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The curve selection lemma and the Morse–Sard theorem

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Abstract. We use an inequality due to Bochnak and Lojasiewicz, which follows from the Curve Selection Lemma of real algebraic geometry in order to prove that, given a C^r function $f : U \subset \mathbb{R}^m \to \mathbb{R}$, we have

$$\lim_{\substack{y \to x \\ y \in \mathrm{crit}(f)}} \frac{|f(y) - f(x)|}{|y - x|^r} = 0, \text{ for all } x \in \mathrm{crit}(f)' \cap U,$$

where $\mathrm{crit}(f) = \{x \in U \mid df(x) = 0\}$. This shows that the so-called *Morse decomposition* of the critical set, used in the classical proof of the Morse–Sard theorem, is not necessary: the conclusion of the Morse decomposition lemma holds for the whole critical set. We use this result to give a simple proof of the classical Morse–Sard theorem (with sharp differentiability assumptions).

0. Introduction

The *Morse–Sard Theorem* is a central result in differential topology. It is the fundamental result that makes transversality theory work, and has applications in many areas of mathematics, in particular to dynamical systems. The classical proof of the Morse–Sard Theorem is based on a fundamental result known as Morse decomposition lemma, which states that a function $f : U \to \mathbb{R}$, of class C^k, has the following property: its critical set $\mathrm{crit}(f) = \{x \in U \mid df(x) = 0\}$ can be decomposed as the union of a countable collection of subsets A_j such that if x is point in A_j and x_n is a sequence of points in A_j, $x_n \neq x$, converging to x, then $\lim_{n\to\infty} \frac{f(x_n) - f(x)}{|x_n - x|^k} = 0$. In the original paper by Morse, the argument of the proof of this result is based on a double induction, in n (the source dimension), and k (the differentiability class of f).

In this paper, we show that it is not necessary to decompose the critical set of f in order that Morse's result becomes true: in fact, if x is a critical point of

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f, and x_n is a non constant sequence of points in crit(f) converging to x, then $\lim_{n\to\infty} \frac{f(x_n)-f(x)}{|x_n-x|^k} = 0$. To prove this result, we use a result of real algebraic geometry, called by Milnor *the Curve Selection Lemma*, which says that if the origin belongs to the closure of a semi-algebraic set X, then there exists an analytic curve p defined in the interval $[0, 1)$, with $p(0) = 0$ such that $p(t)$ is in X for all t in $(0, 1)$. We show how to get from this lemma an inequality due to Bochnak and Lojasiewicz (Theorem (1.2)), which resembles the Mean Value Inequality, except that in Bochnak–Lojasiewicz result, one estimates the absolute value of the difference of a given function in any two points of the domain in terms of the distance between these points and the norm of the derivative in one of these points (and not, as in the classical mean value theorem, in some point of the segment determined by them).

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1. The curve selection lemma

The *Curve Selection Lemma,* stated and proved by Milnor in [7] is a fundamental result in the study of the local structure of semi-algebraic sets. The first versions of this lemma appeared in the last century, around 1950, in the works of Bruhat and Cartan [1, Theorem 1], and Wallace [8, Lemma 18.3].

Let $V \subset \mathbb{R}^m$ be an algebraic subset, that is, the locus of the zeros of a finite number of polynomial equations, and $U \subset \mathbb{R}^m$, an open set defined by a finite number of polynomial inequalities, that is,

$$U = \{x \in \mathbb{R}^m \,|\, g_1(x) > 0, \ldots, g_l(x) > 0\}.$$

The subset $U \cap V$ defined by a finite number of polynomial equalities and inequalities, is a semi-algebraic subset.

Lemma 1.1. (The Curve Selection Lemma). *If $U \cap V$ contains points arbitrarily close to the origin, that is, $0 \in closure(U \cap V)$, then there exists a real analytic curve $p : [0, \epsilon) \to \mathbb{R}^m$, with $p(0) = 0$, and $p(t) \in U \cap V$, for every $t > 0$.*

Milnor's proof has two main steps:

(1) dim $V = 1$. In this case, the argument is simple: the algebraic curve V has a finite number of branches through the origin, one of which necessarily has points in U arbitrarily close of 0. Let $x = p(t)$, $|t| < \epsilon$ be a real analytic parametrization of this branch. For every $i = 1, 2 \ldots, l$, the function $g_i(p(t))$ has positive values for every t in an interval $0 < t < \epsilon'$, or it is ≤ 0 for every $0 < t < \epsilon'$. Then, or the semi-branch $p(0, \epsilon')$ is contained in U, or it is disjoint of U, for all ϵ' sufficiently small. Analogously, the semi-branch $p(-\epsilon', 0)$ is contained in U or does not meet U. But, it follows from the hypothesis that, $p(-\epsilon', \epsilon)$ contains points in U arbitrarily close to 0, hence one of the semi-branches will be necessarily contained in U.

(2) dim $V \geq 2$. The argument in this case is more elaborated and consists in constructing a proper algebraic subset, $V_1 \subset V$ such that $0 \in \text{closure}(U \cap V_1)$. Once such subset V_1 is determined, the procedure can be iterated inductively until we can find an algebraic subset V_q of dimension ≤ 1 such that $0 \in \text{closure}(U \cap V_q)$.

Remark. We can reparametrize the analytic curve $\{x = p(t) : 0 \leq t < \epsilon\}$ of the above proof, to obtain a C^1 parametrization: $\{x = \gamma(s) : 0 \leq s < \epsilon\}$, $\gamma(0) = 0$ and $|\gamma'(s)| = 1$. In fact, let k be the degree of the first non-zero term in the Taylor series at 0 of $x = p(t)$. The change of parameters $\lambda = h(t) = t^k$ is a homeomorphism for $0 \leq t < \epsilon$ and it is easy to verify that $\gamma(\lambda) = p \circ h^{-1}(\lambda)$ is C^1 at $[0, \epsilon)$. Now, it is sufficient to reparametrize γ by the arc length.

The reader may read the elegant and detailed proof of the curve selection lemma given by Milnor [7, Chap. 3].

A more recent proof of this lemma can be found in the book by Coste [4]. In Coste's book, the proof follows as a simple consequence of the theorem on the existence of a triangulation of semi-algebraic sets.

There are several known applications of the curve selection lemma, which also holds when the subset V and the functions g_i are analytic in a neighborhood of 0. The following result, proved by Bochnak and Lojasiewicz in [2] (for analytic functions) is a consequence of Lemma (1.1).

Theorem 1.2. (Lemma 2, [2]). *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a polynomial function such that $f(0) = 0$. Given a constant $C > 1$, there exists a neighborhood W of* 0, *such that $|f(x)| \leq C|x||df(x)|$, $\forall x \in W$.*

Proof. This is the proof given by Bochnak and Lojasiewicz in [2]. Let us suppose that 0 is in the closure of the set $\{C|x||df(x)| < |f(x)|\}$; this is a semi-algebraic set and hence, by the curve selection lemma, it contains a C^1 arc : $\{x = \gamma(t) : 0 < t \leq \epsilon\}$, $\gamma(0) = 0$ and $|\gamma'(t)| = 1$. Let $\phi = f \circ \gamma$; then, it follows that $|t||\phi'(t)| = |t||df(\gamma(t)) \cdot \gamma'(t)| \leq |t||df(\gamma(t))| \leq \frac{1}{C}\frac{|t|}{|\gamma(t)|}|f(\gamma(t))| \leq \rho|f(\gamma(t))| = \rho|\phi(t)|$ in $[0, \delta]$ for some ρ such that $1/C < \rho < 1$ and δ in $(0, \epsilon)$. If $g(t) = \log|\phi(t)|$, for $t \in (0, \delta)$, then $g'(t) = \frac{<\phi'(t), \phi(t)>}{|\phi(t)|^2}$, and hence $|g'(t)| \leq \frac{\rho}{t}$, $t \in (0, \delta)$. Given $s \in (0, \delta)$, integrating $g'(t)$ from s to δ, and using the above inequality, we obtain $\log\left(\frac{|\phi(\delta)|}{|\phi(s)|}\right) = g(\delta) - g(s) \leq \rho \log(\frac{\delta}{s})$, and hence $|\phi(s)| \geq (\frac{s}{\delta})^\rho |\phi(\delta)|$, for every s in $(0, \delta)$, which is a contradiction, since $0 < \rho < 1$ and ϕ is C^1. □

Remark. This theorem does not hold without the assumption that f is analytic. In fact, we can construct C^∞ functions $f : \mathbb{R} \to \mathbb{R}$ with the following property: there exists a sequence of critical points $x_n \in \mathbb{R}$, $f(x_n) \neq 0$, x_n converging to some point x in $\mathbb{R}$ and such that $f(x) = 0$ (exercise!). In this case, inequalities like the one in Theorem (1.2) do not hold.

2. The Morse–Sard theorem

In this section, we use Theorem (1.2) to give a proof of Morse–Sard's Theorem. We start with some lemmas which we will use in the proof of Morse Theorem

(Corollary (2.4) below). In the following discussion, we will use the notation X' for the set of accumulation points of a subset X of an Euclidean space.

In the classical proof by Morse of Corollary (2.4) below, given a C^k function $f: U \subset \mathbb{R}^m \to \mathbb{R}$, the set $\mathrm{crit}(f) := \{x \in U \mid df(x) = 0\}$ is decomposed as a countable union of subsets A_i such that, for every i and every $x \in A_i' \cap A_i$, $\lim\limits_{\substack{y\to x\\ y\in A_i}} \frac{|f(y)-f(x)|}{|y-x|^k} = 0$. We shall use Theorem (1.2) to show that we don't need to decompose $\mathrm{crit}(f)$:

Lemma 2.1. *Given a C^r function $f: U \subset \mathbb{R}^m \to \mathbb{R}$, we have* $\lim\limits_{\substack{y\to x\\ y\in crit(f)}} \frac{|f(y)-f(x)|}{|y-x|^r} = 0$*, for all $x \in crit(f)' \cap U$.*

Proof. We argue by contradiction: let us suppose that there exists $\varepsilon > 0$ such that $|f(y_k) - f(x)| \geq \varepsilon|y_k - x|^r$ for some sequence (y_k) converging to x, with $df(y_k) = 0$ for every k. Let $\tilde{f}$ be the Taylor polynomial of degree r of f at the point x, $\tilde{f}(y) = \sum_{k=0}^{r} \frac{1}{k!} f^{(k)}(x)\cdot(y-x)^k$. Then $f(y) = \tilde{f}(y) + o(|y-x|^r)$, and hence, $|\tilde{f}(y_k) - \tilde{f}(x)| \geq \varepsilon/2 \cdot |y_k - x|^r$ for k sufficiently big. On the other hand, $df(y) = d\tilde{f}(y) + o(|y-x|^{r-1})$ (since $d\tilde{f}$ is the Taylor polynomial of degree $r-1$ of df at the point x), then, since $df(y_k) = 0$ for all k, $d\tilde{f}(y_k) = o(|y_k - x|^{r-1})$. From Theorem (1.2), $|\tilde{f}(y_k)| \leq 2|y_k - x||d\tilde{f}(y_k)|$ for big enough k, and hence $|\tilde{f}(y_k) - \tilde{f}(x)| = o(|y_k - x|\cdot|y_k - x|^{r-1}) = o(|y_k - x|^r)$, which is a contradiction. □

We will use the following result which is a particular case of Vitali's Covering Lemma:

Lemma 2.2. *Let $U \subset \mathbb{R}^m$ be an open subset with volume $a < +\infty$, and $X \subset U$ such that for every $x \in X$, one associates a ball $B(x, \delta_x) \subset U$. Then, there exists a (finite or countable) subset $(x_i) \subset X$ such that $X \subset \bigcup_i B(x_i, \delta_{x_i})$ and $\sum_i vol(B(x_i, \delta_{x_i})) \leq 3^m \cdot a$.*

Proof. For each integer $i \geq 1$, we choose (recursively) a point $x_i \in X$ such that $B(x_i, \delta_{x_i}/3) \cap \bigcup_{j<i} B(x_j, \delta_{x_j}/3) = \emptyset$ and $\delta_{x_i} > \frac{1}{2}\sup\{\delta > 0 \mid \exists x \in X, \delta_x = \delta$ and $B(x, \delta/3) \cap \bigcup_{j<i} B(x_j, \delta_{x_j}/3) = \emptyset\}$ (if, for some $s \geq 2$, the set $\{x \in X \mid B(x, \delta/3) \cap \bigcup_{j<s} B(x_j, \delta_{x_j}/3) = \emptyset\}$ is empty, the collection of the points $x_i \in X$ we constructed is finite, equal to $x_1, x_2, \ldots, x_{s-1}$; otherwise, it is infinite, indexed for the positive integers). Since the open balls $B(x_i, \delta_{x_i}/3)$ are disjoint, we get that $\sum_i \mathrm{vol}(B(x_i, \delta_{x_i}/3)) \leq a$, and then $\sum_i \mathrm{vol}(B(x_i, \delta_{x_i})) \leq 3^m \cdot a$. We claim that $X \subset \bigcup_i B(x_i, \delta_{x_i})$. In fact, giving $x \in X$, one necessarily gets that $B(x, \delta_x/3) \cap \bigcup_i B(x_i, \delta_{x_i}/3) \neq \emptyset$, otherwise x should be included in the set $\{x_i\}$ at some point. Moreover, if i_0 is the smallest i such that $B(x, \delta_x/3) \cap B(x_{i_0}, \delta_{x_{i_0}}/3) \neq \emptyset$, because of our choice of x_{i_0}, it follows that $\delta_{x_{i_0}} > \delta_x/2$. Hence, $|x - x_{i_0}| < \delta_x/3 + \delta_{x_{i_0}}/3 < 2\delta_{x_{i_0}}/3 + \delta_{x_{i_0}}/3 = \delta_{x_{i_0}}$, and thus $x \in B(x_{i_0}, \delta_{x_{i_0}})$. □

Theorem 2.3. *Let $f: U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a C^p function. Then, if $p \geq m/n$, $f(C(f))$ has measure zero in $\mathbb{R}$, where $C(f) := \{x \in U \mid df(x) = 0\}$.*

Proof. Since U is a countable union of bounded sets, and countable unions of sets of measure zero also have measure zero, we can assume without loss of generality that U is bounded (and hence has finite volume).

Let $c_k = \text{vol } B(0, 1)$ be the volume of the unit ball in $\mathbb{R}^k$, for each positive integer k. Lemma (2.1) applied to each coordinate of f implies that we have $\lim\limits_{\substack{y \to x \\ y \in \text{crit}(f)}} \frac{|f(y)-f(x)|}{|y-x|^p} = 0$, for all $x \in \mathrm{C}(f)' \cap U$. It follows that for every $x \in \mathrm{C}(f)$ and $\varepsilon > 0$ there exists $\delta_x \in (0, 1)$ such that $y \in \mathrm{C}(f) \cap B(x, \delta_x) \Rightarrow |f(y) - f(x)| < (\frac{\varepsilon \cdot c_m}{c_n \cdot 3^m \text{vol}(U)})^{1/n} \cdot |y - x|^p$. Then, $f(\mathrm{C}(f) \cap B(x, \delta_x)) \subset D_x$, where $D_x = B(f(x), (\frac{\varepsilon \cdot c_m}{c_n \cdot 3^m \text{vol}(U)})^{1/n} \cdot \delta_x^p)$ is a ball in $\mathbb{R}^n$ whose volume is

$$\begin{aligned} \text{vol}(D_x) &= c_n \cdot \left(\frac{\varepsilon \cdot c_m}{c_n \cdot 3^m \text{vol}(U)}\right) \cdot \delta_x^{pn} = \left(\frac{\varepsilon \cdot c_m}{3^m \text{vol}(U)}\right) \cdot \delta_x^{pn} \\ &\leq \left(\frac{\varepsilon \cdot c_m}{3^m \text{vol}(U)}\right) \cdot \delta_x^{m} = \left(\frac{\varepsilon}{3^m \text{vol}(U)}\right) \cdot \text{vol}(B(x, \delta_x)). \end{aligned}$$

From the above lemma, it follows that there exists a countable set $(x_i) \subset \mathrm{C}(f)$ with $\mathrm{C}(f) \subset \bigcup_i B(x_i, \delta_{x_i})$ and $\sum_i \text{vol } B(x_i, \delta_{x_i}) \leq 3^m \text{vol}(U)$. We then have $f(\mathrm{C}(f)) = \bigcup_i f(\mathrm{C}(f) \cap B(x_i, \delta_{x_i})) \subset \bigcup_i D_{x_i}$, but $\sum_i \text{vol}(D_{x_i}) < \frac{\varepsilon}{3^m \text{vol}(U)} \cdot \sum_i \text{vol } B(x_i, \delta_{x_i}) \leq \frac{\varepsilon}{3^m \text{vol}(U)} \cdot 3^m \text{ vol}(U) = \varepsilon$.

This shows that $f(\mathrm{C}(f))$ has measure zero in $\mathbb{R}$. □

Taking $n = 1$ in the above Theorem we get the following

Corollary 2.4. (Morse Theorem): *Let $f : U \subset \mathbb{R}^m \to \mathbb{R}$ be a C^m function. Then, $f(crit(f))$ has measure zero in $\mathbb{R}$, where $crit(f) := \{x \in U \mid df(x) = 0\}$.*

We now show (as did Sard) that Morse Theorem implies the general Morse–Sard Theorem. We begin with the following Lemma, which is a particular case of Fubini's Theorem.

Lemma 2.5. *Let $K \subset \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}$ be a compact set such that for every $x \in \mathbb{R}^p$, $K_x = \{y \in \mathbb{R}^{n-p} \mid (x, y) \in K\}$ has measure zero (in $\mathbb{R}^{n-p}$). Then K has measure zero (in $\mathbb{R}^n$).*

Proof. There exists $R > 0$ such that $K \subset B(0, R) \times \mathbb{R}^{n-p}$. For every $x \in B(0, R)$, K_x has measure zero, hence, given $\varepsilon > 0$ there exist balls $B(y_i, r_i) \subset \mathbb{R}^{n-p}$ such that $K_x \subset \bigcup_i B(y_i, r_i)$ and $\sum_i \text{vol } B(y_i, r_i) < \varepsilon$. Since K is compact, there is $\delta > 0$ such that $K \cap (B(x, \delta) \times \mathbb{R}^{n-p}) \subset \bigcup_i B(x, \delta) \times B(y_i, r_i)$. In fact, if there were a sequence $(u_n, v_n) \in K$ with $\lim u_n = x$ and $v_n \notin \bigcup_i B(y_i, r_i)$, taking, if necessary a convergent subsequence, we would have $\lim v_n = v \notin \bigcup_i B(y_i, r_i)$ and then $\lim(u_n, v_n) = (x, v) \in K$, and then $v \in K_x \setminus \bigcup_i B(y_i, r_i)$, which is a contradiction.

Using Lemma (2.2), we can cover $B(0, R)$ with open balls $B(x^{(j)}, \delta^{(j)})$ with

$$\sum \text{vol}(B(x^{(j)}, \delta^{(j)})) \leq 3^n \text{vol } B(0, R),$$

which gives a covering of K by the corresponding $B(x^{(j)}, \delta^{(j)}) \times B(y_i^{(j)}, r_i^{(j)})$. Since we have

$$\sum_{i,j} \mathrm{vol}(B(x^{(j)}, \delta^{(j)}) \times B(y_i^{(j)}, r_i^{(j)})) < \varepsilon \sum_j \mathrm{vol}\, B(x^{(j)}, \delta^{(j)}) \leq 3^n \varepsilon \cdot \mathrm{vol}\, B(0, R),$$

and ε can be chosen arbitrarily small, we get that K has measure zero. □

Theorem 2.6. (Morse–Sard) *Let $m \geq n$ and $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ a differentiable function of class C^{m-n+1}. Let crit$(f) := \{x \in U \mid Df(x) : \mathbb{R}^m \to \mathbb{R}^n$ fails to be surjective$\}$. Then $f(\mathrm{crit}(f))$ has measure zero in $\mathbb{R}^n$.*

Proof. Let $C_p = \{x \in U \mid \mathrm{rank}\, Df(x) = p\}$. Then, $\mathrm{crit}(f) = \bigcup_{p=0}^{n-1} C_p$. It is sufficient to show that $f(C_p)$ has measure zero for every $p \leq n-1$. We fix $p \leq n-1$. Since the domain U is a countable union of compact sets, we may assume without loss of generality that C_p, and so $f(C_p)$, are compact sets. We will show that for every $x \in C_p$ there exists $\delta > 0$ such that $f(B(x, \delta) \cap C_p)$ has measure zero in $\mathbb{R}^n$ (we can cover C_p by a countable collection of the balls $B(x, \delta)$). If rank $Df(x)) = p$, there exists a neighborhood V of x such that rank $(Df(y)) \geq p$ for every $y \in V$. We can write (interchanging, if necessary, the order of the vectors of the standard basis of $\mathbb{R}^m$) $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}$ such that $\pi_1(Df(x)) : \mathbb{R}^m \to \mathbb{R}^p$ becomes surjective.

There exists a neighborhood W of x in $\mathbb{R}^m$ such that the function $g : W \to \mathbb{R}^n$, $g(W) = \pi_1 f(W)$ is a submersion.

It follows from the implicit function theorem that there exists a local diffeomorphism (of class C^n) $h : U \times V \to \tilde{W}$ with $U \subset \mathbb{R}^p$, $V \subset \mathbb{R}^{m-p}$ and $\tilde{W} \subset W \subset \mathbb{R}^n$, such that $g \circ h(u, v) \equiv u$, and then $f \circ h(u, v) = (u, \tilde{f}(u, v))$. Then we can assume that f has this normal form. Now, $(u, v) \in C_p \Leftrightarrow D_v \tilde{f}(u, v) = 0$. Thus, given $u \in \mathbb{R}^p$, $F_u := \{(u, v), v \in \mathbb{R}^{m-p}\}$, $f(F_u \cap C_p) \subset \{u\} \times \tilde{f}_u(N)$, where $\tilde{f}_u(v) := \tilde{f}(u, v)$ and $N = \{v \in V \mid D\tilde{f}_u(v) = 0\}$. From Theorem (2.3), $\tilde{f}_u(N)$ has measure zero in $\mathbb{R}^{n-p}$, $\forall u \in U$, since $m - n + 1 \geq \frac{m-p}{n-p} = \frac{m-n}{n-p} + 1$, for $p \leq n-1$. It follows from the previous Lemma that $f(C_p)$ has measure zero in $\mathbb{R}^n$. □

3. Examples

We give here an idea of how to construct examples as the ones in [5] which show that the differentiability assumption in Morse–Sard's Theorem is necessary. To do this, we define for $0 < \alpha < 1$, the *homogeneous Cantor set* K_α: we first exclude from the interval $[0, 1]$ the central interval $U_{11} = \left(\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right)$ of length α. After this, we exclude from the two remaining intervals, $\left(\left[0, \frac{1-\alpha}{2}\right] \text{ and } \left[\frac{1+\alpha}{2}, 1\right]\right)$, the two central intervals of proportion α,

$$U_{21} = \left(\left(\frac{1-\alpha}{2}\right)^2, \frac{1-\alpha^2}{4}\right) \text{ and}$$

$$U_{22} = \left(\frac{1+\alpha}{2} + \left(\frac{1-\alpha}{2}\right)^2, \frac{1+\alpha}{2} + \frac{1+\alpha^2}{2}\right),$$

respectively. In general, after the r-th step of this construction, there are 2^r intervals of length $\left(\frac{1-\alpha}{2}\right)^r$ left. We exclude from the i-th interval, the open central interval $U_{(r+1)i}$ of proportion α (that is, of size $\alpha\left(\frac{1-\alpha}{2}\right)^r$) for $1 \leq i \leq 2^r$. The Cantor set K_α is the set of remaining points. We have

$$K_\alpha = \left\{ \sum_{j=1}^{\infty} \sigma_j \left(\frac{1-\alpha}{2}\right)^n \middle| \sigma_j \in \left\{0, \frac{1+\alpha}{1-\alpha}\right\},\ \forall\, j \right\}.$$

Notice that $K_{1/3}$ is the usual ternary Cantor set.

We now fix an increasing homeomorphism of class C^∞, $\psi\colon [0,1] \to [0,1]$ such that $\psi^{(j)}(0) = \psi^{(j)}(1) = 0$ for every $j \geq 1$ (where $\psi^{(j)}$ indicates the j-th derivative of ψ). Given α and β in $(0, 1)$, we can define the homeomorphism $f_{\alpha,\beta}$ from $[0, 1]$ onto $[0, 1]$ such that $f_{\alpha,\beta}(K_\alpha) = K_\beta$ of the following form: we define $f_{\alpha,\beta}(0) = 0$, $f_{\alpha,\beta}(1) = 1$ and, if $U_{ij}^{(\alpha)} = (a, b)$ and $U_{ij}^{(\beta)} = (c, d)$ are correspondent excluded intervals of the constructions of K_α and K_β, we define $f_{\alpha,\beta}(x) = (d-c)\psi\left(\frac{x-a}{b-a}\right) + c, \forall\, x \in (a, b)$. We extend $f_{\alpha,\beta}$ to K_α by continuity. It is not difficult to show that, if $\left(\frac{1-\beta}{2}\right) \Big/ \left(\frac{1-\alpha}{2}\right)^k < 1$ then $\lim_{r\to\infty} \left(\frac{1-\beta}{2}\right)^r \Big/ \left(\left(\frac{1-\alpha}{2}\right)^r\right)^k = 0$, and, in this case, $f_{\alpha,\beta}$ is a function of class C^k, with $f_{\alpha,\beta}^{(j)}(x) = 0$, for every $x \in K_\alpha$ and $1 \leq j \leq k$. See [3] for more details.

Now let $\beta = 1 - \frac{1}{2^{m-1}}$. We have

$$K_\beta = \left\{ \sum_{j=1}^{\infty} \sigma_j \cdot \frac{1}{2^{mj}},\ \sigma_j \in \{0, 2^m - 1\},\ \forall j \right\},$$

that is, K_β is the set of all real numbers in $[0, 1]$ whose representation in base to 2^m only uses the digits 0 and $2^m - 1$. Notice that

$$\begin{aligned} K_\beta + 2K_\beta + \cdots + 2^{m-1}K_\beta &:= \{x_1 + 2x_2 + \cdots + 2^{m-1}x_m | x_i \in K_\beta,\ 1 \leq i \leq m\} \\ &= (2^m - 1)\left\{ \sum_{j=1}^{\infty} \frac{\left(\sigma_j^{(1)} + 2\sigma_j^{(2)} + 4\sigma_j^{(3)} + \cdots + 2^{m-1}\sigma_j^{(m)}\right)}{2^{mj}}, \right. \\ &\qquad\qquad \left. \sigma_j^{(i)} \in \{0, 1\},\ \forall\, i, j \right\} \\ &= (2^m - 1)\left\{ \sum_{j=1}^{\infty} \frac{\mu_j}{2^{mj}},\ \mu_j \in \{0, 1, 2, \ldots, 2^m - 1\},\ \forall j \right\} \\ &= [0, 2^m - 1]. \end{aligned}$$

We now choose $\alpha \in (0, 1)$ such that $\left(\frac{1-\alpha}{2}\right)^{m-1} > \frac{1}{2^m} = \left(\frac{1-\beta}{2}\right)$. Then $f_{\alpha,\beta}\colon [0, 1] \to [0, 1]$ is a C^{m-1} function with $f_{\alpha,\beta}(K_\alpha) = K_\beta$ and $f'_{\alpha,\beta}(x) = 0$, $\forall\, x \in K_\alpha$. We extend $f_{\alpha,\beta}$ to $\mathbb{R}$ defining $f_{\alpha,\beta}(x) = 0$, $\forall\, x \leq 0$ and $f_{\alpha,\beta}(x) =$

$1, \forall x \geq 1$. The function $f_{\alpha,\beta}$ is differentiable of class C^{m-1}. We now define $F_{\alpha,\beta} \colon \mathbb{R}^m \to \mathbb{R}$ by $F_{\alpha,\beta}(x_1, x_2, \ldots, x_m) = \sum_{j=1}^{m} 2^{j-1} f_{\alpha,\beta}(x_j)$. Clearly, $K_\alpha \times K_\alpha \times \cdots \times K_\alpha = K_\alpha^m$ is contained in the set of critical points of $F_{\alpha,\beta}$, and $F_{\alpha,\beta}(K_\alpha^m) = K_\beta + 2K_\beta + \cdots + 2^{m-1}K_\beta = [0, 2^m - 1]$, which shows that the conclusion of Morse Theorem is false if we assume that F is of class C^{m-1}, instead of assuming that F is of class C^m.

We consider $G \colon \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R} \times \mathbb{R}^p$ defined by $G(x, y) = (F_{\alpha,\beta}(x), y)$, G is C^{m-1}, but the image of its critical points contains $[0, 2^m - 1] \times \mathbb{R}^p$, which obviously does not have measure zero in $\mathbb{R} \times \mathbb{R}^p$. Hence, we cannot lower the hypothesis of the class of differentiability of the function in Morse–Sard Theorem (which implies in this case that the image of the critical points of C^m functions has measure zero), for any choice of m, n with $m \geq n$.

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