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The stability of m-fold circles in the curve shortening problem

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Abstract. The stability of m-fold circles in the curve shortening problem (CSP) is studied in this paper. It turns out that a suitable perturbation of m-fold circle will shrink to a point asymptotically like an m-fold circle under the curve shortening flow.

1. Introduction and formulation

The curve shortening problem (CSP) for simple convex curves was settled by Gage and Hamilton [8] who show that the flow shrinks to a round point in finite time. For locally convex curves it is routine to show that the flow exists for small time. However, simple examples show that singularities may develop before it can shrink to a point. For example, taking the initial curve to be a perturbation of a 2-fold circle so that there is a smaller circle C_1 contained in a larger one C_2, see Fig. 1. It is rather clear that a singularity would occur when the smaller circle contracts to a point, and forms a cusp, while the larger circle still has positive length. In view of this example, it is not expected that Gage–Hamilton's theorem holds for immersed, locally convex curves. Nevertheless, when the initial curve is nice, for example, when it has certain rotational symmetry, the CSP may still drive this curve to a point. Thus it still makes sense to study the asymptotic shape of the flow.

Same as in the embedded case, one expects the asymptotic shape to be a self-similar solution as the circle is a self-similar solution. (In fact, the circle is the only self-similar curve which is embedded.) All self-similar curves with total curvature $2m\pi$ and n many leaves (that's, it is symmetric under $2\pi/n$-rotation) have been classified completely by Abresch–Langer [1]. They proved

Abresch–Langer Theorem. *Let n and m be mutually prime. There exists a smooth, strictly locally convex, non-circular immersed self-similar solution $\gamma_{m,n}$ to the CSP with total curvature $2m\pi$ and n maxima of curvature if and only if*

$$\frac{1}{2} < \frac{m}{n} < \sqrt{\frac{1}{2}}. \tag{1.1}$$

Remark 1. Any circle or multi-fold circle is also a self-similar solution to the CSP.

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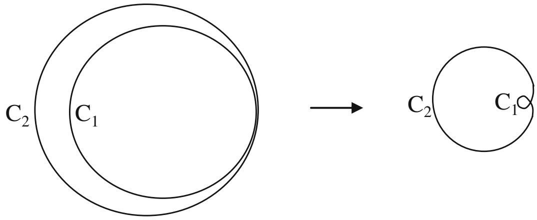

Fig. 1. The cusp appearing in the CSP

In this paper, we will show that a small $2m\pi/n$-periodic perturbation of m-fold circle will shrink to a point asymptotically like an m-fold circle under the CSP. Here and after, m and n always denote two mutually prime numbers satisfying (1.1). We note that the stability of an m-fold circle under the CSP has been investigated by many authors. The earliest one can be dated back to Epstein–Weinstein [7] and later, Abresch–Langer [1], Chou–Zhu [6]. However, they just established the linear stability of an m-fold circle under $2m\pi/n$-periodic perturbations. Our result in fact has disclosed the asymptotic stability of an m-fold circle under the same class of perturbations.

Let γ_m denote the m-fold circle whose enclosed algebraic area is $m\pi$. Our main theorem reads as follows:

Theorem 1. *Consider the CSP with initial data $\gamma_{0\delta} = \gamma_m + \delta\phi\boldsymbol{n}_0$ with $\boldsymbol{n}_0$ being the outward unit normal vector of γ_m, where ϕ is any given smooth $2m\pi/n$-periodic function of normal angle of γ_m such that $\gamma_{0\delta}$ is convex for all small δ. In the case that $|\delta|$ is small enough, the solution shrinks to a point asymptotically like an m-fold circle.*

About the stability of other shrinking self-similar solutions, one may refer to [1,4] and [5]. For the references of relative problems in the generalized CSP, see [2,3,12] and etc.

In the following, we will formulate our problem in terms of support function. Let γ_0 be a given immersed closed plane curve parameterized by p. We study the CSP with an initial data γ_0, i.e.,

$$\begin{cases} \dfrac{\partial \gamma}{\partial t} = -k\mathbf{n}, \\ \gamma(p, 0) = \gamma_0(p) \end{cases} \tag{1.2}$$

where $\gamma(p, t)$ (also denoted by $\gamma(t)$) is a family of curves and $k(p, t)$ denotes the curvature with respect to outward normal vector field $\mathbf{n}(p, t)$.

For each locally convex curve $\gamma(p)$, the support function is defined as

$$h(\theta) = <\gamma(p), \mathbf{n}(p)>,$$

where θ is the angle of the outward normal vector $\mathbf{n} = (\cos\theta, \sin\theta)$ and $\theta \in [0, 2m\pi]$, with $2m\pi$ being the total curvature of γ. The position of the curve γ as a function of θ in terms of h is determined by

$$\gamma(\theta) = (h(\theta)\cos\theta - h_\theta(\theta)\sin\theta, h(\theta)\sin\theta + h_\theta(\theta)\cos\theta). \tag{1.3}$$

The curvature k is given by

$$k(\theta) = \frac{1}{h(\theta) + h_{\theta\theta}(\theta)}. \tag{1.4}$$

The formulation in terms of the support function is very useful for the discussion of convex plane curves. The relationship between locally convex, closed curves and their support functions is contained in the following proposition [6].

Proposition 1.1. *Any $2m\pi$-periodic function h with $h_{\theta\theta}+h>0$ determines a closed, locally convex curve by (1.3) whose support function is h and is with positive curvature given by (1.4). Two such curves with corresponding support functions h and $\bar{h}$, respectively, differ by a translation if and only if the difference of h and $\bar{h}$ is equal to $C_1 \cos\theta + C_2 \sin\theta$ for some C_1 and C_2. Moreover, γ is embedded if and only if h is 2π-periodic.*

Assume that the total curvature of the locally convex curve γ is $2m\pi$. Then the algebraic area of γ can be determined by its support function h, i.e.,

$$\frac{1}{2}\int_0^{2m\pi} h(h + h_{\theta\theta})\, d\theta.$$

And the length of the curve γ can be determined by the formula,

$$\int_0^{2m\pi} h\, d\theta.$$

Now, in terms of the support functions, we can reformulate the problem (1.2). Here and after we use the notation

$$I = [0, 2m\pi/n]$$

to denote the circle formed by identifying the endpoints of $[0, 2m\pi/n]$. We denote by h_m the support function of the curve γ_m. Obviously, $h_m = 1$.

Set the initial data in the problem (1.2) to be $\gamma_{0\delta}$ just as defined in Theorem 1. Then the problem (1.2) is reformulated as the following initial value problem for $h(\theta, t)$,

$$\begin{cases} h_t = -\dfrac{1}{h + h_{\theta\theta}}, & (\theta, t) \in I \times (0, T], \\ h(\theta, 0) = h_m(\theta) + \delta\phi, & \theta \in I. \end{cases} \tag{1.5}$$

For the convenience, we replace $h_m + \delta\phi$ with

$$h_{0\delta}(\theta) = \frac{h_m(\theta) + \delta\phi}{\Psi(\phi)^{1/2}},$$

where $\Psi(\phi) = 1 + \fint_0^{2m\pi}(2\delta\phi + \delta^2\phi^2 - \delta^2\phi_\theta^2)\,d\theta$. The corresponding curve supported by $h_{0\delta}$ is still denoted by $\gamma_{0\delta}$, that is,

$$\gamma_{0\delta} = \frac{\gamma_m + \delta\phi\mathbf{n}_0}{\Psi(\phi)^{1/2}}.$$

This replacement implies that the enclosed algebraic area of $\gamma_{0\delta}$ is $m\pi$. In the following, we will prove the main theorem by considering the problem (1.2) with this initial data $\gamma_{0\delta}$. Such a replacement of initial data will not affect the truth of the main theorem since it does not change asymptotic shape of the solution to the problem (1.2). In the sequel, we always use the notations '$h_{0\delta}$' and '$\gamma_{0\delta}$' to denote such initial data.

Denote by $A(t)$ the algebraic area enclosed by the solution $\gamma(t)$ to the problem (1.2). A direct computation shows that $dA/dt = -2m\pi$. Note that $A(0) = m\pi$. Hence, $A(t) = m\pi(1-2t)$. To examine the ultimate shape of the solution $\gamma(t)$, we normalize it so that its enclosed area is always equal to $m\pi$. Specifically, we let

$$\tilde{\gamma}(\tau) = (m\pi/A(t))^{1/2}\gamma(t)$$

where $d\tau/dt = m\pi/A(t)$. Then $\tilde{\gamma}(\tau)$ solves the problem

$$\begin{cases} \dfrac{\partial\tilde{\gamma}}{\partial\tau} &= -\tilde{k}\mathbf{n} + \tilde{\gamma}, \\ \tilde{\gamma}(0) = \gamma_{0\delta}, \end{cases} \tag{1.6}$$

where $\tilde{k}$ is the curvature of $\tilde{\gamma}$ with respect to the outward unit normal vector of $\tilde{\gamma}$, still denoted by $\mathbf{n}$. Note that this problem is equivalent to problem (1.2) as long as $A(t) > 0$.

In terms of the normalized support function

$$\tilde{h} = (m\pi/A(t))^{1/2}h,$$

we reformulate the problem (1.6) as

$$\begin{cases} \tilde{h}_\tau &= -\dfrac{1}{\tilde{h} + \tilde{h}_{\theta\theta}} + \tilde{h}, \quad (\theta,\tau) \in I \times (0,T], \\ \tilde{h}(\theta,0) = h_{0\delta}, \quad \theta \in I. \end{cases} \tag{1.7}$$

Our problem can also be formulated in terms of the curvature function. This formulation will be useful in our proof. Let $k(\theta,t)$ denote the curvature of the curve $\gamma(t)$. Then $k(\theta,t)$ solves the equation

$$k_t = k^2(k_{\theta\theta} + k),$$

with initial data

$$k(\theta,0) = (h_{0\delta} + (h_{0\delta})_{\theta\theta})^{-1} := k_{0\delta}.$$

In terms of the normalized curvature

$$\tilde{k} = (A(t)/m\pi)^{1/2}k,$$

we have

$$\begin{cases} \tilde{k}_\tau = \tilde{k}^2[\tilde{k}_{\theta\theta} + \tilde{k}] - \tilde{k}, & (\theta, \tau) \in I \times (0, T], \\ \tilde{k}(\theta, 0) = k_{0\delta}, & \theta \in I, \end{cases} \tag{1.8}$$

and $d\tau/dt = m\pi/A(t)$. As the equivalence between the problems (1.6) and (1.8) is concerned, one may keep the following fact in mind, that is, if the curvature k of a uniformly convex curve γ is known as a function of the normal angle θ, then the curve γ is completely determined by, up to a translation,

$$\gamma(\theta) = \gamma(0) + \left(\int_0^\theta \frac{\cos\theta}{k(\theta)}\, d\theta, \quad \int_0^\theta \frac{\sin\theta}{k(\theta)}\, d\theta \right).$$

In the remaining part of this paper, we will prove our main theorem in three steps. First, some a priori estimates are obtained for solutions to the problem (1.7) in Sect. 1. Subsequently, the global existence result is established in Sect. 2. Finally, the convergence result is proved in Sect. 3.

2. A priori estimate

In this section, some a priori estimate is obtained for solutions to the problem (1.7). This estimate will be crucial to yield global existence in the subsequent section. As a byproduct, a stability result is also obtained.

As δ tends to 0, the initial support function $h_{0\delta}(\theta)$ tends to 1 on $[0, 2m\pi]$. It is natural to link the asymptotic behavior of $\tilde{h}$ to the asymptotic stability of the m-fold circle in this case.

The linearized problem for the Eq. (1.7) at $\tilde{h} \equiv 1$ is given by

$$\varphi_t = \varphi_{\theta\theta} + 2\varphi.$$

For those perturbation φ which preserves the enclosed area, we have

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_0^{2m\pi} (\tilde{h} + \varepsilon\varphi)(\tilde{h}_{\theta\theta} + \varepsilon\varphi_{\theta\theta} + \tilde{h} + \varepsilon\varphi)\, d\theta \\ &= \int_0^{2m\pi} (\varphi_{\theta\theta} + \varphi)\tilde{h} + (\tilde{h}_{\theta\theta} + \tilde{h})\varphi\, d\theta \\ &= \int_0^{2m\pi} \varphi\, d\theta \end{aligned}$$

at $\tilde{h} \equiv 1$. By Abresch–Langer Theorem, it can be easily checked that (see Lemma 2.3 below), the eigenvalue problem

$$-(\varphi_{\theta\theta} + 2\varphi) = \lambda\varphi, \ \ \varphi \in X$$

with

$$X = \left\{ \varphi \in L^2(I) : \int\limits_0^{2m\pi} \varphi \, d\theta = 0 \right\}$$

satisfies $\lambda > 0$ and hence suggests the asymptotic stability of the m-fold circle. However, we cannot use the known result such as those in Henry [9] for this conclusion for two reasons: First, the principle part of our equation, $-(\tilde{h}_{\theta\theta} + \tilde{h})^{-1}$, is fully nonlinear; and, second, our area constraint, $\int_0^{2m\pi} \tilde{h}(\tilde{h}_{\theta\theta} + \tilde{h}) \, d\theta = 2m\pi$, is nonlinear.

To overcome these difficulties we shall represent the equation in polar coordinates. To this end, we let α be the polar angle, and set

$$\hat{\gamma}(\alpha, \tau) = \tilde{\gamma}(\theta, \tau),$$

where, as the normal angle θ runs from 0 to $2m\pi$, the polar angle α may also be assumed to run from 0 to $2m\pi$. We still denote by $\mathbf{n}$ the outward unit normal vector of $\hat{\gamma}(\alpha, \tau)$ or $\tilde{\gamma}(\theta, \tau)$. Then we have,

$$\begin{aligned} \frac{\partial}{\partial \tau} \hat{\gamma}(\alpha, \tau) &= \tilde{\gamma}_\theta(\theta, \tau) \frac{\partial \theta}{\partial \tau} + \tilde{\gamma}_\tau(\theta, \tau) \\ &= \tilde{\gamma}_\theta(\theta, \tau) \frac{\partial \theta}{\partial \tau} + \tilde{k}\mathbf{n} + \tilde{\gamma}. \end{aligned}$$

Taking product with $\mathbf{n}$ we get

$$\left\langle \frac{\partial \hat{\gamma}}{\partial \tau}, \mathbf{n} \right\rangle \Big|_{(\alpha,\tau)} = (-\tilde{k} + \tilde{h})\big|_{(\theta,\tau)} = (-\hat{k} + \hat{h})\big|_{(\alpha,\tau)},$$

where $\hat{h}$ and $\hat{k}$ are the support function and curvature function of $\hat{\gamma}$ respectively.

Denote the m-circle by $\Gamma(\alpha) := (\cos\alpha, \sin\alpha)$ with $\alpha \in [0, 2m\pi]$, and its outward unit normal vector by $\mathbf{N}(\alpha) = (\cos\alpha, \sin\alpha)$. Set

$$\hat{\gamma}(\alpha, \tau) = \Gamma(\alpha) - d(\alpha, \tau)\mathbf{N}(\alpha) = (1 - d)\mathbf{N}(\alpha).$$

Then the Eq. (1.6) for the normalized flow is equivalent to the equation

$$\langle \mathbf{N}(\alpha), \mathbf{n}(\alpha, \tau) \rangle d_\tau = \hat{k} - \hat{h}, \quad \alpha \in I, \tau > 0.$$

A direct computation yields,

$$\begin{aligned} \mathbf{n}(\alpha, \tau) &= \frac{(-d_\alpha \sin\alpha + (1-d)\cos\alpha, d_\alpha \cos\alpha + (1-d)\sin\alpha)}{\sqrt{(1-d)^2 + d_\alpha^2}}, \\ \tilde{k}(\alpha, \tau) &= \frac{(1-d)d_{\alpha\alpha} + 2d_\alpha^2 + (1-d)^2}{[(1-d)^2 + d_\alpha^2]^{3/2}} \end{aligned}$$

and

$$\tilde{h}(\alpha, \tau) = (1-d)^2[(1-d)^2 + d_\alpha^2]^{-1/2}.$$

Now the equation becomes

$$d_\tau = \frac{d_{\alpha\alpha} + 2d_\alpha^2/(1-d) + (1-d)}{(1-d)^2 + d_\alpha^2} - (1-d), \quad \alpha \in I, \tau > 0.$$

We write it into the following form

$$d_\tau = d_{\alpha\alpha} + 2d + F(d, d_\alpha, d_{\alpha\alpha}), \quad \alpha \in I, \tau > 0,$$

where

$$F(z, p, q) = \frac{1}{(1-z)^2 + p^2}\left\{(2z - z^2)q - qp^2 + \frac{2p^2}{1-z} - (1+z)p^2 + (1-z)z^2\right\}.$$

One observes that the area enclosed by $\tilde{\gamma}(\tau)$ keeps unchanged. Hence, if the area enclosed by $\hat{\gamma}(.,0)$ is $m\pi$, then it holds that

$$\int_I (1 - d(\alpha,\tau))^2\, d\alpha = m\pi/n, \quad \forall\, \tau \geq 0,$$

that is

$$\int_I (2d - d^2)\, d\alpha = 0, \quad \forall \tau \geq 0.$$

To study the stability of $d \equiv 0$, we need to introduce a new function defined by

$$u = 2d - d^2.$$

Note that d can be resolved from u by the formula $d = u/(1+\sqrt{1-u})$. Then u satisfies the equation

$$u_\tau - u_{\alpha\alpha} - 2u = G(u, u_\alpha, u_{\alpha\alpha}), \tag{2.1}$$

where

$$\begin{aligned} G(z, p, q) &= \frac{1}{4(1-z)^2 + p^2}\left\{4(1-z)zq - p^2 q + 2zp^2 - \frac{p^4}{2(1-z)} \right. \\ &\quad \left. + \frac{2z^2p^2}{(1+\sqrt{1-z})^2} + \frac{4(1-z)^2 z^2}{(1+\sqrt{1-z})^2}\right\} + \frac{p^2}{2(1-z)} - \frac{2z^2}{(1+\sqrt{1-z})^2}. \end{aligned}$$

We have the following structural condition of G

$$\begin{aligned} &|G(z,p,q)| \leq C_0(|z|^2 + |p|^2 + |q|^2), \\ &\quad \forall (z,p,q) \in D := \{(z,p,q) : |z| + |p| + |q| < 1/2\}, \end{aligned} \tag{2.2}$$

for some constant $C_0 > 0$. Note that the solution $u(\alpha, t)$ to (2.1) satisfies that $\int_I u(x,t)\,dx = 0$ as long as the initial data u_0 satisfies that $\int_I u_0\,dx = 0$.

Define

$$X_1 = C^{2,\sigma}(I),$$

and

$$X_2 = \left\{ w \in X_1, \int_I w\,d\alpha = 0 \right\}.$$

Then X_1 and X_2 are the Banach spaces with the norm $||\cdot||_{C^{2,\sigma}(I)}$ for some $\sigma \in (0,1)$. Recall that I is the circle obtained by identifying 0 and $2m\pi/n$ in $[0, 2m\pi/n]$. Similarly, we define three more Banach spaces

$$\widetilde{X}_1 = \widetilde{C}^{2,\sigma}(I \times [0,T])$$

and

$$\widetilde{X}_2 = \left\{ w \in \widetilde{X}_1, \int_I w(\alpha,\tau)\,d\alpha = 0 \right\},$$

with the parabolic Hölder norm $\widetilde{C}^{2,\sigma}(I \times [0,T])$ for some $T > 0$; and

$$\widetilde{X}_3 = \widetilde{C}^{\sigma}(I \times [0,T])$$

with the norm $||\cdot||_{\widetilde{C}^{\sigma}(I\times[0,T])}$.

The implicit function theorem gives the following existence result for a special nonlinear parabolic equation. In the following lemmas, we may take

$$T = 1$$

in the definitions of Banach spaces $\widetilde{X}_1$, $\widetilde{X}_2$ and $\widetilde{X}_3$.

Lemma 2.1. *Consider the equation*

$$\begin{cases} w_\tau &= w_{\alpha\alpha} + 2w + g(w, w_\alpha, w_{\alpha\alpha}), \\ w(0) &= w_0 \in X_1, \end{cases} \tag{2.3}$$

on $I \times [0,1]$ where $g(z,p,q)$ is a $C^{1,\sigma}$-function on the domain D (defined in (2.2)) satisfying

$$g(0,0,0) = \frac{\partial g}{\partial z}(0,0,0) = \frac{\partial g}{\partial p}(0,0,0) = \frac{\partial g}{\partial q}(0,0,0) = 0.$$

Then there exists $\varepsilon_0 \in (0,1)$ such that for any initial data w_0 with $||w_0||_{X_1} \le \varepsilon_0$ the problem (2.3) has a unique solution $w \in \widetilde{X}_1$ satisfying

$$||w||_{\widetilde{X}_1} \le C_1$$

where ε_0 and C_1 depend only on $||g||_{C^{1,\sigma}(D)}$.

Proof. Letting $v = w - w_0$, problem (2.3) is turned into the equivalent problem for v:

$$\begin{cases} v_\tau &= v_{\alpha\alpha} + 2v + w_{0\alpha\alpha} + 2w_0 + g(v + w_0, v_\alpha + w_{0\alpha}, v_{\alpha\alpha} + w_{0\alpha\alpha}), \\ v(0) = 0. \end{cases} \tag{2.4}$$

Set

$$\mathcal{F}[v] = v_{\alpha\alpha} + 2v + w_{0\alpha\alpha} + 2w_0 + g(v + w_0, v_\alpha + w_{0\alpha}, v_{\alpha\alpha} + w_{0\alpha\alpha}) - v_\tau$$

and regard it as a mapping from $\widetilde{X}_1$ to $\widetilde{X}_3$. We compute its Frechét derivative at $v \equiv 0$:

$$D\mathcal{F}[0]\varphi = \varphi'' + 2\varphi + \frac{\partial g}{\partial z}\varphi + \frac{\partial g}{\partial p}\varphi' + \frac{\partial g}{\partial q}\varphi'' - \varphi_t$$

where $\nabla g = (\frac{\partial g}{\partial z}, \frac{\partial g}{\partial p}, \frac{\partial g}{\partial q})$ is evaluated at $(w_0, w_{0\alpha}, w_{0\alpha\alpha})$. By our assumption on g, for w_0 with $||w_0||_{X_1} \le 1/2$, the linearized problem

$$D\mathcal{F}[0]\varphi = f$$

is invertible by the theory of the linear parabolic equation:

$$||\varphi||_{\widetilde{X}_1} \le C_0||f||_{\widetilde{X}_3}$$

where the constants C_0 depends only on $||g||_{C^{1,\sigma}(D)}$. By the implicit function theorem, there exist $r_0, \rho_0 > 0$ (depending on $||g||_{C^{1,\sigma}(D)}$) such that the equation

$$\mathcal{F}[v] = f, \quad ||f - \mathcal{F}[0]||_{\widetilde{X}_3} \le r_0$$

is uniquely solvable in the class of v satisfying $||v - 0||_{\widetilde{X}_1} \le \rho_0$.

Now observe that $\mathcal{F}[0] = w_{0\alpha\alpha} + 2w_0 + g(w_0, w_{0\alpha}, w_{0\alpha\alpha})$ tends to zero as $||w_0||_{X_1}$ tends to zero. It follows that there exists $\varepsilon_0 \in (0, 1)$ such that $||\mathcal{F}[0]||_{\widetilde{X}_3} \le r_0$ whenever $||w_0||_{X_1} \le \varepsilon_0$. It means that

$$\mathcal{F}[v] = 0$$

has a solution v satisfying $||v||_{\widetilde{X}_1} \le \rho_0$. Taking $C_1 = \rho_0 + \varepsilon_0$, we complete the proof. □

Now, a specific dependence on the initial data u_0 is obtained for the solution $u(\alpha, t)$ to the Eq. (2.1),

$$u_\tau - u_{\alpha\alpha} - 2u = G(u, u_\alpha, u_{\alpha\alpha}).$$

Lemma 2.2. *Consider the Eq. (2.1) with initial data $u_0 \in X_1$. There exist $\varepsilon_0 > 0$ and C_1 such that*

$$||u||_{\widetilde{X}_1} \le C_1||u_0||_{X_1}$$

where $||u_0||_{X_1} \le \varepsilon_0$. Here, ε_0 and C_1 depend on $||G||_{C^{1,\sigma}(D)}$.

Proof. It is clear that the term G in the Eq. (2.1) satisfies the assumption for g in Lemma 2.1. Suppose on the contrary, there exists $u_0^j \in X_1$, $||u_0^j||_{X_1} \neq 0$ but $||u_0^j||_{X_1} \to 0$ as $j \to \infty$, and

$$||u^j||_{\widetilde{X}_1} \geq j||u_0^j||_{X_1}. \tag{2.5}$$

Let $\hat{u}^j = \varepsilon_0 u^j/||u_0^j||_{X_1}$, where ε_0 is given in Lemma 2.1. Then $||\hat{u}^j(0)||_{X_1} \leq \varepsilon_0$ and $\hat{u} = \hat{u}^j$ satisfies

$$\hat{u}_\tau = \hat{u}_{\alpha\alpha} + 2\hat{u}_\alpha + \hat{G}(\hat{u}, \hat{u}_\alpha, \hat{u}_{\alpha\alpha}),$$

where

$$\begin{aligned}
&\hat{G}(\hat{u}, \hat{u}_\alpha, \hat{u}_{\alpha\alpha}) \\
&\quad = \varepsilon_0 ||u_0^j||_{X_1}^{-1} G(u, u_\alpha, u_{\alpha\alpha}) \\
&\quad = \varepsilon_0 ||u_0^j||_{X_1}^{-1} G(\varepsilon_0^{-1}||u_0^j||_{X_1}\hat{u}, \varepsilon_0^{-1}||u_0^j||_{X_1}\hat{u}_\alpha, \varepsilon_0^{-1}||u_0^j||_{X_1}\hat{u}_{\alpha\alpha}).
\end{aligned}$$

By the structure condition (2.2) of G, $||\hat{G}||_{C^{1,\sigma}(D)}$ is uniformly bounded in j. It follows from Lemma 2.1 that

$$||\hat{u}^j||_{\widetilde{X}_1} \leq C_1$$

and C_1 only depends on $\|G\|_{C^{1,\sigma}(D)}$. But then

$$\varepsilon_0 ||u^j||_{\widetilde{X}_1} \leq ||u_0^j||_{X_1} ||\hat{u}^j||_{\widetilde{X}_1} \leq C_1 ||u_0^j||_{X_1},$$

which contradicts to (2.5). □

To proceed further, we need a linearized stability result.

Lemma 2.3. *Denote by* (λ_j, φ_j) $(j = 1, 2, \ldots)$ *the corresponding eigenvalues and eigenfunctions to the following problem*

$$-(\varphi_{\alpha\alpha} + 2\varphi) = \lambda\varphi, \quad \varphi \in X, \tag{2.6}$$

where the eigenfunctions are normalized in the sense that $||\varphi_j||_{L^2(I)} = 1$. *Then the smallest eigenvalue* λ_1 *satisfies* $\lambda_1 > \delta$ *for some* $\delta > 0$.

Proof. It's easy to see that when $\lambda = -2$, the corresponding solution to problem (2.6) could only be the zero. The first eigenvalue of (2.6) is then given by $\lambda_1 := \left(\frac{n}{m}\right)^2 - 2$ with the eigenfunctions $\cos\left(\frac{n}{m}\alpha\right)$ and $\sin\left(\frac{n}{m}\alpha\right)$. By Abresch–Langer Theorem, $m/n < \sqrt{2}/2$, it follows that $\lambda_1 > 0$. □

We denote by $H^l(I)$ $(l \in \mathbf{N})$ the usual Sobolev space. Let $f \in H^l(I)$ and satisfy $\int_I f\, d\alpha = 0$. Then we would have

$$||f^{(l)}||_{L^2(I)}^2 = \sum_{j=1}^{\infty} j^{2l} a_j^2 \left(\frac{n}{m}\right)^{2l}, \tag{2.7}$$

where $\{a_j\}_{n=1}^{\infty}$ is given by

$$a_j = \int_I \varphi_j f^{(l)} \, d\alpha.$$

We note that the existence result in the beginning of this section ensures that the Eq. (2.1) has a smooth solution u. Obviously, $u \in H^l(I)$ for any $l \in \mathbf{N}$. Since u also belongs to the space $\widetilde{X}_2$, the identity (2.7) holds for u. Hence, we could define for u a norm equivalent to its usual H^l norm as follows

$$||u||_{H^l(I)}^2 = \sum_{j=1}^{\infty} j^{2l} a_j^2 \left(\frac{n}{m}\right)^{2l}, \quad a_j = \int_I \varphi_j u^{(l)} \, d\alpha. \tag{2.8}$$

Now, we begin to consider the Eq. (2.1) in the space $\widetilde{X}_2$ with Hölder exponent $\sigma = 1/2$ in particular.

Lemma 2.4. (A stability result) *Consider the Eq. (2.1) in the space $\widetilde{X}_2$ with $\sigma = 1/2$. There exists a positive ε_1 such that*

$$||u(\cdot, 1)||_{H^3} \le \varepsilon, \text{ if } ||u_0||_{H^3} \le \varepsilon, \forall\, \varepsilon \in (0, \varepsilon_1),$$

where $||\cdot||_{H^3}$ is defined as in (2.8). As a consequence, for any time $n \in \mathbf{N}$,

$$||u(\cdot, n)||_{H^3} \le \varepsilon, \text{ if } ||u_0||_{H^3} \le \varepsilon, \forall\, \varepsilon \in (0, \varepsilon_1).$$

Proof. First, consider the linear problem in the space $\widetilde{X}_2$,

$$\psi_\tau = \psi_{\alpha\alpha} + 2\psi, \quad \psi(0) = u_0. \tag{2.9}$$

We expand u_0 in the Fourier series with respect to the eigenfunction φ_j of (2.6),

$$u_0 = \sum_{j=1}^{\infty} a_j \varphi_j.$$

By separation of variable,

$$\psi(\alpha, \tau) = \sum_{j=1}^{\infty} a_j(\tau) \varphi_j, \quad a_j(\tau) = e^{-\lambda_j \tau} a_j.$$

Since

$$a_j^2(\tau) = e^{-2\lambda_j \tau} a_j^2 \le e^{-2\delta\tau} a_j^2, \quad \forall\, j \in \mathbf{N},$$

we have

$$\begin{aligned} ||\psi(\tau)||_{H^l(I)}^2 &= \sum_{j=1}^{\infty} j^{2l} a_j^2(\tau) \left(\frac{n}{m}\right)^{2l} \\ &\le e^{-2\delta\tau} \sum_{j=1}^{\infty} j^{2l} a_j^2 \left(\frac{n}{m}\right)^{2l}. \end{aligned}$$

That is,

$$||\psi(\tau)||_{H^l(I)} \le e^{-\delta\tau}||u_0||_{H^l(I)}, \quad \forall\, l \in \mathbf{N}. \tag{2.10}$$

Set $v = u - \psi$. Then v satisfies the equation

$$v_\tau = v_{\alpha\alpha} + 2v + G(u, u_\alpha, u_{\alpha\alpha}), \quad v(0) = 0, \tag{2.11}$$

with $v \in \widetilde{X}_2$. By virtue of standard Schauder type estimate, we have

$$\begin{aligned} ||v||_{\widetilde{X}_2} &\le C_2||G(u, u_\alpha, u_{\alpha\alpha})||_{\widetilde{X}_3} \\ &\le C_3||u||^2_{\widetilde{X}_2} \end{aligned}$$

for some constants $C_2, C_3 > 0$. Due to Lemma 2.2, we have

$$||v||_{\widetilde{X}_2} \le C_3C_1||u_0||^2_{X_2} := C_5||u_0||^2_{X_2}, \tag{2.12}$$

if $||u_0||_{X_2} \le \varepsilon_0$.

Now we assume that $||u_0||_{H^3(I)} \le \varepsilon$. The Sobolev embedding theorem tells that $||u_0||_{X_2} \le C_6||u_0||_{H^3(I)} \le C_6\varepsilon$. With such a u_0 as initial data, the solution $u(x, t)$ to the Eq. (2.1) could be written as $\psi + v$, where ψ is a solution to the Eq. (2.9) and v is a solution to the Eq. (2.11). By the estimate (2.10),

$$||\psi(\tau)||_{H^3(I)} \le e^{-\delta\tau}\varepsilon, \ \tau > 0.$$

On the other hand side, by (2.12),

$$||v(1)||_{X_2} \le C_5(C_6\varepsilon)^2 := C_7\varepsilon^2$$

if $\varepsilon \le C_6^{-1}\varepsilon_0$. The Poinare inequality implies that

$$||v(1)||^2_{H^3} \le C_8||v^{(3)}(1)||^2_{L^2(I)}.$$

Integration by parts yields

$$||v(1)||^2_{H^3} \le -C_8\int_I v^{(4)}(1)v^{(2)}(1)\,d\alpha \le C_9\varepsilon^2\left(\int_I (v^{(4)}(1))^2\,d\alpha\right)^{1/2}.$$

Note that the initial data u_0 being considered are bounded in $C^\infty(I)$. Therefore, we have the estimate

$$\left(\int_I (v^{(6)}(1))^2\,d\alpha\right)^{1/2} \le C^*,$$

for some constant C^* depending on C^6-norm of u_0. Then the integration by parts yields

$$\begin{aligned} \int_I (v^{(4)}(1))^2\,d\alpha &\le \left(\int_I (v^{(6)}(1))^2\,d\alpha\right)^{1/2}\left(\int_I (v^{(2)}(1))^2\,d\alpha\right)^{1/2} \\ &\le C_{10}\varepsilon^2, \end{aligned}$$

and C_{10} depends on C^* and C_7. Thus at last we get

$$||v(1)||_{H^3} \leq \sqrt{C_9\sqrt{C_{10}}}\varepsilon^{3/2} := C_{11}\varepsilon^{3/2}.$$

Combining the estimates for ψ and v, we have

$$\begin{aligned} ||u(1)||_{H^3} &\leq ||\psi(1)||_{H^3} + ||v(1)||_{H^3} \\ &\leq e^{-\delta}\varepsilon + C_{11}\varepsilon^{3/2}. \end{aligned}$$

Taking an $\varepsilon_1 \in (0, C_6^{-1}\varepsilon_0)$ such that

$$e^{-\delta}\varepsilon + C_{11}\varepsilon^{3/2} \leq \varepsilon, \quad \forall\, \varepsilon \in (0, \varepsilon_1),$$

we have

$$||u(1)||_{H^3} \leq \varepsilon, \quad \forall\, \varepsilon \in (0, \varepsilon_1),$$

under the assumption that $||u_0||_{H^3} \leq \varepsilon$. □

We note that Lemma 2.4 not only yields a stability result, but also implies the following a priori estimate for solutions $\tilde{h}$ to the problem (1.7).

Lemma 2.5. (Two-sided bound for $\tilde{h}$) *When $|\delta|$ is small enough, there is a two-sided bound for solutions $\tilde{h}(\theta, \tau)$ to the problem (1.7),*

$$1/2 \leq \tilde{h}(\theta, \tau) \leq 3/2, \quad \forall\, (\theta, \tau) \in I \times [0, T].$$

3. Global existence

In view of the two-sided bound on the support function, some regularity results for the problem (1.7) will be shown in the subsequent several lemmas. The technique is from [2]. In the sequel, we denote two-sided bound for $\tilde{h}$ by r and R respectively. That is,

$$r = 1/2, \quad R = 3/2.$$

First, we show that the normalized curvature $\tilde{k}$ is bounded from above.

Lemma 3.1. (Upper bound for $\tilde{k}$) *When $|\delta|$ is small enough, the solution $\tilde{k}$ to the problem (1.8) is bounded from above, i.e., there exists a constant M_1 independent of T such that*

$$\tilde{k}(\theta, \tau) \leq M_1, \quad \forall(\theta, \tau) \in I \times [0, T].$$

Proof. Consider the quantity $Q = \tilde{k}/(\tilde{h} - a)$ where $\tilde{h} \geq 2a$. According to Lemma 2.5, one can choose $a = r/2$. Let the maximum of Q over $I \times [0, T]$ attain at (θ_0, τ_0), $\tau_0 > 0$. At the point (θ_0, τ_0), we have

$$\frac{\partial Q}{\partial \theta} = 0, \quad \frac{\partial Q}{\partial \tau} \geq 0, \quad \text{and } \frac{\partial^2 Q}{\partial \theta^2} \leq 0.$$

Since

$$\begin{aligned} 0 \leq \frac{\partial Q}{\partial \tau} &= \tilde{k}^2 Q_{\theta\theta} + \frac{2\tilde{k}^2 \tilde{h}_\theta Q_\theta}{\tilde{h} - a} + \frac{2\tilde{k}^2}{(\tilde{h} - a)^2} - \frac{a\tilde{k}^3}{(\tilde{h} - a)^2} - \frac{2\tilde{h} - a}{(\tilde{h} - a)^2}\tilde{k} \\ &\leq \tilde{k}^2 Q_{\theta\theta} + \frac{2\tilde{k}^2 \tilde{h}_\theta Q_\theta}{\tilde{h} - a} - Q^2[a^2 Q - 2] \\ &\leq -Q^2[a^2 Q - 2] \end{aligned}$$

(where the inequality $\tilde{h} - a \geq a$ is used), we deduce that

$$Q(\theta_0, \tau_0) \leq \frac{2}{a^2}.$$

When the maximum of Q attains at the initial time, we have

$$Q(\theta, \tau) \leq \frac{k_{0\delta}(0)}{h_{0\delta}(m\pi/n) - a}.$$

Hence, we have

$$Q(\theta, \tau) \leq \max\left\{\frac{2}{a^2}, \frac{k_{0\delta}(0)}{h_{0\delta}(m\pi/n) - a}\right\} := M.$$

That is

$$\frac{\tilde{k}(\theta, \tau)}{\tilde{h}(\theta, \tau) - a} \leq M.$$

It follows that

$$\begin{aligned} \tilde{k}(\theta, \tau) &\leq M(\tilde{h}(\theta, \tau) - a) \\ &\leq M(R - r/2). \end{aligned}$$ □

After obtaining an upper bound for $\tilde{k}$, we estimate $\tilde{k}$ from the below.

Lemma 3.2. (Lower bound for $\tilde{k}$) *When $|\delta|$ is small enough, the solution $\tilde{k}$ to the problem (1.8) is bounded from below, i.e., there exists a constant $M_2 > 0$ independent of T such that*

$$\tilde{k}(\theta, \tau) \geq M_2, \quad \forall (\theta, \tau) \in I \times [0, T].$$

Proof. We first claim that

$$\sup_{I\times[0,T]}\left[(\tilde{k}_\theta)^2+\tilde{k}^2\right]\le\max\left\{\sup_I[(\tilde{k}_\theta)^2+\tilde{k}^2](\theta,0),\ \sup_{I\times[0,T]}\tilde{k}^2\right\}. \tag{3.1}$$

Consider the function $\Phi=(\tilde{k}_\theta)^2+\tilde{k}^2$. Suppose that there is $(\theta_0,\tau_1)\in I\times(0,T]$ such that

$$\Phi(\theta_0,\tau_1)=\max_{I\times[0,T]}\left[(\tilde{k}_\theta)^2+\tilde{k}^2\right].$$

We claim that $\tilde{k}_\theta$ must vanish at (θ_0,τ_1). For, if this is not true, then we must have $(\tilde{k}_{\theta\theta}+\tilde{k})(\theta_0,\tau_1)=0$ since $\Phi_\theta=2\tilde{k}_\theta(\tilde{k}_{\theta\theta}+\tilde{k})=0$ at (θ_0,τ_1). This tells us that at (θ_0,τ_1),

$$\frac{1}{2}\Phi_\tau=-\tilde{k}^2+\tilde{k}^2\tilde{k}_\theta(\tilde{k}_{\theta\theta\theta}+\tilde{k}_\theta)-(\tilde{k}_\theta)^2$$

and

$$\frac{1}{2}\Phi_{\theta\theta}=\tilde{k}_\theta(\tilde{k}_{\theta\theta\theta}+\tilde{k}_\theta).$$

So, $\frac{1}{2}\Phi_\tau=-\tilde{k}^2+\frac{1}{2}\tilde{k}^2\Phi_{\theta\theta}-(\tilde{k}_\theta)^2$ at (θ_0,τ_1). On the other hand, at (θ_0,τ_1), we have $\Phi_{\theta\theta}\le 0$, $\Phi_\tau\ge 0$. As a consequence

$$0\le\frac{1}{2}\Phi_\tau\le-\tilde{k}^2-(\tilde{k}_\theta)^2,$$

which is impossible. Thus, $\tilde{k}_\theta(\theta_0,\tau_1)=0$. This yields the estimate (3.1).

Combining (3.1) with the upper bound for $|\tilde{k}|$ on $I\times[0,T]$ from Lemma 3.1, we conclude that $|\tilde{k}_\theta|$ is bounded on $I\times[0,T]$. Note that $|\tilde{k}_\theta|=\tilde{k}^{-1}|\tilde{k}_s|$ (s denotes the unit arc length). Therefore

$$\left|\frac{d}{ds}\ln\tilde{k}\right|\le C, \tag{3.2}$$

for some constant $C>0$.

The length of $\tilde{\gamma}(\cdot,\tau)$ is equal to

$$\begin{aligned}\tilde{L}(\tau)&=\int_0^{2m\pi}ds=\int_0^{2m\pi}\tilde{k}^{-1}\,d\theta\\&=\int_0^{2m\pi}(\tilde{h}+\tilde{h}_{\theta\theta})\,d\theta=\int_0^{2m\pi}\tilde{h}\,d\theta,\quad\tau\in[0,T].\end{aligned}$$

Since $\tilde{h}\le R$ on $I\times[0,T]$, then $\tilde{L}(\tau)\le 2m\pi R$, for $\tau\in[0,T]$. Thus (3.2) implies that the oscillation of $(\ln\tilde{k})(\theta,\tau)$ on $[0,2m\pi]$ is uniformly bounded for all $\tau\in[0,T]$. It will follow that $\tilde{k}(\theta,\tau)$ has a uniform positive lower bound for all $\tau \in [0, T]$, if we could show that $\tilde{k}(0, \tau) > 1$ for all $\tau \in [0, T]$. Indeed, if $\tilde{k}(0, \tau_1) < 1$ for some $\tau_1 \in [0, T]$, then $\tilde{k}(\theta, \tau_1) < 1$ on $[0, 2m\pi/n]$. Thus we have

$$\begin{aligned} m\pi &= \frac{1}{2}\int_0^{2m\pi} \frac{\tilde{h}}{\tilde{k}}\, d\theta > \frac{1}{2}\int_0^{2m\pi} \tilde{h}\, d\theta = \frac{1}{2}\int_0^{2m\pi} (\tilde{h} + \tilde{h}_{\theta\theta})\, d\theta \\ &= \frac{1}{2}\int_0^{2m\pi} \frac{d\theta}{\tilde{k}} > \frac{1}{2}\int_0^{2m\pi} d\theta = m\pi, \end{aligned}$$

which is impossible. □

Lemma 3.3. (Smooth estimates for $\tilde{k}$) *Let $\tilde{k}(\theta, \tau)$ be the solution to the problem (1.8) when $|\delta|$ is small enough. Then for every $l \in \mathbf{N}$, there exists a constant C_l depending only on r, R and $k_{0\delta}$ such that*

$$\sup_{I \times [0,T]} |\tilde{k}^{(l)}(\theta, \tau)| \le C_l.$$

Proof. Recall that the Eq. (1.8) satisfied by $\tilde{k}$ is given by,

$$\tilde{k}_\tau = \tilde{k}^2(\tilde{k}_{\theta\theta} + \tilde{k}) - \tilde{k}, \quad (\theta, \tau) \in I \times [0, T].$$

From the proof of Lemma 3.2, we derive the maximum estimate for $\tilde{k}_\theta$. Then the maximum estimate for $\tilde{k}_{\theta\theta}$ could be derived by the Schauder estimate since two-sided bound for $\tilde{k}$ guarantees that the above equation is a uniformly parabolic equation. By differentiating the above equation, we can obtain all higher order estimates for $\tilde{k}$ similarly, see [10] or [11]. □

From the two-sided bound and all higher order estimates for $\tilde{k}$ in the above lemmas, we can deduce global existence of the flow (1.6) for all time.

Theorem 2. (Global existence of $\tilde{h}$) *For the initial data $h_{0\delta}$ when $|\delta|$ is small enough, there is a global solution $\tilde{h}(\cdot, \tau)$ to the problem (1.7).*

Proof. Lemmas 3.1 and 3.2 show that the equation in (1.7) is uniformly parabolic. Lemma 3.3 guarantees that the equation in (1.7) can be written as a linear uniformly parabolic equation with smooth coefficients. Therefore, Schauder estimate gives the estimates on all the derivatives of the solution $\tilde{h}$ independent of τ. This regularity result together with fixed point theorem or implicit function theorem yields a global existence of the solution $\tilde{h}$ to the problem (1.7). □

Note that there is another equivalent way to show the global existence of $\tilde{\gamma}(\tau)$, via showing the global existence of solution $\tilde{k}$ for the problem (1.8), see [4].

4. Convergence

First, a Lyapunov functional is introduced to show the convergence of $\tilde{\gamma}$ to self-similar solutions.

Lemma 4.1. *Let* $\mathcal{J}(\tilde{h}(\cdot,\tau)) = \fint_0^{2m\pi} \log \tilde{h}\, d\theta$. *Then*

$$\frac{d}{d\tau}\mathcal{J}(\tilde{h}(\cdot,\tau)) \leq 0$$

and the equality holds if and only if $\tilde{h}$ *solves*

$$h_{\theta\theta} + h = \frac{1}{h}, \quad h > 0. \tag{4.1}$$

Proof. Since

$$\int_0^{2m\pi} \frac{\tilde{h}}{\tilde{k}}\, d\theta = 2m\pi,$$

and

$$\left(\int_0^{2m\pi} 1\, d\theta\right)^2 \leq \int_0^{2m\pi} \frac{\tilde{h}}{\tilde{k}}\, d\theta \int_0^{2m\pi} \frac{\tilde{k}}{\tilde{h}}\, d\theta, \tag{4.2}$$

we have

$$\int_0^{2m\pi} \frac{\tilde{k}}{\tilde{h}}\, d\theta \geq 2m\pi.$$

Hence,

$$\frac{d}{d\tau}\fint_0^{2m\pi} \log \tilde{h}\, d\theta = 1 - \fint_0^{2m\pi} \frac{\tilde{k}}{\tilde{h}}\, d\theta \leq 0.$$

Moreover, the above equality holds if and only if the equality holds in (4.2). This implies that $\tilde{h} = C\tilde{k}$ for some constant C. Since $\int_0^{2m\pi} (\tilde{h}/\tilde{k})\, d\theta = 2m\pi$, we have $C = 1$. □

Theorem 3. (Convergence of $\tilde{\gamma}$) *Let* δ *be small enough. For any sequence of time* $\{\tau_j\}_{j=1}^{\infty}$ *tending to infinity, there is a subsequence* $\{\tau_{j_k}\}_{k=1}^{\infty}$ *such that the normalized solution* $\tilde{\gamma}(\tau)$ *to the CSP starting from* $\gamma_{0\delta}$ *converges smoothly to either* $\gamma_{m,n}$ *or* m*-fold circle along the subsequence* $\{\tau_{j_k}\}_{k=1}^{\infty}$.

Proof. Note that for any $\tau > 0$

$$\int_0^\tau \left(\frac{d}{d\tau}\mathcal{J}(\tilde{h}(\cdot,\tau))\right) d\tau = \mathcal{J}(\tilde{h}(\cdot,\tau)) - \mathcal{J}(\tilde{h}(\cdot,0)).$$

Since we have known that $\tilde{h}$ is bounded for all time, then the integral

$$\int_0^\tau \left(\frac{d}{d\tau}\mathcal{J}(\tilde{h}(\cdot,\tau))\right) d\tau$$

must be bounded from below. Recall that $\frac{d}{d\tau}\mathcal{J}(\tilde{h}(\cdot,\tau)) \leq 0$ from Lemma 4.1. We have

$$\int_0^\infty \left(\frac{d}{d\tau}\mathcal{J}(\tilde{h}(\cdot,\tau))\right) d\tau > -\infty. \tag{4.3}$$

We claim that

$$\frac{d}{d\tau}\mathcal{J}(\tilde{h}(\cdot,\tau)) \to 0, \quad \text{as } \tau \to \infty. \tag{4.4}$$

If not, then there exists a constant $C_0 > 0$ and a sequence $\{\tau_j\}_{j=1}^\infty$ tending to infinity such that

$$\frac{d}{d\tau}\mathcal{J}(\tilde{h}(\cdot,\tau_j)) \leq -C_0.$$

As noted in the proof of Theorem 2, we have the estimates for all the derivatives of the solutions $\tilde{h}$ independent of τ. Thus we can find a ρ_0 (independent of τ_j) such that

$$\left|\frac{d}{d\tau}\mathcal{J}(\tilde{h}(\cdot,\tau))\right| \geq \frac{C_0}{2} \text{ on } [\tau_j, \tau_j + \rho_0].$$

It follows that

$$\int_{\tau_j}^{\tau_j+\rho_0} \frac{d}{d\tau}\mathcal{J}(\tilde{h}(\cdot,\tau))\, d\tau \leq -\frac{C_0\rho_0}{2}. \tag{4.5}$$

On the other hand, from (4.3) we know that

$$\lim_{j\to\infty} \int_{\tau_j}^\infty \frac{d}{d\tau}\mathcal{J}(\tilde{h}(\cdot,\tau))\, d\tau = 0,$$

which contradicts to (4.5) since $\frac{d}{d\tau}\mathcal{J}(\tilde{h}(\cdot,\tau)) \leq 0$.

Now, for any given sequence $\{\tau_j\}_{j=1}^\infty$, the estimates on all the derivatives of the solutions $\tilde{h}$ permit us to extract a subsequence of $\{\tau_j\}_{j=1}^\infty$, denoted by $\{\tau_{j_k}\}_{k=1}^\infty$, such that $\{\tilde{h}(\cdot, \tau_{j_k})\}_{k=1}^{\infty}$ converges smoothly to a function $\tilde{h}^*$ which is smooth and strictly positive. In addition, by (4.4), $\tilde{h}^*$ satisfies

$$\frac{d}{d\tau}\mathcal{J}(\tilde{h}^*) = 0.$$

So, Lemma 4.1 tells us that $\tilde{h}^*$ is a solution to the Eq. (4.1). Alternatively speaking, $\tilde{\gamma}(\cdot, \tau_{j_k})$ must converge to one shrinking self-similar solution as $k\to\infty$. □

Now, we could conclude the proof of Theorem 1.

Proof. It's easy to observe that $h_{0\delta}$ tends to 1 smoothly on I as $\delta\to 0$. So, for δ with $|\delta|$ small enough, $h_{0\delta}$ will be sufficiently close to 1 so that the initial data u_0 for the Eq. (2.1) is sufficiently close to 0. As a consequence, the stability result in Lemma 2.4 ensures that the solution $\tilde{h}$ to the problem (1.7) will be close to 1 forever, which excludes the possibility of $\tilde{h}$'s convergence to nontrivial solution of (4.1). Finally, we conclude that $\tilde{\gamma}$ must converge to m-fold circle as $\tau\to\infty$. □

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