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A general form of Gelfand–Kazhdan criterion

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Abstract. We formalize the notion of matrix coefficients for distributional vectors in a representation of a real reductive group, which consist of generalized functions on the group. As an application, we state and prove a Gelfand–Kazhdan criterion for a real reductive group in very general settings.

1. Tempered generalized functions and Casselman–Wallach representations

In this section, we review some basic terminologies in representation theory, which are necessary for this article. The two main ones are tempered generalized functions and Casselman–Wallach representations. We refer the readers to [10,11] as general references.

Let G be a real reductive Lie group, by which we mean that

(a) the Lie algebra $\mathfrak{g}$ of G is reductive;
(b) G has finitely many connected components; and
(c) the connected Lie subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ has a finite center.

We say that a (complex valued) function f on G is of moderate growth if there is a continuous group homomorphism

$$\rho: \ G \to \mathrm{GL}_n(\mathbb{C}), \ \text{for some } n \geq 1,$$

such that

$$|f(x)| \leq \mathrm{tr}(\overline{\rho(x)}^t \rho(x)) + \mathrm{tr}(\overline{\rho(x^{-1})}^t \rho(x^{-1})), \quad x \in G.$$

Here "$^-$" stands for the complex conjugation, and "$\ ^t$" the transpose, of a matrix. A smooth function $f \in \mathrm{C}^\infty(G)$ is said to be tempered if Xf has moderate growth for all X in the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_\mathbb{C})$. Here and as usual, $\mathfrak{g}_\mathbb{C}$ is the complexification of $\mathfrak{g}$, and $\mathrm{U}(\mathfrak{g}_\mathbb{C})$ is identified with the space of all left invariant differential operators on G. Denote by $\mathrm{C}^\xi(G)$ the space of all tempered functions on G.

A smooth function $f \in \mathrm{C}^\infty(G)$ is called Schwartz if

$$|f|_{X,\phi} := \sup_{x\in G} \phi(x)\, |(Xf)(x)| < \infty$$

for all $X \in \mathrm{U}(\mathfrak{g}_\mathbb{C})$, and all positive functions ϕ on G of moderate growth. Denote by $\mathrm{C}^\varsigma(G)$ the space of Schwartz functions on G, which is a nuclear Fréchet space under the seminorms

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$\{|\cdot|_{X,\phi}\}$. (See [9] or [8] for the notion as well as basic properties of nuclear Fréchet spaces.) We define the nuclear Fréchet space $\mathrm{D}^{\varsigma}(G)$ of Schwartz densities on G similarly. Fix a Haar measure dg on G, then the map

$$\begin{aligned} \mathrm{C}^{\varsigma}(G) &\to \mathrm{D}^{\varsigma}(G), \\ f &\mapsto f\,dg \end{aligned}$$

is a topological linear isomorphism. We define a tempered generalized function on G to be a continuous linear functional on $\mathrm{D}^{\varsigma}(G)$. Denote by $\mathrm{C}^{-\xi}(G)$ the space of all tempered generalized functions on G, equipped with the strong dual topology. This topology coincides with the topology of uniform convergence on compact subsets of $\mathrm{D}^{\varsigma}(G)$, due to the fact that every bounded subset of a complete nuclear space is relatively compact. Note that $\mathrm{C}^{\xi}(G)$ is canonically identified with a dense subspace of $\mathrm{C}^{-\xi}(G)$:

$$\mathrm{C}^{\xi}(G) \hookrightarrow \mathrm{C}^{-\xi}(G).$$

Remark 1.1. In [11], the space $\mathrm{C}^{\varsigma}(G)$ is denoted by $\mathcal{S}(G)$ and is called the space of rapidly decreasing functions on G. Note that $\mathrm{C}^{\varsigma}(G)$ (or $\mathcal{S}(G)$) is different from Harish–Chandra's Schwartz space of G, which is traditionally denoted by $\mathcal{C}(G)$.

By a representation of G, or just a representation when G is understood, we mean a continuous linear action of G on a complete, locally convex, Hausdorff, complex topological vector space. When no confusion is possible, we do not distinguish a representation with its underlying space. Let V be a representation. It is said to be smooth if the action map $G \times V \to V$ is smooth as a map of infinite dimensional manifolds. Recall that in general, if E and F are two complete, locally convex, Hausdorff, real topological vector spaces, and U_E and U_F are open subsets of E and F respectively, a continuous (C^0) map $f : U_E \to U_F$ is said to be C^1 if the differential

$$\begin{aligned} df : U_E \times E &\to F, \\ x, v &\mapsto \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t} \end{aligned}$$

exists and is C^0. Inductively f is said to be C^k ($k \geq 2$) if it is C^1 and df is C^{k-1}. We say that f is smooth if it is C^k for all $k \geq 0$. With this notion of smoothness, we define (smooth) manifolds and smooth maps between them as in the finite dimension case.

Denote by $\mathrm{C}(G; V)$ the space of V-valued continuous functions on G. It is a complete locally convex space under the topology of uniform convergence on compact sets. Similarly, denote by $\mathrm{C}^{\infty}(G; V)$ the (complete locally convex) space of smooth V-valued functions, with the usual smooth topology (which is defined by the seminorms $|f|_{\Omega,D,\mu} := \sup_{x\in\Omega}|Df(x)|_{\mu}$, where Ω is a compact subset of G, D is a differential operator on G, and $|\cdot|_{\mu}$ is a continuous seminorm on V).

Define

$$V(\infty) := \{v \in V \;:\; c_v \in \mathrm{C}^{\infty}(G; V)\},$$

where $c_v \in \mathrm{C}(G; V)$ is given by $c_v(g) := gv$. This is a G-stable subspace of V. It is easy to check that the linear map

$$V(\infty) \to \mathrm{C}^{\infty}(G; V), \quad v \mapsto c_v \tag{1}$$

is injective and has closed image. Identify $V(\infty)$ with the image of (1), and equip on it the subspace topology of $\mathrm{C}^{\infty}(G; V)$. Then $V(\infty)$ becomes a smooth representation of G, which is called the smoothing of V. The inclusion map $V(\infty) \to V$ is continuous since it can be identified with the map of evaluating at the identity $1 \in G$. If V is smooth, then this inclusion map is a homeomorphism, and hence $V(\infty) = V$ as a representation of G. In this case, its differential is defined to be the continuous $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$ action given by

$$Xv = (Xc_v)(1), \quad X \in \mathrm{U}(\mathfrak{g}_{\mathbb{C}}), v \in V.$$

The representation V is said to be $\mathrm{Z}(\mathfrak{g}_{\mathbb{C}})$ finite if a finite codimensional ideal of $\mathrm{Z}(\mathfrak{g}_{\mathbb{C}})$ annihilates $V(\infty)$, where $\mathrm{Z}(\mathfrak{g}_{\mathbb{C}})$ is the center of $\mathrm{U}(\mathfrak{g}_{\mathbb{C}})$. It is said to be admissible if every irreducible representation of a maximal compact subgroup K of G has finite multiplicity in V. A representation of G which is both admissible and $\mathrm{Z}(\mathfrak{g}_{\mathbb{C}})$ finite is called a Harish–Chandra representation.

The representation V is said to be of moderate growth if for every continuous seminorm $|\ \ |_\mu$ on V, there is a positive function ϕ on G of moderate growth, and a continuous seminorm $|\ \ |_\nu$ on V such that

$$|gv|_\mu \le \phi(g)|v|_\nu, \quad \text{for all } g \in G, v \in V.$$

The representation V is called a Casselman–Wallach representation if the space V is Fréchet, and the representation is smooth and of moderate growth, and Harish–Chandra. Following Wallach [11], the category of all such V is denoted by $\mathcal{FH}$ (the morphisms being G-intertwining continuous linear maps). The strong dual of a Casselman–Wallach representation is again a representation which is smooth and Harish–Chandra. Representations which are isomorphic to such strong duals form a category, which is denoted by $\mathcal{DH}$. By the Casselman–Wallach globalization theorem, both the category $\mathcal{FH}$ and $\mathcal{DH}$ are equivalent to the category $\mathcal{H}$ of admissible finitely generated $(\mathfrak{g}_{\mathbb{C}}, K)$-modules [3], [11, Chap. 11].

From the theory of real Jacquet modules (by Casselman and Wallach), and the Casselman–Wallach globalization theorem, every Casselman–Wallach representation is the smoothing of a Hilbert representation. In addition, all representation spaces in $\mathcal{FH}$ are automatically nuclear Fréchet (and in particular are reflexive), and all morphisms in $\mathcal{FH}$ and $\mathcal{DH}$ are automatically topological homomorphisms with closed image. See [11, Chap. 11]. Recall that in general, a linear map $\lambda : E \to F$ of topological vector spaces is called a topological homomorphism if the induced linear isomorphism $E/\mathrm{Ker}(\lambda) \to \mathrm{Im}(\lambda)$ is a topological isomorphism, where $E/\mathrm{Ker}(\lambda)$ is equipped with the quotient topology of E, and the image $\mathrm{Im}(\lambda)$ is equipped with the subspace topology of F.

2. Statement of results

Let U^∞, V^∞ be a pair of Casselman–Wallach representations of G which are contragredient to each other, i.e., we are given a G-invariant nondegenerate continuous bilinear map

$$\langle , \rangle : U^\infty \times V^\infty \to \mathbb{C}. \tag{2}$$

Note that U^∞ is the only Casselman–Wallach representation whose underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module is the contragredient of that of V^∞, and vice versa.

Denote by $U^{-\infty}$ the strong dual of V^∞. This is the only representation in $\mathcal{DH}$ which has the same underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-module as that of U^∞. Similarly, denote by $V^{-\infty}$ the strong dual of U^∞. For any $u \in U^\infty$, $v \in V^\infty$, the (usual) matrix coefficient $c_{u\otimes v}$ is defined by

$$c_{u\otimes v}(g) := \langle gu, v\rangle, \quad g \in G. \tag{3}$$

By the moderate growth conditions of U^∞ and V^∞, one easily checks that a matrix coefficient $c_{u\otimes v}$ is a tempered function on G.

The following theorem, which defines the notion of matrix coefficients for distributional vectors, is in a sense well-known. See the work of Shilika [7, Sect. 3] in the context of unitary representations, and the work of Kostant [6, Sect. 6.1] or Yamashita [12, Sect. 2.3] in the context of Hilbert representations. With the benefit of the Casselman–Wallach theorem, it is of interest and most natural to state the result in the context of Casselman–Wallach representations. This is also partly justified by the increasing use of these representations, due to the recent progress in restriction problems for classical groups. One purpose of this note is to provide a detailed proof of this result.

Theorem 2.1. *Let G be a real reductive group. Denote by $\mathrm{C}^\xi(G)$ (resp. $\mathrm{C}^{-\xi}(G)$) the space of all tempered functions (resp. tempered generalized functions) on G. Let (U^∞, V^∞) be a pair of Casselman–Wallach representations of G which are contragredient to each other. Then the matrix coefficient map*

$$\begin{aligned} U^\infty \times V^\infty &\to \mathrm{C}^\xi(G), \\ (u, v) &\mapsto c_{u\otimes v} \end{aligned} \tag{4}$$

extends to a continuous bilinear map

$$U^{-\infty} \times V^{-\infty} \to \mathrm{C}^{-\xi}(G),$$

and the induced $G \times G$ intertwining continuous linear map

$$c : U^{-\infty}\widehat{\otimes} V^{-\infty} \to \mathrm{C}^{-\xi}(G) \tag{5}$$

is a topological homomorphism with closed image.

Here "$\widehat{\otimes}$" stands for the completed projective tensor product of Hausdorff locally convex topological vector spaces. In our case, this coincides with the completed epsilon tensor product as the spaces involved are nuclear. Recall again that a linear map $\lambda : E \to F$ of topological vector spaces is called a topological homomorphism if the induced linear isomorphism $E/\mathrm{Ker}(\lambda) \to \mathrm{Im}(\lambda)$ is a topological isomorphism, where $E/\mathrm{Ker}(\lambda)$ is equipped with the quotient topology of E, and the image $\mathrm{Im}(\lambda)$ is equipped with the subspace topology of F. The action of $G \times G$ on $\mathrm{C}^{-\xi}(G)$ is obtained by continuously extending its action on $\mathrm{C}^\xi(G)$:

$$((g_1, g_2)f)(x) := f(g_2^{-1} x g_1).$$

Remark 2.2. (A) Denote by t_{U_∞} the paring $V^\infty \times U^\infty \to \mathbb{C}$, and view it as an element of $U^{-\infty}\widehat{\otimes} V^{-\infty}$. Then $c(t_{U_\infty}) \in \mathrm{C}^{-\xi}(G)$ is the character of the representation U^∞.
(B) Let $U_1^\infty, U_2^\infty, \ldots, U_k^\infty$ be pairwise inequivalent irreducible Casselman–Wallach representations of G. Let V_i^∞ be a Casselman–Wallach representation of G which is contragredient to U_i^∞, $i = 1, 2, \ldots, k$. Then the second assertion of Theorem 2.1 implies that the sum

$$\bigoplus_{i=1}^k U_i^{-\infty}\widehat{\otimes} V_i^{-\infty} \to \mathrm{C}^{-\xi}(G)$$

of the matrix coefficient maps is a topological embedding with closed image.

A second purpose of this note (and additional reason for writing down a proof of Theorem 2.1) is to prove the following generalized form of the Gelfand–Kazhdan criterion. For applications towards uniqueness of certain degenerate Whittaker models, it is highly desirable (and in fact necessary) to have the most general form of the Gelfand–Kazhdan criterion. We refer the reader to [5] for one such application.

Theorem 2.3. *Let S_1 and S_2 be two closed subgroups of G, with continuous characters*

$$\chi_i : S_i \to \mathbb{C}^\times, \quad i = 1, 2.$$

(a) *Assume that there is a continuous anti-automorphism σ of G such that for every $f \in \mathrm{C}^{-\xi}(G)$ which is an eigenvector of $\mathrm{U}(\mathfrak{g}_\mathbb{C})^G$, the conditions*

$$f(sx) = \chi_1(s) f(x), \quad s \in S_1,$$

and

$$f(xs) = \chi_2(s)^{-1} f(x), \quad s \in S_2$$

imply that

$$f(x^\sigma) = f(x).$$

Then for any pair of irreducible Casselman–Wallach representations (U^∞, V^∞) of G which are contragredient to each other, one has that

$$\dim \mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1}) \, \dim \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2}) \le 1.$$

(b) *Assume that for every $f \in \mathrm{C}^{-\xi}(G)$ which is an eigenvector of $\mathrm{U}(\mathfrak{g}_\mathbb{C})^G$, the conditions*

$$f(sx) = \chi_1(s) f(x), \quad s \in S_1,$$

and

$$f(xs) = \chi_2(s)^{-1} f(x), \quad s \in S_2$$

imply that

$$f = 0.$$

Then for any pair of irreducible Casselman–Wallach representations (U^∞, V^∞) of G which are contragredient to each other, one has that

$$\dim \mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1}) \, \dim \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2}) = 0.$$

Here and as usual, $\mathrm{U}(\mathfrak{g}_\mathbb{C})^G$ is identified with the space of bi-invariant differential operators on G, $\mathbb{C}_{\chi_i}$ is the one dimensional representation of S_i given by the character χ_i, and "Hom_{S_i}" stands for continuous S_i homomorphisms. The equalities in the theorem are to be understood as equalities of generalized functions. For example, $f(sx)$ denotes the left translation of f by s^{-1}. Similar notations apply throughout this article.

Remark 2.4. The original Gelfand–Kazhdan criterion is in [4] (for the non-archimedean case), and their idea has been very influential ever since. Various versions for real reductive groups have appeared in the literature, including [7] and [6] (for the study of Whittaker models, but both implicitly). Later works which state some versions of Gelfand–Kazhdan criterion explicitly include that of Yamashita [12, Theorem 2.10], and of Aizenbud et al. [1, Sect. 2]. The papers [7] and [12] have been particularly instructive for the current article.

As a consequence of Part (a) of Theorem 2.3, we have the following criterion of a strong Gelfand pair.

Corollary 2.5. *Let G' be a reductive closed subgroup of the real reductive group G. Let σ be a continuous anti-automorphism of G such that $\sigma(G') = G'$. Assume that for every $f \in \mathrm{C}^{-\xi}(G)$, the condition*

$$f(gxg^{-1}) = f(x) \quad \textit{for all } g \in G'$$

implies that

$$f(x^{\sigma}) = f(x).$$

Then for all irreducible Casselman–Wallach representation V of G, and V' of G', the space of G'-invariant continuous bilinear functionals on $V \times V'$ is at most one dimensional.

3. Proof of Theorem 2.1

Let $(U, \langle , \rangle_U)$ be a Hilbert space which carries a continuous representation of G so that its smoothing coincides with U^{∞}. Denote by V the strong dual of U, which carries a representation of G. (Its topology is given by the inner product

$$\langle \bar{u_1}, \bar{u_2} \rangle_V := \langle u_2, u_1 \rangle_U, \quad u_1, u_2 \in U,$$

where $\bar{u_i} \in V$ is the linear functional $\langle \cdot, u_i \rangle_U$ on U.) Note that the smoothing of V coincides with V^{∞}. Recall, as is well-known, that the three pairs U and V, U^{∞} and $V^{-\infty}$, and $U^{-\infty}$ and V^{∞}, are strong duals of each other as representations of G. For $u \in U$, $v \in V$, set

$$|u|_U := \sqrt{\langle u, u \rangle_U} \quad \text{and} \quad |v|_V := \sqrt{\langle v, v \rangle_V}.$$

Lemma 3.1. *There is a continuous seminorm $|\cdot|_G$ on $\mathrm{D}^{\varsigma}(G)$ such that*

$$\int_G |f(g)|\,|gu|_U\, dg \le |\omega|_G\, |u|_U, \quad \omega = f\, dg \in \mathrm{D}^{\varsigma}(G),\ u \in U.$$

Proof. This is well known, and follows easily from the facts that

(a) U is (automatically) of moderate growth [10, Lemma 2.A.2.2], and
(b) there is a positive continuous function ϕ on G of moderate growth so that $1/\phi$ is integrable. See [10, Lemma 2.A.2.4].

□

By Lemma 3.1, for any $\omega \in \mathrm{D}^{\varsigma}(G)$ and $u \in U$, the integral (in the sense of Riemann)

$$\omega u := \int_G \omega(g)\, gu \tag{6}$$

converges absolutely, and thus defines a vector in U. Furthermore, the bilinear map

$$\mathrm{D}^{\varsigma}(G) \times U \to U, \quad (\omega, u) \mapsto \omega u \tag{7}$$

is continuous.

Lemma 3.2. *For $\omega \in \mathrm{D}^{\varsigma}(G)$ and $u \in U$, we have $\omega u \in U^{\infty}$.*

Proof. Denote by L the representation of G on $\mathrm{D}^{\varsigma}(G)$ by left translations. Thus for $g \in G$ and $\omega \in \mathrm{D}^{\varsigma}(G)$, $L_g(\omega)$ is the push forward of ω via the map

$$G \to G, \quad x \mapsto gx.$$

It is routine to check that

$$L : G \times \mathrm{D}^{\varsigma}(G) \to \mathrm{D}^{\varsigma}(G)$$

is a smooth representation. For $X \in \mathrm{U}(\mathfrak{g}_{\mathbb{C}})$, denote by

$$L_X : \ \mathrm{D}^{\varsigma}(G) \to \mathrm{D}^{\varsigma}(G)$$

its differential. Trivially we have

$$c_{\omega u}(g) = (L_g(\omega))u, \quad g \in G, u \in U. \tag{8}$$

This implies that $c_{\omega u} \in \mathrm{C}^{\infty}(G; U)$, namely $\omega u \in U^{\infty}$. □

The following two lemmas are refinements of [7, Proposition 3.2].

Lemma 3.3. *The bilinear map*

$$\begin{aligned} \Phi_U : \ \mathrm{D}^{\varsigma}(G) \times U &\to U^{\infty}, \\ (\omega, u) &\mapsto \omega u. \end{aligned}$$

is continuous.

Proof. By the defining topology on U^{∞}, we need to show that the map

$$\mathrm{D}^{\varsigma}(G) \times U \to \mathrm{C}^{\infty}(G; U), \quad (\omega, u) \mapsto c_{\omega u}.$$

is continuous. In view of the topology on $\mathrm{C}^{\infty}(G; U)$, this is equivalent to showing that the bilinear map

$$\mathrm{D}^{\varsigma}(G) \times U \to \mathrm{C}(G; U), \quad (\omega, u) \mapsto X(c_{\omega u}),$$

is continuous for all $X \in \mathrm{U}(\mathfrak{g}_{\mathbb{C}})$. This is clearly true by observing that

$$X(c_{\omega u}) = c_{(L_X(\omega))u}, \quad \omega \in \mathrm{D}^{\varsigma}(G), \ u \in U.$$ □

For any $\omega \in \mathrm{D}^{\varsigma}(G)$, denote by $\omega^{\vee}$ its push forward via the map

$$G \to G, \quad g \mapsto g^{-1}.$$

Applying Lemma 3.3 to V, we get a continuous bilinear map

$$\Phi_V : \ \mathrm{D}^{\varsigma}(G) \times V \to V^{\infty}, \quad (\omega, v) \mapsto \omega v.$$

Now for any $\omega \in \mathrm{D}^{\varsigma}(G)$, we define the continuous linear map

$$U^{-\infty} \to U, \quad u \mapsto \omega u$$

to be the transpose of

$$V \to V^{\infty}, \quad v \mapsto \omega^{\vee} v,$$

i.e.,

$$\langle \omega u, v \rangle = \langle u, \omega^{\vee} v \rangle, \quad u \in U^{-\infty}, \ v \in V. \tag{9}$$

Lemma 3.4. *The bilinear map*

$$\begin{aligned} \Phi_V^\vee:\ \ \mathrm{D}^\varsigma(G)\times U^{-\infty} &\to U,\\ (\omega,u)&\mapsto \omega u \end{aligned} \tag{10}$$

is separately continuous and extends (7).

Proof. It is routine to check that (10) extends (7). We already know that (10) is continuous in the second variable.

Fix $u\in U^{-\infty}$, then the continuity of the bilinear map

$$\theta_u:\mathrm{D}^\varsigma(G)\times V\to\mathbb{C},\quad (\omega,v)\mapsto\langle u,\omega^\vee v\rangle.$$

clearly implies the continuity of the map

$$\mathrm{D}^\varsigma(G)\to U,\quad \omega\mapsto\omega u=\theta_u(\omega,\cdot). \qquad \square$$

Lemma 3.5. *The image of $\Phi_V^\vee$ is contained in U^∞, and the induced bilinear map*

$$\begin{aligned} \Phi_V^\vee:\ \ \mathrm{D}^\varsigma(G)\times U^{-\infty} &\to U^\infty,\\ (\omega,u)&\mapsto \omega u \end{aligned} \tag{11}$$

is separately continuous.

Proof. By chasing the definition of ωu, we see that the equality (8) still holds for all $\omega\in\mathrm{D}^\varsigma(G)$ and $u\in U^{-\infty}$. Again, this implies that $\omega u\in U^\infty$.

The proof for the separate continuity of $\Phi_V^\vee$ is similar to that of Lemma 3.3. We need to prove that the map

$$\mathrm{D}^\varsigma(G)\times U^{-\infty}\to\mathrm{C}^\infty(G;U),\quad(\omega,u)\mapsto c_{\omega u}.$$

is separately continuous. This is the same as that the bilinear map

$$\mathrm{D}^\varsigma(G)\times U^{-\infty}\to\mathrm{C}(G;U),\quad(\omega,u)\mapsto X(c_{\omega u}),$$

is separately continuous for all $X\in\mathrm{U}(\mathfrak{g}_\mathbb{C})$. This is again true by checking that

$$X(c_{\omega u})=c_{(L_X(\omega))u},\quad \omega\in\mathrm{D}^\varsigma(G),\ u\in U^{-\infty}. \qquad\square$$

To summarize, we get a separately continuous bilinear map

$$\mathrm{D}^\varsigma(G)\times U^{-\infty}\to U^\infty,\quad(\omega,u)\mapsto\omega u,$$

which extends the action map

$$\mathrm{D}^\varsigma(G)\times U\to U^\infty,\quad(\omega,u)\mapsto\omega u.$$

Similarly, we have a separately continuous bilinear map

$$\mathrm{D}^\varsigma(G)\times V^{-\infty}\to V^\infty,\quad(\omega,v)\mapsto\omega v,$$

which extends the action map

$$\mathrm{D}^\varsigma(G)\times V\to V^\infty,\quad(\omega,v)\mapsto\omega v.$$

Now define the (distributional) matrix coefficient map by

$$\begin{aligned} &c:\ U^{-\infty}\times V^{-\infty}\to\mathrm{C}^{-\xi}(G),\\ &c_{u\otimes v}(\omega):=\langle\omega u,v\rangle=\langle u,\omega^\vee v\rangle,\quad\omega\in\mathrm{D}^\varsigma(G). \end{aligned}\tag{12}$$

The last equality is implied by (9) and the afore-mentioned separate continuity statements.

Lemma 3.6. *The matrix coefficient map c defined in (12) is continuous.*

Proof. Note that the spaces $U^{-\infty}$, $V^{-\infty}$ and $\mathrm{C}^{-\xi}(G)$ are all strong duals of reflexive Fréchet spaces. Therefore by [9, Theorem 41.1], it suffices to show that the map (12) is separately continuous. First fix $u \in U^{-\infty}$, then the map

$$V^{-\infty} \to \mathrm{C}^{-\xi}(G), \quad v \mapsto c_{u\otimes v},$$

is continuous since it is the transpose of the continuous linear map

$$\mathrm{D}^{\varsigma}(G) \to U^{\infty}, \quad \omega \mapsto \omega u.$$

Similarly, fix $v \in V^{-\infty}$, the map

$$U^{-\infty} \to \mathrm{C}^{-\xi}(G), \quad u \mapsto c_{u\otimes v},$$

is continuous since it is the transpose of the continuous linear map

$$\mathrm{D}^{\varsigma}(G) \to V^{\infty}, \quad \omega \mapsto \omega^{\vee} v.$$

It is straightforward to check that (12) extends the usual matrix coefficient map (3). The proof of the first assertion of Theorem 2.1 is now complete.

To prove the second assertion of Theorem 2.1 (the generalized matrix coefficient map (5) is a topological homomorphism with closed image), we need two elementary lemmas.

Lemma 3.7. *Let $\lambda : E \to F$ be a G intertwining continuous linear map of representations of G. Assume that E is Fréchet, smooth, and of moderate growth, and F is a Casselman–Wallach representation. Then $E/\mathrm{Ker}(\lambda)$ is a Casselman–Wallach representation, and λ is a topological homomorphism.*

Proof. The quotient representation $E/\mathrm{Ker}(\lambda)$ is clearly Fréchet, smooth, and of moderate growth. It is $\mathrm{Z}(\mathfrak{g}_{\mathbb{C}})$ finite and K finite since it is mapped injectively to F. Therefore it is a Casselman–Wallach representation. The second assertion is a consequence of Casselman–Wallach globalization Theorem. □

Lemma 3.8. *Let $\lambda : E \to F$ be a continuous linear map of nuclear Fréchet spaces. Equip the dual spaces E' and F' with the strong dual topologies. Then λ is a topological homomorphism if and only if its transpose $\lambda^t : F' \to E'$ is. When this is the case, both λ and λ^t have closed images.*

Proof. The first assertion is a special case of [2, Sect. IV.2, Theorem 1]. (Recall that every bounded set in a complete nuclear space is relatively compact.)

Now assume that λ is a topological homomorphism. Then as a Hausdorff quotient of a Fréchet space, $E/\mathrm{Ker}(\lambda)$ is complete, and so is $\mathrm{Im}(\lambda)$, which implies that $\mathrm{Im}(\lambda)$ is closed in F. By the Extension Theorem of continuous linear functionals, the image of λ^t consists of all elements in E' which vanish on $\mathrm{Ker}(\lambda)$. This is closed in E'. □

Recall that both $V^{\infty}\widehat{\otimes}U^{\infty}$ and $\mathrm{D}^{\varsigma}(G)$ are nuclear Fréchet spaces. In particular, they are both reflexive. The map c of (5) is the transpose of a $G \times G$ intertwining continuous linear map

$$c^t : \mathrm{D}^{\varsigma}(G) \to V^{\infty}\widehat{\otimes}U^{\infty}.$$

(Here we have used the canonical isomorphism $E'\widehat{\otimes}F' \simeq (E\widehat{\otimes}F)'$, for nuclear Fréchet spaces E and F. See [9, Proposition 50.7].) Lemma 3.7 for the group $G \times G$ implies that c^t is a topological homomorphism. Lemma 3.8 then implies that c is a topological homomorphism with closed image. This completes the proof of the second assertion of Theorem 2.1.

4. Proof of Theorem 2.3

The argument is standard (cf. [4] or [7]). We use the notation and the assumption of Theorem 2.3. As before, $U^{-\infty}$ is the strong dual of V^{∞}, and $V^{-\infty}$ is the strong dual of U^{∞}. Suppose that both $\mathrm{Hom}_{S_1}(U^{\infty}, \mathbb{C}_{\chi_1})$ and $\mathrm{Hom}_{S_2}(V^{\infty}, \mathbb{C}_{\chi_2})$ are non-zero. Pick

$$0 \neq u_0 \in \mathrm{Hom}_{S_2}(V^{\infty}, \mathbb{C}_{\chi_2}) \subset U^{-\infty}$$

and

$$0 \neq v_0 \in \mathrm{Hom}_{S_1}(U^{\infty}, \mathbb{C}_{\chi_1}) \subset V^{-\infty}.$$

Then the matrix coefficient $c_{u_0 \otimes v_0} \in \mathrm{C}^{-\xi}(G)$ satisfies the followings:

$$\begin{cases} \text{it is an eigenvector of } \mathrm{U}(\mathfrak{g}_{\mathbb{C}})^G, & \text{(by the irreducibility hypothesis)} \\ c_{u_0 \otimes v_0}(sx) = \chi_1(s)\, c_{u_0 \otimes v_0}(x), & s \in S_1, \text{ and} \\ c_{u_0 \otimes v_0}(xs) = \chi_2(s)^{-1}\, c_{u_0 \otimes v_0}(x), & s \in S_2. \end{cases}$$

By the assumption of the theorem, we have

$$c_{u_0 \otimes v_0}(x^{\sigma}) = c_{u_0 \otimes v_0}(x). \tag{13}$$

Lemma 4.1. *Let $\omega \in \mathrm{D}^{\varsigma}(G)$. Denote by*

$$\sigma_* : \mathrm{D}^{\varsigma}(G) \to \mathrm{D}^{\varsigma}(G)$$

the push forward map by σ. Then

$$\omega u_0 = 0 \quad \textit{if and only if} \quad (\sigma_*(\omega))^{\vee} v_0 = 0.$$

Proof. As a consequence of (13), we have

$$c_{u_0 \otimes v_0}(\omega) = 0 \quad \text{if and only if} \quad c_{u_0 \otimes v_0}(\sigma_*(\omega)) = 0. \tag{14}$$

By the irreducibility of U^{∞}, $\omega u_0 = 0$ if and only if

$$\langle g(\omega u_0), v_0 \rangle = 0 \quad \text{for all } g \in G,$$

i.e.,

$$\langle (L_g \omega) u_0, v_0 \rangle = 0 \quad \text{for all } g \in G.$$

By (14), this is equivalent to saying that

$$\langle (\sigma_*(L_g \omega)) u_0, v_0 \rangle = 0 \quad \text{for all } g \in G.$$

Now the lemma follows from the following elementary identity and the irreducibility of V^{∞}:

$$\langle (\sigma_*(L_g \omega)) u_0, v_0 \rangle = \langle (\sigma_* \omega)(g^{\sigma} u_0), v_0 \rangle = \langle (g^{\sigma} u_0), (\sigma_* \omega)^{\vee} v_0 \rangle.$$ □

End of proof of Theorem 2.3. Let

$$0 \neq u_0' \in \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi 2}) \subset U^{-\infty}$$

be another element. Applying Lemma 4.1 twice, we get that for all $\omega \in \mathrm{D}^\varsigma(G)$,

$$\omega u_0 = 0 \quad \text{if and only if} \quad \omega u_0' = 0.$$

Therefore the two continuous G homomorphisms

$$\Phi : \omega \mapsto \omega u_0, \quad \text{and} \quad \Phi' : \omega \mapsto \omega u_0',$$

from $\mathrm{D}^\varsigma(G)$ to U^∞ have the same kernel, say J. Here and as before, we view $\mathrm{D}^\varsigma(G)$ as a representation of G via left translations. Both Φ and Φ' induce nonzero G homomorphisms into the irreducible Casselman–Wallach representation U^∞

$$\bar{\Phi}, \bar{\Phi}' : \mathrm{D}^\varsigma(G)/J \to U^\infty,$$

without kernel. Lemma 3.7 says that $\mathrm{D}^\varsigma(G)/J$ is a Casselman–Wallach representation of G. By Schur's lemma for Casselman–Wallach representations, $\bar{\Phi}'$ is a scalar multiple of $\bar{\Phi}$, which implies that u_0' is a scalar multiple of u_0. This proves that

$$\dim \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi 2}) = 1.$$

Similarly,

$$\dim \mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi 1}) = 1.$$

This finishes the proof of Part (a) of Theorem 2.3. Part (b) of Theorem 2.3 is immediate as the matrix coefficient $c_{u_0 \otimes v_0}$ would have to be zero if there were nonzero $u_0 \in \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi 2})$ and nonzero $v_0 \in \mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi 1})$. □

5. Proof of Corollary 2.5

Let G' be a reductive closed subgroup of the real reductive group G, and let σ be a continuous anti-automorphism of G such that $\sigma(G') = G'$, as in Corollary 2.5. Assume that for every $f \in \mathrm{C}^{-\xi}(G)$, the condition

$$f(gxg^{-1}) = f(x), \quad g \in G'$$

implies that

$$f(x^\sigma) = f(x).$$

Set

$$H := G \times G',$$

which contains G as a subgroup. Denote by $S \subset H$ the group G' diagonally embedded in H. For any $x = (g, g') \in H$, set

$$x^\sigma := (g^\sigma, g'^\sigma).$$

Lemma 5.1. *If $f \in \mathrm{C}^{-\xi}(H)$ is invariant under the adjoint action, then it is σ-invariant.*

Proof. The assumption at the beginning of this section trivially implies that every invariant tempered generalized function on G is σ-invariant. Since both G and G' are unimodular, it also implies that every invariant tempered generalized function on G' is σ-invariant. The lemma follows easily from these two facts. □

Lemma 5.2. *If $f \in \mathrm{C}^{-\xi}(H)$ is a bi S-invariant, then it is σ-invariant.*

Proof. The multiplication map

$$\begin{aligned} m_H : S \times G \times S &\to H \\ (s_1, g, s_2) &\mapsto s_1 g s_2 \end{aligned}$$

is a surjective submersion. Let f be a bi S-invariant generalized function on H. Then its pull back has the form

$$m_H^*(f) = 1 \otimes f_G \otimes 1, \quad \text{with } f_G \in \mathrm{C}^{-\xi}(G).$$

We use "Ad" to indicate the adjoint action. By considering the commutative diagram

$$\begin{array}{ccc} S \times G \times S & \xrightarrow[m_H]{} & H \\ {\scriptstyle \mathrm{Ad}_s \times \mathrm{Ad}_s \times \mathrm{Ad}_s}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_s} \\ S \times G \times S & \xrightarrow[m_H]{} & H, \end{array}$$

for all $s \in S$, we conclude that f_G is invariant under the adjoint action of G'. Therefore f_G is σ-invariant by assumption.

Set

$$(s_1, g, s_2)^\sigma := (s_2^\sigma, g^\sigma, s_1^\sigma), \quad (s_1, g, s_2) \in S \times G \times S.$$

Then $1 \otimes f_G \otimes 1 \in \mathrm{C}^{-\xi}(S \times G \times S)$ is also σ-invariant. We conclude that f is σ-invariant by appealing to the commutative diagram

$$\begin{array}{ccc} S \times G \times S & \xrightarrow[m_H]{} & H \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ S \times G \times S & \xrightarrow[m_H]{} & H. \end{array}$$

□

Let (V_H, ρ) be an irreducible Casselman–Wallach representation of H.

Lemma 5.3. *Set*

$$\rho_{-\sigma}(h) := \rho(h^{-\sigma}).$$

Then $(V_H, \rho_{-\sigma})$ is an irreducible Casselman–Wallach representation of H which is contragredient to (V_H, ρ).

Proof. Denote by

$$\chi_\rho \in \mathrm{C}^{-\xi}(H)$$

the character of (V_H, ρ). Then its contragredient representation has character $\chi_\rho(h^{-1})$.

It is clear that $(V_H, \rho_{-\sigma})$ is an irreducible Casselman–Wallach representation, with character $\chi_\rho(h^{-\sigma})$. Since a character is always invariant under the adjoint action, Lemma 5.1 implies that

$$\chi_\rho(h^{-1}) = \chi_\rho(h^{-\sigma}).$$

The lemma then follows from the well-known fact that an irreducible Casselman–Wallach representation is determined by its character. □

Lemma 5.4. *We have that*

$$\dim \mathrm{Hom}_S(V_H, \mathbb{C}) \leq 1.$$

Proof. Denote by U_H the irreducible Casselman–Wallach representation which is contragredient to V_H. Lemma 5.2 and Part (a) of Theorem 2.3 imply that

$$\dim \mathrm{Hom}_S(U_H, \mathbb{C}) \dim \mathrm{Hom}_S(V_H, \mathbb{C}) \leq 1.$$

Lemma 5.3 implies that

$$\dim \mathrm{Hom}_S(U_H, \mathbb{C}) = \dim \mathrm{Hom}_S(V_H, \mathbb{C}).$$

We therefore conclude that $\dim \mathrm{Hom}_S(V_H, \mathbb{C}) \leq 1$. □

We finish the proof of Corollary 2.5 by taking $V_H = V \widehat{\otimes} V'$ in Lemma 5.4.

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