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Local Shalika models and functoriality

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Abstract. We prove, over a p-adic local field F, that an irreducible supercuspidal representation of $\mathrm{GL}_{2n}(F)$ is a local Langlands functorial transfer from $\mathrm{SO}_{2n+1}(F)$ if and only if it has a nonzero Shalika model (Corollary 5.2, Proposition 5.4 and Theorem 5.5). Based on this, we verify (Sect. 6) in our cases a conjecture of Jacquet and Martin, a conjecture of Kim, and a conjecture of Speh in the theory of automorphic forms.

1. Introduction

Shalika models or periods for irreducible cuspidal automorphic representations π of $\mathrm{GL}_{2n}(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of a number field k was first introduced in [22] to characterize the pole at $s = 1$ of the partial exterior square L-function $L^S(s, \pi, \Lambda^2)$. More precise relations among Shalika periods, the pole at $s = 1$ of the exterior square L-function $L^S(s, \pi, \Lambda^2)$, and the Langlands functorial transfer property has been discussed in detail in Theorem 2.2 of [23], based on the precise results about the Langlands functorial transfer from k-split SO_{2n+1} to GL_{2n} for irreducible generic cuspidal automorphic representations [6,13,26,27].

The objective of this paper is to discuss the characterizations of the local Langlands functorial property for irreducible unitary supercuspidal representations τ of $\mathrm{GL}_{2n}(F)$, where F is a p-adic local field of characteristic zero, in terms of invariance properties of τ as representations of $\mathrm{GL}_{2n}(F)$. The invariance properties of τ considered in this paper are

(1) the local Shalika model attached to τ, which will be defined in Sect. 2;
(2) the local exterior square L-function $L(s, \tau, \Lambda^2)$ has a pole at $s = 0$;
(3) the local exterior square γ-factor $\gamma(s, \tau, \Lambda^2, \psi)$ has a pole at $s = 1$;
(4) the unitarily induced representation $\mathrm{I}(s, \tau)$ of $\mathrm{SO}_{4n}(F)$, which will be defined in Sect. 2, is reducible at $s = 1$.

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It is clear that if irreducible unitary supercuspidal representations τ and τ' of $\mathrm{GL}_{2n}(F)$ are isomorphic, then τ and τ' have the same local exterior square L-function

$$L(s, \tau, \Lambda^2) = L(s, \tau', \Lambda^2),$$

and the same local exterior square γ-factor

$$\gamma(s, \tau, \Lambda^2, \psi) = \gamma(s, \tau', \Lambda^2, \psi),$$

and share the same Shalika model, which is unique ([20], see also [36]), and the same reducibility at $s = 1$. It is clear that Property (2) and Property (3) are equivalent by the relation between the local L-factor and the corresponding local gamma factor. The equivalence between Property (3) and Property (4) was proved by F. Shahidi [41,42]. In [26] and [27], D. Soudry and the first named author proved that Property (3) is equivalent to that the irreducible unitary supercuspidal representations τ of $\mathrm{GL}_{2n}(F)$ is the local Langlands functorial transfer from $\mathrm{SO}_{2n+1}(F)$. The characterization of Property (4) in terms of nonvanishing of certain orbital integrals was established by F. Shahidi in [42], which implies that the irreducible unitary supercuspidal representations τ of $\mathrm{GL}_{2n}(F)$ is the local Langlands functorial transfer from $\mathrm{SO}_{2n+1}(F)$, assuming the validity of the local endoscopy transfer, i.e the fundamental lemma. This aspect was also discussed in the recent work of G. Chenevier and L. Clozel [5]. For irreducible cuspidal representations of $\mathrm{GL}_{2n}(F)$ of level zero, the relation between Property (4) and the local Langlands transfer property has also been established by P. Kutzko and L. Morris [31] and G. Savin [39]. We refer to [39] and [28] for related arithmetic application.

In this paper, we prove that Property (1) characterizes the local Langlands functorial transfer property, and hence is equivalent to each of the properties discussed above (Corollary 5.2 and Proposition 5.4 and finally Theorem 5.5). First, we prove, by a purely local method, that Property (1) implies Property (4) (Theorem 2.2). Then by applying [41] and [26] and [27], we obtain Corollary 5.2, which says that Property (1) implies the local Langlands functorial transfer property of τ. The converse (Proposition 5.4) is proved by a global argument, which is given in Sect. 5. We expect to have a purely local proof for the converse, but we omit the details here. One also expect that Theorem 5.5 can be stated for more general, nonsupercuspidal representations, which will be considered in our future work. As a consequence, we show (Theorem 5.6) that for irreducible supercuspidal representations of $\mathrm{GL}_{2n}(F)$, the local Shalika model is equivalent to the linear model. It was proved by H. Jacquet and S. Rallis [20] that if τ is an irreducible admissible representation of $\mathrm{GL}_{2n}(F)$, then the existence of nonzero local Shalika model for τ implies the existence of a nonzero local linear model for τ. It is interesting to point out that for an irreducible cuspidal automorphic representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$, the nonvanishing of the Shalika period of τ is equivalent to the nonvanishing of the linear period of τ and the nonvanishing of the central value of the standard L-function, $L\left(\frac{1}{2}, \tau\right)$ [7]. See Sect. 5.2 for details.

It should be mentioned that our local method to prove Theorem 2.2 is based on a general result on $\mathrm{SO}_{4n}(F)$ (Proposition 2.3 and Theorem 3.1), which states that any irreducible admissible representation of $\mathrm{SO}_{4n}(F)$ can not have both nonzero generalized Shalika model and a nonzero degenerate Whittaker model of certain type. In particular, it shows that any irreducible admissible representation of $\mathrm{SO}_{4n}(F)$ can not have both nonzero generalized Shalika model and a nonzero Whittaker model. It is interesting to note that the proof is done by using the following fact: *An irreducible admissible representation of* $\mathrm{GL}_{2n}(F)$ *can not have both nonzero Whittaker models and nonzero symplectic models* (*refer to [14]*). We call this property "*disjointness of Whittaker models and symplectic models*".

In [25], we discovered that via the parabolic induction, the Shalika model for an irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(F)$ is closely related to the generalized Shalika model of the Langlands quotient $\mathrm{J}(1,\tau)$ on $\mathrm{SO}_{4n}(F)$. This was also established in [25] for automorphic forms. The automorphic counter-part of $\mathrm{J}(1,\tau)$ is a residual representation of $\mathrm{SO}_{4n}(\mathbb{A})$. This residual representation is on one hand the kernel function for the construction of the Ginzburg–Rallis–Soudry descent from cuspidal automorphic forms on $\mathrm{GL}_{2n}(\mathbb{A})$ to $\mathrm{SO}_{2n+1}(\mathbb{A})$, which is the inverse map of the corresponding Langlands functorial transfer [13]. On the other hand, this residual representation has a nonzero generalized Shalika period (as proved in [25]). The nonvanishing of the generalized Shalika model for this residual representation is expected to produce a different argument to establish the Ginzburg–Rallis–Soudry descent in this case. Instead of establishing the global descent based on the generalized Shalika period, it is our on-going project to establish the local descent from irreducible supercuspidal representations of $\mathrm{GL}_{2n}(F)$ to $\mathrm{SO}_{2n+1}(F)$, based on the generalized Shalika model on $\mathrm{SO}_{4n}(F)$. It should be mentioned that the inverse map of the local Langlands functorial transfer from $\mathrm{GL}_{2n}(F)$ to $\mathrm{SO}_{2n+1}(F)$ was established by using the local Ginzburg–Rallis–Soudry descent from $\mathrm{GL}_{2n}(F)$ to the metaplectic double cover of $\mathrm{Sp}_{2n}(F)$ and the local theta correspondence between $\mathrm{SO}_{2n+1}(F)$ and the metaplectic double cover of $\mathrm{Sp}_{2n}(F)$ [26]. Our on-going project is to show that these two constructions coincide. From this perspective, Theorem 2.3 is one of the key steps to establish local Ginzburg–Rallis–Soudry descent from $\mathrm{GL}_{2n}(F)$ to $\mathrm{SO}_{2n+1}(F)$ based on the generalized Shalika model on $\mathrm{SO}_{4n}(F)$.

The local result (Theorem 5.5) has three interesting consequences in the theory of automorphic forms, which will be discussed in Sect. 6. In Sect. 6.1, we discuss the compatibility of the global Shalika periods with the global Jacquet–Langlands correspondence (Theorem 6.1), which is a conjecture of H. Jacquet and K. Martin [32]. Further discussions can be found in a recent work of W.-T Gan and S. Takeda [8], and also in [24]). In Sect. 6.2, we prove (Theorems 6.4 and 6.5) special cases of Conjecture 8.3 in [29], which deduces the existence of the pole of a relevant Eisenstein series from the local supercuspidal reducibility of the cuspidal datum. Finally in Sect. 6.3, we show (Theorems 6.7 and 6.9) in our case that the existence of a pole at $s = s_0 > 0$ of an Eisenstein series implies the local reducibility at $s = s_0$ of the corresponding unitarily induced representation, which are special cases of a general conjecture of B. Speh [45].

2. Shalika models and generalizations

Let F be a local field which is a finite extension of the p-adic field $\mathbb{Q}_p$ for some prime number p. We recall briefly the Shalika models and its nonsplit variant in Sect. 2.1, and study the generalized Shalika model for irreducible admissible representations of $\mathrm{SO}_{4n}(F)$ in Sect. 2.2.

2.1. Shalika model and its nonsplit variant

Let $P_{n,n} = M_{n,n}N_{n,n}$ be the maximal parabolic subgroup of GL_{2n}, with

$$M_{n,n} = \mathrm{GL}_n \times \mathrm{GL}_n,$$

and

$$N_{n,n} = \left\{ n(X) = \begin{pmatrix} \mathrm{I}_n & X \\ 0 & \mathrm{I}_n \end{pmatrix} \in \mathrm{GL}_{2n} \right\}.$$

Let ψ be a nontrivial character of F. Define a character

$$\psi_{N_{n,n}}(n(X)) = \psi(\mathrm{tr}(X)).$$

The stabilizer of $\psi_{N_{n,n}}$ in $M_{n,n}$ is GL_n^Δ, the diagonal embedding of GL_n into $M_{n,n}$. Denote by

$$\mathcal{S}_n = \mathrm{GL}_n^\Delta \ltimes N_{n,n} \tag{2.1}$$

the Shalika subgroup. Denote by $\psi_{\mathcal{S}_n}$ the extension of $\psi_{N_{n,n}}$ from $N_{n,n}$ to the Shalika subgroup $\mathcal{S}_n$, such that $\psi_{\mathcal{S}_n}$ is trivial on GL_n^Δ. The Shalika functionals of an irreducible admissible representation (τ, V_τ) of $\mathrm{GL}_{2n}(F)$ is a nonzero functional in the following space

$$\mathrm{Hom}_{\mathcal{S}_n(F)}(V_\tau, \psi_{\mathcal{S}_n}).$$

Equivalently, a Shalika functional is a nontrivial functional f on V_τ satisfying

$$f(\tau(s)v) = \psi_{\mathcal{S}_n}(s) f(v) \text{ for all } s \in \mathcal{S}_n, v \in V_\tau.$$

Therefore V_τ allows a nontrivial embedding into the full induction $\mathrm{Ind}_{\mathcal{S}_n(F)}^{\mathrm{GL}_{2n}(F)}(\psi_{\mathcal{S}_n})$, since by reciprocity

$$\mathrm{Hom}_{\mathcal{S}_n(F)}(V_\tau, \psi_{\mathcal{S}_n}) \cong \mathrm{Hom}_{\mathrm{GL}_{2n}(F)}\left(V_\tau, \mathrm{Ind}_{\mathcal{S}_n(F)}^{\mathrm{GL}_{2n}(F)}(\psi_{\mathcal{S}_n})\right).$$

By the local uniqueness of the Shalika model ([20] and also [36]), the dimension of the space $\mathrm{Hom}_{\mathcal{S}_n(F)}(V_\tau, \psi_{\mathcal{S}_n})$ is at most one. If it is nonzero, we say that τ has a Shalika model. More precisely, if $\ell_{\psi_{\mathcal{S}_n}}$ is a nonzero Shalika functional of (τ, V_τ), the Shalika model of τ consists of all the functions of form

$$S_{\psi_{\mathcal{S}_n}, v}(g) := \ell_{\psi_{\mathcal{S}_n}}(\tau(g)(v)) \tag{2.2}$$

for all $v \in V_\tau$. It is clear that $S_{\psi_{\mathcal{S}_n}, v}(g)$ belongs to the space $\mathrm{Ind}_{\mathcal{S}_n(F)}^{\mathrm{GL}_{2n}(F)}(\psi_{\mathcal{S}_n})$.

One may also define nonsplit version of Shalika model as follows. Let D be a division F-algebra of degree n. The Shalika subgroup $\mathcal{S}_D$ of $\mathrm{GL}_2(D)$ is defined by

$$\left\{\begin{pmatrix} a & \\ & a\end{pmatrix}\begin{pmatrix} \mathrm{I}_D & X \\ 0 & \mathrm{I}_D\end{pmatrix} \in \mathrm{GL}_2(D) \mid a \in D^\times, X \in D\right\} = (D^\times)^\Delta \ltimes U(D), \quad (2.3)$$

where $U(D) = \left\{\begin{pmatrix} \mathrm{I}_D & X \\ 0 & \mathrm{I}_D\end{pmatrix} \in \mathrm{GL}_2(D) \mid X \in D\right\}$. The corresponding character $\psi_{U(D)}$ is given by

$$\psi_{U(D)}\left(\begin{pmatrix} \mathrm{I}_D & X \\ 0 & \mathrm{I}_D\end{pmatrix}\right) = \psi(\mathrm{tr}_{D/F}(X)).$$

Denote by $\psi_{\mathcal{S}_D}$ the extension of $\psi_{U(D)}$ from $U(D)$ to the Shalika subgroup $\mathcal{S}_D$, such that $\psi_{\mathcal{S}_D}$ is trivial on $(D^\times)^\Delta$. The $\psi_{\mathcal{S}_D}$-Shalika functionals of an irreducible admissible representation (τ^D, V_{τ^D}) of $\mathrm{GL}_2(D)$ is a nonzero functional in the following space

$$\mathrm{Hom}_{\mathcal{S}_D(F)}(V_{\tau^D}, \psi_{\mathcal{S}_D}).$$

By the local uniqueness of the $\psi_{\mathcal{S}_D}$-Shalika model ([37] and also [36]), the dimension of the space $\mathrm{Hom}_{\mathcal{S}_D(F)}(V_{\tau^D}, \psi_{\mathcal{S}_D})$ is at most one. If it is nonzero, we say that τ^D has a $\psi_{\mathcal{S}_D}$-Shalika model.

The Shalika periods will be discussed in Sect. 4.

2.2. *Generalized Shalika model*

Let $\nu_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ and inductively define

$$\nu_{2n} = \begin{pmatrix} & & 1 \\ & \nu_{2n-2} & \\ 1 & & \end{pmatrix}, \text{ for } n \geq 2, n \in \mathbb{N}. \quad (2.4)$$

Let SO_{4n} be the even special orthogonal group attached to the nondegenerate $4n$-dimensional quadratic vector space over F with respect to ν_{4n}. That is

$$\mathrm{SO}_{4n} = \{g \in \mathrm{GL}_{4n} |\, {}^t g \cdot \nu_{4n} \cdot g = \nu_{4n}\}.$$

Let $P = \bar{M}\bar{N}$ be the Siegel parabolic subgroup of SO_{4n} consists of elements of the following form:

$$(g, X) = \begin{pmatrix} g & 0 \\ 0 & \nu_{2n}\, {}^t g^{-1} \nu_{2n}\end{pmatrix}\begin{pmatrix} \mathrm{I}_n & X \\ & \mathrm{I}_n\end{pmatrix}, g \in \mathrm{GL}_{2n} \text{ and } {}^t X = -\nu_{2n} X \nu_{2n}. \quad (2.5)$$

The *generalized Shalika subgroup* $\mathcal{H}_{2n}$ of SO_{4n} was introduced in [25], which is the subgroup of P consisting of elements (g, X) with $g \in \mathrm{Sp}_{2n}$. The symplectic group is given by

$$\mathrm{Sp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid {}^t g \cdot J_{2n} \cdot g = J_{2n}\},$$

where J_{2n} is defined inductively by

$$J_{2n} = \begin{pmatrix} & & 1 \\ & J_{2n-2} & \\ -1 & & \end{pmatrix}, \ J_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Define a character $\psi_{\mathcal{H}}$ of $\mathcal{H}_{2n}(F)$ (We write $\mathcal{H} = \mathcal{H}_{2n}$, when n is understood.) by letting

$$\psi_{\mathcal{H}}((g, X)) = \psi(\mathrm{tr}(J_{2n} X \nu_{2n})) \tag{2.6}$$
$$= \psi\left(\mathrm{tr}\left(\begin{pmatrix} -\mathrm{I}_n & \\ & \mathrm{I}_n \end{pmatrix} X\right)\right). \tag{2.7}$$

It is well defined. The *generalized Shalika functional* or *$\psi_{\mathcal{H}}$-functional* of an irreducible admissible representation (σ, V_σ) of $\mathrm{SO}_{4n}(F)$ is a nonzero functional in the following space

$$\mathrm{Hom}_{\mathrm{SO}_{4n}(F)}(V_\sigma, \mathrm{Ind}_{\mathcal{H}_{2n}(F)}^{\mathrm{SO}_{4n}(F)}(\psi_{\mathcal{H}})) = \mathrm{Hom}_{\mathcal{H}_{2n}(F)}(V_\sigma, \psi_{\mathcal{H}}).$$

Similarly, one can use a nonzero generalized Shalika functional to define a generalized Shalika model for σ. On the other hand, it was proved in [25] that the Shalika model on $\mathrm{GL}_{2n}(F)$ and the generalized Shalika model on $\mathrm{SO}_{4n}(F)$ are compatible with respect to a unitarily parabolic induction, which is a special case of model-comparison principle for liftings of irreducible admissible representations. More precise description of this local result obtained in [25] can be given as follows.

For an irreducible, unitary, supercuspidal representation (τ, V_τ) of $\mathrm{GL}_{2n}(F)$, we consider the unitarily induced representation $\mathrm{I}(s, \tau)$ of $\mathrm{SO}_{4n}(F)$ from the Siegel parabolic subgroup $P = \bar{M}\bar{N}$, where the Levi part $\bar{M} \cong \mathrm{GL}_{2n}$ via the following bijection

$$a \in \mathrm{GL}_{2n} \mapsto m(a) := \begin{pmatrix} a & \\ & \nu_{2n}^t a^{-1} \nu_{2n} \end{pmatrix} \in \bar{M}.$$

More precisely, a section $\phi_{\tau,s}$ in $\mathrm{I}(s, \tau)$ is a smooth function from $\mathrm{SO}_{4n}(F)$ to V_τ, such that

$$\phi_{\tau,s}(m(a)ng) = |\det a|^{\frac{s}{2}+\frac{2n-1}{2}} \tau(a)\phi_{\tau,s}(g),$$

where $m(a) \in M$ with $a \in \mathrm{GL}_{2n}(F)$. In other words, one has

$$\mathrm{I}(s, \tau) = \mathrm{Ind}_{P(F)}^{\mathrm{SO}_{4n}(F)}(|\det|^{\frac{s}{2}} \cdot \tau).$$

We recall from [25] the following result.

Theorem 2.1. (Theorem 3.1, [25]) *The induced representation* $\mathrm{I}(s, \tau)$ *admits no nonzero generalized Shalika functionals except when* $s = 1$. *When* $s = 1$, $\mathrm{I}(1, \tau)$ *admits a nonzero generalized Shalika functional if and only if the supercuspidal datum* τ *admits a nonzero Shalika functional. In this case, the generalized Shalika functionals of* $\mathrm{I}(1, \tau)$ *is unique and the nonzero generalized Shalika functionals of* $\mathrm{I}(1, \tau)$ *must factor through the unique Langlands quotient* $\mathrm{J}(1, \tau)$.

One of the key local results we prove in this paper is

Theorem 2.2. *Let τ be an irreducible unitary supercuspidal representation of* $\mathrm{GL}_{2n}(F)$. *If τ has a nonzero Shalika model, then the unitarily induced representation* $\mathrm{I}(s,\tau)$ *of* $\mathrm{SO}_{4n}(F)$ *reduces at $s=1$.*

For the proof, we need

Proposition 2.3. *Any irreducible admissible representation of* $\mathrm{SO}_{4n}(F)$ *can not admit both a nonzero Whittaker functional and a nonzero generalized Shalika functional (i.e. $\psi_{\mathcal{H}}$-functional).*

This is a special case of Theorem 3.1 in Sect. 3, which addresses the relation between the generalized Shalika model and a certain family of degenerate Whittaker models. Based on Proposition 2.3, we can prove Theorem 2.2.

If τ has a nonzero Shalika model, then by Theorem 2.1, the unique Langlands quotient $\mathrm{J}(1,\tau)$ of the unitarily induced representation $\mathrm{I}(1,\tau)$ has a nonzero $\psi_{\mathcal{H}}$-functional, i.e. a nonzero generalized Shalika model. On the other hand, since τ is generic, the unitarily induced representation $\mathrm{I}(1,\tau)$ has a nonzero Whittaker functional. If $\mathrm{I}(1,\tau)$ is not reducible, then we have

$$\mathrm{I}(1,\tau)=\mathrm{J}(1,\tau).$$

It follows that the unique Langlands quotient, which is irreducible, has both a nonzero Whittaker functional and a nonzero $\psi_{\mathcal{H}}$-functional. This is impossible by Proposition 2.3. Hence $\mathrm{I}(1,\tau)$ must reduces at $s=1$. This proves Theorem 2.2.

Remark 2.4. We will prove the converse of Theorem 2.2 in Sect. 5, by using a global argument. A purely local argument is also expected. However, it involves a detailed calculations of local Shalika functionals and will be considered in a future work. The purely local argument is important if one wants to extend the main result (Theorem 5.3) to cover more general representations of $\mathrm{GL}_{2n}(F)$.

Remark 2.5. According to a general result of Shahidi [41], when an irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(F)$ has a nonzero Shalika functional, then for $Re(s)\geq 0$, $s=1$ is the only reducible point of $\mathrm{I}(s,\tau)$.

3. Generalized Shalika models and certain degenerate Whittaker models

In this section we prove a more general version of Proposition 2.3, which addresses the relation between the generalized Shalika model and the Bessel models of certain type.

3.1. A family of degenerate Whittaker models

We consider a family of degenerate Whittaker models on $\mathrm{SO}_{4n}(F)$, which are related to the family of Bessel models considered in [11] for construction of automorphic L-functions of orthogonal groups, and in [12] for construction of the

Ginzburg–Rallis–Soudry global descents. More precisely, for $1 \leq k \leq 2n$, we take a family of unipotent subgroups N_k of SO_{4n}, which consists of elements of following type

$$n = n(u, b, z) = \begin{pmatrix} u & b & z \\ & \mathrm{I}_{2k} & b' \\ & & u' \end{pmatrix} \in \mathrm{SO}_{4n}, \tag{3.1}$$

where $u = (u_{i,j}) \in \mathrm{U}_{2n-k}$, the maximal unipotent subgroup of GL_{2n-k} consisting of all upper triangular unipotent matrices in GL_{2n-k}, $b = (b_{i,j})$ is of size $(2n-k) \times (2k)$ and b', u' are determined by b, u such that n belongs to SO_{4n}. We define a character ψ_k on N_k

$$\psi_k(n) := \psi(u_{1,2} + \cdots + u_{2n-k-1,2n-k})\psi(b_{2n-k,k} + b_{2n-k,k+1}). \tag{3.2}$$

When $k = 1$, N_1 coincides with the unipotent radical N of the Borel subgroup of SO_{4n}, and ψ_1 is the generic character of N. Let σ be an irreducible admissible representation (σ, V_σ) of $\mathrm{SO}_{4n}(F)$. Then σ has a nonzero ψ_k-functional if the following space

$$\mathrm{Hom}_{\mathrm{SO}_{4n}(F)}(V_\sigma, \mathrm{Ind}_{N_k(F)}^{\mathrm{SO}_{4n}(F)}(\psi_k)) \cong \mathrm{Hom}_{N_k(F)}(V_\sigma, \psi_k) \neq 0. \tag{3.3}$$

In this case, a nonzero element in $\mathrm{Hom}_{N_k(F)}(V_\sigma, \psi_k)$ is called a ψ_k-functional of V_σ, or more precisely, a ψ_k-degenerate Whittaker functional of V_σ. For each ψ_k-functional ℓ_{ψ_k}, we define

$$\mathcal{W}_{\psi_k,v}(g) := \ell_{\psi_k}(\sigma(g)(v)) \tag{3.4}$$

for $v \in V_\sigma$, which yields a ψ_k-degenerate Whittaker model (also refer to as N_k-model) for V_σ. In particular, when $k = 1$, it produces a Whittaker model for V_σ. By a theorem of Shalika [40], the Whittaker model of V_σ is unique. However, by a theorem of Moeglin and Waldspurger [33], these ψ_k-degenerate Whittaker models are not unique in general.

The main result in this section is

Theorem 3.1. *For $1 \leq k \leq n$, an irreducible admissible representation of $\mathrm{SO}_{4n}(F)$ can not admit both a nonzero ψ_k-functional and a nonzero generalized Shalika functional (i.e. $\psi_{\mathcal{H}}$-functional).*

We prove this theorem by showing that there are no nonzero distributions on $\mathrm{SO}_{4n}(F)$ which satisfy the required left and right quasi-invariant property determined by these models. By using Bernstein's localization principle, we have to show that there are no admissible double cosets in the decomposition $N_k(F)\backslash\mathrm{SO}_{4n}(F)/\mathcal{H}_{2n}(F)$. The admissibility of double cosets are defined as follows.

Definition 3.2. We say that a double coset $N_k(F)w\mathcal{H}_{2n}(F)$, $w \in \mathrm{SO}_{4n}(F)$ is *admissible* if

$$\psi_{\mathcal{H}}((h, X)) = \psi_k(w(h, X)w^{-1}) \tag{3.5}$$

for all $(h, X) \in w^{-1}N_k(F)w \cap \mathcal{H}_{2n}(F)$.

We claim the following

Proposition 3.3. *For $1 \leq k \leq n$, there is no admissible double coset in $N_k\backslash \mathrm{SO}_{4n}/\mathcal{H}_{2n}$.*

The proposition is a key step to prove Theorem 3.1. The proof of Proposition 3.3 involves tedious calculations of the double cosets, which will be given in Sect. 4.

3.2. Proof of Theorem 3.1

In order to prove Theorem 3.1, we extend the notion of generalized Shalika groups as follows.

Let A be a nonsingular skew-symmetric matrix of size $2n \times 2n$. Define

$$\mathcal{H}_A = \mathrm{Sp}_{2n}(A)\bar{N}.$$

Then elements in $\mathcal{H}_A$ are $(h, X) \in P$ with $h \in \mathrm{Sp}_{2n}(A)$, where

$$\mathrm{Sp}_{2n}(A) = \{g \in \mathrm{GL}_{2n} | gA\,{}^t g = A\}.$$

Define a character $\psi_{\mathcal{H}_A}$ on $\mathcal{H}_A(F)$ by

$$\psi_{\mathcal{H}_A}(h, X) = \psi(\mathrm{tr}(-A^{-1}X\nu_{2n})), \quad (h, X) \in \mathcal{H}_A(F). \tag{3.6}$$

This is a well defined character: for $h \in \mathrm{Sp}_{2n}(A, F)$, since

$$\begin{aligned}\psi_{\mathcal{H}_A}((h,0)(1,X)(h^{-1},0)) &= \psi_{\mathcal{H}_A}(1, hX\,{}^t h)\\ &= \psi(\mathrm{tr}(-A^{-1}hX\nu_{2n}\,{}^t h))\\ &= \psi(\mathrm{tr}(-{}^t hA^{-1}hX\nu_{2n}))\\ &= \psi(\mathrm{tr}(-(h^{-1}A\,{}^t h^{-1})^{-1}X\nu_{2n}))\\ &= \psi(\mathrm{tr}(-A^{-1}X\nu_{2n})).\end{aligned}$$

It is clear that when $A = J_{2n}$, $\mathcal{H}_A = \mathcal{H}_{2n}$ is the generalized Shalika group defined before, and $\psi_{\mathcal{H}_A} = \psi_{\mathcal{H}}$. Similarly, we say that a double coset $N_k(F)w\mathcal{H}_A(F)$, $w \in \mathrm{SO}_{4n}(F)$ is *admissible* if

$$\psi_{\mathcal{H}_A}((h, X)) = \psi_k(w(h, X)w^{-1}) \tag{3.7}$$

for all $(h, X) \in w^{-1}N_k(F)w \cap \mathcal{H}_A(F)$.

We recall Bernstein's localization principle for our case below, and refer to [30] and [46] for the notation and known results.

Let $C_c^\infty(X)$ denote the space of smooth, compactly supported functions on a p-adic space X, and $\mathfrak{D}(X)$ denotes the space of complex-valued linear functionals on $C_c^\infty(X)$. Elements of $\mathfrak{D}(X)$ are called distributions. Given a Lie group G, define the left and right translations l_g and r_g on G; $C_c^\infty(G)$ and $\mathfrak{D}(G)$ as the following:

$$\begin{aligned}&l_g \cdot x = gx;\ \ r_g \cdot x = xg^{-1};\\ &(l_g \cdot f)(x) = f(g^{-1}x);\ \ (r_g \cdot f)(x) = f(xg);\\ &(l_g \cdot T)(f) = T(l_{g^{-1}} \cdot f);\ (r_g \cdot T)(f) = T(r_{g^{-1}} \cdot f),\end{aligned}$$

where $g,\ x \in G$; $f \in C_c^\infty(G)$ and $T \in \mathfrak{D}(G)$.

If G acts on a p-adic space X, we define the action of $l_g,\ g \in G$ on X, $C_c^\infty(X)$ and $\mathfrak{D}(G)$ in a similar manner.

Lemma 3.4. (Bernstein's localization principle, Theorem 6.9, [3]) *Assume that a p-adic group G acts on a p-adic space X constructively, which means that the graph $\{(x, gx) | g \in G, x \in X\}$ of G is the union of finitely many locally closed subsets of $X \times X$. If there are no non-zero G-invariant distributions on any G-orbit of X, then there are no non-zero G-invariant distributions on X.*

The result of Bernstein's localization principle can be extend to quasi-invariant distributions with slight modification to its proof (see [43] for instance.).

Now we show that Proposition 3.3 implies Theorem 3.1.[1]

Let π be an irreducible representation of $G = \mathrm{SO}_{4n}(F)$. Assume that π has nontrivial embeddings in both $\mathrm{Ind}_{N_k}^G \psi_k$ and $\mathrm{Ind}_{\mathcal{H}}^G \psi_{\mathcal{H}}$. By the result of [40], the contragradient $\tilde{\pi}$ of π also admits a Whittaker model. The dual of $\mathrm{Hom}_G(\tilde{\pi}, \mathrm{Ind}_{N_k}^G \psi_k) \neq 0$ gives $\mathrm{Hom}_G(\mathrm{ind}_{N_k}^G \psi_k^{-1}, \pi) \neq 0$ (refer to [9] or [3]). The composition of nontrivial

$$T_1 \in \mathrm{Hom}_G(\mathrm{ind}_{N_k}^G \psi_k^{-1}, \pi) \text{ and } T_2 \in \mathrm{Hom}_G(\pi, \mathrm{Ind}_{\mathcal{H}}^G \psi_{\mathcal{H}})$$

produces a nontrivial intertwining operator (since π is irreducible) in

$$\mathrm{Hom}_G(\mathrm{ind}_{N_k}^G \psi_k^{-1}, \mathrm{Ind}_{\mathcal{H}}^G \psi_{\mathcal{H}}).$$

Consider $T \in \mathfrak{D}(N_k \backslash G)$. The right action of $\mathcal{H}$ on $N_k \backslash G$ is constructive by Theorem A, 6.15, [3]. The restriction of T to the coset $N_k w \mathcal{H}$ is associated with $\mathrm{ind}_{\mathcal{H} \cap w^{-1} N_k w}^{\mathcal{H}} \psi_k^{-w}$, where $\psi_k^{-w}(g) = \psi_k^{-1}(wgw^{-1})$, for $g \in \mathcal{H} \cap w^{-1} N_k w$. Frobenius reciprocity gives

$$\mathrm{Hom}_{\mathcal{H}}(\mathrm{ind}_{\mathcal{H} \cap w^{-1} N_k w}^{\mathcal{H}} \psi_k^{-w}, \psi_{\mathcal{H}}) \cong \mathrm{Hom}_{\mathcal{H} \cap w^{-1} N_k w}(\psi_k^{-w}, \psi_{\mathcal{H}}).$$

By Proposition 3.3, $\mathrm{Hom}_{\mathcal{H} \cap w^{-1} N_k w}(\psi_k^{-w}, \psi_{\mathcal{H}}) = 0$ for all $w \in G$. Hence by Bernstein's localization principle

$$\mathrm{Hom}_G(\mathrm{ind}_{N_k}^G \psi_k^{-1}, \mathrm{Ind}_{\mathcal{H}}^G \psi_{\mathcal{H}}) = 0,$$

which contradicts our assumption. Therefore, for $1 \le k \le n$, π cannot possess both N_k-model and generalized Shalika model. Especially, π cannot possess both Whittaker model and generalized Shalika model. This proves Theorem 3.1.

4. Proof of Proposition 3.3

Let $W = W(\mathrm{SO}_{4n})$ and $W(P)$ denote the Weyl group of $\mathrm{SO}_{4n}(F)$ and $P(F)$, respectively. Then the generalized Bruhat decomposition

$$N(F) \backslash \mathrm{SO}_{4n}(F) / P(F)$$

can be parametrized by $W/W(P)$ with suitable chosen representatives.

[1] Although the proof is routine (refer to Theorem 3.1 and Theorem 3.2.2, [14]), we keep it here for the sake of completion.

Lemma 4.1. ([10], Lemma 5.1) *The set Ω of elements having minimal length in each coset of $W/W(P)$ comprises a complete set of representatives of $W/W(P)$. Elements of Ω can be described as follows. For $w \in \Omega$, there is some even number $0 \le k_w \le 2n$, a sequence of k_w numbers: $\iota_1 < \cdots < \iota_{k_w}$ and a sequence of $2n - k_w$ numbers $\iota_{k+1} > \cdots > \iota_{2n}$ such that*

$$\begin{aligned} w(e_j) &= e_{\iota_j}, \text{ if } j \le k_w; \\ w(e_j) &= -e_{\iota_j}, \text{ if } j > k_w. \end{aligned} \tag{4.1}$$

Then in each generalized Bruhat cell $N(F)wP(F)$ with $w \in \Omega$, we study the double coset decomposition

$$N(F)\backslash N(F)wP(F)/\mathcal{H}_{2n}(F).$$

It is clear that each double coset in the above decomposition has a representative wg, for some $w \in \Omega, g \in \mathrm{GL}_{2n}(F)$. Moreover, the representative wg has the following property.

Lemma 4.2. *If $w, w' \in \Omega$, $g, g' \in \mathrm{GL}_{2n}(F)$, then the double cosets*

$$N(F)wg\mathcal{H}_{2n}(F) = N(F)w'g'\mathcal{H}_{2n}(F)$$

if and only if

$$w = w', \quad \mathrm{U}_{2n}(F)g\mathrm{Sp}_{2n}(F) = \mathrm{U}_{2n}(F)g'\mathrm{Sp}_{2n}(F).$$

Proof. We firs prove that if $N(F)wg\mathcal{H}_{2n}(F) = N(F)w'g'\mathcal{H}_{2n}(F)$, then $w = w'$ and $\mathrm{U}_{2n}(F)g\mathrm{Sp}_{2n}(F) = \mathrm{U}_{2n}(F)g'\mathrm{Sp}_{2n}(F)$.

For simplicity, we write N, $\mathcal{H}$, P, U, GL_{2n}, Sp_{2n} for $N(F)$, $\mathcal{H}_{2n}(F)$, $P(F)$, $\mathrm{U}_{2n}(F)$, $\mathrm{GL}_{2n}(F)$, and $\mathrm{Sp}_{2n}(F)$, respectively.

If $Nwg\mathcal{H} = Nw'g'\mathcal{H}$, then $NwgP = Nw'g'P$. Since $g, g' \in P$,

$$NwP = Nw'P.$$

Note that $w, w' \in \Omega$ which is a complete set of representatives of $N\backslash \mathrm{SO}_{4n}(F)/P$, hence $w = w'$.

Assume w is of the form in (4.1). Since $Nwg\mathcal{H} = Nwg'\mathcal{H}$, there is some $u \in N, h \in \mathcal{H}$ such that

$$uwgh = wg'.$$

Hence $g' = w^{-1}uwgh$, and

$$w^{-1}uw = g'h^{-1}g^{-1} \in P. \tag{4.2}$$

By the action of w on roots of $\mathrm{SO}_{4n}(F)$, we see that

$$w^{-1}Nw \cap P = (w^{-1}Nw \cap \mathrm{GL}_{2n})(w^{-1}Nw \cap \bar{N}).$$

One deduces from (4.2) that

$$(w^{-1}Nw \cap P)g'\mathcal{H} = (w^{-1}Nw \cap P)g\mathcal{H}.$$

This is equivalent to

$$(w^{-1}Nw \cap \mathrm{GL}_{2n})(w^{-1}Nw \cap \bar{N})g'\mathcal{H} = (w^{-1}Nw \cap \mathrm{GL}_{2n})(w^{-1}Nw \cap \bar{N})g\mathcal{H}.$$

That is,

$$\begin{aligned}&(w^{-1}Nw \cap \mathrm{GL}_{2n})(w^{-1}Nw \cap \bar{N})g'\mathrm{Sp}_{2n}(F)\bar{N}\\ &= (w^{-1}Nw \cap \mathrm{GL}_{2n})(w^{-1}Nw \cap \bar{N})g\mathrm{Sp}_{2n}\bar{N}.\end{aligned}$$

Note that elements in GL_{2n} normalize $\bar{N}$, hence

$$\begin{aligned}(w^{-1}Nw \cap \mathrm{GL}_{2n})g'\mathrm{Sp}_{2n}\bar{N} &= (w^{-1}Nw \cap \mathrm{GL}_{2n})g\mathrm{Sp}_{2n}\bar{N},\\ (w^{-1}Nw \cap \mathrm{GL}_{2n})g'\mathrm{Sp}_{2n} &= (w^{-1}Nw \cap \mathrm{GL}_{2n})g\mathrm{Sp}_{2n}.\end{aligned}$$

In the following we compute $w^{-1}Nw \cap \mathrm{GL}_{2n}$. The set of roots of SO_{4n} is

$$\Phi = \{e_i - e_j, e_i + e_j | 1 \le i, j \le 2n, i \ne j\}.$$

The maximal upper triangular unipotent subgroup N of SO_{4n} determines a set of positive roots of SO_{4n}, denoted by

$$\Phi^+ = \{e_i - e_j, e_i + e_j | 1 \le i < j \le 2n\}.$$

For $w \in W$, its action on Φ is given by

$$\begin{aligned}w(e_i - e_j) &= w(e_i) - w(e_j),\\ w(e_i + e_j) &= w(e_i) + w(e_j).\end{aligned}$$

The set of roots of GL_{2n} is

$$\Phi' = \{e_i - e_j, \quad \text{for } 1 \le i, j \le 2n, i \ne j\},$$

and the set of its positive roots determined by U_{2n} is

$$\Phi'^+ = \{e_i - e_j, \quad \text{for } 1 \le i < j \le 2n\}.$$

For $\alpha \in \Phi'$, let U_α be the root group corresponding to α. Then

$$w^{-1}Nw \cap \mathrm{GL}_{2n} = \prod_{\alpha \in \Phi', w(\alpha) \in \Phi^+} \mathrm{U}_\alpha.$$

Next we compute the set

$$C = \{\alpha \in \Phi' |\ w(\alpha) \in \Phi^+\}.$$

Let $\alpha = e_i - e_j$, then $w(\alpha) = w(e_i) - w(e_j)$. According to the formula of w in (4.1), we have the following four cases:

(1) $i, j \le k_w$. Then $w(e_i - e_j) = e_{\iota_i} - e_{\iota_j}$. In this case, $\iota_i < \iota_j$ if and only if $i < j$. Hence

$$e_i - e_j \in C, \text{ for } i < j \le k_w.$$

(2) $i \le k_w$, $j > k_w$. Then $w(e_i - e_j) = e_{\iota_i} + e_{\iota_j} \in \Phi^+$. Hence

$$e_i - e_j \in C, \text{ for } i \le k_w, j > k_w.$$

(3) $i > k_w$, $j \le k_w$. Then $w(e_i - e_j) = -e_{\iota_i} - e_{\iota_j} \notin \Phi^+$.
(4) $i > k_w$, $j > k_w$. Then $w(e_i - e_j) = -e_{\iota_i} + e_{\iota_j}$. In this case $\iota_i > \iota_j$ if and only if $i < j$, by (4.1). Hence

$$e_i - e_j \in C, \text{ for } k_w < i < j.$$

The above discussion shows that

$$w^{-1} N w \cap \mathrm{GL}_{2n} = \mathrm{U}_{2n}.$$

The proof for the reverse direction is similar and we omit it. □

Lemma 4.2 reduces the computation of $N \backslash \mathrm{SO}_{4n}(F)/\mathcal{H}$ to the computation of $\mathrm{U}_{2n} \backslash \mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$, whose representatives were calculated by Jacquet and Rallis in [19]. Let

$$\mathcal{A} = \{A \in \mathrm{GL}_{2n} |\, {}^t A = -A\}$$

be the set of nonsingular skew-symmetric matrices of size $2n \times 2n$. The group GL_{2n} operates on $\mathcal{A}$ by:

$$A \mapsto g A\, {}^t g.$$

The stabilizer of J_{2n} is Sp_{2n}. Then U_{2n} operates on $\mathcal{A}$ and its orbits is the set of double cosets $\mathrm{U}_{2n} \backslash \mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$.

Lemma 4.3. ([19], Lemma 2) *Every nonsingular skew symmetric matrix of degree $2n$ can be written in the form*

$$s = u w' \lambda\, {}^t u$$

with $u \in \mathrm{U}_{2n}$, λ is a diagonal matrix in GL_{2n}, $w' \in W(\mathrm{GL}_{2n})$ the Weyl group of GL_{2n} such that

$$w'^2 = 1, \;\; w' \lambda w'^{-1} = -\lambda.$$

Let $A = w'\lambda = (a_{i,j})$, $\lambda = (\lambda_1, \ldots, \lambda_{2n})$. Then A is a nonsingular skew symmetric matrix with one and only one nonzero element at each row and column:

$$a_{i,j} = \begin{cases} \lambda_j, & \text{if } w'(i) = j; \\ 0, & \text{otherwise.} \end{cases}$$

Let $g \in \mathrm{GL}_{2n}$ be such that $A = g J_{2n}\, {}^t g$. Consider the double coset $N w g \mathcal{H}$, for $w \in \Omega$. By Lemma 4.2, the double coset is independent of the choice of g.

Recall the following:

For $\omega \in \Omega$, there is an even number k_ω, $0 \le k_\omega \le 2n$, such that

$$\iota_1 < \cdots < \iota_{k_\omega}, \iota_{k_\omega+1} > \cdots > \iota_{2n}, \text{ and } \omega : \begin{cases} e_1 & \mapsto & e_{\iota_i} \\ & \cdots & \\ e_{k_\omega} & \mapsto & e_{\iota_{k_\omega}} \\ e_{k_\omega+1} & \mapsto & -e_{\iota_{k_\omega+1}} \\ & \cdots & \\ e_{2n} & \mapsto & -e_{k_{2n}} \end{cases}. \tag{4.3}$$

The action of $\omega \in \Omega$ on Φ is given by

$$\omega(e_i + e_j) = \begin{cases} e_{\iota_i} + e_{\iota_j}, & \text{if } i \le k_\omega,\ j \le k_\omega; \\ e_{\iota_i} - e_{\iota_j}, & \text{if } i \le k_\omega,\ j > k_\omega; \\ -e_{\iota_i} + e_{\iota_j}, & \text{if } i > k_\omega,\ j \le k_\omega; \\ -e_{\iota_i} - e_{\iota_j}, & \text{if } i > k_\omega,\ j > k_\omega; \end{cases} \tag{4.4}$$

$$\omega(e_i - e_j) = \begin{cases} e_{\iota_i} - e_{\iota_j}, & \text{if } i \le k_\omega,\ j \le k_\omega; \\ e_{\iota_i} + e_{\iota_j}, & \text{if } i \le k_\omega,\ j > k_\omega; \\ -e_{\iota_i} - e_{\iota_j}, & \text{if } i > k_\omega,\ j \le k_\omega; \\ -e_{\iota_i} + e_{\iota_j}, & \text{if } i > k_\omega,\ j > k_\omega. \end{cases} \tag{4.5}$$

Lemma 4.4. *The subgroup $\omega^{-1} N_k \omega \cap P$ consists of elements (g, X) of P, where*

$$g = \begin{pmatrix} u & b & z \\ & \mathrm{I}_k & b' \\ & & u' \end{pmatrix}, \quad X = \begin{pmatrix} C & Y & Z \\ 0 & 0 & Y' \\ 0 & 0 & C' \end{pmatrix}, \tag{4.6}$$

$u \in \mathrm{U}_{i_0},\ u' \in \mathrm{U}_{j_0},\ Z \in \mathrm{M}_{i_0 \times i_0},\ Y \in \mathrm{M}_{i_0 \times k},$

$$i_0 = \max_{i \le k_\omega}\{\iota_i \le 2n - k\}, \quad j_0 = 2n - \min_{j > k_\omega}\{\iota_j \le 2n - k\} + 1 = 2n - k - i_0, \tag{4.7}$$

and $C_{i,j} = 0$ for $\iota_i \ge \iota_{2n-j}$.

Proof. Let Φ_k be the set of roots of N_k. That is $\Phi_k \subset \Phi$ consists of roots

$$e_i - e_j \quad (i < j \text{ and } i \le 2n - k); \tag{4.8}$$

$$e_i + e_j \quad (i \le 2n - k \text{ or } j \le 2n - k). \tag{4.9}$$

By (4.4), we see that $\omega(e_i + e_j) \in \Phi_k$ if and only if one of the following conditions holds.

(1) $i \le k_\omega$, $j \le k_\omega$ such that $\iota_i \le 2n - k$ or $\iota_j \le 2n - k$;
(2) $i \le k_\omega$, $j > k_\omega$ such that $\iota_i \le 2n - k$, and $\iota_i < \iota_j$;
(3) $i > k_\omega$, $j \le k_\omega$ such that $\iota_j \le 2n - k$ and $\iota_i > \iota_j$.

By (4.5), we see that $\omega(e_i - e_j) \in \Phi_k$ if and only if one of the following conditions holds.

(1) $i < j \le k_\omega$ and $\iota_i \le 2n - k$;
(2) $i \le k_\omega$, $j > k_\omega$ such that $\iota_i \le 2n - k$ or $\iota_j \le 2n - k$;

(3) $j > i > k_\omega$ and $\iota_j \leq 2n - k$.

The lemma follows. □

Lemma 4.5. *Any representatives of double cosets for $N_k \backslash \mathrm{SO}_{4n}/\mathcal{H}$ can be chosen to be in the form of*

$$u\omega g, \tag{4.10}$$

for some $\omega \in \Omega$, $g \in \mathrm{U}_{2n}\backslash \mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$ such that $A = gJ_{2n}\,{}^t g$ have one and only one nonzero element in each row and column, and $u \in \mathrm{U}_{2k} \cap \mathrm{SO}_{2k}$ is embedded in SO_{4n} by

$$\begin{pmatrix} \mathrm{I}_{2n-k} & & \\ & u & \\ & & \mathrm{I}_{2n-k} \end{pmatrix}. \tag{4.11}$$

Lemma 4.6. *Let $x = uwg$, where notations u, w, g are as in Lemma 4.5. Then the followings are equivalent*:

(1) *$N_k x \mathcal{H}$ is admissible;*
(2) *$N_k u\omega \mathcal{H}_A$ is admissible.*

Proof. $N_k wg\mathcal{H}$ is admissible if and only if

$$\psi_{\mathcal{H}}(h, X) = \psi_k(wg(h, X)g^{-1}w^{-1}) \tag{4.12}$$

for all $(h, X) \in g^{-1}w^{-1}N_k wg \cap \mathcal{H}$. Let

$$(h', X') = g(h, X)g^{-1} = (ghg^{-1}, gX\,{}^t g).$$

Since

$$\begin{aligned} &(h, X) \in g^{-1}w^{-1}N_k wg \cap \mathcal{H} \\ \Longleftrightarrow\ & g(h, X)g^{-1} \in w^{-1}N_k w \cap g\mathcal{H}g^{-1} = w^{-1}N_k w \cap \mathcal{H}_A, \end{aligned}$$

and

$$\begin{aligned} \psi_{\mathcal{H}}(h, X) &= \psi(\mathrm{tr}(J_{2n}X\nu_{2n})) = \psi(\mathrm{tr}(J_{2n}g^{-1}X'\nu_{2n}\,{}^t g^{-1}) \\ &= \psi(\mathrm{tr}({}^t g^{-1}J_{2n}g^{-1}X'\nu_{2n})) \\ &= \psi(\mathrm{tr}((-gJ_{2n}\,{}^t g)^{-1}X'\nu_{2n})) \\ &= \psi(\mathrm{tr}(-A^{-1}X'\nu_{2n})) \\ &= \psi_{\mathcal{H}_A}(h', X'), \end{aligned}$$

the conclusion follows. □

Lemma 4.7. *Let X be a matrix in the form of (4.6). Then $(1, X) \in \mathrm{SO}_{4n}$ if and only if*

$$Z = -\nu_{i_0}\,{}^t Z\nu_{i_0}, \quad Y = -\nu_{i_0}\,{}^t Y'\nu_k, \quad C = -\nu_{i_0}\,{}^t C'\nu_{j_0}. \tag{4.13}$$

Proof. *The result follows from*

$$ {}^{t}X\nu_{2n} = -\nu_{2n}X. \tag{4.14}$$

□

We record a formula for $\psi_{\mathcal{H}_A}$ on $\bar{N} \cap \omega^{-1}N_k\omega$ here for later use. ($\bar{N}$ is the unipotent radical of P.) Write

$$A^{-1} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ -{}^{t}A_{12} & A_{22} & A_{23} \\ -{}^{t}A_{13} & -{}^{t}A_{23} & A_{33} \end{pmatrix} \tag{4.15}$$

with block size (i_0, k, j_0). The formula of $\psi_{\mathcal{H}_A}$ on $\bar{N} \cap \omega^{-1}N_k\omega$ is then given by:

$$\begin{aligned} \psi_{\mathcal{H}_A}((1, X)) &= \psi \circ \mathrm{tr}(-A^{-1}X\nu_{2n}) \\ &= \psi \circ \mathrm{tr}(-A_{11}Z\nu_{i_0} + A_{12}\,{}^{t}(Y\nu_k) + A_{13}\,{}^{t}(C\nu_{j_0}) \\ &\quad + {}^{t}A_{12}Y\nu_k + {}^{t}A_{13}C\nu_{j_0}) \\ &= \psi \circ \mathrm{tr}(-A_{11}Z\nu_{i_0} + 2A_{12}\,{}^{t}(Y\nu_k) + 2A_{13}\,{}^{t}(C\nu_{j_0})) \end{aligned} \tag{4.16}$$

4.1. Reduction to the case of $i_0 = 0$

For $\alpha \in F, m, l \leq 2n, m \neq l$, we let $X_{m,l}(\alpha)$ denote the $2n \times 2n$ matrix such that the entries of $X_{m,l}(\alpha)\nu_{2n}$ are all zeros except the (m, l)th entry is α and (l, m)th entry is $-\alpha$. Then $(1, X_{m,l}(\alpha)) \in \bar{N}$ belongs to the root group corresponding to $e_m + e_l$. Let $E_{m,l}$ be the $2n \times 2n$ elementary matrix with 1 at the (m, l)-entry and 0 at other entries.

For the rest of this section, we fix an arbitrary double coset $N_k x\mathcal{H}$, $x = uwg$, with u, w, g and $A = gJ_{2n}\,{}^{t}g$ as described in Lemma 4.5.

Lemma 4.8. *Assume that $N_k x\mathcal{H}$ is admissible, and $i_0 \neq 0$. Then*

$$\iota^{-1}(2n - k) = 2n - j_0 + 1. \tag{4.17}$$

Proof. On the contrary, assume that $\iota^{-1}(2n - k) = i_0$. Then $m := \iota^{-1}(2n) > i_0$. It is either $m = k_\omega$ or $m = k_\omega + 1$, and we want to show that both cases are invalid.

For the case of $m = k_\omega$, the element $(1, X_{i_0,k_\omega}(\alpha))$ belongs to $\bar{N} \cap \omega^{-1}N_k\omega$. Note that $\omega(e_{i_0} + e_{k_\omega}) = e_{2n-k} + e_{2n}$,

$$\omega(1, X_{i_0,k_\omega}(\alpha))\omega^{-1} = (1, X_{2n-k,2n}(\alpha)) \in N_k. \tag{4.18}$$

We write $(1, X_{2n-k,2n}(\alpha))$ in blocks as follows

$$\begin{pmatrix} \mathrm{I}_{2n-k} & b & 0 \\ & \mathrm{I}_{2k} & b' \\ & & \mathrm{I}_{2n-k} \end{pmatrix}, \; b = (b_{i,j}). \tag{4.19}$$

Note that

$$u(1, X_{2n-k,2n}(\alpha))u^{-1} = \begin{pmatrix} 1 & & \\ & u & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_{2n-k} & b & 0 \\ & I_{2k} & b' \\ & & I_{2n-k} \end{pmatrix} \begin{pmatrix} 1 & & \\ & u^{-1} & \\ & & 1 \end{pmatrix}$$
$$= \begin{pmatrix} I_{2n-k} & bu^{-1} & 0 \\ & I_{2k} & ub' \\ & & I_{2n-k} \end{pmatrix}.$$

Since $b_{2n-k,k+1} = \alpha$ is the only nonzero entry of b and u is an upper triangular unipotent matrix,

$$\psi_k(u\omega(1, X_{i_0,k_\omega}(\alpha)\omega^{-1}u^{-1})) = \psi(\alpha), \tag{4.20}$$

which is not 1 for suitably chosen α. By admissibility of $N_k x\mathcal{H}$ and the formula (4.16) of $\psi_{\mathcal{H}_A}$, the (i_0, k_ω)-th entry of A is nonzero.

Let $g = 1 + \alpha E_{i_0,k_\omega}$. It is a unipotent element in $\mathrm{Sp}_{2n}(A)$ corresponding to $e_{i_0} - e_{k_\omega}$. Since $\omega(e_{i_0} - e_{k_\omega}) = e_{2n-k} - e_{2n}$,

$$\omega g\omega^{-1} = 1 + \alpha E_{2n-k,2n} \in N_k. \tag{4.21}$$

Write $\omega g\omega^{-1} \in \mathrm{SO}_{4n}$ in blocks as the following

$$\omega g\omega^{-1} = \begin{pmatrix} I_{2n-k} & d & 0 \\ & I_{2k} & d' \\ & & I_{2n-k} \end{pmatrix}, \ d = (d_{i,j}) \tag{4.22}$$

Notice

$$u\omega g\omega^{-1}u^{-1} = \begin{pmatrix} 1 & & \\ & u & \\ & & 1 \end{pmatrix} \begin{pmatrix} I_{2n-k} & d & 0 \\ & I_{2k} & d' \\ & & I_{2n-k} \end{pmatrix} \begin{pmatrix} 1 & & \\ & u^{-1} & \\ & & 1 \end{pmatrix}$$
$$= \begin{pmatrix} I_{2n-k} & du^{-1} & 0 \\ & I_{2k} & ud' \\ & & I_{2n-k} \end{pmatrix}.$$

Since $d_{2n-k,k} = \alpha$ is the only nonzero entry of d and u is an upper triangular unipotent matrix,

$$\psi_k(u\omega g\omega^{-1}u^{-1}) \tag{4.23}$$

is nontrivial for suitably chosen $\alpha \in F$, which contradicts to the fact that $\psi_{\mathcal{H}_A}((g, 0)) = 1$ for all $g \in \mathrm{Sp}_{2n}(A)$. Therefore $N_k x\mathcal{H}$ is not admissible.

Similar arguments also works for the case of $m = k_\omega + 1$ and the conclusion follows. □

Lemma 4.9. *Assume that $N_k x\mathcal{H}$ is admissible. If there is some $s \leq i_0$, $\iota_s < 2n - k$ such that $l = \iota^{-1}(\iota_s + 1) \geq k + i_0$, then $A_{s,l} \neq 0$. Conversely, if $A_{s,l} \neq 0$ for some $s \leq i_0$, $\iota_s < 2n - k$ and $l \geq k + i_0$, then $\iota_l = \iota_s + 1$.*

Proof. The equalities $\iota_l = \iota_s + 1 \le 2n - k$ and $k + i_0 = 2n - j_0$ implies $l > k_\omega$. For $(1, X_{s,l}(\alpha)) \in \omega^{-1} N_k \omega \cap \bar{N}$,

$$u\omega(1, X_{s,l}(\alpha))\omega^{-1}u^{-1} = \omega(1, X_{s,l}(\alpha))\omega^{-1} = 1 + \alpha E_{\iota_s, \iota_s+1} \in N_k. \tag{4.24}$$

Since

$$\psi_k(u\omega(1, X_{s,l}(\alpha))\omega^{-1}u^{-1}) = \psi(\alpha) = \psi(\mathrm{tr} A X_{s,l}(\alpha)\nu_{2n}), \tag{4.25}$$

$A_{s,l}$ must be nonzero for $N_k x \mathcal{H}$ to be admissible. □

Lemma 4.10. *Assume that $N_k x \mathcal{H}$ is admissible. Then*

$$A_{i,j} = 0, \quad \textit{for } i \le i_0, \ j \le k_\omega. \tag{4.26}$$

Proof. Write $A = (a_{ij})$. Since A is a nonsingular anti-symmetric matrix with one and only one elements in each row and column, so does A^{-1} and the nonzero entries of A^{-1} are at the same positions as those of A. If $i_0 = 0$, there is nothing to prove. For $i_0 \ne 0$, assume on the contrary that $A_{i,j} \ne 0$ for some $i \le i_0$, $j \le k_\omega$. Consider $(1, X_{i,j}(\alpha)) \in \bar{N}$. By the definition of $\psi_{\mathcal{H}_A}$,

$$\psi_{\mathcal{H}_A}((1, X_{i,j}(\alpha))) \ne 1 \tag{4.27}$$

for suitably chosen α. Since $\omega(e_i + e_j) = e_{\iota_i} + e_{\iota_j}$,

$$\omega(1, X_{i,j}(\alpha))\omega^{-1} = (1, X_{\iota_i, \iota_j}(\alpha)). \tag{4.28}$$

Because $\iota_{i_0} \ne 2n - k$ by Lemma 4.8, the adjoint action of u on $(1, X_{i,j}(\alpha))$ does not affect the value of ψ_k. That is

$$\psi_k(u\omega(1, X_{\iota_i, \iota_j}(\alpha))\omega^{-1}u^{-1}) = \psi_k(1, X_{\iota_i, \iota_j j}(\alpha)) = 1, \tag{4.29}$$

for all $\alpha \in F$. This contradicts to the admissibility of $N_k x \mathcal{H}$. □

Here we record one simple observation.

Remark 4.11. Let $u = (u_{i,j}) \in \mathrm{U}_{2n}$ and $b = (b_{i,j}) \in \mathrm{M}_{2n \times 2n}$. If we write $b = {}^t(r_1, \ldots, r_{2n})$ and $b = (c_1, \ldots, c_{2n})$ as row vectors and column vectors, respectively, then

$$bu = (c'_1, \ldots, c'_{2n}), \, ub = (r'_1, \ldots, r'_{2n}), \tag{4.30}$$

where

$$c'_j = \sum_{l \le j} c_l u_{lj}, \quad r'_j = \sum_{l \ge j} r_l u_{jl}. \tag{4.31}$$

Lemma 4.12. *Assume that $N_k x \mathcal{H}$ is admissible. Then $i_0 = 0$.*

Proof. If $i_0 \neq 0$, then $\iota_{k+1+i_0} = 2n - k$ by Lemma 4.8. Because $2n$ is the biggest integer between 1 and $2n$, we have either

$$\iota_{k_\omega} = 2n \text{ or } \iota_{k_\omega+1} = 2n. \tag{4.32}$$

We need to show that both cases are invalid. Same argument works for both cases and we will only show the proof for the case of $\iota_{k_\omega} = 2n$.

First, we claim that $\iota_{i_0} \neq 2n - k - 1$. If on the contrary $\iota_{i_0} = 2n - k - 1$, then by Lemma 4.9

$$A_{i_0,k+i_0} \neq 0. \tag{4.33}$$

Assume $A_{j,k_\omega} \neq 0$ for some j. Then $j > i_0$ by Lemma 4.10. Let

$$g = \mathrm{I}_{2n} + \alpha E_{k_\omega,k+i_0} + \beta E_{i_0,j} \tag{4.34}$$

where α and β are chosen such that $g \in \mathrm{Sp}_{2n}(A)$. This kind of g exists, because

$$A_{i_0,k+i_0} \neq 0,\ A_{j,k_\omega} \neq 0. \tag{4.35}$$

This trick is played often throughout the proof of non-admissibility. Since

$$\omega(e_{k_\omega} - e_{k+i_0}) = e_{2n} + e_{2n-k}, \quad \omega(e_{i_0} - e_j) = e_{2n-k-1} \pm e_{\iota_j}, \tag{4.36}$$

these two roots belong to Φ_k. Hence $\omega g \omega^{-1} \in N_k$ and for suitably chosen α

$$\psi_k(u\omega g\omega^{-1}u^{-1}) \neq 1 \tag{4.37}$$

by Remark 4.11. It contradicts to admissibility of $N_k x \mathcal{H}_A$. Hence $\iota_{i_0} \neq 2n - k - 1$ and

$$\iota_{i_0+k+2} = 2n - k - 1. \tag{4.38}$$

Next we use mathematical induction to show

$$\iota_{i_0+k+j} = 2n - k - j + 1, \text{ for } 1 \leq j \leq j_0. \tag{4.39}$$

We have shown the cases for $j = 1, 2$. Assume that Eq. 4.39 is correct for integers less than or equal to j. Then $A_{m,i_0+k+j-1} \neq 0$ for some $m > i_0$ by Lemma 4.10. Now, we assume on the contrary that $\iota_{i_0} = 2n - k - j$. Then by Lemma 4.9, $A_{i_0,2n-k-j+1} \neq 0$. Let

$$g = \mathrm{I}_{2n} + \alpha E_{i_0+k+j-1,i_0+k+j} + \beta E_{i_0,m} \tag{4.40}$$

where α and β are chosen such that $g \in \mathrm{Sp}_{2n}(A)$. Note that

$$\begin{aligned}\omega(e_{i_0+k+j-1} - e_{i_0+k+j}) = e_{2n-k-j+1} - e_{2n-k-j}, \omega(e_{i_0} - e_m)\\ = e_{2n-k-j} \pm e_{\iota_m}.\end{aligned} \tag{4.41}$$

Since $m > i_0$, these two roots belongs to Φ_k. For suitably chosen α,

$$\psi_k(u\omega g\omega^{-1}u^{-1}) \neq 1 \tag{4.42}$$

by Remark 4.11. This contradicts to the admissibility of $N_k x \mathcal{H}_A$ and Eq. 4.39 holds. It follows that $i_0 = 0$. □

We summarize that $i_0 = 0$ for admissible double cosets $N_k x \mathcal{H}_A$, and hence ω must be of the following form

$$\omega : e_{k+j} \mapsto -e_{2n-k-j+1}, \quad j = 1, \ldots, j_0. \tag{4.43}$$

4.2. Non-admissibility for the case of $k \leq n$

For $k \leq n$, we want to show that there are no admissible double cosets in $N_k\backslash \mathrm{SO}_{4n}/\mathcal{H}$. Define by $\eta = \eta_A$ a permutation on $\{1, \dots, 2n\}$ according to A such that

$$A_{t,\eta_t} \neq 0 \text{ for } 1 \leq t \leq 2n. \tag{4.44}$$

Lemma 4.13. *Let $k \leq n$. Assume that $N_k x\mathcal{H}$ is admissible. If $\eta_{t+1} \leq k$ for some $k+2 \leq t$, then $\eta_t \leq k$.*

Proof. Assume on the contrary that $\eta_t \geq k+1$. Consider

$$g = \mathrm{I}_{2n} + \alpha E_{t,t+1} + \beta E_{\eta_{t+1},\eta_t}, \tag{4.45}$$

where α and β are chosen such that $g \in \mathrm{Sp}_{2n}(A)$. Note that

$$\omega(e_t - e_{t+1}) = e_{2n-t} - e_{2n-t+1}, \omega(\eta_{t+1} - \eta_t) = e_{\iota_{\eta_t}} \pm e_{\iota_{\eta_{t+1}}}. \tag{4.46}$$

These two roots belongs to Φ_k, so $\omega g \omega^{-1} \in N_k$. By Remark 4.11, we see that $\psi_{\mathcal{H}_A}(g) = 1$ and

$$\psi_k(u\omega g\omega^{-1}u^{-1}) \neq 1 \text{ for suitably chosen } \alpha, \tag{4.47}$$

which contradicts to the admissibility. □

Lemma 4.14. *$k \leq n$. Assume that $N_k x\mathcal{H}$ is admissible and ω is in the form of (4.43). Let $\iota_m = 2n$. Then $A_{m,l} \neq 0$, for some $l \leq k$.*

Proof. Assume on the contrary that $A_{m,l} \neq 0$ for some $l > k$.

By the definition of ω, m equals either k_ω or $k_\omega + 1$. Same argument works for both cases and we will only show the proof for $m = k_\omega$. Recall that $\iota_{k+1} = 2n - k$. Suppose that $A_{j,k+1} \neq 0$ for some j. Then $j \neq m$. (Otherwise by taking $g = \mathrm{I}_{2n} + \alpha E_{k+1,m} \in \mathrm{Sp}_{2n}(A)$, $\psi_k(u\omega g\omega^{-1}u^{-1}) = \psi(\pm\alpha)$ and $\psi_{\mathcal{H}_A}(g) = 1$ will reach a contradiction.) Consider the element

$$g = \mathrm{I}_{2n} + \alpha E_{m,k+1} + \beta E_{j,l}, \tag{4.48}$$

where α and β are chosen such that $g \in \mathrm{Sp}_{2n}(A)$. Note that

$$\omega(e_m - e_{k+1}) = e_{2n-k} + e_{2n}, \omega(e_j - e_l) = \pm e_{\iota_j} + e_{\iota_l}. \tag{4.49}$$

The above two roots belong to Φ_k, so $\omega g\omega^{-1} \in N_k$. Unless $j = l-1 > k$, (i.e. $\omega(e_j - e_l) = e_{\iota_l} - e_{\iota_l+1}$) by Remark 4.11, we see that

$$\psi_k(u\omega g\omega^{-1}u^{-1}) \neq 1 \text{ for suitably chosen } \alpha. \tag{4.50}$$

If $j = l-1 > k$, then

$$\eta_t = l - t + k - 1 > k \text{ for } t \geq k$$

by similar arguments and mathematics induction. Hence A is in the form of

$$A = \begin{pmatrix} A_1 & \\ & A_0 \end{pmatrix},$$

where A_0 is a nonsingular anti-symmetric matrix of size $2n-k$. We reach a contradiction, since Whittaker models and $\mathrm{Sp}_{4n-2k}(A_0)$-models are disjoint [14]. It completes the proof. □

Proof of Proposition (3.3) Keep the same notations as before and let $k \leq n$. By Lemma 4.13, one of the following statements is true

(1) $\eta_{k+j} \leq k$ for all $1 \leq j \leq 2n-k$. By Lemma 4.14, there are at least $2n-k+1$ t's satisfying $\eta_t \leq k$, which contradicts to $2n-k+1 > k$.
(2) There exists some $1 \leq j_0 < 2n-k-1$ such that

$$\eta_{k+j} \leq k, \text{ for } 1 \leq j \leq j_0, \text{ and } \eta_{k+j} > k \text{ for } j \geq j_0+1.$$

That is

$$A = \begin{pmatrix} A_1 & \\ & A_0 \end{pmatrix} \tag{4.51}$$

for some A_1, A_0 nonsingular anti-symmetric matrices of size $k+j_0$ and $k_0 = 2n-k-j_0$, respectively.

Again, by the disjointness of Sp_{2k_0}-models and Whittaker models, $N_k x\mathcal{H}$ is not admissible. □

5. Local Langlands functorial transfer

We first recall briefly the local Langlands conjecture for $\mathrm{GL}_m(F)$, which is a Theorem of M. Harris and R. Taylor [15], and of G. Henniart [16]. Let $\mathcal{W}_F$ be the Weil group of F. A local Langlands parameter for $\mathrm{GL}_m(F)$ is a group homomorphism

$$\varphi : \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$$

such that the restriction of φ to $\mathcal{W}_F$ is continuous with respect to the topology of the Weil group and the topology of the complex Lie group $\mathrm{GL}_m(\mathbb{C})$ and the restriction of φ to $\mathrm{SL}_2(\mathbb{C})$ is algebraic. By the local Langlands conjecture for $\mathrm{GL}_m(F)$, the set of equivalence classes of all irreducible admissible representations of $\mathrm{GL}_m(F)$ is parametrized by the $\mathrm{GL}_m(\mathbb{C})$-conjugacy classes of all m-dimensional local Langlands parameters.

Let ι be the natural embedding of the complex symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ into $\mathrm{GL}_{2n}(\mathbb{C})$. Note that $\mathrm{Sp}_{2n}(\mathbb{C})$ is the complex dual group of $\mathrm{SO}_{2n+1}(F)$. For any irreducible admissible representation τ of $\mathrm{GL}_{2n}(F)$, let φ_τ be the local Langlands parameter for τ by the local Langlands conjecture for $\mathrm{GL}_{2n}(F)$. Then τ is a local Langlands functorial transfer from $\mathrm{SO}_{2n+1}(F)$ to $\mathrm{GL}_{2n}(F)$ if the local parameter φ_τ have its image in $\mathrm{Sp}_{2n}(\mathbb{C})$, i.e.,

$$\varphi_\tau(\mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C})) \subset \mathrm{Sp}_{2n}(\mathbb{C}).$$

In this case, an irreducible admissible representation π of $\mathrm{SO}_{2n+1}(F)$ was explicitly constructed in [26] and [27] in terms of the local parameter φ_τ, such that τ is the image of the local Langlands functorial transfer from π. In such a circumstance, the local exterior square L-function or the local exterior square gamma factor attached to τ plays crucial roles. Recall from Sect. 7, [41], the relation between these local factors are given by

$$\gamma(s, \pi, \Lambda^2, \psi) = \epsilon(s, \pi, \Lambda^2, \psi)\frac{L(1-s, \pi, \Lambda^2)}{L(s, \pi, \Lambda^2)},$$

where $\epsilon(s, \pi, \Lambda^2, \psi)$ is the exterior square local ϵ-factor of τ. It follows that the local exterior square L-function $L(s, \pi, \Lambda^2)$ has a pole at $s = 0$ if and only if the exterior square local gamma factor $\gamma(s, \pi, \Lambda^2, \psi)$ has a pole at $s = 1$.

Theorem 5.1. ([26]) *An irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(F)$ is a local Langlands functorial transfer from $\mathrm{SO}_{2n+1}(F)$ if and only if the exterior square local gamma factor*

$$\gamma(s, \tau, \Lambda^2, \psi)$$

has a pole at $s = 1$. In this case, τ must be self-dual.

5.1. Characterization in terms of Shalika models

We show that the local Langlands functorial transfer of τ can also be characterized in terms of the existence of a nonzero local Shalika model of τ.

In fact, by Theorem 2.2, if an irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(F)$ has a nonzero Shalika model, then the unitarily induced representation $\mathrm{I}(s, \tau)$ reduces at $s = 1$. By [41], $\mathrm{I}(s, \tau)$ may be reducible at either $s = 0$ or $s = 1$, but not both. By Corollary 7.6 of [41], $\mathrm{I}(s, \tau)$ reduces at $s = 1$ if and only if the local exterior square gamma factor

$$\gamma(s, \tau, \Lambda^2, \psi)$$

has a pole at $s = 1$. Then by Theorem 5.1, τ is a local Langlands functorial transfer from $\mathrm{SO}_{2n+1}(F)$.

Corollary 5.2. *An irreducible unitary supercuspidal representation τ of $\mathrm{GL}_{2n}(F)$ is a local Langlands functorial transfer from $\mathrm{SO}_{2n+1}(F)$ if τ has a nonzero Shalika model.*

Remark 5.3. Corollary 5.2 can be proved by a global argument. We sketch it here briefly. By [38], for an irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(F)$ with a nonzero Shalika model, there is a number field k and a local finite place v_0 of k such that $k_{v_0} = F$, and there exists an irreducible unitary cuspidal automorphic representation Σ of $\mathrm{GL}_{2n}(\mathbb{A})$ such that $\Sigma_{v_0} = \tau$ and Σ has a nonzero Shalika period. By [22], and also [23], Σ is a global Langlands functorial transfer from

SO_{2n+1} to GL_{2n}. Finally, by [27], the v_0-local component $\Sigma_{v_0} = \tau$ is a local Langlands functorial transfer from $SO_{2n+1}(F)$ to $GL_{2n}(F)$.

It seems to be a very hard problem to extend the result of [38] to the case when τ is not supercuspidal. However, our local argument seems more accessible when τ is not supercuspidal.

The converse of Corollary 5.2 is also true. We use a global argument, although a purely local argument is also expected.

We assume that an irreducible unitary supercuspidal representation τ of $GL_{2n}(F)$ is a local Langlands functorial transfer from $SO_{2n+1}(F)$. Then by [26], there exists an irreducible generic supercuspidal representation π of $SO_{2n+1}(F)$ such that τ is a local Langlands functorial transfer from π. Again, by [26], there exist a number field k and a local finite place v_0 of k such that $k_{v_0} = F$, and there exists an irreducible generic cuspidal automorphic representation Π of $SO_{2n+1}(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of k, such that $\Pi_{v_0} = \pi$. By [6], Π has a lift Σ to $GL_{2n}(\mathbb{A})$ under the global Langlands functorial transfer from $SO_{2n+1}(\mathbb{A})$ to $GL_{2n}(\mathbb{A})$. Because of the assumption of Π at the local place v_0, Σ must be an irreducible self-dual cuspidal automorphic representation of $GL_{2n}(\mathbb{A})$ with the property that the partial exterior square L-function $L^S(s, \Sigma, \Lambda^2)$ has a pole at $s = 1$. By [27], we must have that $\Sigma_{v_0} = \tau$. It follows from [22] (see also [23]), Σ has a nonzero global Shalika model. Hence the v_0-local component $\Sigma_{v_0} = \tau$ must have a nonzero local Shalika model.

Proposition 5.4. *If an irreducible unitary supercuspidal representation τ of $GL_{2n}(F)$ is a local Langlands functorial transfer from $SO_{2n+1}(F)$, then τ has a nonzero Shalika model.*

From the discussions in previous sections, we state the following theorem which characterizes the local Langlands transfer property from various aspects.

Theorem 5.5. *Let τ be an irreducible supercuspidal representation of $GL_{2n}(F)$. Then the following are equivalent.*

(1) *τ has a nonzero Shalika model.*
(2) *The local exterior square L-factor $L(s, \tau, \Lambda^2)$ has a pole at $s = 0$.*
(3) *The local exterior square γ-factor $\gamma(s, \tau, \Lambda^2, \psi)$ has a pole at $s = 1$.*
(4) *The unitarily induced representation $I(1, \tau)$ of $SO_{4n}(F)$ is reducible.*
(5) *τ is a local Langlands functorial transfer from $SO_{2n+1}(F)$.*

If one of the above holds for τ, then τ is self-dual.

5.2. Linear models

Following the work of D. Bump and S. Friedberg [4], S. Friedberg and H. Jacquet found [7] the connection between the linear period and the Shalika period for cuspidal automorphic representations of $GL_{2n}(\mathbb{A})$. More precisely, for an irreducible cuspidal automorphic representation Σ of $GL_{2n}(\mathbb{A})$, Σ has a nonzero Shalika period if and only if Σ has a nonzero linear period and the central value of the standard L-function attached to Σ is nonzero.

In [20], H. Jacquet and S. Rallis proved the local uniqueness of linear models, and proved that for an irreducible admissible representation τ of $\mathrm{GL}_{2n}(F)$, if τ has a nonzero Shalika model, then τ has a nonzero linear model. In particular, the local Shalika model is unique. The linear model is defined below.

Let $\mathcal{L}_n$ be the Levi subgroup of standard parabolic subgroup $P_{n,n}$, which is isomorphic to $\mathrm{GL}_n \times \mathrm{GL}_n$. For an irreducible admissible representation (τ, V_τ) of $\mathrm{GL}_{2n}(F)$, we say τ has a nonzero $\mathcal{L}_n$-*functional* if the following space

$$\mathrm{Hom}_{\mathcal{L}_n(F)}(V_\tau, 1) \neq 0, \tag{5.1}$$

where 1 is the trivial representation of $\mathcal{L}_n(F)$. By reciprocity, a nonzero $\mathcal{L}_n$-functional is equivalent to a nontrivial embedding of V_τ in $\mathrm{Ind}_{\mathcal{L}_n(F)}^{\mathrm{GL}_{2n}(F)} 1$, which is called a *linear model* for V_τ.

Theorem 5.6. *Let τ be an irreducible supercuspidal representation of $\mathrm{GL}_{2n}(F)$, τ has a nonzero linear model if and only if τ has a nonzero Shalika model.*

It is enough to show that if τ has a nonzero linear model, then it has a nonzero Shalika model. To this end, we consider the unitarily induced representation of $\mathrm{Sp}_{4n}(F)$, which is induced from the Siegel parabolic subgroup $Q = \mathrm{GL}_{2n} \cdot V$:

$$\mathrm{I}^{\mathrm{Sp}_{4n}}(s, \tau) := \mathrm{Ind}_{Q(F)}^{\mathrm{Sp}_{4n}(F)}(\tau \otimes |\det|^s). \tag{5.2}$$

The point is to show

Proposition 5.7. *If an irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(F)$ has a nonzero linear model, then the unitarily induced representation $\mathrm{I}^{\mathrm{Sp}_{4n}}(s, \tau)$ reduces at $s = \frac{1}{2}$.*

The global version of Proposition 5.7 is given in [12] as for the case of Shalika periods given in [25]. The global argument in [12] provides a local argument for the proof of Proposition 5.7, just as the global argument in [25] relates to the proof of Theorem 2.1. We omit the detail here.

Now we prove Theorem 5.6. Assume that an irreducible supercuspidal representation τ of $\mathrm{GL}_{2n}(F)$ has a nonzero linear model, then by Proposition 5.7, the induced representation $\mathrm{I}^{\mathrm{Sp}_{4n}}(s, \tau)$ reduces at $s = \frac{1}{2}$. By Theorem 5.3 in [42], the reducibility of $\mathrm{I}^{\mathrm{Sp}_{4n}}(s, \tau)$ at $s = \frac{1}{2}$ is equivalent to the reducibility of $\mathrm{I}^{\mathrm{SO}_{4n}}(s, \tau)$ at $s = 1$. Note here that $\mathrm{I}^{\mathrm{SO}_{4n}}(s, \tau)$ is the same as $\mathrm{I}(s, \tau)$ in Theorem 5.5 for $\mathrm{SO}_{4n}(F)$. Hence by Theorem 5.5, τ has a nonzero Shalika model. The converse was proved by Jacquet and Rallis in Sect. 6 of [20].

We remark that the local Shalika model is expected to control the construction of the local Ginzburg–Rallis–Soudry descent from $\mathrm{GL}_{2n}(F)$ to $\mathrm{SO}_{2n+1}(F)$, while the local linear model is expected to control the local Ginzburg–Rallis–Soudry descent from $\mathrm{GL}_{2n}(F)$ to the metaplectic double cover of $\mathrm{Sp}_{2n}(F)$ (see [12] for the global case by using linear period). The relation between these two descents are given by the local Howe duality and the local converse theorem for $\mathrm{SO}_{2n+1}(F)$ [26]. This is our on-going working project. The results will be given in our forthcoming work.

6. Three applications

We discuss applications of the local theory developed in previous sections to the theory of automorphic forms.

6.1. The Jacquet–Langlands correspondence: Shalika periods

Let k be a number field and $\mathbb{A}$ the ring of adeles of k. We discuss the first global application of Theorem 5.5.

Let D be a division k-algebra of degree n. Then $G' = \mathrm{GL}_2(D)$ and $G = \mathrm{GL}_{2n}(k)$ are inner k-forms as algebraic k-groups. The Shalika subgroup $\mathcal{S}_n$ and the nonsplit Shalika subgroup $\mathcal{S}_D$ were defined in (2.1) and (2.3), respectively. For a nontrivial character ψ of $k\backslash\mathbb{A}$, we define a one-dimensional representation

$$\theta_n\left(\begin{pmatrix} g & \\ & g \end{pmatrix}\begin{pmatrix} \mathrm{I}_n & X \\ 0 & \mathrm{I}_n \end{pmatrix}\right) = \psi(\mathrm{tr}_{\mathrm{M}_{n\times n}/k}(X)),\ g \in \mathrm{GL}_n,\ X \in \mathrm{M}_{n\times n} \tag{6.1}$$

of $\mathcal{S}_n(\mathbb{A})$, which is trivial on $\mathcal{S}_n(k)$. Also we define a one-dimensional representation

$$\theta_D\left(\begin{pmatrix} a & \\ & a \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \psi(\mathrm{tr}_{D/k}(x)),\ a \in D^*,\ x \in D \tag{6.2}$$

of $\mathcal{S}_D(\mathbb{A})$, which is trivial on $\mathcal{S}_D(k)$. If π is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$, we say that π is $(\mathcal{S}_n, \theta_n)$-distinguished, or has a nonzero Shalika model if the following integral

$$\int_{\mathcal{S}_n(k)\backslash\mathcal{S}_n(\mathbb{A})} \phi_\pi(s)\theta_n^{-1}(s)ds \tag{6.3}$$

is nonzero for some $\phi_\pi \in V_\pi$. Let π^D be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(D)(\mathbb{A})$. We say that π^D is $(\mathcal{S}_D, \theta_D)$-distinguished, or has a nonzero (non-split) Shalika model if the following integral

$$\int_{\mathcal{S}_D(k)\backslash\mathcal{S}_D(\mathbb{A})} \phi_{\pi^D}(s)\theta_D^{-1}(s)ds \tag{6.4}$$

is nonzero for some $\phi_{\pi^D} \in V_{\pi^D}$.

By the work of Arthur and Clozel ([1] and more recent work of Badulescu ([2]), the global Jacquet–Langlands correspondence holds for $\mathrm{GL}_{2n}(\mathbb{A})$ and $\mathrm{GL}_2(D)(\mathbb{A})$. As an application, we prove the following global result.

Theorem 6.1. *Let D be a division k-algebra of degree n. Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ and π^D be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(D)(\mathbb{A})$. Assume that π and π^D are paired up by the global Jacquet–Langlands correspondence between $\mathrm{GL}_{2n}(\mathbb{A})$ and $\mathrm{GL}_2(D)(\mathbb{A})$. Assume that there exists a finite local place v_0 of k such that D splits at v_0 and $\pi^D_{v_0} = \pi_{v_0}$ is supercuspidal. If π^D is $(\mathcal{S}_D, \theta_D)$-distinguished, then π is $(\mathcal{S}_n, \theta_n)$-distinguished.*

Proof. Let π^D be a $(\mathcal{S}_D, \theta_D)$-distinguished irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(D)(\mathbb{A})$. By the assumption, there is a finite local place v_0 such that π_{v_0} is supercuspidal and D_{v_0} is split; and also there is an irreducible unitary cuspidal automorphic representation π of $\mathrm{GL}_{2n}(\mathbb{A})$ such that $\pi^D_{v_0} = \pi_{v_0}$. By the global Jacquet–Langlands correspondence, we know that

$$\pi_v \cong \pi^D_v$$

for almost all local places v of k, where $\mathrm{GL}_2(D)(k_v) \cong \mathrm{GL}_{2n}(k_v)$.

Since π^D is $(\mathcal{S}_D, \theta_D)$-distinguished, one knows that at each local place v, the local component π^D_v has a nonzero Shalika model of type $(\mathcal{S}_{D,v}, \theta_{D,v})$. This means that the following space

$$\mathrm{Hom}_{\mathrm{GL}_2(D)(k_v)}\left(V_{\pi^D_v}, \mathrm{Ind}^{\mathrm{GL}_2(D)(k_v)}_{\mathcal{S}_D(k_v)}(\theta_{D,v})\right) = \mathrm{Hom}_{\mathcal{S}_D(k_v)}(V_{\pi^D_v}, \theta_{D,v}) \quad (6.5)$$

is nonzero. By the local uniqueness of Shalika models of type $(\mathcal{S}_{D,v}, \theta_{D,v})$, the dimension of the space in Eq. (4.7) is one ([20] and [37], and also [36]).

At the finite local places v where $\mathrm{GL}_2(D)(k_v) \cong \mathrm{GL}_{2n}(k_v)$ and π^D_v is spherical (or unramified), that $\pi^D_v = \pi_v$ has a nonzero local Shalika model of type

$$(\mathcal{S}_{D,v}, \theta_{D,v}) = (\mathcal{S}_{n,v}, \theta_{n,v})$$

implies that $\pi^D_v = \pi_v$ is self-dual. Hence the image of π^D under the global Jacquet–Langlands correspondence, π is locally self-dual at almost all local places. This implies π is self-dual as an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ by the strong multiplicity one theorem for irreducible cuspidal automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A})$ ([40] and [21]). Hence π is either of symplectic type, i.e. the partial exterior square L-function $L^S(s, \pi, \Lambda^2)$ has a simple pole at $s = 1$, or of orthogonal type, i.e. the partial symmetric square $L^S(s, \pi, S^2)$ has a simple pole at $s = 1$.

We claim that π can not be of orthogonal type. Assume that π is of orthogonal type. By [44], there exists an irreducible unitary generic cuspidal automorphic representation σ of $\mathrm{SO}_{2n}(\mathbb{A})$ such that π is the image of σ under the weak Langlands functorial transfer from SO_{2n} to GL_{2n}. Then by [6], the Langlands functorial transfer from σ to π is compatible with the local Langlands functorial transfer at every local place. In particular, at the finite local place v_0, the irreducible supercuspidal representation $\pi^D_{v_0} = \pi_{v_0}$ is the image of σ_{v_0} under the local Langlands functorial transfer from SO_{2n} to GL_{2n}. By Theorem 7.3 of [6], the symmetric square local L-function $L(s, \pi_{v_0}, S^2)$ has a simple pole at $s = 0$.

On the other hand, the irreducible supercuspidal representation $\pi^D_{v_0} = \pi_{v_0}$ of $\mathrm{GL}_{2n}(k_{v_0})$ has a nonzero Shalika model of type $(\mathcal{S}_{n,v_0}, \theta_{n,v_0})$. By Theorem 3.3, the local exterior square L-function $L(s, \pi_{v_0}, \Lambda^2)$ has a simple pole at $s = 0$. Consider the following identity

$$L(s, \pi_{v_0} \times \pi_{v_0}) = L(s, \pi_{v_0}, \Lambda^2) \cdot L(s, \pi_{v_0}, S^2), \quad (6.6)$$

which implies the local Rankin–Selberg convolution L-function $L(s, \pi_{v_0} \times \pi_{v_0})$ has a pole at $s = 0$ of order two. This is impossible, since $L(s, \pi_{v_0} \times \pi_{v_0})$ has at most a simple pole at $s = 0$ [17]. This proves our claim above.

Therefore, π must be of symplectic type, i.e. the partial exterior square L-function $L^S(s, \pi, \Lambda^2)$ has a simple pole at $s = 1$. By [22] (see also [23]), the irreducible cuspidal automorphic representation π is $(\mathcal{S}_n, \theta_n)$-distinguished. This proves the theorem. □

More generally, Jacquet and Martin made the following conjecture.

Conjecture 6.2. (Jacquet–Martin [32]) Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ and π^D be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(D)(\mathbb{A})$. Assume that π and π^D are paired up by the global Jacquet–Langlands correspondence between $\mathrm{GL}_{2n}(\mathbb{A})$ and $\mathrm{GL}_2(D)(\mathbb{A})$. Then π has a nonzero Shalika model if and only if π^D has a nonzero nonsplit Shalika model.

It is clear that the assumption that there exists a finite local place v_0 of k such that D splits at v_0 and $\pi^D_{v_0} = \pi_{v_0}$ is supercuspidal can be relaxed after a version of Theorem 5.5 is extended to general irreducible admissible representations of $\mathrm{GL}_{2n}(F)$.

6.2. Poles of certain Eisenstein series

As another global application of Theorem 5.5, we show that the local reducibility implies the existence of the pole of certain Eisenstein series, and hence confirm in this case Conjecture 8.3 of [29], which is an important ingredient to understand the residual spectrum of the space of automorphic forms.

We first recall from [29] a general conjecture which relates the local reducibility at one local place to the existence of the pole of the relevant Eisenstein series.

Let G be a quasi-split reductive algebraic group defined over a number field k, and M is the Levi subgroup of a standard maximal parabolic k-subgroup $P = MN$ of G. Let π be an irreducible unitary cuspidal automorphic representation of $M(\mathbb{A})$. Denote by $E(g; \phi_\pi, s)$ the Eisenstein series attached to the cuspidal datum (P, π) [34]. By the Langlands theory of Eisenstein series, $E(g; \phi_\pi, s)$ has meromorphic continuation to the whole complex plane $\mathbb{C}$. In order to understand the noncuspidal discrete spectrum of the space of automorphic forms on $G(\mathbb{A})$, it is important to determine when an Eisenstein series has a square integrable residue at $s > 0$ in terms of the cuspidal datum (P, π).

Let v_0 be a finite local place of k, and τ be an irreducible supercuspidal representation of $M(k_{v_0})$. Denote by $\mathrm{I}(s, \tau)$ the normalized induced representation of $G(k_{v_0})$ from the supercuspidal datum (P, τ).

Conjecture 6.3. (Conjecture 8.3 [29]) With notation as above, if the unitarily induced representation $\mathrm{I}(s, \tau)$ of $G(k_{v_0})$ is reducible at $s = s_0 > 0$, then there exists an irreducible unitary cuspidal automorphic representation π of $M(\mathbb{A})$ such that $\pi_{v_0} = \tau$ and the Eisenstein series $E(g; \phi_\pi, s)$ has a pole at $s = s_0$.

We prove this conjecture for the case that $G = \mathrm{SO}_{4n}$ and $P = MN$ is the Siegel parabolic subgroup of G. In this case, τ is an irreducible unitary supercuspidal representation of $M(k_{v_0}) = \mathrm{GL}_{2n}(k_{v_0})$. Hence the only possible positive reducibility point of $\mathrm{I}^{\mathrm{SO}_{4n}}(s, \tau)$ is at $s = 1$.

Theorem 6.4. *Let τ be an irreducible unitary supercuspidal representation of* $\mathrm{GL}_{2n}(k_{v_0})$ *such that the unitarily induced representation* $\mathrm{I}^{\mathrm{SO}_{4n}}(s,\tau)$ *reduces at* $s=1$. *Then there exists an irreducible unitary cuspidal automorphic representation π of* $\mathrm{GL}_{2n}(\mathbb{A})$ *such that $\pi_{v_0}=\tau$ and the Eisenstein series $E(g,\phi_\pi,s)$ has a pole at $s=1$.*

The proof goes as follows.

First, by Theorem 5.5, τ admits a nonzero Shalika model if $\mathrm{I}^{\mathrm{SO}_{4n}}(s,\tau)$ reduces at $s=1$ by the assumption. Then by a theorem of [38], there exists an irreducible unitary cuspidal automorphic representation π of $\mathrm{GL}_{2n}(\mathbb{A})$ such that $\pi_{v_0}=\tau$ and π has a nonzero Shalika period, which produces the local Shalika model for $\pi_{v_0}=\tau$. By the global result of [25], the residue at $s=1$ of the Eisenstein series $E(g,\phi_\pi,s)$ has a nonzero generalized Shalika model over $\mathrm{SO}_{4n}(\mathbb{A})$. In particular, the Eisenstein series $E(g,\phi_\pi,s)$ has a pole at $s=1$. This completes the proof.

The same result holds for Sp_{4n} in terms of the linear periods. We state the result here, without giving any further details.

Theorem 6.5. *Let τ be an irreducible unitary supercuspidal representation of* $\mathrm{GL}_{2n}(k_{v_0})$ *such that the unitarily induced representation* $\mathrm{I}^{\mathrm{Sp}_{4n}}(s,\tau)$ *reduces at* $s=\frac{1}{2}$. *Then there exists an irreducible unitary cuspidal automorphic representation π of* $\mathrm{GL}_{2n}(\mathbb{A})$ *such that $\pi_{v_0}=\tau$ and the Eisenstein series $E(g,\phi_\pi,s)$ on* $\mathrm{Sp}_{4n}(\mathbb{A})$ *has a pole at $s=\frac{1}{2}$.*

6.3. Uniform local reducibility

We first state a general conjecture of B. Speh about uniform local reducibility of unitarily induced representations.

Let G be a quasi-split reductive algebraic group defined over a number field k, and let $P=MN$ be a maximal parabolic k-subgroup of G. Let π be an irreducible unitary cuspidal automorphic representation of $M(\mathbb{A})$. Assume that the central character of π is trivial on $A_{M(\mathbb{A})}$ (see [34] for the definition of this notation). Following [34], we define an Eisenstein series $E(g,\phi_\pi,s)$ attached to the cuspidal datum (P,π). It can also be built from a smooth section in the unitarily induced representation $\mathrm{I}(s,\pi)$ (see Sect. 6.2 for a special case).

Conjecture 6.6. (Speh) For $0<s_0\in\mathbb{R}$, if the pole at $s=s_0$ of $E(g,\phi_\pi,s)$ exists, then for all local place v of k, the unitarily induced representation $\mathrm{I}(s,\pi_v)$ of $G(k_v)$ reduces at $s=s_0$.

We show that special cases of Conjecture 6.6 holds.

Theorem 6.7. *Let π be an irreducible unitary cuspidal automorphic representation of* $\mathrm{GL}_{2n}(\mathbb{A})$. *Assume that the pole at $s=1$ of the Eisenstein series $E(g,\phi_\pi,s)$, as in Theorem 6.4, exists. Then for all finite local places v of k, the unitarily induced representation* $\mathrm{I}^{\mathrm{SO}_{4n}}(s,\pi_v)$ *reduces at $s=1$.*

Proof. If the pole at $s=1$ of $E(g,\phi_\pi,s)$ exists, then by computing the constant term of $E(g,\phi_\pi,s)$ along P, we deduce that the exterior square L-function

$L(s, \pi, \Lambda^2)$ has a pole at $s = 1$. By [22] and [23], π has a nonzero Shalika model. By [25], the residue

$$\mathrm{Res}_{s=1} E(g, \phi_\pi, s)$$

has a nonzero generalized Shalika model. Hence for every local place v of k, the unitarily induced representation $\mathrm{I}(s, \pi_v)$ has a nonzero local generalized Shalika model.

On the other hand, for v finite, if $\mathrm{I}^{\mathrm{SO}_{4n}}(s, \pi_v)$ is irreducible, then $\mathrm{I}^{\mathrm{SO}_{4n}}(s, \pi_v)$ has a nonzero Whittaker model. This is impossible by Proposition 2.3.

Hence $\mathrm{I}^{\mathrm{SO}_{4n}}(s, \pi_v)$ must be reducible at $s = 1$ at all finite local places. □

Remark 6.8. Under the assumption of Theorem 6.7, we expect that $\mathrm{I}^{\mathrm{SO}_{4n}}(s, \pi_v)$ is also reducible at $s = 1$ at any infinite local place of k. But we omit further discussions here.

Similarly, by using linear period, we can prove

Theorem 6.9. *Let π be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. Assume that the pole at $s = \frac{1}{2}$ of the Eisenstein series $E(g, \phi_\pi, s)$ on $\mathrm{Sp}_{4n}(\mathbb{A})$, as in Theorem 6.5, exists. Then for all finite local places v of k, the unitarily induced representation $\mathrm{I}^{\mathrm{Sp}_{4n}}(s, \pi_v)$ reduces at $s = \frac{1}{2}$.*

We omit the details of the proof here.

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