\frac{\gamma}{2}}.$$

Hölder's inequality on the right hand side then gives

$$\begin{aligned}\alpha\frac{\gamma}{2}&\int_{\{|u_n|\geq k\}} \frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}} \\ &\leq \left[\int_{\{|u_n|\geq k\}} |f|^{\frac{\gamma+2}{2}}\right]^{\frac{2}{\gamma+2}} \left[\int_{\{|u_n|\geq k\}} (1+|u_n|)^{\frac{\gamma+2}{2}}\right]^{\frac{\gamma}{\gamma+2}} \\ &\leq \left[\int_{\{|u_n|\geq k\}} |f|^{\frac{\gamma+2}{2}}\right]^{\frac{2}{\gamma+2}} \left[\int_\Omega (1+|u_n|)^{\frac{\gamma+2}{2}}\right]^{\frac{\gamma}{\gamma+2}},\end{aligned}$$

so that, by (2.10),

$$\alpha\frac{\gamma}{2}\int\limits_{\{|u_n|\geq k\}}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}}\leq C\left[\int\limits_{\{|u_n|\geq k\}}|f|^{\frac{\gamma+2}{2}}\right]^{\frac{2}{\gamma+2}},$$

which is (2.6).

Then, again by Hölder's inequality, and by (2.6) and (2.10),

$$\begin{aligned}\int\limits_{\{|u_n|\geq k\}}|\nabla u_n| &= \int\limits_{\{|u_n|\geq k\}}\frac{|\nabla u_n|}{(1+|u_n|)^{\frac{\gamma+2}{4}}}(1+|u_n|)^{\frac{\gamma+2}{4}}\\ &\leq\left[\int\limits_{\{|u_n|\geq k\}}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}}\right]^{\frac12}\left[\int\limits_{\{|u_n|\geq k\}}(1+|u_n|)^{\frac{\gamma+2}{2}}\right]^{\frac12}\\ &\leq C\left[\int\limits_{\{|u_n|\geq k\}}|f|^{\frac{\gamma+2}{2}}\right]^{\frac{1}{\gamma+2}}\left[\int\limits_{\Omega}(1+|u_n|)^{\frac{\gamma+2}{2}}\right]^{\frac12}\\ &\leq C\left[\int\limits_{\{|u_n|\geq k\}}|f|^{\frac{\gamma+2}{2}}\right]^{\frac{1}{\gamma+2}},\end{aligned}\tag{2.11}$$

so that (2.7) is proved.

Finally, choosing $T_k(u_n)$ as a test function in (2.4) we get, dropping the non-negative linear term, and using (1.2),

$$\alpha\int\limits_{\Omega}|\nabla T_k(u_n)|^2\leq k(1+k)^{\gamma}\int\limits_{\Omega}|f|,$$

which is (2.8). □

**Lemma 2.5.** *If $\{u_n\}$ is the sequence of solutions to (2.4), there exists a subsequence, still denoted by $\{u_n\}$, and a function $u$ in $L^{\frac{\gamma+2}{2}}(\Omega)$, with $T_k(u)$ belonging to $H_0^1(\Omega)$ for every $k>0$, such that $u_n$ almost everywhere converges to $u$ in $\Omega$, and $T_k(u_n)$ weakly converges to $T_k(u)$ in $H_0^1(\Omega)$.*

*Proof.* Consider (2.6) written for $k=0$:

$$\int\limits_{\Omega}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}}\leq C\,\|f\|_{L^{\frac{\gamma+2}{2}}(\Omega)}.\tag{2.12}$$

Since (if $\gamma\neq 2$)

$$\frac{|\nabla u_n|^2}{(1+|u_n|)^{\frac{\gamma+2}{2}}}=\frac{16}{(2-\gamma)^2}\left|\nabla\left[(1+|u_n|)^{\frac{2-\gamma}{4}}-1\right]\right|^2,$$

the sequence $v_n = \frac{4}{2-\gamma}[(1+|u_n|)^{\frac{2-\gamma}{4}} - 1]\mathrm{sgn}(u_n)$ is bounded in $H_0^1(\Omega)$ by (2.12). If $\gamma = 2$ we have

$$\frac{|\nabla u_n|^2}{(1+|u_n|)^2} = |\nabla \log(1+|u_n|)|^2,$$

so that $v_n = [\log(1+|u_n|)]\mathrm{sgn}(u_n)$ is bounded in $H_0^1(\Omega)$. In both cases, up to a subsequence still denoted by $v_n$, $v_n$ converges to some function $v$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in $\Omega$. If $\gamma < 2$, define

$$u(x) = \left[\left(\frac{2-\gamma}{4}|v(x)| + 1\right)^{\frac{4}{2-\gamma}} - 1\right]\mathrm{sgn}(v(x)),$$

if $\gamma > 2$ define

$$u(x) = \begin{cases} \left[\left(\frac{2-\gamma}{4}|v(x)|+1\right)^{\frac{4}{2-\gamma}} - 1\right]\mathrm{sgn}(v(x)) & \text{if } |v(x)| < \frac{4}{\gamma-2}, \\ +\infty & \text{if } v(x) = \frac{4}{\gamma-2}, \\ -\infty & \text{if } v(x) = -\frac{4}{\gamma-2}, \end{cases}$$

while if $\gamma = 2$, define

$$u(x) = [\mathrm{e}^{|v(x)|} - 1]\mathrm{sgn}(v(x)).$$

Thus, $u_n$ almost everywhere converges, up to a subsequence still denoted by $u_n$, to $u$. From now on, we will consider this particular subsequence, for which it holds that $u_n$ almost everywhere converges to $u$.

We use now (2.5) written for $k = 0$:

$$\int_\Omega |u_n|^{\frac{\gamma+2}{2}} \le \int_\Omega |f|^{\frac{\gamma+2}{2}} \le C.$$

Since $u_n$ almost everywhere converges to $u$, we have from Fatou's lemma that

$$\int_\Omega |u|^{\frac{\gamma+2}{2}} \le C.$$

Hence $u$ belongs to $L^{\frac{\gamma+2}{2}}(\Omega)$, which implies that $u$ is almost everywhere finite (note that if $\gamma > 2$ this fact did not follow from the definition of $u$, since $|v|$ could have assumed the value $\frac{4}{\gamma-2}$ on a set of positive measure).

Let now $k > 0$; since from (2.8) it follows that the sequence $\{T_k(u_n)\}$ is bounded in $H_0^1(\Omega)$, there exists a subsequence $T_k(u_{n_j})$ which weakly converges to some function $v_k$ in $H_0^1(\Omega)$. Using the almost everywhere convergence of $u_n$ to $u$, we have that $v_k = T_k(u)$. Since the limit is independent on the subsequence, then the whole sequence $\{T_k(u_n)\}$ weakly converges to $T_k(u)$, for every $k > 0$. □

*Remark 2.6.* Using the fact that $T_k(u)$ is in $H_0^1(\Omega)$ for every $k > 0$, and the results of [3], we have that there exists a unique measurable function $v$ with values in $\mathbb{R}^N$, such that

$$\nabla T_k(u) = v\,\chi_{\{|u|\le k\}} \qquad \text{almost everywhere in } \Omega, \text{ for every } k > 0.$$

Following again [3], we will *define* $\nabla u = v$, the approximate gradient of $u$.

*Remark 2.7.* We emphasize that if $\gamma = 2$, then (2.11), written for $k = 0$, becomes

$$\int_\Omega |\nabla u_n| \le \left[\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^2}\right]^{\frac{1}{2}} \left[\int_\Omega (1+|u_n|)^2\right]^{\frac{1}{2}}.$$

Since

$$\frac{|\nabla u_n|^2}{(1+|u_n|)^2} = |\nabla \log(1+|u_n|)|^2,$$

a nonlinear interpolation result follows: let $A$ be in $\mathbb{R}^+$ and let $v$ in $L^2(\Omega)$ be such that $\log(A+|v|)$ belongs to $H_0^1(\Omega)$. Then $v$ belongs to $W_0^{1,1}(\Omega)$, and

$$\int_\Omega |\nabla v| \le \|\log(A+|v|)\|_{H_0^1(\Omega)} \left[\int_\Omega (A+|v|)^2\right]^{\frac{1}{2}}.$$

Our next result deals with the strong convergence of $T_k(u_n)$ in $H_0^1(\Omega)$.

**Proposition 2.8.** *Let $u_n$ and $u$ be the sequence of solutions to problems (2.4) and the function in $L^{\frac{\gamma+2}{2}}(\Omega)$ given by Lemma 2.5. Then, for every fixed $k > 0$, $T_k(u_n)$ strongly converges to $T_k(u)$ in $H_0^1(\Omega)$, as $n$ tends to infinity.*

*Proof.* We follow the proof of [17].

Let $h > k$ and choose $T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)]$ as a test function in (2.4). We have

$$\int_\Omega \frac{a(x,\nabla u_n)\cdot \nabla T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)]}{(1+|u_n|)^\gamma}$$
$$= -\int_\Omega (u_n - f_n)\, T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)]. \qquad (2.13)$$

We observe that the right hand side converges to zero as first $n$ and then $h$ tend to infinity, since $u_n$ converges to $u$ almost everywhere in $\Omega$ and $u_n$ and $f_n$ are bounded in $L^{\frac{\gamma+2}{2}}(\Omega)$. Thus, if we define $\varepsilon(n,h)$ as any quantity such that

$$\lim_{h\to+\infty}\ \lim_{n\to+\infty}\ \varepsilon(n,h) = 0,$$

then

$$\int_\Omega (u_n - f_n)\, T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)] = \varepsilon(n,h).$$

Let $M = 4k + h$. Observing that $\nabla T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)] = 0$ if $|u_n| \geq M$, by (2.13) we have

$$\varepsilon(n,h) = \int_{\{|u_n|<k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla[u_n - T_h(u_n) + T_k(u_n) - T_k(u)]}{(1+|u_n|)^\gamma} + \int_{\{|u_n|\geq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla[u_n - T_h(u_n) + T_k(u_n) - T_k(u)]}{(1+|u_n|)^\gamma}.$$

Since $u_n - T_h(u_n) = 0$ in $\{|u_n| \leq k\}$ and $\nabla T_k(u_n) = 0$ in $\{|u_n| \geq k\}$, we have, using that $a(x,0) = 0$,

$$\varepsilon(n,h) = \int_\Omega \frac{a(x, \nabla T_k(u_n)) \cdot \nabla[T_k(u_n) - T_k(u)]}{(1+|u_n|)^\gamma} + \int_{\{|u_n|\geq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla[u_n - T_h(u_n)]}{(1+|u_n|)^\gamma} - \int_{\{|u_n|\geq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla T_k(u)}{(1+|u_n|)^\gamma}.$$

The second term of the right hand side is positive, so that

$$\varepsilon(n,h) \geq \int_\Omega \frac{[a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla[T_k(u_n) - T_k(u)]}{(1+k)^\gamma} + \int_\Omega \frac{a(x, \nabla T_k(u)) \cdot \nabla[T_k(u_n) - T_k(u)]}{(1+|u_n|)^\gamma} - \int_{\{|u_n|\geq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla T_k(u)}{(1+|u_n|)^\gamma} = I_n + J_n - K_n\,.$$

The last two terms tend to zero as $n$ tends to infinity. Indeed

$$\lim_{n\to+\infty} J_n = \lim_{n\to+\infty} \int_\Omega \frac{a(x, \nabla T_k(u)) \cdot \nabla[T_k(u_n) - T_k(u)]}{(1+|u_n|)^\gamma} = 0,$$

since $T_k(u_n)$ converges to $T_k(u)$ weakly in $H_0^1(\Omega)$ and $\frac{a(x,\nabla T_k(u))}{(1+|u_n|)^\gamma}$ is strongly compact in $(L^2(\Omega))^N$ by the growth assumption (1.3) on $a$. The last term can be rewritten as

$$K_n = \int_\Omega \frac{a(x, \nabla T_M(u_n)) \cdot \nabla T_k(u)\, \chi_{\{|u_n|\geq k\}}}{(1+|u_n|)^\gamma}.$$

Since $M$ is fixed with respect to $n$, then the sequence $\{a(x,\nabla T_M(u_n))\}$ is bounded in $(L^2(\Omega))^N$. Hence, there exists $\sigma$ in $(L^2(\Omega))^N$, and a subsequence $\{a(x,\nabla T_M(u_{n_j}))\}$, such that

$$\lim_{j\to+\infty} a(x,\nabla T_M(u_{n_j})) = \sigma,$$

weakly in $(L^2(\Omega))^N$. On the other hand,

$$\lim_{n\to+\infty} \frac{\nabla T_k(u)\chi_{\{k\le|u_n|\}}}{(1+|u_n|)^\gamma} = \frac{\nabla T_k(u)\chi_{\{k\le|u|\}}}{(1+|u|)^\gamma} = 0,$$

strongly in $(L^2(\Omega))^N$, and so

$$\lim_{j\to+\infty} K_{n_j} = \lim_{j\to+\infty} \int\limits_{\{|u_{n_j}|\ge k\}} \frac{a(x,\nabla T_M(u_{n_j}))\cdot\nabla T_k(u)}{(1+|u_{n_j}|)^\gamma} = 0.$$

Since the limit does not depend on the subsequence, we have

$$\lim_{n\to+\infty} K_n = \lim_{n\to+\infty} \int\limits_{\{|u_n|\ge k\}} \frac{a(x,\nabla T_M(u_n))\cdot\nabla T_k(u)}{(1+|u_n|)^\gamma} = 0,$$

as desired. Therefore,

$$\varepsilon(n,h) \ge I_n = \int\limits_\Omega \frac{[a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))]\cdot\nabla[T_k(u_n)-T_k(u)]}{(1+k)^\gamma},$$

so that, thanks to (1.4),

$$\lim_{n\to+\infty} \int\limits_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))]\cdot\nabla[T_k(u_n)-T_k(u)] = 0.$$

Using this formula, (1.4) and the results of [8,10], we then conclude that $T_k(u_n)$ strongly converges to $T_k(u)$ in $H_0^1(\Omega)$, as desired. □

**Corollary 2.9.** *Let $u_n$ and $u$ be as in Proposition 2.8. Then $\nabla u_n$ converges to $\nabla u$ almost everywhere in $\Omega$, where $\nabla u$ has been defined in Remark 2.6.*

**Lemma 2.10.** *Let $u_n$ and $u$ be as in Proposition 2.8. Then $\nabla u_n$ strongly converges to $\nabla u$ in $(L^1(\Omega))^N$. Moreover $u_n$ strongly converges to $u$ in $L^{\frac{\gamma+2}{2}}(\Omega)$.*

*Proof.* We begin by proving the convergence of $\nabla u_n$ to $\nabla u$. Let $\varepsilon > 0$, and let $k > 0$ be sufficiently large such that

$$\left[\int\limits_{\{|u_n|\ge k\}} |f|^{\frac{\gamma+2}{2}}\right]^{\frac{1}{\gamma+2}} < \varepsilon, \tag{2.14}$$

uniformly with respect to $n$. This can be done thanks to (2.10) and to the absolute continuity of the integral. Let $E$ be a measurable set. Writing

$$\int_E |\nabla u_n| = \int_E |\nabla T_k(u_n)| + \int_{E\cap\{|u_n|\geq k\}} |\nabla u_n|$$

we have, by (2.7), and by (2.14),

$$\int_E |\nabla u_n| \leq \int_E |\nabla T_k(u_n)| + C\varepsilon.$$

Using Hölder's inequality and (2.8), we obtain

$$\int_E |\nabla u_n| \leq C\,\text{meas}(E)^{\frac{1}{2}} k^{\frac{1}{2}}\,(1+k)^{\frac{\gamma}{2}} \left(\int_\Omega |f|\right)^{\frac{1}{2}} + C\varepsilon.$$

Choosing $\text{meas}(E)$ small enough (recall that $k$ is now fixed) we have

$$\int_E |\nabla u_n| \leq C\varepsilon,$$

uniformly with respect to $n$, where $C$ does not depend on $n$ or $\varepsilon$. Since $\nabla u_n$ almost everywhere converges to $\nabla u$ by Corollary 2.9, we can apply Vitali's theorem to obtain the strong convergence of $\nabla u_n$ to $\nabla u$ in $(L^1(\Omega))^N$.

As for the second convergence, by (2.5) we have

$$\begin{aligned}\int_E |u_n|^{\frac{\gamma+2}{2}} &\leq \int_{E\cap\{|u_n|\leq k\}} |u_n|^{\frac{\gamma+2}{2}} + \int_{E\cap\{|u_n|\geq k\}} |u_n|^{\frac{\gamma+2}{2}} \\ &\leq k^{\frac{\gamma+2}{2}}\text{meas}(E) + \int_{\{|u_n|\geq k\}} |f|^{\frac{\gamma+2}{2}}.\end{aligned}$$

As before, we first choose $k$ such that the second integral is small, uniformly with respect to $n$, and then the measure of $E$ small enough such that the first term is small. The almost everywhere convergence of $u_n$ to $u$, and Vitali's theorem, then imply that $u_n$ strongly converges to $u$ in $L^{\frac{\gamma+2}{2}}(\Omega)$. □

*Remark 2.11.* Since we have proved that $\nabla u_n$ strongly converges to $\nabla u$ in $(L^1(\Omega))^N$, so that $u$ belongs to $W_0^{1,1}(\Omega)$, then the approximate gradient $\nabla u$ of $u$ is nothing but the distributional gradient of $u$ (see [3]).

*Proof of Theorem 2.1.* Using the previous results, we pass to the limit, as $n$ tends to infinity, in the weak formulation of (2.4). Starting from

$$\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla\varphi}{(1+|u_n|)^\gamma} + \int_\Omega u_n\,\varphi = \int_\Omega f_n\,\varphi, \quad \varphi\in W_0^{1,\infty}(\Omega),$$

the limit of the second and the last integral is easy to compute; indeed, recall that by Lemma 2.10, and by definition of $f_n$, the sequences $\{u_n\}$ and $\{f_n\}$ strongly converge to $u$ and $f$ respectively in $L^{\frac{\gamma+2}{2}}(\Omega)$, hence in $L^1(\Omega)$. For the first integral, we have that $a(x, \nabla u_n)$ converges almost everywhere in $\Omega$ to $a(x, \nabla u)$ thanks to Corollary 2.9, and the continuity assumption on $a(x, \cdot)$; furthermore, (1.3) implies that

$$|a(x, \nabla u_n)| \leq \beta |\nabla u_n|,$$

and the right hand side is compact in $L^1(\Omega)$ by Lemma 2.10. Thus, by Vitali's theorem $a(x, \nabla u_n)$ strongly converges to $a(x, \nabla u)$ in $(L^1(\Omega))^N$, so that

$$\lim_{n\to+\infty} \int_\Omega \frac{a(x, \nabla u_n) \cdot \nabla \varphi}{(1 + |u_n|)^\gamma} = \int_\Omega \frac{a(x, \nabla u) \cdot \nabla \varphi}{(1 + |u|)^\gamma},$$

where we have also used that $u_n$ almost everywhere converges to $u$, and Lebesgue's theorem. Thus, we have that

$$\int_\Omega \frac{a(x, \nabla u) \cdot \nabla \varphi}{(1 + |u|)^\gamma} + \int_\Omega u\,\varphi = \int_\Omega f\,\varphi, \quad \forall \varphi \in W_0^{1,\infty}(\Omega),$$

i.e., $u$ satisfies (2.1). □

## 3. Uniqueness of the solution obtained by approximation

Let $f \in L^{\frac{\gamma+2}{2}}(\Omega)$, let $f_n$ be a sequence of $L^\infty(\Omega)$ functions converging to $f$ in $L^{\frac{\gamma+2}{2}}(\Omega)$, with $|f_n| \leq |f|$, and let $u_n$ be a solution of (2.4). In Sect. 2 we proved the existence of a distributional solution $u$ in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma+2}{2}}(\Omega)$ to (1.1), such that, up to a subsequence,

$$\lim_{n\to+\infty} \|u_n - u\|_{W_0^{1,1}(\Omega)\cap L^{\frac{\gamma+2}{2}}(\Omega)} = 0. \tag{3.1}$$

Now, let $g \in L^{\frac{\gamma+2}{2}}(\Omega)$, let $g_n$ be a sequence of $L^\infty(\Omega)$ functions converging to $g$ in $L^{\frac{\gamma+2}{2}}(\Omega)$, with $|g_n| \leq |g|$, and let $z_n$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$ be a weak solution of

$$\begin{cases} -\mathrm{div}\left(\dfrac{a(x, \nabla z_n)}{(1 + |z_n|)^\gamma}\right) + z_n = g_n & \text{in } \Omega, \\ z_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Then, up to a subsequence, we can assume that

$$\lim_{n\to+\infty} \|z_n - z\|_{W_0^{1,1}(\Omega)\cap L^{\frac{\gamma+2}{2}}(\Omega)} = 0, \tag{3.3}$$

where $z$ in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma+2}{2}}(\Omega)$ is a distributional solution of

$$\begin{cases} -\text{div}\left(\dfrac{a(x, \nabla z)}{(1+|z|)^\gamma}\right) + z = f & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Our result, which will imply the uniqueness of the solution by approximation (see [12]) of (1.1), is the following.

**Theorem 3.1.** *Assume that $u_n$ and $z_n$ are solutions of (2.4) and (3.2) respectively, and that (3.1) and (3.3) hold true, with $u$ and $z$ solutions of (1.1) and (3.4) respectively. Then*

$$\int_\Omega |u - z| \le \int_\Omega |f - g|. \tag{3.5}$$

*Moreover,*

$$f \le g \text{ a.e. in } \Omega \text{ implies } u \le z \text{ a.e. in } \Omega. \tag{3.6}$$

*Proof.* Substracting (3.2) from (2.4) we obtain

$$-\text{div}\left(\frac{a(x, \nabla u_n)}{(1+|u_n|)^\gamma} - \frac{a(x, \nabla z_n)}{(1+|z_n|)^\gamma}\right) + u_n - z_n = f_n - g_n.$$

Choosing $T_h(u_n - z_n)$ as a test function we have

$$\int_\Omega \left[\frac{a(x, \nabla u_n)}{(1+|u_n|)^\gamma} - \frac{a(x, \nabla z_n)}{(1+|z_n|)^\gamma}\right] \cdot \nabla T_h(u_n - z_n)$$
$$+ \int_\Omega (u_n - z_n) T_h(u_n - z_n) = \int_\Omega (f_n - g_n) T_h(u_n - z_n).$$

This equality can be written in an equivalent way as

$$\int_\Omega \frac{[a(x, \nabla u_n) - a(x, \nabla z_n)] \cdot \nabla T_h(u_n - z_n)}{(1+|u_n|)^\gamma}$$
$$+ \int_\Omega (u_n - z_n) T_h(u_n - z_n) = \int_\Omega (f_n - g_n) T_h(u_n - z_n)$$
$$- \int_\Omega \left[\frac{1}{(1+|u_n|)^\gamma} - \frac{1}{(1+|z_n|)^\gamma}\right] a(x, \nabla z_n) \cdot \nabla T_h(u_n - z_n).$$

By (1.4), the first term of the left hand side is nonnegative, so that it can be dropped; using Lagrange's theorem on the last term of the right hand side, we therefore have, since the absolute value of the derivative of the function $s \mapsto \frac{1}{(1+|s|)^{\gamma}}$ is bounded by $\gamma$,

$$\int_{\Omega} (u_n - z_n) T_h(u_n - z_n) \leq \int_{\Omega} (f_n - g_n) T_h(u_n - z_n) + \gamma h \int_{\Omega} |a(x, \nabla z_n)| |\nabla T_h(u_n - z_n)|.$$

Dividing by $h$ we obtain

$$\int_{\Omega} (u_n - z_n) \frac{T_h(u_n - z_n)}{h} \leq \int_{\Omega} |f_n - g_n| \frac{|T_h(u_n - z_n)|}{h} + \gamma \int_{\Omega} |a(x, \nabla z_n)| |\nabla T_h(u_n - z_n)|.$$

Since, for every fixed $n$, $u_n$ and $z_n$ belong to $H_0^1(\Omega)$, and $a(x, \xi)$ satisfies (1.3), the limit as $h$ tends to zero gives

$$\int_{\Omega} |u_n - z_n| \leq \int_{\Omega} |f_n - g_n|, \tag{3.7}$$

which then yields (3.5) passing to the limit and using the second part of Lemma 2.10.

The use of $T_h(u_n - z_n)^+$ as a test function and the same technique as above imply that

$$\int_{\Omega} (u_n - z_n)^+ \leq \int_{\{u_n \geq z_n\}} (f_n - g_n).$$

Hence, passing to the limit as $n$ tends to infinity, we obtain, if we suppose that $f \leq g$ almost everywhere in $\Omega$,

$$\int_{\Omega} (u - z)^+ \leq \int_{\{u \geq z\}} (f - g) \leq 0,$$

so that (3.6) is proved. □

Thanks to (3.5), we can prove that problem (1.1) has a unique solution obtained by approximation.

**Corollary 3.2.** *There exists a unique solution obtained by approximation of (1.1), in the sense that the solution $u$ in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma+2}{2}}(\Omega)$ obtained as limit of the sequence $u_n$ of solutions of (2.4) does not depend on the sequence $f_n$ chosen to approximate the datum $f$ in $L^{\frac{\gamma+2}{2}}(\Omega)$.*

*Remark 3.3.* Note that (3.7) implies the uniqueness of the solution of (2.2), while (3.6) implies that if $f \geq 0$, then the solution $u$ of (1.1) is nonnegative.

*Remark 3.4.* Corollary 3.2, together with estimates (3.5) and (2.5), implies that the map $S$ from $L^{\frac{\gamma+2}{2}}(\Omega)$ into itself defined by $S(f) = u$, where $u$ is the solution of (1.1) with datum $f$, is well defined and satisfies

$$\|S(f) - S(g)\|_{L^1(\Omega)} \le \|f - g\|_{L^1(\Omega)}, \quad \|S(f)\|_{L^{\frac{\gamma+2}{2}}(\Omega)} \le \|f\|_{L^{\frac{\gamma+2}{2}}(\Omega)}.$$

## 4. A non existence result

As stated in the Introduction, we prove here a non existence result for solutions of (1.1) if the datum is a bounded Radon measure concentrated on a set $E$ of zero harmonic capacity.

**Theorem 4.1.** *Assume that $\gamma > 1$, and let $\mu$ be a nonnegative Radon measure, concentrated on a set $E$ of zero harmonic capacity. Then there is no solution to*

$$\begin{cases} -\mathrm{div}\left(\dfrac{a(x,\nabla u)}{(1+u)^\gamma}\right) + u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

*More precisely, if $\{f_n\}$ is a sequence of nonnegative $L^\infty(\Omega)$ functions which converges to $\mu$ in the tight sense of measures, and if $u_n$ is the sequence of solutions to (2.4), then $u_n$ tends to zero almost everyhwere in $\Omega$ and*

$$\lim_{n\to+\infty} \int_\Omega u_n\,\varphi = \int_\Omega \varphi\,d\mu \qquad \forall \varphi \in W_0^{1,\infty}(\Omega).$$

*Remark 4.2.* A similar non existence result for the case $\gamma \le 1$ is much more complicated to obtain. Indeed, if for example $a(x,\xi) = \xi$, and $\gamma = 1$, the change of variables $v = \log(1+u)$ yields that $v$ is a solution to

$$\begin{cases} -\Delta v + \mathrm{e}^v - 1 = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Existence and non existence of solutions for such a problem has been studied in [9] (where the concept of "good measure" was introduced) and in [20] (if $N = 2$) and [2] (if $N \ge 3$).

*Proof.* Let $\mu$ be as in the statement. Then (see [13]) for every $\delta > 0$ there exists a function $\psi_\delta$ in $C_0^\infty(\Omega)$ such that

$$0 \le \psi_\delta \le 1, \qquad \int_\Omega |\nabla \psi_\delta|^2 \le \delta, \qquad \int_\Omega (1-\psi_\delta)d\mu \le \delta.$$

Note that, as a consequence of the estimate on $\psi_\delta$ in $H_0^1(\Omega)$, and of the fact that $0 \le \psi_\delta \le 1$, $\psi_\delta$ tends to zero in the weak* topology of $L^\infty(\Omega)$ as $\delta$ tends to zero.

If $f_n$ is a sequence of nonnegative functions which converges to $\mu$ in the tight convergence of measures, that is, if

$$\lim_{n\to+\infty}\int_\Omega f_n\,\varphi=\int_\Omega \varphi\,d\mu,\quad \forall\varphi\in C^0(\overline{\Omega}),$$

then

$$0\le\lim_{n\to+\infty}\int_\Omega f_n\,(1-\psi_\delta)=\int_\Omega (1-\psi_\delta)\,d\mu\le\delta. \tag{4.1}$$

Let $u_n$ be the nonnegative solution to the approximating problem (2.4). If we choose $1-(1+u_n)^{1-\gamma}$ as a test function in (2.4), we have, by (1.2), and dropping the nonnegative lower order term,

$$\alpha(\gamma-1)\int_\Omega\left|\frac{\nabla u_n}{(1+u_n)^\gamma}\right|^2\le(\gamma-1)\int_\Omega\frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+u_n)^{2\gamma}}\le\int_\Omega f_n.$$

Recalling (1.3), we thus have

$$\int_\Omega\left|\frac{a(x,\nabla u_n)}{(1+u_n)^\gamma}\right|^2\le\beta\int_\Omega\left|\frac{\nabla u_n}{(1+u_n)^\gamma}\right|^2\le C\int_\Omega f_n,$$

with $C$ depending on $\alpha$, $\beta$ and $\gamma$. Therefore, up to a subsequence, there exist $\sigma$ in $(L^2(\Omega))^N$ and $\rho$ in $L^2(\Omega)$ such that

$$\lim_{n\to+\infty}\frac{a(x,\nabla u_n)}{(1+u_n)^\gamma}=\sigma,\quad \lim_{n\to+\infty}\left|\frac{\nabla u_n}{(1+u_n)^\gamma}\right|=\rho, \tag{4.2}$$

weakly in $(L^2(\Omega))^N$ and $L^2(\Omega)$ respectively.

The choice of $[1-(1+u_n)^{1-\gamma}](1-\psi_\delta)$ as a test function in (2.4) gives

$$\begin{aligned}(\gamma-1)&\int_\Omega\frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+u_n)^{2\gamma}}(1-\psi_\delta)\\&+\int_\Omega u_n[1-(1+u_n)^{1-\gamma}](1-\psi_\delta)\\&=\int_\Omega f_n[1-(1+u_n)^{1-\gamma}](1-\psi_\delta)\\&+\int_\Omega\frac{a(x,\nabla u_n)\cdot\nabla\psi_\delta}{(1+u_n)^\gamma}[1-(1+u_n)^{1-\gamma}]\\&\le\int_\Omega f_n(1-\psi_\delta)\\&+\int_\Omega\frac{a(x,\nabla u_n)\cdot\nabla\psi_\delta}{(1+u_n)^\gamma}[1-(1+u_n)^{1-\gamma}].\end{aligned} \tag{4.3}$$

We study the right hand side. For the first term, (4.1) implies that

$$\lim_{\delta\to 0^+}\lim_{n\to+\infty}\int_\Omega f_n\,(1-\psi_\delta)=0,$$

while for the second one, we have, using (4.2), and the boundedness of $[1-(1+u_n)^{1-\gamma}]$,

$$\lim_{n\to+\infty}\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla\psi_\delta}{(1+u_n)^\gamma}[1-(1+u_n)^{1-\gamma}]=\int_\Omega \sigma\cdot\nabla\psi_\delta[1-(1+u)^{1-\gamma}].$$

Recalling that $\sigma$ is in $(L^2(\Omega))^N$, that $\psi_\delta$ tends to zero in $H_0^1(\Omega)$, and using the boundedness $[1-(1+u)^{1-\gamma}]$, we have

$$\lim_{\delta\to 0^+}\lim_{n\to+\infty}\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla\psi_\delta}{(1+u_n)^\gamma}[1-(1+u_n)^{1-\gamma}]=0.$$

Therefore, since both terms of the left hand side of (4.3) are nonnegative, we obtain

$$\lim_{\delta\to 0^+}\lim_{n\to+\infty}\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+u_n)^{2\gamma}}(1-\psi_\delta)=0.$$

Assumption (1.2) then gives

$$\begin{aligned}\lim_{\delta\to 0^+}\lim_{n\to+\infty}\alpha\int_\Omega\left|\frac{\nabla u_n}{(1+u_n)^\gamma}\right|^2(1-\psi_\delta)\\ \le\lim_{\delta\to 0^+}\lim_{n\to+\infty}\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla u_n}{(1+u_n)^{2\gamma}}(1-\psi_\delta)=0.\end{aligned}$$

Since the functional

$$v\in L^2(\Omega)\mapsto\int_\Omega |v|^2(1-\psi_\delta)$$

is weakly lower semicontinuous on $L^2(\Omega)$, we have

$$\int_\Omega|\rho|^2=\lim_{\delta\to 0^+}\int_\Omega|\rho|^2(1-\psi_\delta)\le\lim_{\delta\to 0^+}\lim_{n\to+\infty}\int_\Omega\left|\frac{\nabla u_n}{(1+u_n)^\gamma}\right|^2(1-\psi_\delta)=0,$$

which implies that $\rho=0$. Thus, since

$$\frac{\nabla u_n}{(1+u_n)^\gamma}=\frac{1}{\gamma-1}\nabla\left(1-(1+u_n)^{1-\gamma}\right),$$

by the second limit of (4.2) the sequence $1-(1+u_n)^{1-\gamma}$ weakly converges to zero in $H_0^1(\Omega)$, and so (up to subsequences) it strongly converges to zero in $L^2(\Omega)$. Therefore $u_n$ (up to subsequences) tends to zero almost everywhere in $\Omega$. Since the limit does not depend on the subsequence, the whole sequence $u_n$ tends to zero almost everywhere in $\Omega$.

We now have, for $\Phi$ in $(L^2(\Omega))^N$, and by (1.3),

$$\left|\int_\Omega \frac{a(x,\nabla u_n)}{(1+|u_n|)^\gamma}\cdot\Phi\right| \le \int_\Omega \left|\frac{a(x,\nabla u_n)}{(1+|u_n|)^\gamma}\right| |\Phi| \le \beta\int_\Omega \frac{|\nabla u_n|}{(1+|u_n|)^\gamma}|\Phi|.$$

Thus, by (4.2),

$$\left|\int_\Omega \sigma\cdot\Phi\right| = \lim_{n\to+\infty}\left|\int_\Omega \frac{a(x,\nabla u_n)}{(1+|u_n|)^\gamma}\cdot\Phi\right| \le \beta\int_\Omega \rho\,|\Phi| = 0,$$

which implies that $\sigma = 0$. Therefore, passing to the limit in (2.4), that is, in

$$\int_\Omega \frac{a(x,\nabla u_n)\cdot\nabla\varphi}{(1+u_n)^\gamma} + \int_\Omega u_n\,\varphi = \int_\Omega f_n\,\varphi, \quad \varphi\in W_0^{1,\infty}(\Omega),$$

we get, since the first term tends to zero,

$$\lim_{n\to+\infty}\int_\Omega u_n\,\varphi = \int_\Omega \varphi\,d\mu,$$

for every $\varphi$ in $W_0^{1,\infty}(\Omega)$, as desired. □

*Remark 4.3.* With minor technical changes (see [13]) one can prove the same result if $\mu$ is a signed Radon measure concentrated on a set $E$ of zero harmonic capacity.

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