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Deformations of smooth toric surfaces

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Abstract. For a complete, smooth toric variety Y, we describe the graded vector space T_Y^1. Furthermore, we show that smooth toric surfaces are unobstructed and that a smooth toric surface is rigid if and only if it is Fano. For a given toric surface we then construct homogeneous deformations by means of Minkowski decompositions of polyhedral subdivisions, compute their images under the Kodaira-Spencer map, and show that they span T_Y^1.

1. Introduction

The deformation theory of toric singularities has been rather well studied; see for example [1]. It is possible to calculate the space of infinitesimal deformations, construct certain homogeneous deformations, and in some cases, construct a versal deformation all based on the combinatorics of the polyhedral cone corresponding to the toric singularity. On the other hand, with the exception of [8,9], and some rigidity results there has, to our knowledge, been very little work done on deformations of complete toric varieties.

The goal of this paper is to start understanding the deformation theory of complete toric varieties. For simplicity's sake, we will assume that all varieties are smooth; we hope to be able to remove this assumption in a later paper. In Sect. 2 we will compute the vector space of infinitesimal deformations of an arbitrary smooth, complete toric variety. This essentially boils down to counting the number of connected components in certain graphs coming from hyperplane sections of the fan defining the toric variety. The formula takes a particularly simple form for toric surfaces. As an application we shall see that a complete, smooth toric surface is rigid if and only if it is Fano, as well as that weakly Fano varieties in higher dimensions are quite often rigid.

In Sect. 3, we then concentrate on constructing homogeneous one-parameter deformations of complete, smooth toric surfaces by means of Minkowski decompositions of polyhedral subdivisions. This is analogous to the construction of deformations of toric singularities in terms of Minkowski decompositions of polytopes. For each such deformation we construct, we also compute its image under the

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Kodaira-Spencer map. This enables us to show that the homogeneous deformations we construct in fact form a basis for the space of infinitesimal deformations.[1] The deformations we construct have a natural fiberwise $\mathbb{C}^*$ action, and the fibers can in fact be described by divisorial fans.

We assume that the reader is familiar with toric varieties as presented in [5]. Throughout this paper, we will be using the following notation. As usual, N will be a rank n lattice, with dual lattice M. $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$ shall denote the associated $\mathbb{Q}$-vector spaces. For any M-graded group G and $u \in M$, let $G(u)$ be the grade u component of G. $Y = \mathrm{TV}(\Sigma)$ will be a complete, smooth, n-dimensional toric variety corresponding to the complete smooth fan Σ on $N_{\mathbb{Q}}$. By $\Sigma^{(k)}$ we denote the set of all k-dimensional cones in Σ. Let $\rho_1, \ldots, \rho_l \in \Sigma^{(1)}$ be all rays in the fan Σ. In the case $n = 2$, we number these rays in some counterclockwise order; by ρ_{l+1} we then mean ρ_1, etc. By $\nu(\rho_i)$ we shall denote the minimal generator of the ray ρ_i. Furthermore, by D_i we denote the invariant divisor corresponding to ρ_i.

2. Infinitesimal deformations

Let Y be a smooth toric variety; we are interested in the vector space of infinitesimal deformations, which we term T_Y^1. Since Y is smooth, we simply have $T_Y^1 = H^1(Y, \mathcal{T}_Y)$. This vector spaces carries an M grading, so we can compute it by computing each homogeneous piece $T_Y^1(u)$. Furthermore, the cohomology of $\mathcal{T}_Y$ is closely intertwined with the cohomology of the boundary divisors of Y. We first shall show how to compute the cohomology of the boundary divisors and then proceed to use this to compute T_Y^1. We then use these computations to prove several rigidity results.

2.1. Cohomology of boundary divisors

Fix some smooth, complete variety $Y = \mathrm{TV}(\Sigma)$ and choose some weight $u \in M$; for $1 \leq i \leq l$ we wish to calculate $H^1(Y, \mathcal{O}(D_i))(u)$. Using a result of Demazure, we show that this can be done by counting connected components in a certain graph. Let $\Gamma_i(u)$ be the graph with vertex set $V_i(u) = \{\nu(\rho_j),\ j \neq i \mid \langle \nu(\rho_j), u\rangle < 0\}$ and two vertices $\nu(\rho_j)$ and $\nu(\rho_k)$ joined by an edge if and only if ρ_j and ρ_k are rays in some common cone of Σ.

Proposition 2.1. *For any smooth, complete toric variety Y we have* $H^1(Y, \mathcal{O}(D_i))(u) = 0$ *if* $\langle \nu(\rho_i), u\rangle \neq -1$. *Otherwise,*

$$\dim H^1(Y, \mathcal{O}(D_i))(u) = \max\{0, \dim H^0(\Gamma_i(u), \mathbb{C}) - 1\}.$$

Proof. Let $U_i(u) = \{v \in N_{\mathbb{Q}} | \langle v, u\rangle < h(v)\}$, where h is the piecewise linear function on Σ given by $h(\nu(\rho_i)) = -1$, $h(\nu(\rho_j)) = 0$ for $j \neq i$. Then by [4],

[1] As pointed out by A. Mavlyutov, the construction of one-parameter deformations as found in [8] can be extended to construct deformations spanning T_Y^1 as well.

$H^p(Y, \mathcal{O}(D_i))(u) \cong H^p(N_{\mathbb{Q}}, U_i(u))$ for all $p \geq 0$. Thus, we have the following exact sequence coming from relative cohomology:

$$0 \to H^0(Y, \mathcal{O}(D_i))(u) \to \mathbb{C} \to H^0(U_i(u), \mathbb{C}) \to H^1(Y, \mathcal{O}(D_i))(u) \to 0.$$

Suppose that $\langle \nu(\rho_i), u \rangle \neq -1$. If u=0 then $U_i(u) = \emptyset$ and thus $H^1(Y, \mathcal{O}(D_i))(u)$=0. If instead $u \neq 0$, then we also have $H^1(Y, \mathcal{O}(D_i))(u) = 0$. Indeed, a direct calculation shows that $H^0(Y, \mathcal{O}(D_i))(u) = 0$. Furthermore, $U_i(\lambda u)$ is homeomorphic to $U_i(u)$ for all $\lambda \in \mathbb{N}$. Since $H^1(Y, \mathcal{O}(D_i))$ is finite dimensional, it follows from the above exact sequence that $\dim H^0(U_i, \mathbb{C})(\lambda u) = \dim H^0(U_i, \mathbb{C})(u) = 1$ and thus that $H^1(Y, \mathcal{O}(D_i))(u) = 0$.

We can now assume that $\langle \nu(\rho_i), u \rangle = -1$. In this case, $H^0(U_i(u), \mathbb{C}) = H^0(\Gamma_i(u), \mathbb{C})$. Indeed, $U_i(u)$ can be retracted to $\widetilde{U}_i(u) = U_i(u) \cap \mathrm{roof}(\Sigma)$, where $\mathrm{roof}(\Sigma)$ is the simplicial complex with vertices being the primitive generators of the rays of Σ, and any such vertices forming a simplex if the corresponding rays belong to a common cone. Now for each simplex S of dimension larger than one in $\mathrm{roof}(\Sigma)$, $\widetilde{U}_i(u) \cap S$ can be replaced in $\widetilde{U}_i(u)$ with $\widetilde{U}_i(u) \cap \partial S$ without changing the connectivity of the set. Thus, we can replace $\widetilde{U}_i(u)$ by its intersection with the union of all one-simplices of $\mathrm{roof}(\Sigma)$. This is nothing other than $\Gamma_i(u)$.

Now, one easily checks that $H^0(Y, \mathcal{O}(D_i)) = 0$ unless $\Gamma_i(u) = \emptyset$. The formula then follows from the exact sequence. □

In dimensions two and three, we can calculate the first cohomology groups by counting connected components of an even simpler graph. Indeed, let $\Gamma_i^\circ(u)$ be the subgraph of $\Gamma_i(u)$ induced by those vertices whose corresponding rays share a common cone with ρ_i.

Lemma 2.2. *For $u \in M$ such that $\langle \nu(\rho_i), u \rangle = -1$, we have*

$$H^0(\Gamma_i(u), \mathbb{C}) \leq H^0(\Gamma_i^\circ(u), \mathbb{C})$$

with equality holding if $n = \dim N \leq 3$.

Proof. Any connected component of $\Gamma_i(u)$ has non-empty intersection with $\Gamma_i^\circ(u)$. Indeed, let $\widetilde{\Gamma}_i(u)$ be the graph with vertex set $V_i(u) \cup \{\nu(\rho_i)\}$ and edges induced by common cone membership as before. Then the vertex set of $\widetilde{\Gamma}_i(u)$ consists of exact those $\nu(\rho_j)$ such that $\langle \nu(\rho_j), u \rangle < 0$. Thus, $\widetilde{\Gamma}_i(u)$ has the same number of connected components as a convex set, namely one. The inequality then follows.

If $n = 2$ the equality is immediately obvious. For $n = 3$, we can consider $\Gamma_i(u)$ as a planar graph embedded in the plane $\langle \cdot, u \rangle = -1$ by identifying each $\nu(\rho_j)$ with the intersection of ρ_j with this plane. If a ray ρ_k sharing a common cone with ρ_i does not intersect this plane, it is not in $\Gamma_i^\circ(u) \subset \Gamma_i(u)$. The two-dimensional cone generated by ρ_i and ρ_k intersects the above plane in a ray τ_k, which no edges in $\Gamma_i(u)$ intersect. One easily sees that such τ_k partition both $\Gamma_i^\circ(u)$ and $\Gamma_i(u)$ into their connected components, so that the desired equality must hold. □

2.2. The generalized Euler sequence

We will now use our knowledge of the first cohomology groups of the boundary divisors to compute T^1_Y. Indeed, for a smooth, complete toric variety, the higher cohomology groups of the tangent bundle are isomorphic to the sum of those of the boundary divisors via the generalized Euler sequence:

Lemma 2.3. ([7]) *Let Y be a smooth, complete toric variety with boundary divisors $D_1, \ldots, D_l$. Then $H^i(Y, \mathcal{T}_Y) \cong \bigoplus_{i=1}^{l} H^i(Y, \mathcal{O}(D_i))$ as M-graded groups for $i \geq 1$.*

Proof. Taking the long exact cohomology sequence coming from the generalized Euler sequence

$$0 \mapsto N_1(Y) \otimes \mathcal{O}_Y \to \bigoplus_{i=1}^{l} \mathcal{O}(D_i) \to \mathcal{T}_Y \to 0\,, \tag{1}$$

and using the fact that $H^i(Y, N_1 \otimes \mathcal{O}_Y)$ vanishes for $i \geq 1$ gives us

$$H^i(Y, \mathcal{T}_Y) \cong H^i\left(Y, \bigoplus_{i=1}^{l} \mathcal{O}(D_i)\right) \cong \bigoplus_{i=1}^{l} H^i(Y, \mathcal{O}(D_i))$$

for $i \geq 1$. □

Combining this with the cohomology computations of the previous section yields the following:

Proposition 2.4. *For a smooth, complete toric variety Y,*

$$\dim T^1_Y(u) = \dim H^1(Y, \mathcal{T}_Y)(u) = \sum_{\langle \nu(\rho_i), u\rangle = -1} \max\{0, \dim H^0(\Gamma_i(u), \mathbb{C}) - 1\},$$

where if $\dim n = 2, 3$ *we can replace the* $\Gamma_i(u)$ *with* $\Gamma_i^\circ(u)$.

In the case of a toric surface, this becomes even more explicit:

Corollary 2.5. *For a smooth, complete toric surface $Y = \mathrm{TV}(\Sigma)$*

$$\dim T^1_Y(u) = \#\left\{\rho_i \in \Sigma^{(1)} \;\middle|\; \begin{array}{l} \langle \nu(\rho_i), u\rangle = -1 \\ \langle \nu(\rho_{i\pm1}), u\rangle < 0 \end{array}\right\}$$

for all $u \in M$. Furthermore, $T^2_Y = 0$, i.e. Y is unobstructed.

Proof. The first claim follows from the proposition and the fact that $\dim H^0(\Gamma_i^\circ(u), \mathbb{C}) \leq 2$, with equality if and only if $\langle \nu(\rho_i), u\rangle = -1$ and $\langle \nu(\rho_{i\pm1}), u\rangle < 0$. For the second, consider that

$$H^2(Y, \mathcal{T}_Y) \cong \bigoplus H^2(Y, \mathcal{O}(D_i)) \cong \bigoplus H^0\left(Y, \mathcal{O}\left(-D_i - \sum_{j=1}^{l} D_j\right)\right) = 0$$

by Serre duality. □

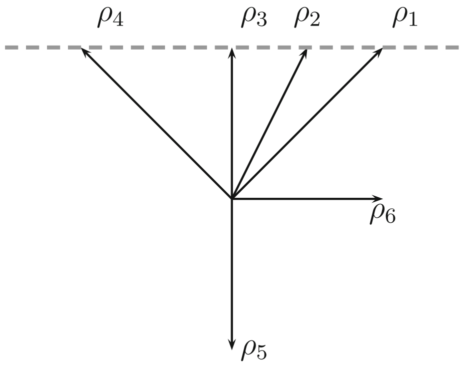

Fig. 1. $\mathcal{F}_1$ with two successive blowups

The following corollary is then immediate:

Corollary 2.6. *Let Y be a smooth, complete toric surface. Let $\widetilde{Y}$ be an equivariant blow-up of Y. Then $T^1_Y \subset T^1_{\widetilde{Y}}$.*

Example. We consider the toric surface Y whose fan Σ has rays through $(1, 1)$, $(1, 2)$, $(0, 1)$, $(-1, 1)$, $(0, -1)$, and $(1, 0)$ as pictured in Fig. 1. This is simply the first Hirzebruch $\mathcal{F}_1$ with two successive blowups. One easily checks that the only degrees u for which $T^1_Y(u)$ isn't trivial are $[0, -1]$, $[-1, 0]$, and $[1, -1]$. For $u = [0, -1]$, the dashed gray line in Fig. 1 marks the hyperplane $\langle \cdot, u\rangle = -1$. One thus sees that only the ray ρ_3 satisfies the conditions $\langle \nu(\rho_i), u\rangle = -1$, $\langle \nu(\rho_{i\pm 1}), u\rangle < 0$ and thus, $\dim T^1_Y([0, -1]) = 1$. Likewise, for $u = [-1, 0]$ and $u = [1, -1]$ the rays ρ_1 and ρ_3 respectively satisfy the conditions and so $\dim T^1_Y([-1, 0]) = 1$ and $\dim T^1_Y([-1, 1]) = 1$ as well.

Example. We also consider a three-dimensional example. Let $\rho_1 = \langle(1, 0, 1)\rangle$, $\rho_2 = \langle(1, 1, 0)\rangle$, $\rho_3 = \langle(0, 1, 1)\rangle$, $\rho_4 = \langle(-1, 0, 0)\rangle$, $\rho_5\langle(-1, -1, 1)\rangle$, $\rho_6 = \rho_0 = \langle(0, -1, 0)\rangle$, $\rho_7 = \langle(0, 0, 1)\rangle$, and $\rho_8 = -\rho_7$. Now, let Σ be the fan with top-dimensional cones generated by $\rho_i, \rho_{i+1}, \rho_7$ or by $\rho_i, \rho_{i+1}, \rho_8$ for $0 \le i < 6$. If we set $-R = [0, 0, -1]$, the affine slices $\langle \Sigma, R\rangle = \pm 1$ are pictured in Fig. 2. We shall consider infinitesimal deformations of the threefold $Y = \mathrm{TV}(\Sigma)$.

Note that the graph $\Gamma_7(-R) = \Gamma^\circ_7(-R)$ consists of the three non-connected vertices $\nu(\rho_1)$, $\nu(\rho_3)$, and $\nu(\rho_5)$. Thus, $\dim H^1(Y, \mathcal{O}(D_7))(-R) = 2$. One also easily checks that this is the only degree/ray combination for which the first cohomology doesn't vanish. Thus, $\dim T^1_Y = \dim T^1_Y(-R) = 2$.

2.3. Rigidity results

In [3] it was shown that a smooth, complete toric Fano variety is rigid. It is possible to use our explicit cohomology calculations to generalize this slightly; in many cases, it suffices to assume weakly Fano:

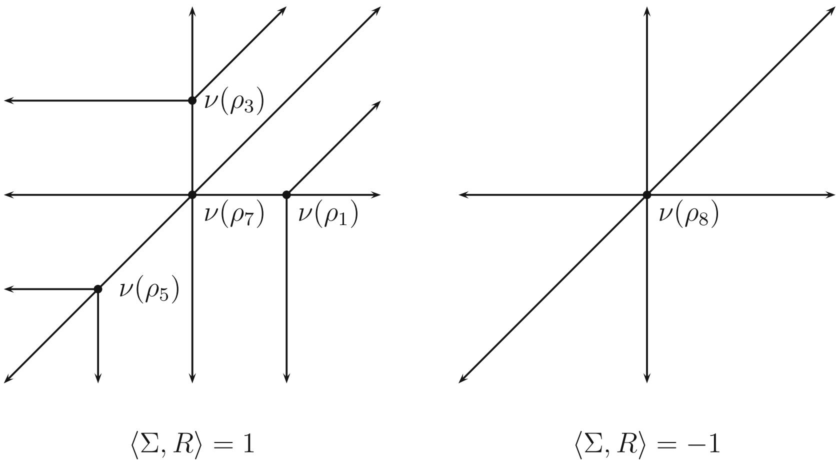

Fig. 2. Hyperplane sections of three-dimensional fan

Corollary 2.7. *Let Y be a complete, smooth, weakly Fano toric variety, and assume that there is no equivariant embedding $\widetilde{A}_1 \times (\mathbb{C}^*)^{n-2} \hookrightarrow Y$, where $\widetilde{A}_1$ is the minimal resolution of a toric A_1 singularity. Then Y is rigid.*

Proof. Consider $u \in M$ and $1 \leq i \leq l$ such that $\langle \nu(\rho_i), u\rangle = -1$ and take $\nu(\rho_j), \nu(\rho_k) \in \Gamma_i^\circ(u)$ with $j \neq k$. We shall show that $\nu(\rho_j)$ and $\nu(\rho_k)$ are in fact connected in $\Gamma_i(u)$; the claim then follows from Proposition 2.4 and Lemma 2.2.

Let γ be the line segment connecting $\nu(\rho_j)$ and $\nu(\rho_k)$ and let $\widetilde{\gamma}$ be the projection of γ to the roof of Σ. Now, since the roof of Σ is concave,

$$\langle v, u\rangle \leq \max\{\langle \nu(\rho_j), u\rangle, \langle \nu(\rho_k), u\rangle\} \leq -1$$

for all $v \in \widetilde{\gamma}$. Thus, if γ doesn't intersect ρ_i, $\widetilde{\gamma}$ is in $U_i(u)$ and $\nu(\rho_j)$ and $\nu(\rho_k)$ are connected in $U_i(u)$ and thus also in $\Gamma_i(u)$. Suppose on the other hand that γ intersects ρ_i, that is, that $\nu(\rho_i) \in \widetilde{\gamma}$. Then from the concavity of the roof and $\langle \nu(\rho_i), u\rangle = -1$, it follows that $\langle \nu(\rho_j), u\rangle = \langle \nu(\rho_k), u\rangle = -1$. Since however both ρ_j and ρ_k share common cones with ρ_i, one easily sees that the subfan of Σ with rays ρ_i, ρ_j, and ρ_k corresponds to the toric variety $\widetilde{A}_1 \times (\mathbb{C}^*)^{n-2}$, a contradiction. □

Now, any weakly Fano smooth, complete toric surface which isn't Fano does in fact admit an embedding of $\widetilde{A}_1$, so the above result doesn't provide anything new for $n = 2$. However, we can show that non-Fano surfaces are in fact never rigid:

Corollary 2.8. *A smooth, complete toric surface Y is rigid if and only if Y is Fano.*

Proof. Every smooth, complete toric surface Y can be constructed as an equivariant blow-up of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or the Hirzebruch surface $\mathcal{F}_r$, cf. [5, p. 43]. Of these varieties, an application of Corollary 2.5 shows that only $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{F}_1$ are rigid. One then checks case by case rigidity for blow-ups of these surfaces. Once a blow-up is no longer rigid, the above corollary tells us that further blow ups are also non-rigid; it thus turns out that we only have to check finitely many cases and that the rigid surfaces are exactly those which are Fano. □

Example. Consider rays $\rho_0 \dots \rho_4$ generated by $(0,0,1)$, $(-1,0,1)$, $(0,-1,1)$, $(1,1,1)$, and $(0,0,-1)$, respectively. Let Σ be the fan with six top-dimensional cones spanned by ρ_i, ρ_j, ρ_k for $1 \leq i, j \leq 3$ and $k \in \{0,4\}$. Then the complete toric threefold $Y = \mathrm{TV}(\Sigma)$ is rigid. Indeed, although it isn't Fano, it is weakly Fano, and one easily checks that there is no equivariant embedding of $\widetilde{A}_1 \times (\mathbb{C}^*)$.

3. Homogeneous one-parameter deformations

In [1], Altmann introduced so-called toric deformations of affine toric varieties to study the deformation theory of toric singularities. We wish to construct an analogon in the global setting.

Recall that a deformation $\pi : X \to S$ of an affine toric variety Y is called toric if X is a toric variety, and the natural inclusion $Y \hookrightarrow X$ is torus equivariant and induces an isomorphism on the closed orbits. Let Y correspond to a cone σ in $N_\mathbb{Q}$ and X to a cone $\widetilde{\sigma}$ in $\widetilde{N}_\mathbb{Q}$ with $\mathrm{codim}(Y, X) = 1$. It turns out that the inclusion $Y \hookrightarrow X$ is given by a single binomial $\chi^{s^1} - \chi^{s^2}$, with $s^1, s^2 \in \widetilde{\sigma}^\vee \cap \widetilde{M}$; let R be the common image of s^1, s^2 in $\sigma^\vee \cap M \setminus 0$. Setting the parameter $t = \chi^{s^1} - \chi^{s^2}$ thus gives a natural map $X \to \mathbb{A}^1$, which is a 1-parameter deformation of Y; we say that it has degree $-R$. This deformation is not necessarily equal to the deformation π, but they share certain properties.

Ideally, in the global setting we would like to consider deformations $\pi : X \to S$ of a complete, smooth toric surface Y which are locally one-parameter toric deformations in a fixed degree $-R$. For Y non-complete and non-smooth, the author employed exactly such a construction in [6] to describe certain simultaneous resolutions. However, when Y is complete, this is simply not possible, since we can never find an $R \neq 0$ living in all $\sigma^\vee$, $\sigma \in \Sigma$. Instead, we will consider deformations which are locally one-parameter toric deformations of degree $-R$ only on those open sets $\mathrm{TV}(\sigma)$ with $R \in \sigma^\vee$. We first describe how to construct such deformations and then compute their images under the Kodaira-Spencer map.

3.1. Construction

Let $Y = \mathrm{TV}(\Sigma)$ be a smooth, complete toric surface and let $R \in M$ with R primitive. We can choose a basis of N such that $R = [0, 1]$ in the corresponding basis of M. We then consider the polyhedral subdivision

$$\Xi_0 = \Sigma \cap \{v \in N_\mathbb{Q} | \langle v, [0,1] \rangle = 1\}.$$

Let $\mathbb{L} = \{v \in N | \langle v, [0,1] \rangle = 1\}$ be the lattice attained when considering the point $(0,1) \in N$ as the origin. We can thus consider Ξ_0 to be a polyhedral subdivision of $\mathbb{Q}$. Let $\Delta_0^0, \dots, \Delta_0^{m+1}$ be the line segments in Ξ_0, ordered such that $\Delta_0^i \geq \Delta_0^{i+1}$. We choose our ordering of the rays $\rho_i \in \Sigma^{(1)}$ such that ρ_1 passes between Δ_0^0 and Δ_0^1.

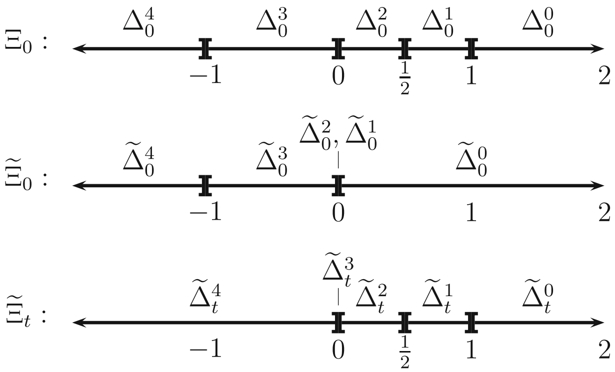

Fig. 3. A possible subdivision decomposition

Definition. A subdivision decomposition $(\widetilde{\Xi}_0, \widetilde{\Xi}_t)$ of Ξ_0 consists of the sets $\widetilde{\Xi}_0 = \{\widetilde{\Delta}_0^0, \dots, \widetilde{\Delta}_0^{m+1}\}$ and $\widetilde{\Xi}_t = \{\widetilde{\Delta}_t^0, \dots, \widetilde{\Delta}_t^{m+1}\}$ such that

1. $\widetilde{\Delta}_0^i, \widetilde{\Delta}_t^i$ are polytopes in $\mathbb{Q}$;
2. $\widetilde{\Delta}_0^i \geq \widetilde{\Delta}_0^{i+1}, \widetilde{\Delta}_t^i \geq \widetilde{\Delta}_t^{i+1}$ for all i and both $\widetilde{\Xi}_0$ and $\widetilde{\Xi}_t$ cover $\mathbb{Q}$;
3. $\Delta_0^i = \widetilde{\Delta}_0^i + \widetilde{\Delta}_t^i$ for all i.

Furthermore, we say that $(\widetilde{\Xi}_0, \widetilde{\Xi}_t)$ is admissible if for each $1 \leq i \leq m$ either $\widetilde{\Delta}_0^i$ or $\widetilde{\Delta}_t^i$ is a lattice point.

Remark. To each admissible subdivision decomposition $(\widetilde{\Xi}_0, \widetilde{\Xi}_t)$ we can associate the $m+2$-tuples $a = (a_0, a_1, \dots, a_m, a_{m+1})$ and $\lambda = (\lambda_0, \dots, \lambda_{m+1})$ with $a_i \in \{-1, 1\}$ and $\lambda_i \in \mathbb{Z}$ as follows: For $0 \leq i \leq m+1$ we choose $\lambda_i \in \mathbb{Z}$ such that $\Delta_0^i = \widetilde{\Delta}_0^i + \lambda_i$ if possible, in which case we set $a_i = 1$, otherwise $a_i = -1$ and we choose $\lambda_i \in \mathbb{Z}$ such that $\Delta_0^i = \widetilde{\Delta}_t^i + \lambda_i$. Note that $\lambda_i = \lambda_{i+1}$ if $a_i = a_{i+1}$ and $\lambda_i + \lambda_{i+1} = \Delta_0^i \cap \Delta_0^{i+1}$ if $a_i \neq a_{i+1}$.

Conversely, an integer λ_0 and a binary $m+2$-tuple a define a subdivision decomposition of Ξ_0. Such a decomposition is admissible if and only if $\Delta_0^i \cap \Delta_0^{i+1}$ is a lattice point for every i such that $a_i \neq a_{i+1}$.

Example. We continue the example from Fig. 1 in Sect. 2.2. If we take the standard basis, then we attain the polyhedral subdivision induced by Σ on the dashed gray line in Fig. 1. We picture this subdivision Ξ_0 in Fig. 3. A possible subdivision decomposition is pictured as well; in this decomposition, $\widetilde{\Delta}_0^1$, $\widetilde{\Delta}_0^2$, and $\widetilde{\Delta}_t^3$ are all equal to the lattice point 0. For this decomposition, we have $(a_0, \dots, a_4) = (1, -1, -1, 1, 1)$ and $(\lambda_0, \dots, \lambda_4) = (1, 0, 0, 0, 0)$. Note that modulo the choice of λ_0, there are exactly four possible decompositions, corresponding to the tuples $a = (1, 1, 1, 1, 1)$, $a = (1, -1, -1, -1, 1)$, $a = (1, -1, -1, 1, 1)$, and $a = (1, 1, 1, -1, 1)$.

From each admissible decomposition $(\widetilde{\Xi}_0, \widetilde{\Xi}_t)$ of Ξ_0 we will construct a one-parameter deformation of Y. We will do this locally on $\mathrm{TV}(\sigma)$ for each $\sigma \in \Sigma^{(2)}$

and then glue together to attain a global family. We first set up some notation. Let σ_i be the cone in Σ spanned by ρ_i, ρ_{i+1}. Note that for $1 \le i \le m$, $\mathrm{Cone}(\Delta_0^i) = \sigma_i$. For each $1 \le i \le l$, let $w_i^1 = [r_i^1, s_i^1]$, $w_i^2 = [r_i^2, s_i^2]$ be positively oriented minimal lattice generators of $\sigma_i^\vee$ and set $Z_{i,j} = x^{r_i^j} y^{s_i^j}$ for $j = 1, 2$. Then $\mathrm{TV}(\sigma_i) = \mathrm{Spec}\, A_i$, where

$$A_i = \mathbb{C}[\sigma_i^\vee \cap M] = \mathbb{C}[Z_{i,1}, Z_{i,2}].$$

Note furthermore that $w_i^1 = -w_{i+1}^2$.

Let a and λ be as in the above remark. For $i > m+1$ or $i < 0$ we set $a_i = 1$ and $\lambda_i = 0$. Then for $0 \le i \le l-1$ and $j \in \{1, 2\}$ define polynomials

$$\widetilde{Z}_{i,j} = \begin{cases} x^{r_i^j} y^{s_i^j + \lambda_i r_i^j} (y-t)^{-\lambda_i r_i^j} & a_i = 1 \\ x^{r_i^j} y^{-\lambda_i r_i^j} (y-t)^{s_i^j + \lambda_i r_i^j} & a_i = -1 \end{cases}$$

Likewise, define rings

$$\begin{array}{ll} \widetilde{A}_i = \mathbb{C}[t, \widetilde{Z}_{i,1}, \widetilde{Z}_{i,2}] & \text{if } \; 0 \le i \le m+1; \\ \widetilde{A}_i = \mathbb{C}[t, \widetilde{Z}_{i,1}, \widetilde{Z}_{i,2}, y(y-t)^{-1}] & \text{if } \; i > m+1. \end{array}$$

Then the natural map $\pi_i : \mathrm{Spec}\, \widetilde{A}_i \to \mathrm{Spec}\, \mathbb{C}[t]$ defines a one-parameter deformation of $\mathrm{Spec}\, A_i = \mathrm{TV}(\sigma_i)$. The maps π_i glue naturally to give a map $\pi : X \to \mathbb{A}^1$. Indeed, for some $0 \le i, j < l$ identify $\mathrm{Spec}\, \widetilde{A}_i$ with $\mathrm{Spec}\, \widetilde{A}_j$ on all open subsets $\mathrm{Spec}\, B$, where B is a localization of both $\widetilde{A}_i$ and $\widetilde{A}_j$.

Theorem 3.1. *The family $\pi : X \to \mathbb{A}^1$ is a one-parameter deformation of $Y =$ $\mathrm{TV}(\Sigma)$.*

Proof. We just need to show that $\pi^{-1}(0) = \mathrm{TV}(\Sigma)$; this amounts to showing that the gluing of the Spec $\widetilde{A}_i$ restricted to the special fiber is equivalent to the gluing of the Spec A_i present in $\mathrm{TV}(\Sigma)$. Now, it is immediate that any one of the gluings of the Spec $\widetilde{A}_i$ restricted to the special fiber induces a gluing on the Spec A_i already present in $\mathrm{TV}(\Sigma)$. Thus, we just need to show that any gluing between the A_i lifts to a gluing between the $\widetilde{A}_i$.

Each ray ρ_i corresponds to the gluing induced by the diagram

$$A_{i-1} \to (A_{i-1})_{Z_{i-1,1}} = (A_i)_{Z_{i,2}} \leftarrow A_i.$$

For $i \neq 0, m+2$ this gluing can be lifted to that induced by the diagram

$$\widetilde{A}_{i-1} \to (\widetilde{A}_{i-1})_{\widetilde{Z}_{i-1,1}} = (\widetilde{A}_i)_{\widetilde{Z}_{i,2}} \leftarrow \widetilde{A}_i.$$

Indeed, the equality in the middle follows from the fact that $\widetilde{Z}_{i-1,1} = \widetilde{Z}_{i,2}^{-1}$ for $i \neq 0, m+2$.

On the other hand, for $i = m+2, l$ this gluing can be lifted to that induced by the diagram

$$\widetilde{A}_{i-1} \to (\widetilde{A}_{i-1})_{\widetilde{Z}_{i-1,1}, y^{-1}(y-t)} = (\widetilde{A}_i)_{\widetilde{Z}_{i,2}, y^{-1}(y-t)} \leftarrow \widetilde{A}_i;$$

The equality in the middle is easily checked. All other gluings in $\mathrm{TV}(\Sigma)$ are induced by the above. □

Remark. Let $\Sigma_+ = \{\sigma \in \Sigma | R \in (\sigma^\vee)^\circ\}$. The deformation π restricted to the open subset $\mathrm{TV}(\Sigma_+) \subset Y$ can then be completely described by a fan in a three-dimensional lattice $\widetilde{N} = \mathbb{Z}^3$. Indeed, for $1 \le i \le m$ set

$$\widetilde{\sigma}_i = \mathrm{Conv}\{(\widetilde{\Delta}_0^i, 1, 0), (\widetilde{\Delta}_t^i, 0, 1)\} \subset \widetilde{N}_{\mathbb{Q}}.$$

The cones $\widetilde{\sigma}_i$ fit together to induce a fan $\widetilde{\Sigma}_+$ in $N_{\mathbb{Q}}$. Let $\pi_+ : \mathrm{TV}(\widetilde{\Sigma}_+) \to \mathbb{A}^1$ be the map given by $t = \chi^{[0,1,0]} - \chi^{[0,0,1]}$ with t the parameter of $\mathbb{A}^1$. Then π_+ is a deformation of $\mathrm{TV}(\Sigma_+)$ which is a toric deformation on each $\mathrm{TV}(\sigma_i)$; this follows from the local description in [1]. Furthermore, calculation shows that it is simply the restriction of π to $\mathrm{TV}(\Sigma_+)$. Note that the inclusion of the special fiber $\mathrm{TV}(\Sigma_+) \hookrightarrow \mathrm{TV}(\widetilde{\Sigma}_+)$ is induced by the unique map $N \to \widetilde{N}$ sending $v \in \mathbb{L}$ to $(v, 1, 1)$.

Remark. It is easy to see that the total space X and fibers of π retain a $\mathbb{C}^*$-action given by the torus Spec$[\mathbb{Z}]$, where we consider the monomorphism of tori given by the projection of $M = \mathbb{Z}^2$ onto its first factor. In the corresponding grading, x has degree 1 and y and t degree 0. We see immediately that the $\widetilde{Z}_{i,j}$ are homogeneous with respect to this grading, and thus the rings $\widetilde{A}_i$ are as well. Since the gluings are compatible with this grading, we get a $\mathbb{C}^*$-action on X, and since the deformation parameter t has degree 0, this torus acts fiberwise. Thus, the fibers of π are complete $\mathbb{C}^*$ surfaces.

Remark. The construction of the deformation $\pi : X \to \mathbb{A}^1$ may appear somewhat technical; the same deformation can be described quite elegantly using the theory of T-varieties [2]. Indeed, if we set $\Xi_\infty = \Sigma \cap \{v \in N_{\mathbb{Q}} | \langle v, [0, 1]\rangle = -1\}$ (with proper labeling) and let $\Xi = \Xi_0 \otimes \{0\} + \Xi_\infty \otimes \{\infty\}$ be a divisorial fan on $\mathbb{P}^1$, $Y = \mathrm{TV}(\Sigma)$ is equal to the T-variety associated to Ξ. Now, consider the divisors $D_0 = V(y)$, $D_t = V(y-t)$, and $D_\infty = V(y^{-1})$ on $\mathbb{P}^1 \times \mathbb{A}^1$ with t the natural $\mathbb{A}^1$ coordinate and y the natural $\mathbb{P}^1$ coordinate. The divisorial fan $\widetilde{\Xi} = \widetilde{\Xi}_0 \otimes D_0 + \widetilde{\Xi}_t \otimes D_t + \widetilde{\Xi}_\infty \otimes D_\infty$ corresponds to a T-variety equal to the total space X. The map π corresponds to the projection onto the $\mathbb{A}^1$ factor.

The local theory of deformations of T-varieties is being developed by Robert Vollmert in [12]. Hendrik Süß has also constructed an example of a smoothing of a Fano threefold using the language of T-varieties [11]. The author of the present paper plans an upcoming paper outlining the general theory of deformations of (not necessarily smooth) complete T-varieties.

The above description of the deformation π has the further advantage that the fibers $\pi^{-1}(t)$ for $t \neq 0$ can be identified with the T-variety over $\mathbb{P}^1$ coming from the divisorial fan with coefficients $\widetilde{\Xi}_0$ at 0, $\widetilde{\Xi}_t$ at t, and $\widetilde{\Xi}_\infty$ at ∞. While this description would take us to far afield for the present article, we do demonstrate its usefulness in the example at the end of Sect. 3.2, where we analyze the fibers of deformations of Hirzebruch surfaces.

3.2. The Kodaira-Spencer map

Let $\pi : X \to \mathbb{A}^1$ be the deformation coming from the admissible decomposition $(\widetilde{\Xi}_0, \widetilde{\Xi}_t)$ or equivalently the integer λ_0 and admissible binary $m + 2$-tuple

a. As above, For $l > i > m+1$ we set $a_i = 1$ and $\lambda_i = 0$. We now pull back to π an infinitesimal first order deformation and compute the corresponding cohomology class in $H^1(Y, \mathcal{T}_Y)$; this can be considered to be the image of π under the Kodaira-Spencer map, see [10, Sect. 1.2.4]. Note that when we compute the image of the Kodaira-Spencer map, we will be expressing cohomology classes as Čech cocycles with respect to the natural affine invariant open covering $\mathfrak{U} = \left\{\mathrm{TV}(\tau) | \tau \in \Sigma^{(2)}\right\}$; to describe such a cocycle, it will be in fact sufficient to give sections $d_{i-1,i} \in \Gamma(TV(\rho_i), \mathcal{T}_Y)$ for $1 \le i \le l$.

Theorem 3.2. *The image of π in $H^1(Y, \mathcal{T}_Y)$ can be described as the Čech cocycle induced by $d_{i-1,i} \in \Gamma(\mathrm{TV}(\rho_i), \mathcal{T}_Y)$, where*

$$d_{i-1,i} = \begin{cases} 0 & i \notin \{m+2, 0\} \text{ and } a_i = a_{i-1} \\ a_{i-1}(\lambda_{i-1} - \lambda_i)xy^{-1}\frac{\partial}{\partial x} & i \in \{m+2, 0\} \text{ and } a_i = a_{i-1}. \\ a_{i-1}((\lambda_i + \lambda_{i-1})xy^{-1}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) & a_i \neq a_{i-1} \end{cases}$$

In particular, the deformation is homogeneous of degree $[0, -1] = -R$.

Proof. We proceed to compute the image of π under the Kodaira-Spencer map as described in [10]. We shall thus be considering π as a deformation over $\operatorname{Spec} \mathbb{C}[t]/t^2$ instead of over $\operatorname{Spec} \mathbb{C}[t]$. We first claim that π is trivial on the sets $\mathrm{TV}(\sigma_i)$ for $\sigma_i \in \Sigma^{(2)}$, that is, there are isomorphisms

$$\theta_i^{\#} : \widetilde{A}_i \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/t^2 \to A_i \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/t^2$$

respecting the $\mathbb{C}[t]/t^2$ structure. Indeed, define $\theta_i^{\#}$ by

$$\theta_i^{\#} : \widetilde{Z}_{i,k} \mapsto Z_{i,k}$$

for $k = 1, 2$. This uniquely defines $\theta_i^{\#}$ since for $m+1 < i < 0$, $y(y-t)^{-1} \in \mathbb{C}[t, \widetilde{Z}_{i,1}, \widetilde{Z}_{i,2}]/t^2$. Indeed, $y(y-t)^{-1} = 1 + ty^{-1}$ modulo t^2. One easily checks that $\theta_i^{\#}$ is an isomorphism.

Now we set $\theta_{i,j}^{\#} := \theta_j^{\#}(\theta_i^{\#})^{-1}$; this corresponds to an automorphism

$$\theta_{i,j} \in \operatorname{Aut}\left(\mathrm{TV}(\sigma_i \cap \sigma_j) \times \operatorname{Spec} \mathbb{C}[t]/t^2\right).$$

To compute the image of the Kodaira-Spencer map, it will be sufficient to only consider the automorphisms $\theta_{i-1,i}$ that is, those corresponding to ${\theta_{i-1,i}}^{\#}$. We actually want to calculate for each i the derivation $d_{i-1,i}$ defined by

$$\theta_{i-1,i}^{\#} = \mathrm{id} + t \cdot d_{i-1,i}.$$

It's time to start calculating. We first note that

$$\frac{y-t}{y} = \left(1 - a_i \widetilde{Z}_{i,1}^{r_i^2} \widetilde{Z}_{i,2}^{-r_i^1} t\right)^{a_i}.$$

Now, let b_1^k, b_2^k be such that $w_{i-1}^k = b_1^k w_i^1 + b_2^k w_i^2$. We then have

$$\begin{aligned}\theta_{i-1,i}^{\#}(Z_{i-1,k}) &= \theta_i^{\#}\left(\widetilde{Z}_{i-1,k}\right)\\ &= \theta_i^{\#}\left(\widetilde{Z}_{i,1}^{b_1^k}\widetilde{Z}_{i,2}^{b_2^k}\cdot\left(\frac{y-t}{y}\right)^{\frac{1}{2}(a_i-a_{i-1})s_{i-1}^k+(a_i\lambda_i-a_{i-1}\lambda_{i-1})r_{i-1}^k}\right)\\ &= Z_{i-1,k}\cdot\left(1-a_i y^{-1}t\right)^{\frac{1}{2}a_i(a_i-a_{i-1})s_{i-1}^k+a_i(a_i\lambda_i-a_{i-1}\lambda_{i-1})r_{i-1}^k}\end{aligned}$$

Thus, we have that

$$\begin{aligned}d_{i-1,i}(Z_{i-1,k}) = &-\frac{1}{2}(a_i-a_{i-1})s_{i-1}^k\cdot y^{-1}Z_{i-1,k}\\ &-(a_i\lambda_i-a_{i-1}\lambda_{i-1})r_{i-1}^k\cdot y^{-1}Z_{i-1,k},\end{aligned}$$

that is,

$$d_{i-1,i} = \frac{1}{2}(a_{i-1}-a_i)\frac{\partial}{\partial y} + (a_{i-1}\lambda_{i-1}-a_i\lambda_i)xy^{-1}\frac{\partial}{\partial x}.$$

The expressions for $d_{i-1,i}$ in the statement of the theorem then follow from the facts that $a_{i-1}, a_i \in \{-1, 1\}$ and that for $i \notin \{0, m-2\}$, the equality $a_i = a_{i-1}$ implies that $\lambda_{i-1} = \lambda_i$. □

Remark. The cocycle condition tells us that we should have $\sum_{i=1}^l d_{i-1,i} = 0$. This amounts to the equalities

$$\sum_{i=1}^{l}(a_{i-1}-a_i) = 0$$

$$\sum_{i=1}^{l}(a_{i-1}\lambda_{i-1}-a_i\lambda_i) = 0$$

which are easily seen to hold.

The following corollary tells us how the image of the Kodaira-Spencer map looks after applying the isomorphism from the Euler sequence.

Corollary 3.3. *The image of π in $\bigoplus_{i=1}^l H^1(Y, \mathcal{O}(D_i))$ under the isomorphism $\bigoplus H^1(Y, \mathcal{O}(D_i)) \cong H^1(Y, \mathcal{T}_Y)$ can be described as the Čech cocycle induced by*

$$g_{j-1,j} \in \bigoplus_{i=1}^{l}\Gamma(\mathrm{TV}(\rho_j), \mathcal{O}(D_i))\cdot e_{D_i},$$

where

$$\begin{aligned}g_{m+1,m+2} &= \sum_{\substack{1\le j\le m+1\\ a_j\ne a_{j+1}}} -(a_{j-1}y^{-1})\cdot e_{D_j};\\ g_{j-1,j} &= (a_{j-1}y^{-1})\cdot e_{D_j} \quad \text{for } j\notin\{m+2,0\} \text{ and } a_i\ne a_{i-1};\\ g_{j-1,j} &= 0 \quad \text{otherwise.}\end{aligned}$$

Proof. We first recall that the map $\mathcal{O}(D_i) \to \mathcal{T}_Y$ is locally given by

$$\chi^u \mapsto \left(\nu(\rho_i)_1 x \frac{\partial}{\partial x} + \nu(\rho_i)_2 y \frac{\partial}{\partial y}\right) \cdot \chi^u$$

for $u \in M$. Suppose now that $m+2 = l$. In this case, one easily checks for $j \neq m+2$ that the image of $g_{j-1,j}$ is exactly $d_{j-1,j}$ as in the above theorem. It follows that the image of $g_{m+1,m+2}$ is $d_{m+1,m+2}$ as above. Indeed, $g_{m+1,m+2} = -\sum_{j\neq m+2} g_{j-1,j}$ and $d_{m+1,m+2} = -\sum_{j\neq m+2} d_{j-1,j}$.

Suppose instead that $l > m+2$. For $m+2 \leq i \leq l-1$ define $f_i = \sum_k \alpha_k y^{-1} e_{D_k}$, and for other i set $f_i = 0$. For each i we have $f_i \in \Gamma(\mathrm{TV}(\sigma_i), \bigoplus \mathcal{O}(D_j))$ and thus this defines a homogeneous degree $-R$ element in the zeroth term of the Čech complex for $\bigoplus \mathcal{O}(D_j)$. The image of f in the first term of the Čech complex is the cocycle induced by $f_{m+1,m+2} = \sum_k \alpha_k y^{-1} e_{D_k}$, $f_{l-1,l} = -f_{m+1,m+2}$. Now, it is not difficult to choose numbers α_k such that the image of $g_{l-1,l} + f_{l-1,l}$ is $d_{l-1,l}$ as above. Similar to the previous paragraph it then also follows that the image of $g_{m+1,m+2} + f_{m+1,m+2}$ is $d_{m+1,m+2}$. Since however the cocycle induced by the $f_{j-1,j}$ lies in the image of the first term of the Čech complex, it is cohomologous to 0. □

Fix now some $R \in M$ primitive. For $2 \leq i \leq m$ with $\langle \nu(\rho_i), R\rangle = 1$, let $\pi(i)$ be the deformation corresponding to $\lambda_0 = 0$ and $a_j = 1$ for $j < i$ and $a_j = -1$ for $j \geq i$.

Corollary 3.4. *The deformations* $\{\pi(i)\}_{\substack{2\leq i\leq m \\ \langle \nu(\rho_i), R\rangle = 1}}$ *form a basis of* $T^1_Y(-R)$.

Proof. From the above corollary we know that the image of $\pi(i)$ in $\bigoplus_{j=1}^{l} H^1(Y, \mathcal{O}(D_j))$ is the Čech cocycle induced by $g_{i-1,i} = y^{-1} \cdot e_{D_i}$ and $g_{m+1,m+2} = -y^{-1} \cdot e_{D_i}$. This cocycle is not cohomologous to 0. Indeed, since $\Gamma(\mathrm{TV}(\sigma_i), \mathcal{O}(D_i))(-R) = \Gamma(\mathrm{TV}(\sigma_{i-1}), \mathcal{O}(D_i))(-R) = 0$, there is no element in the zeroth term of the Čech complex of $\mathcal{O}(D_i)$ whose image is the above cocycle. We have thus seen that for each $2 \leq i \leq m$ with $\langle \nu(\rho_i), R\rangle = 1$ that the image of $\pi(i)$ is a non-trivial cocycle. Clearly these cocycles are linearly independent, since each is supported on a different bundle $\mathcal{O}(D_i)$. Finally, from Corollary 2.5 we see that they must span $T^1_Y(-R)$. □

Example. We shall conclude with another example, namely the Hirzebruch surface $Y = \mathcal{F}_r$. Here everything is already known; if $r = 1$, then Y is Fano and thus rigid. Otherwise, $\mathcal{F}_r$ can be deformed to exactly those Hirzebruch surfaces $\mathcal{F}_s$ with $0 \leq s < r$ and $s \equiv r \mod 2$, where by $\mathcal{F}_0$ we mean $\mathbb{P}^1 \times \mathbb{P}^1$. We shall construct these deformations using subdivision decompositions, but first we turn our attention to T^1_Y.

First note that $\mathcal{F}_r = \mathrm{TV}(\Sigma)$ where Σ has rays ρ_1, ρ_2, ρ_3, and ρ_4 generated by $(1, 0)$, $(0, 1)$, $(-1, r)$, and $(0, -1)$, respectively. We immediately see that $\dim T^1_Y(u) = 1$ for $u = [-\alpha, -1]$ with $r > \alpha > 0$ and otherwise zero. Indeed, it is exactly for these weights u that $\langle \nu(\rho_2), u\rangle = -1$, $\langle \nu(\rho_1), u\rangle < 0$, and $\langle \nu(\rho_3), u\rangle < 0$. In particular, we see that $\dim T^1_Y = r - 1$.

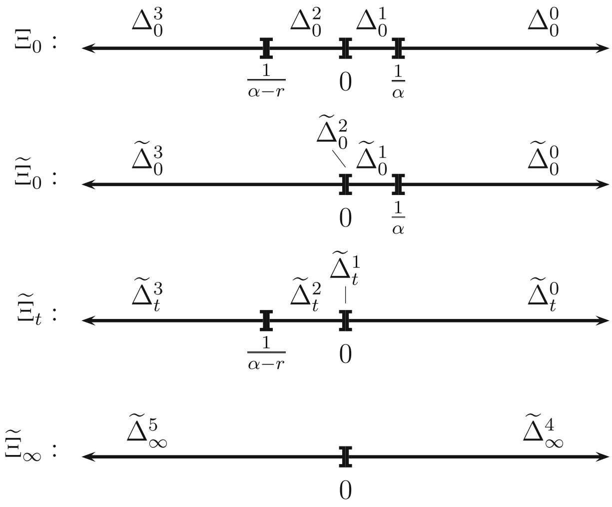

Fig. 4. A subdivision decomposition for $\mathcal{F}_r$

Now fix some degree $-R = [-\alpha, -1]$ with $r > \alpha > 0$; the fan Σ induces a polyhedral subdivision Ξ_0 on the hyperplane $\langle \cdot, R\rangle = 1$. If we take $\nu(\rho_2)$ to be the origin, then Ξ_0 is $\mathbb{Q}$ with subdivisions at $1/(\alpha - r)$, 0, and $1/\alpha$. Ignoring shifts with λ_0, possible subdivision decompositions are $a = (1, 1, -1, -1)$ and $a = (-1, -1, 1, 1)$; the decomposition corresponding to $a = (1, 1, -1, -1)$ is pictured in Fig. 4. The images of the corresponding deformations by the Kodaira-Spencer map differ only by sign.

Using the language of T-varieties we can see what the general fiber of the above deformation is. If we take $a = (1, 1, -1, -1)$, then the general fiber is the T-variety corresponding to the divisorial fan $\widetilde{\Xi} = \widetilde{\Xi}_0 \otimes \{0\} + \widetilde{\Xi}_t \otimes \{t\} + \widetilde{\Xi}_\infty \otimes \{\infty\}$ on $\mathbb{P}^1$. Since the coefficient $\widetilde{\Xi}_\infty$ is trivial, this in fact corresponds to the toric variety whose fan $\widetilde{\Sigma}$ is induced by the decomposition $\widetilde{\Xi}_0$ embedded in height one and $\widetilde{\Xi}_t$ embedded in height minus one. In other words, $\widetilde{\Sigma}$ has rays through the points $(0, 1)$, $(1, \alpha)$, $(0, -1)$, and $(-1, \alpha - r)$. Using a lattice automorphism, we see that this is the fan for $\mathcal{F}_{r-2\alpha}$ when $r \geq 2\alpha$ and the fan for $\mathcal{F}_{2\alpha-r}$ when $r \leq 2\alpha$. Thus, we see that we can deform $\mathcal{F}_r$ exactly to those other Hirzebruch surfaces mentioned above. Furthermore, for $0 < s < r$ with $s \equiv r \mod 2$ there are exactly two degrees in which there are homogeneous deformations deforming $\mathcal{F}_r$ to $\mathcal{F}_s$, namely $[-(r + s)/2, -1]$ and $[-(r - s)/2, -1]$; if $s = 0$ then the single degree in which such a deformation exists is $[-r/2, -1]$.

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