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Gamma-convergence of nonlocal perimeter functionals

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Abstract. Given $\Omega \subset \mathbb{R}^n$ open, connected and with Lipschitz boundary, and $s \in (0, 1)$, we consider the functional

$$\mathcal{J}_s(E,\Omega) = \int_{E\cap\Omega}\int_{E^c\cap\Omega}\frac{dxdy}{|x-y|^{n+s}} + \int_{E\cap\Omega}\int_{E^c\cap\Omega^c}\frac{dxdy}{|x-y|^{n+s}} + \int_{E\cap\Omega^c}\int_{E^c\cap\Omega}\frac{dxdy}{|x-y|^{n+s}},$$

where $E \subset \mathbb{R}^n$ is an arbitrary measurable set. We prove that the functionals $(1-s)\mathcal{J}_s(\cdot,\Omega)$ are equi-coercive in $L^1_{\mathrm{loc}}(\Omega)$ as $s \uparrow 1$ and that

$$\Gamma - \lim_{s\uparrow 1}(1-s)\mathcal{J}_s(E,\Omega) = \omega_{n-1}P(E,\Omega), \quad \text{for every } E \subset \mathbb{R}^n \text{ measurable},$$

where $P(E,\Omega)$ denotes the perimeter of E in Ω in the sense of De Giorgi. We also prove that as $s \uparrow 1$ limit points of local minimizers of $(1-s)\mathcal{J}_s(\cdot,\Omega)$ are local minimizers of $P(\cdot,\Omega)$.

1. Introduction

For a measurable set $E \subset \mathbb{R}^n$, $n \geq 1$, $0 < s < 1$, and a connected open set $\Omega \Subset \mathbb{R}^n$ with Lipschitz boundary (or simply $\Omega = (a,b) \Subset \mathbb{R}$ if $n = 1$), we consider the functional

$$\mathcal{J}_s(E,\Omega) := \mathcal{J}_s^1(E,\Omega) + \mathcal{J}_s^2(E,\Omega),$$

where

$$\mathcal{J}_s^1(E,\Omega) := \int_{E\cap\Omega}\int_{E^c\cap\Omega}\frac{1}{|x-y|^{n+s}}dxdy,$$
$$\mathcal{J}_s^2(E,\Omega) := \int_{E\cap\Omega}\int_{E^c\cap\Omega^c}\frac{1}{|x-y|^{n+s}}dxdy + \int_{E\cap\Omega^c}\int_{E^c\cap\Omega}\frac{1}{|x-y|^{n+s}}dxdy.$$

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The functional $\mathcal{J}_s(E,\Omega)$ can be thought of as a fractional perimeter of E in Ω which is non-local in the sense that it is not determined by the behaviour of E in a neighbourhood of $\partial E\cap\Omega$, and which can be finite even if the Hausdorff dimension of ∂E is $n-s>n-1$. Notice that the term $\mathcal{J}_s^1(E,\Omega)$ is simply half of the fractional Sobolev space seminorm $|\chi_E|_{W^{s,1}(\Omega)}$, where χ_E denotes the characteristic function of E. Roughly speaking this term represents the $(n-s)$-dimensional fractional perimeter of E *inside* Ω, while $\mathcal{J}_s^2$ is the contribution near $\partial\Omega$. This can be made precise when letting $s\uparrow 1$. We also recall the following elementary scaling property:

$$\mathcal{J}_s^i(\lambda E,\lambda\Omega)=\lambda^{n-s}\mathcal{J}_s^i(E,\Omega)\qquad\text{for } \lambda>0,\ i=1,2. \tag{1}$$

This functional has already been investigated by several authors. In [15] Visintin studied some basic properties of $\mathcal{J}_s$, and in particular he showed that $\mathcal{J}_s$ satisfies a suitable co-area formula, see Lemma 10 below. Caffarelli et al. [5] studied the behavior of minimizers of $\mathcal{J}_s$, proving that if E is a local minimizer of $\mathcal{J}_s(\cdot,\Omega)$, i.e.

$$\mathcal{J}_s(E,\Omega)\le\mathcal{J}_s(F,\Omega)\quad\text{whenever } E\Delta F\Subset\Omega,$$

then $(\partial E)\cap\Omega$ is of class $C^{1,\alpha}$ up to a set of Hausdorff codimension in $\mathbb{R}^n$ at least 2.

As it is well-known (see for instance [10] and the references therein), for minimizers E of the classical De Giorgi's perimeter, which we shall denote $P(E,\Omega)$, the regularity results are stronger. The boundary of a local minimizer E of $P(\cdot,\Omega)$ is analytic if $n\le 7$, it has (at most) isolated singularities when $n=8$ and it is analytic up to a set of codimension at least 8 in $\mathbb{R}^n$ if $n\ge 9$. This suggests that the results of [5] might not be optimal for s close to 1. Motivated by this, Caffarelli and Valdinoci [4] studied the limiting properties of minimal sets for the s-perimeter as $s\to 1^-$.

Partly motivated by their work, we make a complete analysis of the limiting properties, in the sense of Γ-convergence, of $\mathcal{J}_s$ as $s\to 1^-$, under no other assumption than the measurability of the sets considered. Our proofs differ in particular from those in [4] because they do not rely on uniform (as $s\to 1^-$) regularity estimates on s-minimal boundaries borrowed from [5]. The only result we need from [5], in the proof of our Lemma 14, is the local minimality of halfspaces, whose proof is reproduced in Sect. 4.

We start by proving a coercivity result.

Theorem 1. (Equi-coercivity) *Assume that $s_i\uparrow 1$ and that E_i are measurable sets satisfying*

$$\sup_{i\in\mathbb{N}}(1-s_i)\mathcal{J}_{s_i}^1(E_i,\Omega')<\infty\qquad\forall\Omega'\Subset\Omega.$$

Then (E_i) is relatively compact in $L^1_{\mathrm{loc}}(\Omega)$, any limit point E has locally finite perimeter in Ω.

Notice the scaling factor $(1-s)$, which accounts for the fact that $\mathcal{J}_1^1(E,\Omega)=+\infty$ unless $E\subset\Omega^c$, or $\Omega\subset E$, as already shown by Brézis [2,3].

Let ω_k denote the volume of the unit ball in $\mathbb{R}^k$ for $k\ge 1$, and set $\omega_0:=1$.

Theorem 2. (Γ-convergence) *For every measurable set $E \subset \mathbb{R}^n$ we have*

$$\Gamma - \liminf_{s\uparrow 1}(1-s)\mathcal{J}_s^1(E,\Omega) \geq \omega_{n-1}P(E,\Omega),$$
$$\Gamma - \limsup_{s\uparrow 1}(1-s)\mathcal{J}_s(E,\Omega) \leq \omega_{n-1}P(E,\Omega), \tag{2}$$

with respect to the local convergence in measure, i.e. the $L^1_{\rm loc}$ convergence of the corresponding characteristic functions in $\mathbb{R}^n$.

We recall that (2) means that

$$\liminf_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}^1(E_i,\Omega) \geq \omega_{n-1}P(E,\Omega)$$
$$\text{whenever } \chi_{E_i} \to \chi_E \text{ in } L^1_{\rm loc}(\mathbb{R}^n),\ \ s_i \uparrow 1,$$

and that for every measurable set E and sequence $s_i \uparrow 1$ there exists a sequence E_i with $\chi_{E_i} \to \chi_E$ in $L^1_{\rm loc}(\mathbb{R}^n)$ such that

$$\limsup_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}(E_i,\Omega) \leq \omega_{n-1}P(E,\Omega).$$

We finally show that as $s \uparrow 1$ local minimizers converge to local minimizers, where by a local minimizer of $\mathcal{J}_s(\cdot,\Omega)$ we mean a Borel set $E \subset \mathbb{R}^n$ such that $\mathcal{J}_s(E,\Omega) \leq \mathcal{J}_s(F,\Omega)$ whenever $E\Delta F \Subset \Omega$. Notice that if E is a local minimizer of $\mathcal{J}_s(\cdot,\Omega)$ and $\Omega' \subset \Omega$, then E is also a local minimizer of $\mathcal{J}_s(\cdot,\Omega')$. A similar definition holds for $P(\cdot,\Omega)$.

Theorem 3. (Convergence of local minimizers) *Assume that $s_i \uparrow 1$, E_i are local minimizer of $\mathcal{J}_{s_i}(\cdot,\Omega)$, and $\chi_{E_i} \to \chi_E$ in $L^1_{\rm loc}(\mathbb{R}^n)$. Then*

$$\limsup_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}(E_i,\Omega') < +\infty \qquad \forall \Omega' \Subset \Omega, \tag{3}$$

E is a local minimizer of $P(\cdot,\Omega)$ and $(1-s_i)\mathcal{J}_{s_i}(E_i,\Omega') \to \omega_{n-1}P(E,\Omega')$ whenever $\Omega' \Subset \Omega$ and $P(E,\partial\Omega')=0$.

We point out that Γ-convergence results for functionals reminiscent of $\mathcal{J}_s^1(\cdot,\mathbb{R}^n)$ have been proven in [13,14].

We fix some notation used throughout the paper:

- we write $x \in \mathbb{R}^n$ as (x',x_n) with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$;
- we denote by H the halfspace $\{x:\ x_n \leq 0\}$ and by $Q=(-1/2,1/2)^n$ the canonical unit cube;
- we denote by $B_r(x)$ the ball of radius r centered at x and, unless otherwise specified, $B_r := B_r(0)$.
- for every $h \in \mathbb{R}^n$ and function u defined on $U \subset \mathbb{R}^n$ we set $\tau_h u(x) := u(x+h)$ for all $x \in U-h$.

For the definition and basic properties of the perimeter $P(E,\Omega)$ in the sense of De Giorgi we refer to the monographs [1,10].

2. Proof of Theorem 1

The proof is a direct consequence of the Frechet-Kolmogorov compactness criterion in L^p_{loc} (applied with $p = 1$), ensuring pre-compactness of any family $\mathcal{G} \subset L^1_{\mathrm{loc}}(\Omega)$ satisfying

$$\lim_{h\to 0} \sup_{u\in\mathcal{G}} \|\tau_h u - u\|_{L^1(\Omega')} = 0 \qquad \forall \Omega' \Subset \Omega,$$

and of the following pointwise upper bound on $\|\tau_h u - u\|_{L^1}$: for all $u \in L^1(\Omega)$, $A \Subset \Omega$, $h \in \mathbb{R}^n$ with $|h| < \mathrm{dist}(A, \partial\Omega)/2$ and $s \in (0, 1)$ we have

$$\|\tau_h u - u\|_{L^1(A)} \leq C(n)|h|^s(1-s)\mathcal{F}_s(u, \Omega), \tag{4}$$

where

$$\mathcal{F}_s(u, \Omega) := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|^{n+s}} dxdy. \tag{5}$$

The functional $\mathcal{F}_s$ is obviously related to $\mathcal{J}^1_s$ by

$$\mathcal{F}_s(\chi_E, \Omega) = 2\mathcal{J}^1_s(E, \Omega).$$

The upper bound (4) is a direct consequence of Proposition 4 below, whose proof can be found in [11]. Since the inequality is not explicitly stated in [11], we repeat it for the reader's convenience.

Proposition 4. *For all $u \in L^1(\Omega)$, $A \Subset \Omega$ and $s \in (0, 1)$ we have*

$$\frac{\|\tau_h u - u\|_{L^1(A)}}{|h|^s} \leq C(n)(1-s) \int_{B_{|h|}} \frac{\|\tau_\xi u - u\|_{L^1(A_{|h|})}}{|\xi|^{n+s}} d\xi \tag{6}$$

whenever $0 < |h| < \mathrm{dist}(A, \partial\Omega)/2$, and $A_{|h|} := \{x \in \mathbb{R}^n : \mathrm{dist}(x, A) < |h|\}$.

We start with two preliminary results.

Proposition 5. *Let $u \in L^1(\Omega)$, $h \in \mathbb{R}^n$ and $A \Subset \Omega$ open with $|h| < \mathrm{dist}(A, \partial\Omega)/2$. Then for any $z \in (0, |h|]$ we have:*

$$\|\tau_h u - u\|_{L^1(A)} \leq C(n) \frac{|h|}{z^{n+1}} \int_{B_z} \|\tau_\xi u - u\|_{L^1(A_{|h|})} d\xi, \tag{7}$$

where $A_{|h|}$ is as in Proposition 4.

Proof. Fix a non-negative function $\varphi \in C^1_c(B_1)$ with $\int_{B_1} \varphi dx = 1$. For $x \in A$ and $z \in (0, |h|]$ we write

$$\begin{aligned} u(x) &= \frac{1}{z^n} \int_{B_z} u(x+y)\varphi\left(\frac{y}{z}\right) dy + \frac{1}{z^n} \int_{B_z} (u(x) - u(x+y))\varphi\left(\frac{y}{z}\right) dy \\ &=: U(x, z) + V(x, z). \end{aligned}$$

Then we have

$$|u(x+h)-u(x)| \le |U(x+h,z)-U(x,z)| + |V(x+h,z)| + |V(x,z)|. \quad (8)$$

The second and third terms can be easily estimated as follows:

$$|V(x+h,z)| + |V(x,z)| \le \frac{\sup|\varphi|}{z^n}\int_{B_z}\left\{|\tau_y u(x)-u(x)| + |\tau_y u(x+h)-u(x+h)|\right\} dy.$$

For the first one instead notice that

$$\begin{aligned}\nabla_x U(x,z) &= -\frac{1}{z^{n+1}}\int_{B_z(x)} u(y)\nabla\varphi\left(\frac{y-x}{z}\right)dy \\ &= -\frac{1}{z^{n+1}}\int_{B_z(x)} (u(y)-u(x))\nabla\varphi\left(\frac{y-x}{z}\right)dy\end{aligned}$$

and so

$$\begin{aligned}|U(x+h,z)-U(x,z)| &\le |h|\int_0^1 |\nabla_x U(x+sh,z)|ds \\ &\le \sup|\nabla\varphi|\frac{|h|}{z^{n+1}}\int_0^1\int_{B_z}|u(y+x+sh)-u(x+sh)|dyds.\end{aligned}$$

Notice now that $z \le |h|$ and so $1 \le |h|/z$, hence from (8) we have:

$$\begin{aligned}|u(x+h)-u(x)| &\le C\left\{\frac{1}{z^n}\int_{B_z}|\tau_y u(x)-u(x)| + |\tau_y u(x+h)-u(x+h)|dy\right. \\ &\quad \left. + \frac{|h|}{z^{n+1}}\int_0^1\int_{B_z}|u(y+x+sh)-u(x+sh)|dyds\right\} \\ &\le C\frac{|h|}{z^{n+1}}\left\{\int_{B_z}|\tau_y u(x)-u(x)| + |\tau_y u(x+h)-u(x+h)|dy\right. \\ &\quad \left. + \int_0^1\int_{B_z}|\tau_y u(x+sh)-u(x+sh)|dyds\right\},\end{aligned}$$

with $C = \sup|\varphi| + \sup|\nabla\varphi|$. Integrating both sides over A we infer (7) with $C(n) = 3C$. □

Recall now the following version of Hardy's inequality:

Proposition 6. *Let $g : \mathbb{R} \to [0,\infty)$ be a Borel function, then for every $s > 0$ we have*

$$\int_0^r \frac{1}{\xi^{n+s+1}}\int_0^\xi g(t)dtd\xi \le \frac{1}{n+s}\int_0^r \frac{g(t)}{t^{n+s}}dt \qquad \forall r \ge 0. \quad (9)$$

Proof. We have

$$\int_0^r \frac{1}{\xi^{n+s+1}} \int_0^\xi g(t)dtd\xi = \int_0^r g(t) \int_t^r \frac{1}{\xi^{n+s+1}} d\xi dt$$
$$= \frac{1}{n+s} \int_0^r g(t) \left(\frac{1}{t^{n+s}} - \frac{1}{r^{n+s}} \right) dt \le \frac{1}{n+s} \int_0^r \frac{g(t)}{t^{n+s}} dt.$$

□

Proof of Proposition 4. Multiply both sides of (7) by z^{-s} and integrate with respect to z between 0 and $|h|$ to obtain

$$\frac{|h|^{(1-s)}}{(1-s)} \|\tau_h u - u\|_{L^1(A)} \le C(n)|h| \int_0^{|h|} \frac{1}{z^{n+s+1}} \int_{B_z} \|\tau_\xi u - u\|_{L^1(A_{|h|})} d\xi dz.$$

Now apply inequality (9) with

$$g(t) := \int_{\partial B_t} \|\tau_\xi u - u\|_{L^1(A_{|h|})} d\mathcal{H}^{n-1}(\xi)$$

and obtain

$$\begin{aligned} \int_0^{|h|} \frac{1}{z^{n+s+1}} \int_{B_z} \|\tau_\xi u - u\|_{L^1(A_{|h|})} d\xi dz &= \int_0^{|h|} \frac{1}{z^{n+s+1}} \int_0^z g(t)dtdz \\ &\le C(n) \int_0^{|h|} \frac{1}{t^{n+s}} g(t)dt \\ &= C(n) \int_{B_{|h|}} \frac{\|\tau_\xi u - u\|_{L^1(A_{|h|})}}{|\xi|^{n+s}} d\xi. \end{aligned} \tag{10}$$

Putting all together

$$\frac{\|\tau_h u - u\|_{L^1(A)}}{(1-s)} \le C(n)|h|^s \int_{B_{|h|}} \frac{\|\tau_\xi u - u\|_{L^1(A_{|h|})}}{|\xi|^{n+s}} d\xi$$

and the thesis follows. □

3. Proof of Theorem 2

In the proof of the lim inf inequality we shall adapt to this framework the blow-up technique introduced, for the first time in the context of lower semicontinuity, by Fonseca and Müller [9]. The proof of the lim sup inequality, which is typically constructive and by density, is slightly different from the analogous results in [4], since we approximate with polyhedra, rather than $C^{1,\alpha}$ sets. Notice also that the natural strategies in the proof of the lim inf and lim sup inequalities produce constants Γ_n, see (11), and $\Gamma_n^* \ge \Gamma_n$, see (17); our final task will be to show that they both coincide with ω_{n-1}.

3.1. The $\Gamma - \liminf$ *inequality*

Let us define

$$\Gamma_n := \inf \left\{ \liminf_{s\uparrow 1} (1-s)\mathcal{J}_s^1(E_s, Q) \;\middle|\; \chi_{E_s} \to \chi_H \text{ in } L^1(Q) \right\}. \tag{11}$$

We denote by $\mathcal{C}$ the family of all n-cubes in $\mathbb{R}^n$

$$\mathcal{C} := \{ R(x + rQ) : \ x \in \mathbb{R}^n, \ r > 0, \ R \in SO(n) \}.$$

Lemma 7. *Given* $s_i \uparrow 1$ *and sets* $E_i \subset \mathbb{R}^n$ *with* $\chi_{E_i} \to \chi_E$ *in* $L^1_{\text{loc}}(\mathbb{R}^n)$ *as* $i \to \infty$, *one has*

$$\liminf_{i\to\infty} (1-s_i)\mathcal{J}^1_{s_i}(E_i, \Omega) \geq \Gamma_n P(E, \Omega). \tag{12}$$

We can assume that the left-hand side of (12) is finite, otherwise the inequality is trivial. Then, passing to the limit as $i \to \infty$ in (6) with $s = s_i$ we get

$$\|\tau_h \chi_E - \chi_E\|_{L^1(\Omega')} \leq C(n)|h| \liminf_{i\to\infty} (1-s_i)\mathcal{J}^1_{s_i}(E_i, \Omega) \qquad \forall \Omega' \Subset \Omega$$

whenever $|h| < \text{dist}(\Omega', \partial\Omega)/2$, hence E has finite perimeter in Ω.

We shall denote by μ the perimeter measure of E, i.e. $\mu(A) = |D\chi_E|(A)$ for any Borel set $A \subset \Omega$, and we shall use the following property of sets of finite perimeter: for μ-a.e. $x \in \Omega$ there exists $R_x \in SO(n)$ such that $(E - x)/r$ locally converge in measure to $R_x H$ as $r \to 0$. In addition,

$$\lim_{r\to 0} \frac{\mu(x + rR_x Q)}{r^{n-1}} = 1, \quad \text{for } \mu\text{-a.e. } x. \tag{13}$$

Indeed this property holds for every $x \in \mathcal{F}E$, where $\mathcal{F}E$ denotes the reduced boundary of E, see Theorem 3.59(b) in [1].

Now, given a cube $C \in \mathcal{C}$ contained in Ω we set

$$\alpha_i(C) := (1-s_i)\mathcal{J}^1_{s_i}(E_i, C)$$

and

$$\alpha(C) := \liminf_{i\to\infty} \alpha_i(C).$$

We claim that, setting $C_r(x) := x + rR_x Q$, where R_x is as in (13), for μ-a.e. x we have

$$\liminf_{r\to 0} \frac{\alpha(C_r(x))}{\mu(C_r(x))} \geq \Gamma_n \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n. \tag{14}$$

Then observing that for all $\varepsilon > 0$ the family

$$\mathcal{A} := \{C_r(x) \subset \Omega \ : \ (1+\varepsilon)\alpha(C_r(x)) \geq \Gamma_n \mu(C_r(x))\}$$

is a fine covering of μ-almost all of Ω, by a suitable variant of Vitali's theorem (see [12]) we can extract a countable subfamily of disjoint cubes

$$\{C_j \subset \Omega : j \in J\}$$

such that $\mu(\Omega \setminus \bigcup_{j\in J} C_j) = 0$, whence

$$\begin{aligned}\Gamma_n P(E,\Omega) &= \Gamma_n \mu\left(\bigcup_{j\in J} C_j\right) = \Gamma_n \sum_{j\in J} \mu(C_j)\\ &\le (1+\varepsilon)\sum_{j\in J}\alpha(C_j) \le (1+\varepsilon)\liminf_{i\to\infty}\sum_{j\in J}\alpha_i(C_j)\\ &\le (1+\varepsilon)\liminf_{i\to\infty}(1-s_i)\mathcal{J}^1_{s_i}(E_i,\Omega).\end{aligned}$$

In the last inequality we used that $\mathcal{J}^1_s$ is superadditive and positive for every $s \in (0,1)$. Since $\varepsilon > 0$ is arbitrary we get the $\Gamma - \liminf$ estimate.

We now prove the inequality in (14) at any point x such that $(E-x)/r$ converges locally in measure as $r \to 0$ to $R_x H$ and (13) holds. Because of (13), we need to show that

$$\liminf_{r\to 0}\frac{\alpha(C_r(x))}{r^{n-1}} \ge \Gamma_n. \tag{15}$$

Since from now on x is fixed, we can assume with no loss of generality (by rotation invariance) that $R_x = I$, so that the limit hyperplane is H and the cubes $C_r(x)$ are the standard ones $x + rQ$. Let us choose a sequence $r_k \to 0$ such that

$$\liminf_{r\to 0}\frac{\alpha(C_r(x))}{r^{n-1}} = \lim_{k\to\infty}\frac{\alpha(C_{r_k}(x))}{r_k^{n-1}}.$$

For $k > 0$ we can choose $i(k)$ so large that the following conditions hold:

$$\begin{cases}\alpha_{i(k)}(C_{r_k}(x)) \le \alpha(C_{r_k}(x)) + r_k^n,\\ r_k^{1-s_{i(k)}} \ge 1 - \frac{1}{k},\\ \displaystyle⨍_{C_{r_k}(x)} |\chi_{E_{i(k)}} - \chi_E|\,dx < \frac{1}{k}.\end{cases}$$

Then we infer

$$\begin{aligned}\frac{\alpha(C_{r_k}(x))}{r_k^{n-1}} &\ge \frac{\alpha_{i(k)}(C_{r_k}(x))}{r_k^{n-1}} - r_k\\ &= \frac{(1-s_{i(k)})\mathcal{J}^1_{s_{i(k)}}((E_{i(k)}-x)/r_k, Q)\,r_k^{n-s_{i(k)}}}{r_k^{n-1}} - r_k\\ &\ge \left(1-\frac{1}{k}\right)(1-s_{i(k)})\mathcal{J}^1_{s_{i(k)}}((E_{i(k)}-x)/r_k, Q) - r_k,\end{aligned}$$

i.e.

$$\lim_{k\to\infty}\frac{\alpha(C_{r_k}(x))}{r_k^{n-1}} \geq \liminf_{k\to\infty}(1-s_{i(k)})\mathcal{J}^1_{s_{i(k)}}((E_{i(k)}-x)/r_k, Q).$$

On the other hand we have

$$\lim_{k\to\infty}\int_Q |\chi_{(E_{i(k)}-x)/r_k} - \chi_{(E-x)/r_k}|dx = 0,$$

and

$$\lim_{k\to\infty}\int_Q |\chi_{(E-x)/r_k} - \chi_H|dx = 0.$$

It follows that $(E_{i(k)} - x)/r_k \to H$ in $L^1(Q)$. Recalling the definition of Γ_n we conclude the proof of (15) and of Lemma 7.

3.2. The $\Gamma - \limsup$ *inequality*

It is enough to prove the $\Gamma - \limsup$ inequality for a collection $\mathcal{B}$ of sets of finite perimeter which is dense in energy, i.e. such that for every set E of finite perimeter there exists $E_k \in \mathcal{B}$ with $\chi_{E_k} \to \chi_E$ in $L^1_{\rm loc}(\mathbb{R}^n)$ as $k\to\infty$ and $\limsup_k P(E_k,\Omega) = P(E,\Omega)$. Indeed, let d be a distance inducing the $L^1_{\rm loc}$ convergence and, for a set E of finite perimeter, let E_k be as above. Given $s_k \uparrow 1$, we can find sets $\hat{E}_k$ with $d(\chi_{\hat{E}_k}, \chi_{E_k}) < 1/k$ and

$$(1-s_k)\mathcal{J}_{s_k}(\hat{E}_k,\Omega) \leq \Gamma_n^* P(E_k,\Omega) + \frac{1}{k}.$$

Then we have $\chi_{\hat{E}_k} \to \chi_E$ in $L^1_{\rm loc}(\mathbb{R}^n)$ and

$$\limsup_{k\to\infty}(1-s_k)\mathcal{J}_{s_k}(\hat{E}_k,\Omega) \leq \limsup_{k\to\infty}\Gamma_n^* P(E_k,\Omega) = \Gamma_n^* P(E,\Omega).$$

We shall take $\mathcal{B}$ to be the collection of polyhedra Π which satisfy $P(\Pi,\partial\Omega) = 0$ (i.e. with faces transversal to $\partial\Omega$, see Proposition 15). Equivalently,

$$\lim_{\delta\to 0} P(\Pi, \Omega_\delta^+ \cup \Omega_\delta^-) = 0,$$

where

$$\begin{aligned}\Omega_\delta^+ &:= \{x \in \Omega^c \mid d(x,\Omega) < \delta\}\\ \Omega_\delta^- &:= \{x \in \Omega \mid d(x,\Omega^c) < \delta\}.\end{aligned} \tag{16}$$

In fact, we have:

Lemma 8. *For a polyhedron* $\Pi \subset \mathbb{R}^n$ *there holds*

$$\limsup_{s\uparrow 1}(1-s)\mathcal{J}_s(\Pi,\Omega) \le \Gamma_n^* P(\Pi,\Omega) + 2\Gamma_n^* \lim_{\delta\to 0} P(\Pi, \Omega_\delta^+ \cup \Omega_\delta^-),$$

where

$$\Gamma_n^* := \limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^1(H,Q). \tag{17}$$

Proof. **Step 1.** We first estimate $\mathcal{J}_s^1(\Pi,\Omega)$. For a fixed $\varepsilon>0$ set

$$(\partial\Pi)_\varepsilon := \{x\in\Omega \mid d(x,\partial\Pi)<\varepsilon\}, \quad (\partial\Pi)_\varepsilon^- := (\partial\Pi)_\varepsilon \cap \Pi.$$

We can find N_ε disjoint cubes $Q_i^\varepsilon \subset \Omega$, $1\le i\le N_\varepsilon$, of side length ε satisfying the following properties:

(i) if $\tilde{Q}_i^\varepsilon$ denotes the dilation of Q_i^ε by a factor $(1+\varepsilon)$, then each cube $\tilde{Q}_i^\varepsilon$ intersects exactly one face Σ of $\partial\Pi$, its barycenter belongs to Σ and each of its sides is either parallel or orthogonal to Σ;

(ii) $\mathcal{H}^{n-1}\left(((\partial\Pi)\cap\Omega)\setminus\bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon\right) = |P(\Pi,\Omega) - N_\varepsilon\varepsilon^{n-1}| \to 0$ as $\varepsilon\to 0$.

For $x\in\mathbb{R}^n$ set

$$I_s(x) := \int_{\Pi^c\cap\Omega} \frac{dy}{|x-y|^{n+s}}.$$

We consider several cases.

Case 1. $x\in(\Pi\cap\Omega)\setminus(\partial\Pi)_\varepsilon^-$. Then for $y\in\Pi^c\cap\Omega$ we have $|x-y|\ge\varepsilon$, hence

$$I_s(x) \le \int_{(B_\varepsilon(x))^c} \frac{1}{|x-y|^{n+s}}dy = n\omega_n\int_\varepsilon^\infty \frac{1}{\rho^{s+1}}d\rho = \frac{n\omega_n}{s\varepsilon^s},$$

since $n\omega_n = \mathcal{H}^{n-1}(S^{n-1})$. Therefore

$$\int_{(\Pi\cap\Omega)\setminus(\partial\Pi)_\varepsilon^-} I_s(x)dx \le \frac{n\omega_n\mathcal{L}^n(\Pi\cap\Omega)}{s\varepsilon^s}. \tag{18}$$

Case 2. $x\in(\partial\Pi)_\varepsilon^- \setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon$. Then

$$\begin{aligned} I_s(x) &\le \int_{(B_{d(x,\Pi^c\cap\Omega)}(x))^c} \frac{1}{|x-y|^{n+s}}dy = n\omega_n\int_{d(x,\Pi^c\cap\Omega)}^\infty \frac{1}{\rho^{s-1}}d\rho \\ &= \frac{n\omega_n}{s[d(x,\Pi^c\cap\Omega)]^s}. \end{aligned} \tag{19}$$

Now write $(\partial\Pi)\cap\Omega = \bigcup_{j=1}^J \Sigma_j$, where each Σ_j is the intersection of a face of $\partial\Pi$ with Ω, and define

$$(\partial\Pi)_{\varepsilon,j}^- := \{x\in(\partial\Pi)_\varepsilon^- : \mathrm{dist}(x,\Pi^c\cap\Omega) = \mathrm{dist}(x,\Sigma_j)\}.$$

Clearly $(\partial\Pi)^-_\varepsilon = \bigcup_{j=1}^J (\partial\Pi)^-_{\varepsilon,j}$. Moreover we have

$$(\partial\Pi)^-_{\varepsilon,j} \subset \{x + t\nu : x \in \Sigma_{\varepsilon,j},\ t \in (0,\varepsilon),\ \nu \text{ is the interior unit normal to } \Sigma_{\varepsilon,j}\},$$

and $\Sigma_{\varepsilon,j}$ is the set of points x belonging to the same hyperplane as Σ_j and with $\mathrm{dist}(x,\Sigma_j) \le \varepsilon$. Clearly $\mathcal{H}^{n-1}(\Sigma_{\varepsilon,j}) \le \mathcal{H}^{n-1}(\Sigma_j) + C\varepsilon$ as $\varepsilon \to 0$. Then from (19) we infer

$$\begin{aligned}
\int_{(\partial\Pi)^-_\varepsilon \setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon} I_s(x)dx &\le \frac{n\omega_n}{s}\sum_{j=1}^J \int_{(\partial\Pi)^-_{\varepsilon,j}\setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon} \frac{1}{[d(x,\Pi^c)]^s}dx \\
&\le \frac{n\omega_n}{s}\sum_{j=1}^J \int_{(\partial\Pi)^-_{\varepsilon,j}\setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon} \frac{1}{[d(x,\Sigma_{\varepsilon,j})]^s}dx \\
&\le \frac{n\omega_n}{s}\sum_{j=1}^J \int_{(\Sigma_{\varepsilon,j})\setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon} \left(\int_0^\varepsilon \frac{dt}{t^s}\right) d\mathcal{H}^{n-1} \\
&= \frac{n\omega_n \varepsilon^{1-s}}{s(1-s)}\mathcal{H}^{n-1}\left(\left(\bigcup_{j=1}^J \Sigma_{\varepsilon,j}\right)\setminus \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon\right) = \frac{\varepsilon^{1-s}o(1)}{s(1-s)},
\end{aligned} \tag{20}$$

with error $o(1) \to 0$ as $\varepsilon \to 0$ and independent of s.

Case 3. $x \in \Pi \cap \bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon$. In this case we write

$$\begin{aligned}
I_s(x) &= \int_{(\Pi^c\cap\Omega)\cap\{y:|x-y|\ge\varepsilon^2\}} \frac{dy}{|x-y|^{n+s}} + \int_{(\Pi^c\cap\Omega)\cap\{y:|x-y|<\varepsilon^2\}} \frac{dy}{|x-y|^{n+s}} \\
&=: I_s^1(x) + I_s^2(x).
\end{aligned}$$

Then, similar to the case 1,

$$I_s^1(x) \le n\omega_n \int_{\varepsilon^2}^\infty \frac{1}{\rho^{s+1}}d\rho = \frac{n\omega_n}{s\varepsilon^{2s}},$$

hence (since all cubes are contained in Ω)

$$\int_{\Pi\cap\bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon} I_s^1(x)dx \le \frac{\mathcal{L}^n(\Omega)n\omega_n}{s\varepsilon^{2s}}. \tag{21}$$

As for $I_s^2(x)$ observe that if $x \in Q_i^\varepsilon$ and $|x-y| \le \varepsilon^2$, then $y \in \tilde{Q}_i^\varepsilon$, where $\tilde{Q}_i^\varepsilon$ is the cube obtained by dilating Q_i^ε by a factor $1+\varepsilon$ (hence the side length of $\tilde{Q}_i^\varepsilon$ is $\varepsilon + \varepsilon^2$). Then

$$\int_{\Pi\cap\bigcup_{i=1}^{N_\varepsilon} Q_i^\varepsilon} I_s^2(x)dx \le \sum_{i=1}^{N_\varepsilon} \int_{\Pi\cap Q_i^\varepsilon} \int_{\Pi^c\cap\tilde{Q}_i^\varepsilon} \frac{1}{|x-y|^{n+s}}dydx$$

$$\begin{aligned}
&\leq \sum_{i=1}^{N_\varepsilon} \int_{\Pi\cap\tilde{Q}_i^\varepsilon} \int_{\Pi^c\cap\tilde{Q}_i^\varepsilon} \frac{1}{|x-y|^{n+s}} dydx \\
&= N_\varepsilon \mathcal{J}_s^1(H, (\varepsilon+\varepsilon^2)Q) \\
&= N_\varepsilon (\varepsilon+\varepsilon^2)^{n-s} \mathcal{J}_s^1(H, Q), \qquad (22)
\end{aligned}$$

where in the last identity we used the scaling property (1). Keeping $\varepsilon > 0$ fixed, letting s go to 1 and putting (18)–(22) together we infer

$$\begin{aligned}
\limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^1(\Pi, \Omega) &\leq o(1) + \limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^1(H, Q) N_\varepsilon (\varepsilon+\varepsilon^2)^{n-1} \\
&= o(1) + \Gamma_n^* P(\Pi, \Omega),
\end{aligned}$$

with error $o(1) \to 0$ as $\varepsilon \to 0$ uniformly in s. Since $\varepsilon > 0$ is arbitrary, we conclude

$$\limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^1(\Pi, \Omega) \leq \Gamma_n^* P(\Pi, \Omega).$$

Step 2. It now remains to estimate $\mathcal{J}_s^2$. Let us start by considering the term

$$\int_{\Pi\cap\Omega} \int_{\Pi^c\cap\Omega^c} \frac{1}{|x-y|^{n+s}} dydx.$$

Case 1. $x \in \Pi \cap (\Omega \setminus \Omega_\delta^-)$. Then for $y \in \Pi^c \cap \Omega^c$ we have $|x-y| \geq \delta$, whence

$$I(x) := \int_{\Pi^c\cap\Omega^c} \frac{dy}{|x-y|^{n+s}} \leq n\omega_n \int_\delta^\infty \frac{d\rho}{\rho^{1+s}} = \frac{n\omega_n}{s\delta^s}.$$

Case 2. $x \in \Pi \cap \Omega_\delta^-$. In this case, using the same argument of case 1 for $y \in \Pi^c \cap (\Omega^c \setminus \Omega_\delta^+)$, we have

$$\begin{aligned}
I(x) &= \int_{\Pi^c\cap\Omega_\delta^+} \frac{dy}{|x-y|^{n+s}} + \int_{\Pi^c\cap(\Omega^c\setminus\Omega_\delta^+)} \frac{dy}{|x-y|^{n+s}} \\
&\leq \int_{\Pi^c\cap\Omega_\delta^+} \frac{dy}{|x-y|^{n+s}} + \frac{n\omega_n}{s\delta^s}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{\Pi\cap\Omega} \int_{\Pi^c\cap\Omega^c} \frac{dydx}{|x-y|^{n+s}} &\leq \frac{2n\omega_n|\Omega|}{s\delta^s} + \int_{\Pi\cap\Omega_\delta^-} \int_{\Pi^c\cap\Omega_\delta^+} \frac{dydx}{|x-y|^{n+s}} \\
&\leq \frac{2n\omega_n|\Omega|}{s\delta^s} + \int_{\Pi\cap(\Omega_\delta^-\cup\Omega_\delta^+)} \int_{\Pi^c\cap(\Omega_\delta^-\cup\Omega_\delta^+)} \frac{dydx}{|x-y|^{n+s}}.
\end{aligned}$$

An obvious similar estimate can be obtained by swapping Π and Π^c, finally yielding

$$\begin{aligned}
\mathcal{J}_s^2(\Pi, \Omega) &\leq \frac{4n\omega_n|\Omega|}{s\delta^s} + 2\int_{\Pi\cap(\Omega_\delta^-\cup\Omega_\delta^+)} \int_{\Pi^c\cap(\Omega_\delta^-\cup\Omega_\delta^+)} \frac{dydx}{|x-y|^{n+s}} \\
&= \frac{4n\omega_n|\Omega|}{s\delta^s} + 2\mathcal{J}_s^1(\Pi, \Omega_\delta^- \cup \Omega_\delta^+).
\end{aligned}$$

Using the result of step 1 we get

$$\limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^2(\Pi,\Omega) \le 2\Gamma_n^* P(\Pi,\Omega_\delta^- \cup \Omega_\delta^+).$$

Since $\delta > 0$ is arbitrary, letting δ go to zero we conclude the proof of the lemma. □

Lemma 9. (Characterization of Γ_n^*) *The limsup in* (17) *is a limit and* $\Gamma_n^* = \omega_{n-1}$.

Proof. The proof is inspired from [4, Lemma 11]. We shall actually prove a slightly stronger statement. Set for $a > 0$

$$Q_a := \{x : |x_i| \le 1/2 \text{ for } 1 \le i \le n-1,\ |x_n| \le a\}.$$

Then we show that

$$\lim_{s\uparrow 1}(1-s)\mathcal{J}_s^1(H,Q_a) = \omega_{n-1}, \qquad \forall a > 0.$$

Let us first consider the case $n \ge 2$. Fix $x \in Q_a \cap H$ and write as usual $x = (x', x_n)$, $y = (y', y_n)$. We consider

$$I_s(x) := \int_{Q_a\cap H^c} \frac{1}{|x-y|^{n+s}}dy = \int_0^a \int_{Q_a\cap\partial H} \frac{1}{|x-y|^{n+s}}dy'dy_n.$$

With the change of variable $z' = (y'-x')/|y_n - x_n|$ and setting

$$\Sigma(x,y_n) := \left\{z' \in \mathbb{R}^{n-1} : \left|z_i' + \frac{x_i'}{|x_n-y_n|}\right| \le \frac{1}{2|x_n-y_n|} \text{ for } 1 \le i \le n-1\right\},$$

we get

$$\begin{aligned} I_s(x) &= \int_0^a \int_{\Sigma(x,y_n)} \frac{1}{|x_n-y_n|^{s+1}(1+|z'|^2)^{(n+s)/2}}dz'dy_n \\ &\le \int_0^a \frac{1}{|x_n-y_n|^{s+1}}dy_n \cdot \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|z'|^2)^{(n+s)/2}}dz' \\ &= \frac{(-x_n)^{-s}-(a-x_n)^{-s}}{s} \cdot (n-1)\omega_{n-1}\int_0^\infty \frac{\rho^{n-2}}{(1+\rho^2)^{(n+s)/2}}d\rho. \end{aligned} \tag{23}$$

Now integrating I with respect to x, observing that $\mathcal{H}^{n-1}(Q_a \cap \partial H) = 1$ and that by dominated convergence one has

$$\begin{aligned} \lim_{s\uparrow 1}\int_0^\infty \frac{\rho^{n-2}}{(1+\rho^2)^{(n+s)/2}}d\rho &= \int_0^\infty \frac{\rho^{n-2}}{(1+\rho^2)^{(n+1)/2}}d\rho \\ &= \left[\frac{\rho^{n-1}}{(n-1)(1+\rho^2)^{(n-1)/2}}\right]_0^\infty = \frac{1}{n-1}, \end{aligned} \tag{24}$$

we get

$$\begin{aligned}\int_{H\cap Q_a} I_s(x)dx &\le \mathcal{H}^{n-1}(Q_a\cap\partial H)\sup_{x'\in Q_a\cap\partial H}\int_{-a}^0 I_s(x',x_n)dx_n\\ &\le \omega_{n-1}(1+o(1))\int_{-a}^0 \frac{(-x_n)^{-s}-(a-x_n)^{-s}}{s}dx_n\\ &= \frac{\omega_{n-1}(1+o(1))a^{1-s}(2-2^{1-s})}{s(1-s)},\end{aligned}$$

with error $o(1)\to 0$ as $s\uparrow 1$ dependent only on s. Therefore

$$\limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^1(H,Q_a) = \limsup_{s\uparrow 1}(1-s)\int_{H\cap Q_a} I_s(x)dx \le \omega_{n-1}. \qquad (25)$$

Now observing that for ε small enough

$$|x_n|\le\varepsilon^2,\quad |y_n|\le\varepsilon^2,\quad |x_i|\le\frac{1}{2}-\varepsilon \quad \text{for } 1\le i\le n-1 \qquad (26)$$

implies that $B_{1/(2\varepsilon)}(0)\subset\Sigma(x,y_n)$, similar to (23) we estimate

$$\begin{aligned} I_s(x) &\ge \int_0^{\varepsilon^2}\int_{Q\cap\partial H}\frac{1}{|x-y|^{n+s}}dy'dy_n\\ &\ge \int_0^{\varepsilon^2}\int_{B_{1/(2\varepsilon)}(0)}\frac{1}{|x_n-y_n|^{s+1}(1+|z'|^2)^{(n+s)/2}}dz'dy_n\\ &= \frac{(-x_n)^{-s}-(\varepsilon^2-x_n)^{-s}}{s}\cdot(n-1)\omega_{n-1}\int_0^{\frac{1}{2\varepsilon}}\frac{\rho^{n-2}}{(1+\rho^2)^{(n+s)/2}}d\rho,\end{aligned}$$

whenever x is as in (26). Integrating with respect to x satisfying (26) one has

$$\begin{aligned}\int_{H\cap Q_a} I_s(x)dx &\ge (1-2\varepsilon)^{n-1}\int_{-\varepsilon^2}^0\frac{(-x_n)^{-s}-(\varepsilon^2-x_n)^{-s}}{s}dx_n\\ &\quad\times(n-1)\omega_{n-1}\int_0^{\frac{1}{2\varepsilon}}\frac{\rho^{n-2}}{(1+\rho^2)^{(n+s)/2}}d\rho\\ &= \frac{(n-1)\omega_{n-1}(1-2\varepsilon)^{n-1}\varepsilon^{2(1-s)}(2-2^{1-s})}{s(1-s)}\int_0^{\frac{1}{2\varepsilon}}\frac{\rho^{n-2}}{(1+\rho^2)^{(n+s)/2}}d\rho.\end{aligned}$$

Letting first $s\uparrow 1$ and then $\varepsilon\to 0$ and using (24) again we conclude

$$\liminf_{s\uparrow 1}(1-s)\mathcal{J}_s^1(H,Q_a)\ge\omega_{n-1},$$

which together with (25) completes the proof when $n\ge 2$.

When $n=1$ one computes explicitly

$$\begin{aligned}\mathcal{J}_s^1(H,Q_a) &= \int_{-a}^0\int_0^a\frac{1}{|x-y|^{1+s}}dydx = \int_{-a}^0\frac{(-x)^{-s}-(a-x)^{-s}}{s}dx\\ &= \frac{a^{1-s}(2-2^{1-s})}{s(1-s)},\end{aligned}$$

hence

$$\lim_{s\uparrow 1}(1-s)\mathcal{J}_s^1(H, Q_a) = 1 = \omega_0. \qquad \square$$

3.3. Gluing construction and characterization of the geometric constants

A key observation in [15], which we shall need, is that $\mathcal{F}$ satisfies a generalized coarea formula, namely $\mathcal{F}_s(u, \Omega) = \int_0^1 \mathcal{F}_s(\chi_{\{u>t\}}, \Omega)\, dt$; we reproduce here the simple proof of this fact and we state the result in terms of $\mathcal{J}_s$.

Lemma 10. (Coarea formula) *For every measurable function $u : \Omega \to [0, 1]$ we have*

$$\frac{1}{2}\mathcal{F}_s(u, \Omega) = \int_0^1 \mathcal{J}_s^1(\{u > t\}, \Omega) dt.$$

Proof. Given $x, y \in \Omega$, the function $t \mapsto \chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)$ takes its values in $\{-1, 0, 1\}$ and it is nonzero precisely in the interval having $u(x)$ and $u(y)$ as extreme points, hence

$$|u(x) - u(y)| = \int_0^1 |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)| dt.$$

Substituting into (5), using Fubini's theorem and observing that

$$|\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)| = \chi_{\{u>t\}}(x)\chi_{\Omega\setminus\{u>t\}}(y) + \chi_{\Omega\setminus\{u>t\}}(x)\chi_{\{u>t\}}(y),$$

we infer

$$\begin{aligned}
\mathcal{F}_s(u, \Omega) &= \int_\Omega\int_\Omega\int_0^1 \frac{|\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|}{|x-y|^{n+s}} dt dx dy \\
&= 2\int_0^1 \int_{\{u>t\}} \int_{\Omega\setminus\{u>t\}} \frac{1}{|x-y|^{n+s}} dx dy dt \\
&= 2\int_0^1 \mathcal{J}_s^1(\{u > t\}, \Omega) dt.
\end{aligned}$$

$\square$

Proposition 11. (Gluing) *Given $s \in (0, 1)$, measurable sets E_1, E_2 in $\mathbb{R}^n$ with $\mathcal{J}_s^1(E_i, \Omega) < \infty$ for $i = 1, 2$ and given $\delta_1 > \delta_2 > 0$ we can find a measurable set F such that*

(a) $\|\chi_F - \chi_{E_1}\|_{L^1(\Omega)} \leq \|\chi_{E_1} - \chi_{E_2}\|_{L^1(\Omega)}$,
(b) $F \cap (\Omega \setminus \Omega_{\delta_1}) = E_1 \cap (\Omega \setminus \Omega_{\delta_1})$, $F \cap \Omega_{\delta_2} = E_2 \cap \Omega_{\delta_2}$, *where*

$$\Omega_\delta := \{x \in \Omega : d(x, \Omega^c) \leq \delta\} \quad \textit{for } \delta > 0,$$

(c) for all $\varepsilon > 0$ *we have*

$$\mathcal{J}_s^1(F,\Omega) \le \mathcal{J}_s^1(E_1,\Omega) + \mathcal{J}_s^1(E_2,\Omega_{\delta_1+\varepsilon}) + \frac{C}{\varepsilon^{n+s}} + C(\Omega,\delta_1,\delta_2)\left[\frac{\|\chi_{E_1}-\chi_{E_2}\|_{L^1(\Omega_{\delta_1}\setminus\Omega_{\delta_2})}}{(1-s)} + \|\chi_{E_1}-\chi_{E_2}\|_{L^1(\Omega)}\right].$$

Proof. Consider a function $\varphi \in C^\infty(\mathbb{R}^n)$ such that $0 \le \varphi \le 1$ in Ω, $\varphi \equiv 0$ in Ω_{δ_2}, $\varphi \equiv 1$ in $\Omega \setminus \Omega_{\delta_1}$, and $|\nabla\varphi| \le 2/(\delta_1-\delta_2)$.

Given two measurable functions $u,\ v : \Omega \to [0,1]$ such that $\mathcal{F}_s(u,\Omega) < \infty$, $\mathcal{F}_s(v,\Omega) < \infty$, define $w : \Omega \to [0,1]$ as $w := \varphi u + (1-\varphi)v$. For $x, y \in \Omega$ we can write

$$\begin{aligned} w(x)-w(y) &= (\varphi(x)-\varphi(y))u(y) + \varphi(x)(u(x)-u(y)) \\ &\quad + (1-\varphi(x))(v(x)-v(y)) - v(y)(\varphi(x)-\varphi(y)) \\ &= (\varphi(x)-\varphi(y))(u(y)-v(y)) + \varphi(x)(u(x)-u(y)) \\ &\quad + (1-\varphi(x))(v(x)-v(y)), \end{aligned}$$

and infer

$$|w(x)-w(y)| \le |\varphi(x)-\varphi(y)||u(y)-v(y)| + \chi_{\{\varphi\neq 0\}}(x)|u(x)-u(y)| + \chi_{\{\varphi\neq 1\}}(x)|v(x)-v(y)|.$$

Observing that $\{\varphi \neq 0\} \subset \Omega \setminus \Omega_{\delta_2}$ and $\{\varphi \neq 1\} \subset \Omega_{\delta_1}$ we get

$$\begin{aligned} \mathcal{F}_s(w,\Omega) &\le \int_\Omega |u(y)-v(y)| \int_\Omega \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}dxdy \\ &\quad + \int_{\Omega\setminus\Omega_{\delta_2}}\int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{n+s}}dxdy + \int_{\Omega_{\delta_1}}\int_\Omega \frac{|v(x)-v(y)|}{|x-y|^{n+s}}dxdy \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

From

$$|\varphi(x)-\varphi(y)| \le |\nabla\varphi(y)||x-y| + \frac{1}{2}\|\nabla^2\varphi\|_\infty |x-y|^2$$

and the inequalities $\int_\Omega |x-y|^{-(n+s-\alpha)}dx \le C(\Omega)/(\alpha-s)$ (with $\alpha=1, \alpha=2$) we have

$$\begin{aligned} I_1 &\le \int_\Omega |u(y)-v(y)| \int_\Omega \left(\frac{|\nabla\varphi(y)|}{|x-y|^{n+s-1}} + \frac{\|\nabla^2\varphi\|_\infty}{2|x-y|^{n+s-2}}\right)dxdy \\ &\le C(\Omega,\delta_1,\delta_2)\left(\frac{\|u-v\|_{L^1(\Omega_{\delta_1}\setminus\Omega_{\delta_2})}}{1-s} + \frac{\|u-v\|_{L^1(\Omega)}}{(2-s)}\right). \end{aligned}$$

Clearly $I_2 \le \mathcal{F}_s(u,\Omega)$. As for I_3, choosing $\varepsilon > 0$ we get

$$\begin{aligned} I_3 &\le \int_{\Omega_{\delta_1}}\int_{\Omega_{\delta_1+\varepsilon}} \frac{|v(x)-v(y)|}{|x-y|^{n+s}}dxdy + \int_{\Omega_{\delta_1}}\int_{\Omega\setminus\Omega_{\delta_1+\varepsilon}} \frac{|v(x)-v(y)|}{|x-y|^{n+s}}dxdy \\ &\le \mathcal{F}_s(v,\Omega_{\delta_1+\varepsilon}) + \frac{2\mathcal{L}^n(\Omega_{\delta_1})\mathcal{L}^n(\Omega\setminus\Omega_{\delta_1+\varepsilon})}{\varepsilon^{n+s}}. \end{aligned}$$

Summing up we obtain

$$\mathcal{F}_s(w,\Omega) \le \mathcal{F}_s(u,\Omega) + \mathcal{F}_s(v,\Omega_{\delta_1+\varepsilon}) + C(\Omega,\delta_1,\delta_2)\frac{\|u-v\|_{L^1(\Omega_{\delta_1}\setminus\Omega_{\delta_2})}}{1-s} + C(\Omega,\delta_1,\delta_2)\|u-v\|_{L^1(\Omega)} + \frac{C(\Omega)}{\varepsilon^{n+s}}. \tag{27}$$

We now apply this with $u=\chi_{E_1}$, $v=\chi_{E_2}$, so that (27) reads as

$$\begin{aligned}\mathcal{F}_s(w,\Omega) \le\ & 2\mathcal{J}_s^1(E_1,\Omega) + 2\mathcal{J}_s^1(E_2,\Omega_{\delta_1+\varepsilon}) \\ & + C(\Omega,\delta_1,\delta_2)\frac{\|\chi_{E_1}-\chi_{E_2}\|_{L^1(\Omega_{\delta_1}\setminus\Omega_{\delta_2})}}{1-s} \\ & + C(\Omega,\delta_1,\delta_2)\|\chi_{E_1}-\chi_{E_2}\|_{L^1(\Omega)} + \frac{C(\Omega)}{\varepsilon^{n+s}},\end{aligned} \tag{28}$$

and by Lemma 10 there exists $t\in(0,1)$ such that $F := \{w>t\}$ satisfies

$$2\mathcal{J}_s^1(F,\Omega) \le \mathcal{F}_s(w,\Omega).$$

By construction we see that F satisfies conditions (a) and (b), and by (28) it follows that also condition (c) is satisfied. □

The following corollary is an immediate consequence of Proposition 11.

Corollary 12. *Given measurable sets $E_s \subset \mathbb{R}^n$ for $s\in(0,1)$, with $\chi_{E_s}\to\chi_E$ in $L^1(\Omega)$ as $s\uparrow 1$ and with $\mathcal{J}_s^1(E_s,\Omega)<\infty$, $\mathcal{J}_s^1(E,\Omega)<\infty$, and given $\delta_1>\delta_2>0$ we can find measurable sets $F_s\subset\mathbb{R}^n$ such that*

(a) $\chi_{F_s}\to\chi_E$ *in* $L^1(\Omega)$ *as* $i\to\infty$,
(b) $F_s\cap(\Omega\setminus\Omega_{\delta_1}) = E_s\cap(\Omega\setminus\Omega_{\delta_1})$, $F_s\cap\Omega_{\delta_2} = E\cap\Omega_{\delta_2}$,
(c) for all $\varepsilon>0$ *we have*

$$\liminf_{s\uparrow 1}(1-s)\mathcal{J}_s^1(F_s,\Omega) \le \liminf_{s\uparrow 1}(1-s)\mathcal{J}_s^1(E_s,\Omega) + \limsup_{s\uparrow 1}(1-s)\mathcal{J}_s^1(E,\Omega_{\delta_1+\varepsilon}).$$

We devote the rest of the section to the proof of the equality of the consants Γ_n and Γ_n^* appearing in the proof of the Γ-liminf and Γ-limsup respectively (we already proved that $\Gamma_n^*=\omega_{n-1}$). We shall introduce an intermediate quantity $\tilde\Gamma_n\in[\Gamma_n,\Gamma_n^*]$ and prove in two steps that $\tilde\Gamma_n=\Gamma_n$ (by the gluing Proposition 11) and then use the local minimality of hyperplanes to show that $\tilde\Gamma_n=\Gamma_n^*$.

Lemma 13. *We have $\Gamma_n=\tilde\Gamma_n$, where*

$$\tilde\Gamma_n := \inf\left\{\liminf_{s\uparrow 1}(1-s)\mathcal{J}_s^1(E_s,Q)\right\},$$

with the infimum taken over all families of measurable sets $(E_s)_{0<s<1}$ with the property that $\chi_{E_s}\to\chi_H$ in $L^1(Q)$ as $s\uparrow 1$ and, for some $\delta>0$, $E_s\cap Q^\delta = H\cap Q^\delta$ for all $s\in(0,1)$, where $Q^\delta=\{x\in Q : d(x,Q^c)<\delta\}$.

Proof. Clearly $\tilde{\Gamma}_n \geq \Gamma_n$. In order to prove the converse consider sets $E_s \subset \mathbb{R}^n$ for $s \in (0,1)$ with $\chi_{E_s} \to \chi_H$ in $L^1(Q)$ as $s \uparrow 1$. Without loss of generality we can assume that $\mathcal{J}_s^1(E_s, \Omega) < \infty$ for all $s \in (0,1)$. Then according to Corollary 12 for any given $\delta > 0$ we can find a family of measurable sets $(F_s)_{0<s<1}$ such that $\chi_{F_s} \to \chi_H$ in $L^1(Q)$ as $s \uparrow 1$, $F_s \cap Q^\delta = H \cap Q^\delta$ and

$$\tilde{\Gamma}_n \leq \liminf_{s\uparrow 1} (1-s)\mathcal{J}_s^1(F_s, \Omega) \leq \liminf_{s\uparrow 1} (1-s)\mathcal{J}_s^1(E_s, Q) + \Gamma_n^* \inf_{\varepsilon>0} P(H, Q^{\delta+\varepsilon}),$$

where we also used Lemma 8. Since $\delta > 0$ is arbitrary and $P(H, Q^\delta) \to 0$ as $\delta \to 0$ we infer

$$\tilde{\Gamma}_n \leq \liminf_{s\uparrow 1} (1-s)\mathcal{J}_s^1(E_s, Q)$$

and, since $(E_s)_{0<s<1}$ is arbitrary, this proves that $\tilde{\Gamma}_n \leq \Gamma_n$. □

Lemma 14. *We have* $\tilde{\Gamma}_n = \Gamma_n^*$.

Proof. Clearly $\tilde{\Gamma}_n \leq \Gamma_n^*$. In order to prove the converse we consider sets $(E_s)_{0<s<1}$ with $\chi_{E_s} \to \chi_H$ in $L^1(Q)$ as $s \uparrow 1$ and with $E_s \cap Q^\delta = H \cap Q^\delta$ for some $\delta > 0$ (here Q^δ is defined as in Lemma 13). Since our goal is to estimate $\mathcal{J}_s^1(E_s, Q)$ from below, possibly modifying E_s outside Q we may assume that

$$E_s \cap (\mathbb{R}^n \setminus Q) = H \cap (\mathbb{R}^n \setminus Q). \tag{29}$$

This implies, according to Proposition 17 in Sect. 4, that $\mathcal{J}_s(H, Q) \leq \mathcal{J}_s(E_s, Q)$ for $s \in (0,1)$. Then, in order to prove that

$$\lim_{s\uparrow 1} (1-s)\mathcal{J}_s^1(H, Q) \leq \liminf_{s\to 1^-} (1-s)\mathcal{J}_s^1(E_s, Q), \tag{30}$$

it is enough to show that

$$\lim_{s\uparrow 1} (1-s)(\mathcal{J}_s^2(H, Q) - \mathcal{J}_s^2(E_s, Q)) = 0. \tag{31}$$

One immediately sees that (29) imples

$$\begin{aligned} |\mathcal{J}_s^2(H, Q) - \mathcal{J}_s^2(E_s, Q)| \leq & \int_{(E_s \Delta H)\cap Q} \int_{H^c \cap Q^c} \frac{1}{|x-y|^{n+s}} dxdy \\ & + \int_{(E_s^c \Delta H^c)\cap Q} \int_{H \cap Q^c} \frac{1}{|x-y|^{n+s}} dxdy =: I + II. \end{aligned}$$

Observing that $(E_s \Delta H) \cap Q^\delta = \emptyset$ we can estimate for $y \in (E_s \Delta H) \cap Q$

$$\int_{H^c \cap Q^c} \frac{1}{|x-y|^{n+s}} dx \leq \int_{\mathbb{R}^n \setminus B_\delta(y)} \frac{1}{|x-y|^{n+s}} dx = \frac{n\omega_n}{s\delta^s},$$

hence $I \leq n\omega_n/(s\delta^s)$. One can bound from above II in the same way. Now (31) follows at once upon multiplying by $1-s$ and letting $s \uparrow 1$. This shows (30), and taking the infimum in (30) over all families $(E_s)_{0<s<1}$ as above shows that $\Gamma_n^* \leq \tilde{\Gamma}_n$. □

4. Proof of Theorem 3

In order to prove (3) define Ω_δ as in Proposition 11 for some small $\delta > 0$ and set $F_i := E_i \cap (\Omega^c \cup \Omega_\delta)$. By the minimality of E_i we then have

$$\begin{aligned}
\limsup_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}(E_i,\Omega\backslash\Omega_\delta) &\le \limsup_{i\to\infty}(1-s_i)\left(\mathcal{J}_{s_i}(E_i,\Omega)-\mathcal{J}^1_{s_i}(E_i,\Omega_\delta)\right)\\
&\le \limsup_{i\to\infty}(1-s_i)\left(\mathcal{J}_{s_i}(F_i,\Omega)-\mathcal{J}^1_{s_i}(E_i,\Omega_\delta)\right)\\
&= \limsup_{i\to\infty}(1-s_i)\Big[\left(\mathcal{J}^1_{s_i}(F_i,\Omega)-\mathcal{J}^1_{s_i}(F_i,\Omega_\delta)\right)\\
&\qquad\qquad +\mathcal{J}^2_{s_i}(F_i,\Omega)\Big].
\end{aligned}$$

Since $F_i \cap (\Omega\backslash\Omega_\delta) = \emptyset$ we have, using Proposition 16 in Sect. 4,

$$\begin{aligned}
\limsup_{i\to\infty}(1-s_i)\left(\mathcal{J}^1_{s_i}(F_i,\Omega)-\mathcal{J}^1_{s_i}(F_i,\Omega_\delta)\right) &\le \limsup_{i\to\infty}(1-s_i)\mathcal{J}^1_{s_i}(\Omega\backslash\Omega_\delta,\Omega)\\
&= \limsup_{i\to\infty}(1-s_i)\frac{\mathcal{F}_{s_i}(\chi_{\Omega\backslash\Omega_\delta},\Omega)}{2}\\
&\le \frac{n\omega_n P(\Omega\backslash\Omega_\delta,\mathbb{R}^n)}{2}.
\end{aligned}$$

Again using Proposition 16 in Sect. 4 we get

$$\limsup_{i\to\infty}(1-s_i)\mathcal{J}^2_{s_i}(F_i,\Omega) \le \limsup_{i\to\infty}(1-s_i)\mathcal{J}^1_{s_i}(\Omega,\mathbb{R}^n) \le \frac{n\omega_n P(\Omega,\mathbb{R}^n)}{2},$$

whence (3) follows for $\Omega' \subset \Omega\backslash\Omega_\delta$, hence for every $\Omega' \Subset \Omega$.

For the sake of simplicity we first consider perturbations in compactly supported balls. The general case will require only minor modifications.

Consider the monotone set function $\alpha_i(A) := (1-s_i)\mathcal{J}^1_{s_i}(E_i, A)$ for every open set $F \subset \Omega$ (see Sect. 4 for the definition and some basic properties of monotone set functions), extended to

$$\alpha_i(F) := \inf\{\alpha_i(A) : F \subset A \subset \Omega,\ A \text{ open}\}$$

for every $F \subset \Omega$. Clearly α_i is regular. Thanks to (3) and Theorem 21, up to extracting a subsequence, α_i weakly converges to a regular monotone set function α, which is regular and super-additive on disjoint open sets. We shall now prove that if $B_R(x) \Subset \Omega$ and $\alpha(\partial B_R(x)) = 0$, then E is a local minimum of the functional $P(\cdot, B_R(x))$, and

$$\lim_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}(E_i, B_R(x)) = P(E, B_R(x)).$$

Indeed consider a Borel set $F \subset \Omega$ such that $E\Delta F \Subset B_R$ (here and in the following x is fixed and $B_r := B_r(x)$ for any $r > 0$). Then we can find $r < R$ such that $E\Delta F \subset B_r$. By Theorem 2 there exist sets F_i such that

$$\lim_{i\to\infty}|(F_i\Delta F)\cap B_R| = 0, \quad \lim_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}(F_i, B_R) = \omega_{n-1}P(F, B_R).$$

According to Proposition 11, given ρ and t with $r < \rho < t < R$, we can find sets G_i such that

$$G_i = E_i \text{ in } \mathbb{R}^n \setminus B_t, \quad G_i = F_i \text{ in } B_\rho,$$

and for all $\varepsilon > 0$ there holds

$$\begin{aligned}\mathcal{J}^1_{s_i}(G_i, B_R) \le \mathcal{J}^1_{s_i}(F_i, B_R) + \mathcal{J}^1_{s_i}(E_i, B_R \setminus \overline{B}_{\rho-\varepsilon}) + \frac{C}{\varepsilon^{n+s_i}} \\ + \frac{C|(E_i \Delta F_i) \cap (B_t \setminus B_\rho)|}{(1-s_i)} + C|(F_i \Delta E_i) \cap B_R|.\end{aligned}$$

By the local minimality of E_i we infer

$$\mathcal{J}_{s_i}(E_i, B_R) \le \mathcal{J}_{s_i}(G_i, B_R).$$

We shall now estimate

$$\begin{aligned}\mathcal{J}^2_{s_i}(G_i, B_R) &= \int_{G_i \cap B_R} \int_{G_i^c \cap B_R^c} \frac{dxdy}{|x-y|^{n+s_i}} + \int_{G_i^c \cap B_R} \int_{G_i \cap B_R^c} \frac{dxdy}{|x-y|^{n+s_i}} \\ &=: I + II\end{aligned}$$

We have

$$\begin{aligned}I &= \int_{G_i \cap B_R} \int_{E_i^c \cap B_R^c} \frac{dxdy}{|x-y|^{n+s_i}} = \int_{G_i \cap B_t} \int_{E_i^c \cap B_R^c} \frac{dxdy}{|x-y|^{n+s_i}} \\ &\quad + \int_{E_i \cap (B_R \setminus B_t)} \int_{E_i^c \cap B_R^c} \frac{dxdy}{|x-y|^{n+s_i}} \\ &\le \frac{C|G_i \cap B_t|}{s_i(R-t)^{s_i}} + \int_{E_i \cap (B_R \setminus B_t)} \int_{E_i^c \cap (B_{R'} \setminus B_R)} \frac{dxdy}{|x-y|^{n+s_i}} \\ &\quad + \int_{E_i \cap (B_R \setminus B_t)} \int_{E_i^c \cap B_{R'}^c} \frac{dxdy}{|x-y|^{n+s_i}} \\ &\le \mathcal{J}^1_{s_i}(E_i, B_{R'} \setminus \overline{B}_t) + \frac{C}{s_i}\left(\frac{1}{(R-t)^{s_i}} + \frac{1}{(R'-R)^{s_i}}\right),\end{aligned}$$

for any $R' \in (R, \operatorname{dist}(x, \partial\Omega))$. Since II can be estimated in a similar way, we infer

$$\mathcal{J}^2_{s_i}(G_i, B_R) \le 2\mathcal{J}^1_{s_i}(E_i, B_{R'} \setminus \overline{B}_t) + \frac{C}{s_i}\left(\frac{1}{(R-t)^{s_i}} + \frac{1}{(R'-R)^{s_i}}\right),$$

hence,

$$\limsup_{i\to\infty}(1-s_i)\mathcal{J}^2_{s_i}(G_i, B_R) \le 2\limsup_{i\to\infty}(1-s_i)\mathcal{J}^1_{s_i}(E_i, B_{R'} \setminus \overline{B}_t).$$

Finally

$$
\begin{aligned}
\omega_{n-1}P(E,B_R) &\le \liminf_{i\to\infty}(1-s_i)\mathcal{J}^1_{s_i}(E_i,B_R) \le \liminf_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}(E_i,B_R)\\
&\le \liminf_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}(G_i,B_R)\\
&\le \liminf_{i\to\infty}(1-s_i)\mathcal{J}^1_{s_i}(G_i,B_R) + \limsup_{i\to\infty}(1-s_i)\mathcal{J}^2_{s_i}(G_i,B_R)\\
&\le \liminf_{i\to\infty}(1-s_i)\mathcal{J}^1_{s_i}(F_i,B_R)\\
&\quad +3\limsup_{i\to\infty}(1-s_i)\mathcal{J}^1_{s_i}(E_i,B_{R'}\setminus\overline{B}_{\rho-\varepsilon})\\
&\quad +C\lim_{i\to\infty}|(E_i\Delta F_i)\cap(B_t\setminus B_\rho)|. \qquad (32)
\end{aligned}
$$

The last term is zero, since $E = F$ in $B_t \setminus B_\rho$ and $|(E_i \Delta E)\cap B_R| \to 0$, $|(F_i\Delta F)\cap B_R| \to 0$ as $i\to\infty$. Using Proposition 22 from Sect. 4, and recalling that $\alpha(\partial B_R) = 0$, we infer

$$
\begin{aligned}
&\lim_{R'\downarrow R,\ \rho\uparrow R,\ \varepsilon\downarrow 0}\ \limsup_{i\to\infty}(1-s_i)\mathcal{J}^1_{s_i}(E_i,B_{R'}\setminus\overline{B}_{\rho-\varepsilon})\\
&\quad = \lim_{\delta\to 0}\limsup_{i\to\infty}\alpha_i(B_{R+\delta}\setminus\overline{B}_{R-\delta}) = 0,
\end{aligned}
$$

and (32) finally yields

$$
\omega_{n-1}P(E,B_R) \le \lim_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}(F_i,B_R) = \omega_{n-1}P(F,B_R),
$$

so E is a local minimizer of $P(\cdot,B_R)$. Choosing $F = E$ the chain of inequalities in (32) gives

$$
\lim_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}(E_i,B_R) = \lim_{i\to\infty}(1-s_i)\mathcal{J}^1_{s_i}(E_i,B_R) = \omega_{n-1}P(E,B_R), \quad (33)
$$

as wished. In order to complete the proof we first remark that the above arguments applies to any open set $\Omega' \Subset \Omega$ with Lipschitz boundary and $\alpha(\partial\Omega') = 0$, upon replacing $B_R(x)$ by Ω', $B_{R+\delta}$ by $N_\delta(\Omega')$ and $B_{R-\delta}$ by $N_{-\delta}(\Omega')$, where

$$
N_\delta(\Omega') := \{x\in\Omega : d(x,\Omega') < \delta\}, \quad N_{-\delta}(\Omega') := \{x\in\Omega' : d(x,\partial\Omega') > \delta\}
$$

for $\delta > 0$ small.

In particular $\alpha(\Omega') = P(E,\Omega')$ for every open set $\Omega' \Subset \Omega$ with Lipschitz boundary and $\alpha(\partial\Omega') = 0$. Since for every $\Omega' \Subset \Omega$ and $\varepsilon > 0$ small enough the set

$$
\{\delta\in(-\varepsilon,\varepsilon) : \alpha(\partial N_\delta(\Omega')) > 0\}
$$

is at most countable (remember that α is super-additive and locally finite), and since both α and $P(E,\cdot)$ are *regular* monotone set functions on Ω, it is not difficult to show that $\alpha = P(E,\cdot)$, and the proof is complete. □

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Appendix: some useful results

We list here some results which we used in the previous sections.

Proposition 15. *Let* $E \subset \mathbb{R}^n$ *be a set with finite perimeter in* Ω*. Then for every* $\varepsilon > 0$ *there exists a polyhedral set* $\Pi \subset \mathbb{R}^n$ *such that*

(i) $|(E \triangle \Pi) \cap \Omega| < \varepsilon$,
(ii) $|P(E, \Omega) - P(\Pi, \Omega)| < \varepsilon$,
(iii) $P(\Pi, \partial\Omega) = 0$.

Proof. Classical theorems (see for example [1,7]) imply that there exists a polyhedral set Π' satisfying (i) and (ii). In order to get (iii) first notice that

$$P(\Pi', \partial\Omega) > 0 \quad \text{if and only if } \mathcal{H}^{n-1}(\partial\Pi' \cap \partial\Omega) > 0,$$

and that the latter condition can be satisfied only if $\partial\Omega$ contains a piece Σ with $\mathcal{H}^{n-1}(\Sigma) > 0$ contained in a hyperplane and $\nu_\Omega = \pm\nu_{\Pi'} = \text{const}$ on Σ (here ν_Ω and $\nu_{\Pi'}$ denote the interior unit normal to $\partial\Omega$ and $\partial\Pi'$ respectively). Since the set

$$\left\{\nu \in S^{n-1} : \mathcal{H}^{n-1}(\{x \in \partial\Omega : \nu_\Omega(x) = \nu\}) > 0\right\}$$

is at most countable, it is easy to see that there exists a rotation $R \in SO(n)$ close enough to the identity so that the polyhedron $\Pi := R(\Pi')$ satisfies (i), (ii) and (iii). □

Proposition 16. *Let* $u \in BV(\Omega)$ *and let* $\Omega' \Subset \Omega$ *be open. Then we have*

$$\begin{aligned}\limsup_{s\uparrow 1}(1-s)\mathcal{F}_s(u, \Omega') &\le n\omega_n \limsup_{|h|\to 0} \int_{\Omega'} \frac{|u(x+h) - u(x)|}{|h|} dx \\ &\le n\omega_n |Du|(\Omega). \qquad (34)\end{aligned}$$

Proof. For $h \in \mathbb{R}^n$ let us define

$$g(h) = \int_{\Omega'} \frac{|u(x+h) - u(x)|}{|h|} dx$$

and fix $L > \limsup_{|h|\to 0} g(h)$. Then there exists $\delta_L > 0$ such that $\Omega' + h \subset \Omega$ for all $h \in B_{\delta_L}$ and $L \ge g(h)$ for $0 < |h| \le \delta_L$. Multiplying by $|h|^{-n-s+1}$ and integrating with respect to h on B_{δ_L} we obtain

$$\frac{n\omega_n \delta_L^{1-s} L}{1-s} \ge \int_{B_{\delta_L}} \frac{g(h)}{|h|^{n+s-1}} dh = \int_{B_{\delta_L}} \int_{\Omega'} \frac{|u(x+h) - u(x)|}{|h|^{n+s}} dx dh. \qquad (35)$$

Now notice that

$$\begin{aligned}
&\int_{\Omega'}\int_{\Omega'}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}dxdy\\
&=\int_{(\Omega'\times\Omega')\cap\{|x-y|\le\delta_L\}}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}dxdy\\
&\quad+\int_{(\Omega'\times\Omega')\cap\{|x-y|\ge\delta_L\}}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}dxdy\\
&\le\int_{B_{\delta_L}}\int_{\Omega'}\frac{|u(x+h)-u(x)|}{|h|^{n+s}}dxdh+\int_{B^c_{\delta_L}}\int_{\Omega'}\frac{|u(x+h)-u(x)|}{|h|^{n+s}}dxdh\\
&\le\int_{B_{\delta_L}}\int_{\Omega'}\frac{|u(x+h)-u(x)|}{|h|^{n+s}}dx+\frac{2n\omega_n}{s\delta_L^s}\|u\|_{L^1(\Omega)}.
\end{aligned}\tag{36}$$

Putting together (35) and (36) we obtain

$$n\omega_n L\ge\limsup_{s\uparrow 1}(1-s)\int_{\Omega'}\int_{\Omega'}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}dxdy,$$

and for $L\to\limsup_{|h|\to 0}g(h)$ the first inequality in (34). The second one is well-known. □

Minimality of H

Proposition 17. *For every $s\in(0,1)$, H is the unique minimizer of $\mathcal{J}_s(\cdot,Q)$, in the sense that $\mathcal{J}_s(H,Q)\le\mathcal{J}_s(F,Q)$ for every set $F\subset\mathbb{R}^n$ with $F\cap Q^c=H\cap Q^c$, with strict inequality if $F\ne H$.*

The proof of Proposition 17 easily follows from a couple of results of [5], which we give here for the sake of completeness.

Proposition 18. (Existence of minimizers) *Given $E_0\subset\Omega^c$ and $s\in(0,1)$ there exists $E\subset\mathbb{R}^n$ such that $E\cap\Omega^c=E_0$ and*

$$\inf_{F\cap\Omega^c=E_0}\mathcal{J}_s(F,\Omega)=\mathcal{J}_s(E,\Omega).\tag{37}$$

Proof. This follows immediately from the lower semicontinuity of $\mathcal{J}_s$ with respect to the L^1_{loc} convergence (a simple consequence of Fatou's lemma) and the coercivity estimate of Proposition 4. □

In general a set E satisfying (37) will be called a *minimizer* of $\mathcal{J}_s(\cdot,\Omega)$. Following the notation of [5], we set $L(A,B):=\int_A\int_B|x-y|^{-n-s}dxdy$ for $s\in(0,1)$ and $A,B\subset\mathbb{R}^n$ measurable. Notice that $L(A\cup B,C)=L(A,C)+L(B,C)$ if $|A\cap B|=0$ and $L(A,B)=L(B,A)$. Now we can write

$$\mathcal{J}_s(E,\Omega)=L(E\cap\Omega,E^c)+L(E\cap\Omega^c,E^c\cap\Omega).$$

It is easy to check that a minimizer E of $\mathcal{J}_s(\cdot, \Omega)$ satisfies

$$L(A, E) \leq L(A, E^c \setminus A) \qquad \text{for } A \subset E^c \cap \Omega \tag{38}$$

$$L(A, E^c) \leq L(A, E \setminus A) \qquad \text{for } A \subset E \cap \Omega. \tag{39}$$

It suffices indeed to compare E with $E \setminus A$ and with $E \cup A$.

Proposition 19. (Comparison principle I) *Let E satisfy (38) with $\Omega = Q$ and assume that $H \cap Q^c \subset E$. Then $H \subset E$ up to a set of measure zero (i.e. $|H \cap E^c| = 0$).*

Proof. Let $T(x', x_n) := (x', -x_n)$ denote the reflection across ∂H and set $A^- := H \cap E^c$, $A^+ := T(A^-) \cap E^c$, $A := A^- \cup A^+ \subset E^c \cap Q$, $A_1 := A^+ \cup T(A^+)$, $A_2 = A^- \setminus T(A^+)$ and $F := T(E^c \setminus A) \subset H$. Then, observing that $L(B, C) = L(T(B), T(C))$, from (38) we infer

$$\begin{aligned} 0 &\geq L(A, E) - L(A, E^c \setminus A) = L(A, E) - L(T(A), F) \\ &= L(A, E) - L(A_1, F) - L(T(A_2), F) \\ &= L(A, E) - L(A, F) + L(A_2, F) - L(T(A_2), F) \\ &= L(A, E \setminus F) + L(A_2, F) - L(T(A_2), F) \\ &= L(A_1, E \setminus F) + L(A_2, E \setminus F) + (L(A_2, F) - L(T(A_2), F)). \end{aligned}$$

The first two terms on the right-hand side are clearly positive. We also have $L(A_2, F) > L(T(A_2), F)$ unless $|A_2| = 0$, since for $y \in F$ and $x \in A_2 \setminus \partial H$ one has $|x - y| < |T(x) - y|$. Therefore the right-hand side must be zero, $|A_2| = 0$ and either $|A_1| = 0$ (and the proof is complete), or $|E \setminus F| = 0$. In the latter case consider for a small $\varepsilon > 0$ the translated set $E_\varepsilon := E + (0, \ldots, 0, \varepsilon)$, which satisfies (38) in $Q_\varepsilon := Q + (0, \ldots, 0, \varepsilon)$, hence also in $\tilde{Q}_\varepsilon := Q_\varepsilon \cap T(Q_\varepsilon)$. Repeating the above procedure for E_ε in $\tilde{Q}_\varepsilon$ we get $|A_{2,\varepsilon}| = 0$ (A_ε^-, A_ε^+, etc. are defined as above with respect to the set E_ε in the domain $\tilde{Q}_\varepsilon$, still reflecting across ∂H; we use also the fact since $H \subset H_\varepsilon := H + (0, \ldots, 0, \varepsilon)$, we have $H \cap \tilde{Q}_\varepsilon^c \subset E_\varepsilon$) and, since $|E_\varepsilon \setminus F_\varepsilon| = \infty$, $|A_{1,\varepsilon}| = 0$. This implies at once that $|A_\varepsilon^-| = 0$ and $|H \setminus E_\varepsilon| = 0$. Since this is true for every small $\varepsilon > 0$, it follows that $H \subset E$ (up to a set of measure 0). □

By a similar argument, the proposition above also holds replacing H by H^c. Also, it is easy to see that if E satisfies (39), then E^c satisfies (38), hence by applying Proposition 19 to E^c and H^c one has the following corollary.

Proposition 20. (Comparison principle II) *Let E satisfy (39) with $\Omega = Q$ and assume that $E \cap Q^c \subset H$. Then $E \subset H$ up to a set of measure zero (i.e. $|H^c \cap E| = 0$).*

Proof of Proposition 17. According to Proposition 18 a minimizer E of $\mathcal{J}_s(\cdot, Q)$ with $E \cap Q^c = H \cap Q^c$ exists. Then E satisfies both (38) and (39), hence by Propositions 19 and 20 we have $H \subset E$ and $E \subset H$ (up to sets of measure 0), i.e. $E = H$. □

Monotone set functions

We report some of the main results of [8], see also [6, Chapter 16] for more general and related results. In the sequel for an open set $\Omega \subset \mathbb{R}^n$, we denote by $\mathcal{P}(\Omega)$ the set of subsets of Ω and by $\mathcal{A}(\Omega)$, $\mathcal{K}(\Omega) \subset \mathcal{P}(\Omega)$, the collection of open and compact subset of Ω respectively. We also define

$$\mathcal{C}(\Omega) := \left\{ \bigcup_{i=1}^{M} Q_i : \; Q_i \in \mathcal{Q}, \; M \in \mathbb{N} \right\},$$

where $\mathcal{Q}$ is *countable* the set of *open* cubes $Q_r(x) := x + rQ \Subset \Omega$ with $x \in \mathbb{Q}^n$ and $0 < r \in \mathbb{Q}$. The collections $\mathcal{A}(\Omega)$, $\mathcal{K}(\Omega)$ and $\mathcal{C}(\Omega)$ satisfy the following property

$$A \in \mathcal{A}(\Omega), \; K \in \mathcal{K}(\Omega), \; K \subset A \;\Rightarrow\; \text{there exists } C \in \mathcal{C}(\Omega) \text{ with } K \subset C \Subset A. \tag{40}$$

We say that a set function $\alpha : \mathcal{P}(\Omega) \to [0, \infty]$ is *monotone* if

$$\alpha(E) \le \alpha(F) \quad \text{wherever } E \subset F,$$

and that a monotone set function is *regular* if the following two conditions hold

$$\alpha(A) = \sup\{\alpha(K) : \; K \subset A, \; K \in \mathcal{K}(\Omega)\} \quad \text{for any } A \in \mathcal{A}(\Omega), \tag{41}$$

$$\alpha(E) = \inf\{\alpha(A) : \; E \subset A, \; A \in \mathcal{A}(\Omega)\} \quad \text{for any } E \in \mathcal{P}(\Omega). \tag{42}$$

Thanks to (40) it is clear that (41) is equivalent to

$$\alpha(A) = \sup\{\alpha(V) : \; V \Subset A, \; V \in \mathcal{A}(\Omega)\} = \sup\{\alpha(C) : \; C \Subset A, \; C \in \mathcal{C}(\Omega)\}. \tag{43}$$

We also say that a monotone set function α is *super-additive* if

$$\alpha(E \cup F) \ge \alpha(E) + \alpha(F), \quad \text{wherever } E, F \in \mathcal{P}(\Omega), \; E \cap F = \emptyset.$$

We say that a sequence of regular monotone set functions α_i *weakly converges* to a monotone set function α if the following two conditions hold:

$$\liminf_{i \to \infty} \alpha_i(A) \ge \alpha(A) \quad \text{for every } A \in \mathcal{A}(\Omega), \tag{44}$$

$$\limsup_{i \to \infty} \alpha_i(K) \le \alpha(K) \quad \text{for every } K \in \mathcal{K}(\Omega). \tag{45}$$

The limit need not be unique, but it is easy to see that a sequence of regular monotone set functions admits at most one *regular* limit.

Theorem 21. (De Giorgi-Letta) *Let* (α_i) *be a sequence of regular monotone set functions such that*

$$\limsup_{i \to \infty} \alpha_i(\Omega') < \infty \quad \textit{for every open set } \Omega' \Subset \Omega.$$

Then there exists a subsequence $(\alpha_{i'})$ *weakly converging to a regular monotone set function* α. *Moreover if each* α_i *is super-additive on disjoint open sets*[1] (*and hence on disjoint compact sets*), *then so is* α.

[1] This means that $\alpha_i(A \cup B) \ge \alpha_i(A) + \alpha_i(B)$ wherever $A, B \in \mathcal{A}(\Omega)$ are disjoint.

Proof. Since the proof is standard we only sketch it.

Step 1. Being $\mathcal{C}(\Omega)$ countable, we can easily extract a diagonal subsequence, still denoted by (α_i) such that,

$$\beta(C) := \lim_{i\to\infty} \alpha_i(C) < \infty \quad \text{for any } C \in \mathcal{C}(\Omega).$$

Step 2. We define

$$\alpha(A) := \sup\{\beta(C) : C \Subset A,\ C \in \mathcal{C}(\Omega)\} \quad \text{for every } A \in \mathcal{A}(\Omega),$$
$$\alpha(E) := \inf\{\alpha(A) : A \supset E,\ A \in \mathcal{A}(\Omega)\} \quad \text{for every } E \in \mathcal{P}(\Omega).$$

Clearly for $C \in \mathcal{C}(\Omega)$ we have $\alpha(C) \le \beta(C)$.

Step 3. The set function α is clearly monotone, and if every α_i is super-additive on disjoint open sets, then so is α. It is also easy to see that (44) is satisfied. As for (45), it is an easy consequence of the identity

$$\alpha(K) = \inf\{\beta(C) : C \supset K,\ C \in \mathcal{C}(\Omega)\}.$$

which follows from (40). Then α_i converges weakly to α.

Step 4. It remains to prove the regularity of α. Identity (42) follows by the definition of α. In order to prove (41) fix any $A \in \mathcal{A}(\Omega)$. Then for $C \in \mathcal{C}(\Omega)$ with $C \Subset A$, we have

$$\beta(C) = \lim_{i\to\infty} \alpha_i(C) \le \limsup_{i\to\infty} \alpha_i(\overline{C}) \le \alpha(\overline{C}) \le \alpha(C') \le \beta(C').$$

From this and the definition of $\alpha(A)$, (43) follows at once, hence also (41). □

Proposition 22. *Let (α_i) be a sequence of regular monotone set functions weakly converging to a regular monotone set function α, and let $K_j \downarrow K$ be a decreasing sequence of compact sets such that $\alpha(K) = 0$. Then*

$$\lim_{j\to\infty} \limsup_{i\to\infty} \alpha_i(K_j) = 0$$

Proof. We have

$$0 = \alpha(K) = \lim_{j\to\infty} \alpha(K_j) \ge \lim_{j\to\infty} \limsup_{i\to\infty} \alpha_i(K_j),$$

where the second equality follows from the regularity of α. Indeed for $A \in \mathcal{A}(\Omega)$ with $A \supset K$, we have by compactness $A \supset K_j$ for j large enough, hence

$$\alpha(A) \ge \lim_{j\to\infty} \alpha(K_j) \ge \alpha(K) = 0,$$

and the claim follows by taking the infimum over all $A \in \mathcal{A}(\Omega)$ with $A \supset K$. □

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