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On the vanishing and the finiteness of supports of generalized local cohomology modules

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring, I an ideal of R and M, N two finitely generated R-modules. The first result of this paper is to prove a vanishing theorem for generalized local cohomology modules which says that $H^j_I(M, N) = 0$ for all $j > \dim(R)$, provided M is of finite projective dimension. Next, we study and give characterizations for the least and the last integer r such that $\mathrm{Supp}(H^r_I(M, N))$ is infinite.

1. Introduction

For an integer $j \geq 0$, the jth generalized local cohomology module $H^j_I(M, N)$ of two R-modules M and N with respect to an ideal I was defined by Herzog [6] as follows:

$$H^j_I(M, N) = \varinjlim_n \mathrm{Ext}^j_R(M/I^n M, N).$$

It is clear that $H^j_I(R, N)$ is just the ordinary local cohomology module $H^j_I(N)$ of N with respect to I. Huneke [8] conjectured that the set of associated primes of $H^j_I(N)$ is finite for all generated modules N and all ideals I. Although Katzman [11] constructed a counter example for this conjecture, the conjecture is still true in many situations (see [9,12–14]). Therefore it is an important problem in the theory of local cohomology to study the question of when the sets $\mathrm{Ass}(H^j_I(N))$ and $\mathrm{Supp}(H^j_I(N))$ are finite. The purpose of this paper is to investigate a similar question as above for the theory of generalized local cohomology.

It should be mentioned here that some basic properties of local cohomology modules cannot extend to generalized local cohomology modules. For example, if N is I-torsion then $H^i_I(N) = 0$ for all $i > 0$, but $H^i_I(M, N) \cong \mathrm{Ext}^i_R(M, N)$ and the later does not vanish in general for $i > 0$; or while the Grothendieck's Vanishing Theorem says that $H^i_I(N) = 0$ for all $i > \dim(N)$, the generalized local cohomology modules $H^i_I(M, N)$ may not vanish in general for infinitely

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many $i \geq 0$. However, we can show in this paper that $\bigcup_{j \leq i} \mathrm{Supp}(H_I^j(M, N)) = \bigcup_{j \leq i} \mathrm{Supp}(\mathrm{Ext}_R^j(M/IM, N))$ for all $i \geq 0$ (Lemma 2.8). It follows that although $H_I^i(M, N)$ may not vanish, we still have $\mathrm{Supp}(H_I^i(M, N)) \subseteq \bigcup_{j \leq \dim(N)} \mathrm{Supp}(H_{I_M}^j(N))$ for all $i \geq 0$, where $I_M = \mathrm{ann}(M/IM)$ is the annihilator of the R-module M/IM. Moreover, we also prove that if M has finite projective dimension then $H_I^j(M, N) = 0$ for all $j > \dim(R)$ (Theorem 3.1). Then we can exploit Lemma 2.8 and Theorem 3.1 in the studying the finiteness of the set of associated primes as well as of the support of generalized local cohomology modules.

Our paper is divided into five sections. In Sect. 2, we prove two auxiliary lemmas (Lemmas 2.7 and 2.8) and its consequence (Corollary 2.9) on the support of generalized local cohomology modules. In Sect. 3, by using spectral sequences, we prove that $H_I^j(M, N) = 0$ for all $j > \dim(R)$, provided M is of finite projective dimension (Theorem 3.1). This generalizes a vanishing result of generalized local cohomology modules with respect to the maximal ideal of Herzog and Zamani [7, Theorem 3.2]. In Sect. 4, we use Lemma 2.8 and the notion of generalized regular sequences introduced by Nhan [14] to characterize the least integer r such that $\mathrm{Supp}(H_I^r(M, N))$ is an infinite set (Theorem 4.1); from this we can describe concretely the finiteness of $\mathrm{Ass}(H_I^r(M, N))$ (Theorem 4.5). In the last section, we study the last integer s such that $\mathrm{Supp}(H_I^s(M, N))$ is an infinite set (Theorem 5.1(a)); and we also give lower and upper bounds for this s (Theorem 5.1(b)).

2. Preliminaries

Throughout this paper M, N are finitely generated modules over a Noetherian local ring $(R, \mathfrak{m})$. Let $\mathrm{pd}_R(M)$ denote the projective dimension of M. For any ideal I of R we denote by $I_M = \mathrm{ann}_R(M/IM)$ the annihilator of the module M/IM and by Γ_I the I-torsion functor. First, we recall some known facts on generalized local cohomology modules.

Lemma 2.1. (cf. [4, Lemmas 2.1, 2.3]) *The following statements are true.*

(i) *Let $E^\bullet$ be an injective resolution of N. Then, for any $j \geq 0$, we have*

$$H_I^j(M, N) \cong H^j(\Gamma_I(\mathrm{Hom}(M, E^\bullet))) \cong H^j(\mathrm{Hom}(M, \Gamma_I(E^\bullet))) \cong H^j(\mathrm{Hom}(M, \Gamma_{I_M}(E^\bullet))).$$

(ii) *If $\Gamma_{I_M}(N) = N$ or $I \subseteq \mathrm{ann}(M)$, then $H_I^j(M, N) \cong \mathrm{Ext}_R^j(M, N)$ for all $j \geq 0$.*

Lemma 2.2. (cf. [4, Theorem 2.4]) *Let $l = \mathrm{depth}(I_M, N)$. Then*

$$\mathrm{Ass}\, H_I^l(M, N) = \mathrm{Ass}\, \mathrm{Ext}_R^l(M/IM, N).$$

Lemma 2.3. (cf. [18, Theorem 3.7]) *If $\mathrm{pd}_R(M) < +\infty$, then $H_I^j(M, N) = 0$ for all $j > \mathrm{pd}_R(M) + \dim(M \otimes_R N)$.*

Lemma 2.4. (cf. [7, Lemma 3.1]) *Let* $d = \dim(R)$. *If* $\mathrm{pd}_R(M) < +\infty$, *then we have* $\dim(\mathrm{Ext}^j_R(M, R)) \le d - j$ *for all* $0 \le j \le \mathrm{pd}_R(M)$.

Lemma 2.5. *Assume that the local ring homomorphism* $f : R \to S$ *is flat. Then* $H^j_I(M, N) \otimes_R S \cong H^j_{IS}(M \otimes_R S, N \otimes_R S)$ *for all* $j \ge 0$.

Lemma 2.6. *Let* $n = \dim(N)$. *Then* $\mathrm{Supp}(H^{n-1}_I(N))$ *is a finite set.*

Proof. Let $\mathfrak{a} = \mathrm{ann}_R(N)$ and $\overline{R} = R/\mathfrak{a}$, then $\dim(\overline{R}) = n$ and N is an $\overline{R}$-module. Hence, by the independence theorem in [2], we have $H^{n-1}_I(N) \cong H^{n-1}_{I\overline{R}}(N)$ as $\overline{R}$-modules. Since $\mathrm{Supp}_{\overline{R}}(H^{n-1}_{I\overline{R}}(N))$ is finite by Marley [13, Corollary 2.5] and $\mathrm{Supp}(H^{n-1}_I(N)) \subseteq \mathrm{Supp}(R/\mathfrak{a})$, $\mathrm{Supp}(H^{n-1}_I(N))$ is finite as required. □

The next two lemmata are important for our further investigations in this paper.

Lemma 2.7. *Let* $\mathbb{N}$ *be the set of all positive integers and* $i \in \mathbb{N} \cup \{+\infty\}$. *Set* $J_i = \bigcap_{j<i} \mathrm{ann}(\mathrm{Ext}^j_R(M/IM, N))$. *Then* $H^j_I(M, N) \cong H^j_{J_i}(M, N)$ *for all* $j < i$.

Proof. We note first that $I_M \subseteq J_i$. Let $E^\bullet : 0 \to E^0 \to \cdots \to E^j \to \cdots$ be a minimal injective resolution of N. For any $j \ge 0$, we have by Brodmann and Sharp [2, 10.1.10] that

$$\Gamma_{I_M}(E^j) = \bigoplus_{I_M \subseteq \mathfrak{p} \in \mathrm{Ass}(E^j)} E(R/\mathfrak{p})^{\mu^j(\mathfrak{p}, N)}$$

and

$$\Gamma_{J_i}(E^j) = \bigoplus_{J_i \subseteq \mathfrak{p} \in \mathrm{Ass}(E^j)} E(R/\mathfrak{p})^{\mu^j(\mathfrak{p}, N)},$$

where $\mu^j(\mathfrak{p}, N) = \dim_{k(\mathfrak{p})}(\mathrm{Ext}^j_{R_\mathfrak{p}}(k(\mathfrak{p}), N_\mathfrak{p}))$ is the jth Bass number of N with respect to $\mathfrak{p}$. Since the sequence $0 \to E^0_\mathfrak{p} \to E^1_\mathfrak{p} \to \cdots \to E^j_\mathfrak{p} \to \cdots$ is a minimal injective resolution of $N_\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Ass}(E^j)$ by Brodmann and Sharp [2, 11.1.6], $N_\mathfrak{p} \ne 0$. We consider now two cases.

Firstly, let $i \in \mathbb{N}$. For any $j < i$ and any $\mathfrak{p} \in \mathrm{Ass}(E^j)$ satisfying $\mathfrak{p} \supseteq I_M$ and $\mathfrak{p} \not\supseteq J_i$, we have $\mathrm{Ext}^l_R(M/IM, N)_\mathfrak{p} = 0$ for all $l < i$. It implies $\mathrm{depth}((I_M)_\mathfrak{p}, N_\mathfrak{p}) \ge i$, and so $\mathrm{depth}(N_\mathfrak{p}) \ge i$. Thus $\mu^j(\mathfrak{p}, N) = 0$, so that $\Gamma_{I_M}(E^j) = \Gamma_{J_i}(E^j)$. Hence we get $H^j_I(M, N) \cong H^j_{J_i}(M, N)$ for all $j < i$ by Lemma 2.1.

Secondly, if $i = +\infty$, then $J_i = \bigcap_{j \ge 0} \mathrm{ann}(\mathrm{Ext}^j_R(M/IM, N))$. For any $j \ge 0$, and any $\mathfrak{p} \in \mathrm{Ass}(E^j)$ such that $\mathfrak{p} \supseteq I_M$, we obtain $(I_M)_\mathfrak{p} N_\mathfrak{p} \ne N_\mathfrak{p}$. Set $\nu = \mathrm{depth}((I_M)_\mathfrak{p}, N_\mathfrak{p})$. Then $\nu < +\infty$ and $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Ext}^\nu_R(M/IM, N))$. It follows that $\mathfrak{p} \supseteq \mathrm{ann}(\mathrm{Ext}^\nu_R(M/IM, N)) \supseteq J_i$, and hence $\Gamma_{I_M}(E^j) = \Gamma_{J_i}(E^j)$. Therefore we obtain $H^j_I(M, N) \cong H^j_{J_i}(M, N)$ for all $j \ge 0$ by Lemma 2.1 again. □

Lemma 2.8. *Let* $i \in \mathbb{N} \cup \{+\infty\}$. *Then we have*

$$\bigcup_{j<i} \mathrm{Supp}(H^j_I(M, N)) = \bigcup_{j<i} \mathrm{Supp}(\mathrm{Ext}^j_R(M/IM, N)).$$

Proof. Let $i \in \mathbb{N} \cup \{+\infty\}$, and $J_i = \bigcap_{j<i} \operatorname{ann}(\operatorname{Ext}^j_R(M/IM, N))$. Then, by Lemma 2.7, we obtain $H^j_I(M, N) \cong H^j_{J_i}(M, N)$ for all $j < i$. Therefore

$$\bigcup_{j<i} \operatorname{Supp}(H^j_I(M, N)) \subseteq \operatorname{Supp}(R/J_i) = \bigcup_{j<i} \operatorname{Supp}(\operatorname{Ext}^j_R(M/IM, N)).$$

Conversely, let $\mathfrak{p} \in \bigcup_{j<i} \operatorname{Supp}(\operatorname{Ext}^j_R(M/IM, N))$. Set $\nu = \operatorname{depth}((I_M)_{\mathfrak{p}}, N_{\mathfrak{p}})$, then $\nu < i$. For each $n > 0$, since $I_M \subseteq \sqrt{\operatorname{ann}(IM/I^nM)}$, $\operatorname{Ext}^j_R(IM/I^nM, N)_{\mathfrak{p}} = 0$ for all $j < \nu$. Thus, from the exact sequence

$$0 \to IM/I^nM \to M/I^nM \to M/IM \to 0$$

we get the following exact sequence

$$0 \to \operatorname{Ext}^{\nu}_R(M/IM, N)_{\mathfrak{p}} \to \operatorname{Ext}^{\nu}_R(M/I^nM, N)_{\mathfrak{p}}$$

for all $n > 0$. This induces an exact sequence

$$0 \to \operatorname{Ext}^{\nu}_R(M/IM, N)_{\mathfrak{p}} \to H^{\nu}_I(M, N)_{\mathfrak{p}}.$$

Since $\nu < i$ and $\operatorname{Ext}^{\nu}_R(M/IM, N)_{\mathfrak{p}} \neq 0$ it follows that $\mathfrak{p} \in \bigcup_{j<i} \operatorname{Supp}(H^j_I(M, N))$ as required. □

Until now one does not know about the last integer i such that $H^i_I(M, N) \neq 0$, even for the case $I = \mathfrak{m}$. For example, assume that R is not regular local ring and $\operatorname{pd}_R(M) = +\infty$, then $H^i_{\mathfrak{m}}(M, R/\mathfrak{m}) = \operatorname{Ext}^i_R(M, R/\mathfrak{m}) \neq 0$ for all $i \geq 0$ (see Suzuki [16, Lemma 3.1]). However, the following result shows that there is a union of only finitely many supports of generalized local cohomology modules so that the other supports can be viewed as its subsets.

Corollary 2.9. *Let* $n = \dim(N)$. *Then we have*

$$\bigcup_{j\geq 0} \operatorname{Supp}(H^j_I(M, N)) = \bigcup_{j\leq n} \operatorname{Supp}(H^j_{I_M}(N)) = \bigcup_{j\leq n} \operatorname{Supp}(H^j_I(M, N)).$$

Proof. For any $i \in \mathbb{N} \cup \{+\infty\}$ we obtain by Lemma 2.8 that

$$\bigcup_{j<i} \operatorname{Supp}(H^j_{I_M}(N)) = \bigcup_{j<i} \operatorname{Supp}(\operatorname{Ext}^j_R(R/I_M, N)).$$

On the other hand, for any $\mathfrak{p} \in \bigcup_{j<i} \operatorname{Supp}(\operatorname{Ext}^j_R(R/I_M, N))$ we set $t = \operatorname{depth}((I_M)_{\mathfrak{p}}, N_{\mathfrak{p}})$. Since

$$i > t = \inf\{l \mid \operatorname{Ext}^l_R(R/I_M, N)_{\mathfrak{p}} \neq 0\} = \inf\{l \mid \operatorname{Ext}^l_R(M/IM, N)_{\mathfrak{p}} \neq 0\},$$

there exists a non-negative integer $j < i$ such that $\mathfrak{p} \in \operatorname{Supp}(\operatorname{Ext}^j_R(M/IM, N))$. Therefore

$$\bigcup_{j<i} \operatorname{Supp}(\operatorname{Ext}^j_R(R/I_M, N)) \subseteq \bigcup_{j<i} \operatorname{Supp}(\operatorname{Ext}^j_R(M/IM, N)).$$

Similarly, we can prove for the converse inclusion. Hence

$$\bigcup_{j<i} \mathrm{Supp}(\mathrm{Ext}^j_R(R/I_M, N)) = \bigcup_{j<i} \mathrm{Supp}(\mathrm{Ext}^j_R(M/IM, N)).$$

So we get by Lemma 2.8 that

$$\bigcup_{j<i} \mathrm{Supp}(H^j_{I_M}(N)) = \bigcup_{j<i} \mathrm{Supp}(H^j_I(M, N))$$

and the corollary follows from Grothendieck's Vanishing Theorem for local cohomology modules. □

3. A vanishing theorem

Herzog and Zamani [7, Theorem 3.2] showed that if $\mathrm{pd}_R(M) < +\infty$, then $H^t_{\mathfrak{m}}(M, N) = 0$ for all $t > \dim(R)$. In this section we extend Herzog–Zamani's result for an arbitrary ideal I as follows.

Theorem 3.1. *Assume that* $\mathrm{pd}_R(M) < +\infty$ *and* $d = \dim(R)$. *Then* $H^t_I(M, N) = 0$ *for all* $t > d$.

Proof. We first claim that $H^t_I(M, R) = 0$ for all $t > d$. Let $x_1, \ldots, x_m$ be a set of generators of I and $K^n_\bullet$ the Koszul complex of R with respect to $x^n_1, \ldots, x^n_m$. We denote by $C^n_\bullet$ the total complex associated to the double complex $K^n_\bullet \otimes_R F_\bullet$, where $F_\bullet$ is a projective resolution of M. Then it is easy to see that $\mathrm{Hom}(C^n_\bullet, R)$ is just isomorphic to the total complex of the double complex $\mathrm{Hom}(K^n_\bullet, \mathrm{Hom}(F_\bullet, R))$. Therefore we have by Rotman [15, Theorem 11.18] the following convergent spectral sequence

$$H^i(\mathrm{Hom}(K^n_\bullet, \mathrm{Ext}^j_R(M, R))) \underset{i}{\Longrightarrow} H^{i+j}(\mathrm{Hom}(C^n_\bullet, R)).$$

Since $H^i_I(M, R) \cong \varinjlim_n H^i(\mathrm{Hom}(C^n_\bullet, R))$ for all $i \geq 0$ by Bijan-Zadeh [1, Theorem 4.2], we obtain by passing to direct limits the following convergent spectral sequence

$$E^{i,j}_2 = H^i_I(\mathrm{Ext}^j_R(M, R)) \underset{i}{\Longrightarrow} H^{i+j} = H^{i+j}_I(M, R).$$

Thus for each $t \geq 0$ there is a finite filtration of the module $H^t = H^t_I(M, R)$

$$0 = \phi^{t+1} H^t \subseteq \phi^t H^t \subseteq \cdots \subseteq \phi^1 H^t \subseteq \phi^0 H^t = H^t$$

such that $E^{i,t-i}_\infty \cong \phi^i H^t / \phi^{i+1} H^t$ for all $0 \leq i \leq t$. It is clear that $E^{i,j}_2 = 0$ for all $j > \mathrm{pd}_R(M)$. If $i + j > d$ then $i > d - j$. Hence $i > \dim(\mathrm{Ext}^j_R(M, R))$ for all $0 \leq j \leq \mathrm{pd}_R(M)$ by Lemma 2.4. So $E^{i,j}_2 = 0$ for all $0 \leq j \leq \mathrm{pd}_R(M)$ and $i + j > d$. Thus, for each $t > d$ we get $E^{i,t-i}_2 = 0$ for all $0 \leq i \leq t$. Moreover, since $E_\infty^{i,t-i}$ is subquotient of $E_2^{i,t-i}$ for all $0 \le i \le t$, it implies that $E_\infty^{i,t-i} = 0$ for all $0 \le i \le t$ and all $t > d$. Therefore from the exact sequences

$$0 \to \phi^{i+1} H^t \to \phi^i H^t \to E_\infty^{i,t-i} \to 0$$

for all $0 \le i \le t$ we get $H_I^t(M, R) = \phi^0 H^t = H^t = 0$ for all $t > d$, and the claim is proved.
Next, since $d + 1 > \dim(M \otimes_R N)$, we get by Lemma 2.3 that $H_I^t(M, N) = 0$ for all $t > \mathrm{pd}_R(M) + d + 1$. Thus, it is enough to prove by descending induction on t that $H_I^t(M, N) = 0$, for all $d < t \le \mathrm{pd}_R(M) + d + 1$. It is clear that the assertion is true for $t = \mathrm{pd}_R(M) + d + 1$. Assume that $d < t < \mathrm{pd}_R(M) + d + 1$ and the assertion is true for $t + 1$. Since N is finitely generated R-module, there exists a non-negative integer n such that the following sequence

$$0 \to L \to R^n \to N \to 0$$

is exact for some finitely generated R-module L. This induces an exact sequence of generalized local cohomology modules

$$H_I^t(M, R^n) \to H_I^t(M, N) \to H_I^{t+1}(M, L).$$

Since $t > d$ and $H_I^{t+1}(M, L) = 0$ by the inductive hypothesis, we get by the claim that $H_I^t(M, R^n) \cong H_I^t(M, R)^n = 0$. Therefore $H_I^t(M, N) = 0$ as required. □

It should be mentioned that the functor $H_I^t(M, -)$ commutes with the direct limits in the category of all R-modules. Therefore as an immediate consequence of Theorem 3.1 we get the following result.

Corollary 3.2. *Assume that* $\mathrm{pd}_R(M) < +\infty$ *and* $d = \dim(R)$. *Then* $H_I^t(M, K) = 0$ *for all* $t > d$ *and all* (*not necessary to be finitely generated*) R*-module* K.

Remark 3.3. Grothendieck's Vanishing and non-Vanishing Theorems in the theory of local cohomology say that $H_I^t(N) = 0$ for all $t > n = \dim(N)$ and $H_{\mathfrak{m}}^n(N) \neq 0$. Therefore, in view of Theorem 3.1, it is natural to ask whether $H_I^t(M, N) = 0$ and $H_{\mathfrak{m}}^n(M, N) \neq 0$ for all $t > n = \dim(N)$ and all finitely generated R-modules of finite projective dimension. Unfortunately, the following example shows that the answer of the above question is negative.
Let k be a field and $R = k[[x, y, u, v]]$. Let $\mathfrak{m} = (x, y, u, v)R$, $M = R/(y)$ and $N = R/(x) \cap (y)$. It is clear that $\dim(R) = 4$, $\mathrm{pd}_R(M) < +\infty$ and $\dim(N) = 3$. Following [7], from the exact sequence $0 \to R \xrightarrow{y} R \to M \to 0$, we get an exact sequence

$$H_{\mathfrak{m}}^3(N) \xrightarrow{y} H_{\mathfrak{m}}^3(N) \to H_{\mathfrak{m}}^4(M, N) \to 0.$$

Thus, we get an isomorphism $H_{\mathfrak{m}}^4(M, N) \cong H_{\mathfrak{m}}^3(N)/yH_{\mathfrak{m}}^3(N)$. By Brodmann and Sharp [2, Theorem 7.3.2], we have $\mathrm{Att}(H_{\mathfrak{m}}^3(N)) = \{(x)R, (y)R\}$. Hence, since $H_{\mathfrak{m}}^3(N)$ is Artinian, we get by Brodmann and Sharp [2, Proposition 7.2.11] that $yH_{\mathfrak{m}}^3(N) \neq H_{\mathfrak{m}}^3(N)$. It follows that $H_{\mathfrak{m}}^4(M, N) \neq 0$.

4. The least integer r such that $\mathrm{Supp}(H^r_I(M, N))$ is infinite

Firstly, we recall the notion of generalized regular sequences introduced in [14]: A sequence $x_1, \ldots, x_r \in I$ is called a *generalized regular sequence* of N in I if for any $j = 1, \ldots, r$, $x_j \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(N/(x_1, \ldots, x_{j-1})N)$ satisfying $\dim(R/\mathfrak{p}) \geq 2$. If $\dim(N/IN) \geq 2$, then any generalized regular sequence of N in I is of length at most $\dim(N) - \dim(N/IN)$. Moreover, in this case all maximal generalized regular sequences of N in I have the same length, and this common length is called generalized depth of N in I and denoted by $\mathrm{gdepth}(I, N)$. Note that if $\dim(N/IN) \leq 1$ then there exists a generalized regular sequence of length r of N in I for any given integer $r > 0$. So, in this case we stipulate $\mathrm{gdepth}(I, N) = +\infty$. Below we show that the generalized depth can be computed by generalized local cohomology modules.

Theorem 4.1. *Set* $r = \mathrm{gdepth}(I_M, N)$ *and* $J_r = \bigcap_{j<r} \mathrm{ann}(\mathrm{Ext}^j_R(M/IM, N))$. *Then* $\dim(R/J_r) \leq 1$ *and*

$$\begin{aligned} r &= \inf\{i \mid \mathrm{Supp}(H^i_I(M, N)) \text{ is not finite}\} \\ &= \inf\{i \mid H^i_I(M, N) \ncong H^i_{J_r}(M, N)\}, \end{aligned}$$

where we use the convenience that $\inf(\emptyset) = +\infty$.

Proof. If $\dim(N/I_M N) \leq 1$ then $r = +\infty$ and $\dim(R/J_r) \leq 1$. Since $\mathrm{Supp}(N/I_M N)$ is finite, $\mathrm{Supp}(H^j_I(M, N))$ is finite for all $j \geq 0$. On the other hand, by Lemma 2.7, $H^j_I(M, N) \cong H^j_{J_r}(M, N)$ for all $j \geq 0$. Therefore the result is true in this case.

If $\dim(N/I_M N) \geq 2$ then $r < +\infty$. Let $x_1, \ldots, x_r$ be a maximal generalized regular sequence of N in I_M. For any $\mathfrak{p} \in \mathrm{Supp}(N/I_M N)$ such that $\dim(R/\mathfrak{p}) \geq 2$, $x_1/1, \ldots, x_r/1$ is an $N_\mathfrak{p}$-regular sequence in $(I_M)_\mathfrak{p}$. It follows that $\mathrm{Ext}^j_R(M/IM, N)_\mathfrak{p} = 0$ for all $j < r$, hence $\dim(\mathrm{Ext}^j_R(M/IM, N)) \leq 1$ for all $j < r$. Thus $\dim(R/J_r) \leq 1$, so that $\bigcup_{j<r} \mathrm{Supp}(\mathrm{Ext}^j_R(M/IM, N)) = \mathrm{Supp}(R/J_r)$ is a finite set. It follows by Lemma 2.8 that $\mathrm{Supp}(H^j_I(M, N))$ is a finite set for all $j < r$. Moreover, by Nhan [14, Proposition 4.4] we have

$$r = \min\{\mathrm{depth}((I_M)_\mathfrak{p}, N_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}(N/I_M N), \dim(R/\mathfrak{p}) \geq 2\}.$$

So $r = \mathrm{depth}((I_M)_\mathfrak{p}, N_\mathfrak{p})$ for some $\mathfrak{p} \in \mathrm{Supp}(N/I_M N)$ with $\dim(R/\mathfrak{p}) \geq 2$. Hence $\mathrm{Ext}^r_R(M/IM, N)_\mathfrak{p} \neq 0$. Thus, we have $\mathfrak{p} \in \bigcup_{j=0}^r \mathrm{Supp}(H^j_I(M, N))$ by Lemma 2.8. On the other hand, since $\mathrm{Supp}(H^j_I(M, N))$ is finite for all $j < r$, $\mathfrak{p} \notin \bigcup_{j<r} \mathrm{Supp}(H^j_I(M, N))$. Thus $\mathfrak{p} \in \mathrm{Supp}(H^r_I(M, N))$, so that $\mathrm{Supp}(H^r_I(M, N))$ is an infinite set by Kaplansky [10, Theorem 144]. Therefore

$$r = \inf\{i \mid \mathrm{Supp}(H^i_I(M, N)) \text{ is not finite}\}.$$

Finally, keep in mind that $\mathrm{Supp}(H^r_{J_r}(M,N))$ is finite, while $\mathrm{Supp}(H^r_I(M,N))$ is infinite by the conclusion above. It implies that $H^r_I(M,N) \not\cong H^r_{J_r}(M,N)$. Therefore, since $H^j_I(M,N) \cong H^j_{J_r}(M,N)$ for all $j < r$ by Lemma 2.7,

$$r = \inf\{i \mid H^i_I(M,N) \not\cong H^i_{J_r}(M,N)\},$$

and the proof of Theorem 4.1 is complete. □

Corollary 4.2. *Let i be a non-negative integer. If* $\mathrm{Supp}(H^j_I(N))$ *is a finite set for all $j \le i$, so is* $\mathrm{Supp}(H^j_I(M,N))$ *for all finitely generated R-module M.*

Proof. As $\mathrm{gdepth}(I,N) \le \mathrm{gdepth}(I_M,N)$, the result follows by Theorem 4.1. □

Note that for an arbitrary R-module K, the condition for $\mathrm{Supp}(K)$ to be a finite set is in general not equivalent to the condition that $\dim(R/\mathfrak{p}) \le 1$ for all $\mathfrak{p} \in \mathrm{Supp}(K)$. However, we have immediate consequences of Theorem 4.1 as follows.

Corollary 4.3. *Let i be a non-negative integer. Then* $\mathrm{Supp}(H^j_I(M,N))$ *is finite for all $j \le i$ if and only if* $\dim(R/\mathfrak{p}) \le 1$ *for all* $\mathfrak{p} \in \mathrm{Supp}(H^j_I(M,N))$ *and all $j \le i$.*

Corollary 4.4. *Set* $r = \mathrm{gdepth}(I,N)$ *and* $J_r = \bigcap_{j<r} \mathrm{ann}(\mathrm{Ext}^j_R(R/I,N))$. *Then we have* $\dim(R/J_r) \le 1$ *and*

$$r = \inf\{i \mid \mathrm{Supp}(H^i_I(N)) \text{ is not finite}\} = \inf\{i \mid H^i_I(N) \not\cong H^i_{J_r}(N)\}.$$

It should be mentioned that the first equality of Corollary 4.4 was proved by Nhan [14, Proposition 5.2].

Theorem 4.5. *Let i be a non-negative integer and* $P_i = \bigcup_{j<i} \mathrm{Supp}(H^j_I(M,N))$, *then*

$$\mathrm{Ass}(H^i_I(M,N)) \bigcup P_i = \mathrm{Ass}(\mathrm{Ext}^i_R(M/IM,N)) \bigcup P_i.$$

In particular, $\mathrm{Ass}(H^r_I(M,N))$ *is a finite set, where* $r = \mathrm{gdepth}(I_M,N)$.

Proof. Let $\mathfrak{p} \in \mathrm{Ass}(H^i_I(M,N))$. Assume that $\mathfrak{p} \notin P_i$. Then, by Lemma 2.8, we have $\mathrm{Ext}^i_R(M/IM,N)_\mathfrak{p} \ne 0$ and $\mathrm{Ext}^j_R(M/IM,N)_\mathfrak{p} = 0$ for all $j < i$. It follows that $i = \mathrm{depth}((I_M)_\mathfrak{p}, N_\mathfrak{p})$. So, by Lemma 2.2, we get

$$\mathrm{Ass}(H^i_I(M,N)_\mathfrak{p}) = \mathrm{Ass}(\mathrm{Ext}^i_R(M/IM,N)_\mathfrak{p}).$$

Thus $\mathfrak{p} \in \mathrm{Ass}(\mathrm{Ext}^i_R(M/IM,N))$, since $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}(H^i_I(M,N)_\mathfrak{p})$. Conversely, let $\mathfrak{p} \in \mathrm{Ass}(\mathrm{Ext}^i_R(M/IM,N))$ and $\mathfrak{p} \notin P_i$. By similar arguments as above, we can show that $i = \mathrm{depth}((I_M)_\mathfrak{p}, N_\mathfrak{p})$. Hence $\mathfrak{p} \in \mathrm{Ass}(H^i_I(M,N))$ by Lemma 2.2. Therefore

$$\mathrm{Ass}(H^i_I(M,N)) \bigcup P_i = \mathrm{Ass}(\mathrm{Ext}^i_R(M/IM,N)) \bigcup P_i.$$

Finally, let $r = \mathrm{gdepth}(I_M,N)$. Then we get by Theorem 4.1 that P_r is a finite set. Hence $\mathrm{Ass}(H^r_I(M,N))$ is a finite set as required. □

It has shown by Khashyarmanesh and Salarian [12, Theorem B] or Nhan [14, Theorem 5.6] that if i is an integer such that $\mathrm{Supp}(H_I^j(N))$ is a finite set for all $j < i$ then $\mathrm{Ass}(H_I^i(N))$ is a finite set. The next corollary gives us a description concretely of this set $\mathrm{Ass}(H_I^i(N))$.

Corollary 4.6. *Let i be a non-negative integer and $P_i = \bigcup_{j<i} \mathrm{Supp}(H_I^j(N))$, then*

$$\mathrm{Ass}(H_I^i(N)) \bigcup P_i = \mathrm{Ass}(\mathrm{Ext}_R^i(R/I, N)) \bigcup P_i.$$

In particular, $\mathrm{Ass}(H_I^r(N))$ *is finite for* $r = \mathrm{gdepth}(I, N)$.

5. The last integer s such that $\mathrm{Supp}(H_I^s(M, N))$ is infinite

The following theorem is the main result in this section.

Theorem 5.1. *Let s be an integer. Assume that* $\mathrm{pd}_R(M) < +\infty$. *Then*

(a) *The following statements are equivalent*:
 (i) $\mathrm{Supp}(H_I^j(M, R/\mathfrak{p}))$ *is finite for all $j > s$ and all* $\mathfrak{p} \in \mathrm{Assm}(N)$, *where* $\mathrm{Assm}(N)$ *denote the set of minimal elements of* $\mathrm{Ass}(N)$;
 (ii) $\mathrm{Supp}(H_I^j(M, N))$ *is finite for all $j > s$*;
 (iii) $\mathrm{Supp}(H_I^{s+1}(M, R/\mathfrak{p}))$ *is finite for all* $\mathfrak{p} \in \mathrm{Supp}(N)$.

(b) *Assume that* $\dim(N/I_M N) \geq 2$. *Set* $d = \dim(R)$, $r = \mathrm{gdepth}(I_M, N)$ *and*

$$\gamma = \sup\{\mathrm{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}(N/I_M N), \dim(R/\mathfrak{p}) \geq 2)\}.$$

Let s be the least integer satisfying one of three equivalent conditions in (a), then

$$\max\{r, \gamma\} \leq s < d - 1.$$

To prove Theorem 5.1(a), we need the following lemma.

Lemma 5.2. *Assume that* $\mathrm{pd}_R(M) < +\infty$. *Let s be a non-negative integer and L a finitely generated R-module such that* $\mathrm{Supp}(L) \subseteq \mathrm{Supp}(N)$. *Then, if* $\mathrm{Supp}(H_I^j(M, N))$ *is a finite set for all $j > s$, so is* $\mathrm{Supp}(H_I^j(M, L))$.

Proof. Let $d = \dim(R)$. Then $H_I^j(M, L) = 0$ for all $j > d$ by Theorem 3.1. Thus we can assume that $s \leq d$. We proceed by descending induction on j. It is clear that the assertion is true for $j \geq d + 1$. Let $j < d + 1$. Since $\mathrm{Supp}(L) \subseteq \mathrm{Supp}(N)$, we get by Vasconcelos [17, Theorem 4.1] that there exists a finite filtration

$$0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_t = L$$

such that for any $i = 1, \ldots, t$, L_i/L_{i-1} is a homomorphic image of N^{n_i} for some integer $n_i > 0$. Using short exact sequences $0 \to L_{i-1} \to L_i \to L_i/L_{i-1} \to 0$ for $i = 1, \ldots, t$, we can reduce the situation to the case $t = 1$. Therefore, there is an exact sequence $0 \to U \to N^n \to L \to 0$ for some $n > 0$ and some finitely generated R-module U. So, we get a long exact sequence

$$\cdots \to H_I^j(M, N^n) \to H_I^j(M, L) \to H_I^{j+1}(M, U) \to \cdots$$

As $\mathrm{Supp}(U) \subseteq \mathrm{Supp}(N)$, we get by induction that $\mathrm{Supp}(H_I^{j+1}(M, U))$ is finite. On the other hand, by the hypothesis, $\mathrm{Supp}(H_I^j(M, N^n)) = \mathrm{Supp}(H_I^j(M, N))$ is finite. Therefore, by the above exact sequence, $\mathrm{Supp}(H_I^j(M, L))$ is finite as required. □

As an immediate consequence of Lemma 5.2, we get the following result.

Corollary 5.3. *Assume that* $\mathrm{pd}_R(M) < +\infty$. *Let s be a non-negative integer and L a finitely generated R-module such that* $\mathrm{Supp}(L) = \mathrm{Supp}(N)$. *Then* $\mathrm{Supp}(H_I^j(M, L))$ *is finite for all $j > s$ if and only if so is* $\mathrm{Supp}(H_I^j(M, N))$.

Now, it is ready to prove Theorem 5.1(a).

Proof of Theorem 5.1(a). $(i) \Rightarrow (ii)$. Assume that $\mathrm{Assm}(N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ and that $\mathrm{Supp}(H_I^j(M, R/\mathfrak{p}_i))$ is finite for all $i = 1, \ldots, t$. Set $L_0 = 0$ and $L_i = \oplus_{j=1}^i (R/\mathfrak{p}_j)$ for each $i \in \{1, \ldots, t\}$. We have $\mathrm{Supp}(L_t) = \mathrm{Supp}(N)$, since $\mathrm{Ass}(L_t) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\} = \mathrm{Assm}(N)$. It follows from Corollary 5.3 that $\mathrm{Supp}(H_I^j(M, N))$ is finite for all $j > s$ if we can show that $\mathrm{Supp}(H_I^j(M, L_t))$ is finite for all $j > s$. Indeed, we proceed by induction on t. It is nothing to prove for $t = 1$. Let $t > 1$. From the exact sequence $0 \to L_{t-1} \to L_t \to R/\mathfrak{p}_t \to 0$ we get a long exact sequence

$$H_I^j(M, L_{t-1}) \to H_I^j(M, L_t) \to H_I^j(M, R/\mathfrak{p}_t).$$

Therefore $\mathrm{Supp}(H_I^j(M, L_t))$ is finite by (i) and the inductive hypothesis.
$(ii) \Rightarrow (iii)$ follows by Lemma 5.2.
$(iii) \Rightarrow (i)$. By inductive method we need only to show that $\mathrm{Supp}(H_I^{s+2}(M, R/\mathfrak{p}))$ is finite for all $\mathfrak{p} \in \mathrm{Supp}(N)$. Let $\mathfrak{p} \in \mathrm{Supp}(N)$. If $\dim(R/\mathfrak{p}) \le 1$ then the finiteness of $\mathrm{Supp}(H_I^{s+2}(M, R/\mathfrak{p}))$ is clear. Assume that $\dim(R/\mathfrak{p}) \ge 2$. We consider two cases $I_M \nsubseteq \mathfrak{p}$ and $I_M \subseteq \mathfrak{p}$, where $I_M = \mathrm{ann}_R(M/IM)$ is the annihilator of the R-module M/IM.

Case 1. $I_M \nsubseteq \mathfrak{p}$. Then there exists an $x \in I_M \setminus \mathfrak{p}$. Set $G = R/(\mathfrak{p} + xR)$, then $\mathrm{Supp}(G) \subseteq \mathrm{Supp}(N)$. We have a finite filtration

$$0 = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_t = G$$

such that, for each $i = 1, \ldots, t$, $G_i/G_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}(N)$. Now, with the same method that used in the proof of $(i) \Rightarrow (ii)$ we can show that

$$\mathrm{Supp}(H_I^{s+1}(M, G)) = \mathrm{Supp}(H_I^{s+1}(M, G_t))$$

is a finite set. On the other hand, we derive from the exact sequence $0 \to R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \to G \to 0$ an exact sequence

$$H_I^{s+1}(M, G) \to (0 : x)_{H_I^{s+2}(M, R/\mathfrak{p})} \to 0.$$

Thus $\mathrm{Supp}((0 : x)_{H_I^{s+2}(M,R/\mathfrak{p})})$ is finite, and so is $\mathrm{Supp}((0 : I_M)_{H_I^{s+2}(M,R/\mathfrak{p})})$. Therefore

$$\mathrm{Supp}(H_I^{s+2}(M, R/\mathfrak{p})) = \mathrm{Supp}((0 : I_M)_{H_I^{s+2}(M,R/\mathfrak{p})})$$

is finite, since $H_I^{s+2}(M, R/\mathfrak{p})$ is I_M-torsion by Lemma 2.1.

Case 2. $I_M \subseteq \mathfrak{p}$. By Lemma 2.1 we have

$$H_I^j(M, R/\mathfrak{p}) \cong \mathrm{Ext}_R^j(M, R/\mathfrak{p})$$

for all $j \geq 0$. For any $\mathfrak{q} \in \mathrm{Supp}(R/\mathfrak{p})$ with $\dim(R/\mathfrak{q}) \geq 2$, we get by the assumption that

$$\mathrm{Supp}(\mathrm{Ext}_R^{s+1}(M, R/\mathfrak{q})) = \mathrm{Supp}(H_I^{s+1}(M, R/\mathfrak{q}))$$

is finite. Therefore

$$\mathrm{Ext}_{R_\mathfrak{q}}^{s+1}(M_\mathfrak{q}, k(\mathfrak{q})) = \mathrm{Ext}_R^{s+1}(M, R/\mathfrak{q})_\mathfrak{q} = 0.$$

Thus $\mathrm{Tor}_{s+1}^{R_\mathfrak{q}}(M_\mathfrak{q}, k(\mathfrak{q})) = 0$ by Bruns and Herzog [3, Proposition 1.3.1]. It follows by Eisenbud [5, Corollary 19.5] that $\mathrm{pd}_{R_\mathfrak{q}}(M_\mathfrak{q}) < s + 1$. Hence

$$\mathrm{Ext}_R^{s+2}(M, R/\mathfrak{p})_\mathfrak{q} = \mathrm{Ext}_{R_\mathfrak{q}}^{s+2}(M_\mathfrak{q}, (R/\mathfrak{p})_\mathfrak{q}) = 0$$

for all $\mathfrak{q} \in \mathrm{Supp}(R/\mathfrak{p})$ satisfying $\dim(R/\mathfrak{q}) \geq 2$. Thus $\dim((\mathrm{Ext}_R^{s+2}(M, R/\mathfrak{p}))) \leq 1$, and so

$$\mathrm{Supp}(H_I^{s+2}(M, R/\mathfrak{p})) = \mathrm{Supp}(\mathrm{Ext}_R^{s+2}(M, R/\mathfrak{p}))$$

is finite. The proof of Theorem 5.1(a) is complete. □

To prove Theorem 5.1(b) we need one lemma more.

Lemma 5.4. *Let* $d = \dim(R)$. *Assume that* $\mathrm{pd}_R(M) < +\infty$. *Then the following statements are true.*

(i) $H_I^d(M, N)$ *is Artinian.*
(ii) $\mathrm{Supp}(H_I^{d-1}(M, N))$ *is a finite set.*

Proof. (i) We claim first that $H_I^d(M, R)$ is Artinian. By using the spectral sequence as in the proof of Theorem 3.1, we obtain a finite filtration

$$0 = \phi^{d+1}H^d \subseteq \phi^d H^d \subseteq \cdots \subseteq \phi^1 H^d \subseteq \phi^0 H^d = H^d$$

of the module $H^d = H_I^d(M, R)$ such that $E_\infty^{i,d-i} \cong \phi^i H^d / \phi^{i+1} H^d$ for all $0 \leq i \leq d$; and for any $i = 0, \ldots, d$ there exists exact sequences

$$0 \to \phi^{i+1} H^t \to \phi^i H^t \to E_\infty^{i,t-i} \to 0.$$

Note that $E_\infty^{i,d-i}$ is a subquotient of $E_2^{i,d-i}$ for all $0 \leq i \leq d$, where

$$E_2^{i,d-i} = H_I^i(\mathrm{Ext}_R^{d-i}(M, R)).$$

Thus in order to prove the Artinianness of $H_I^d(M, R)$, we need only to show the Artinianness of $H_I^i(\mathrm{Ext}_R^{d-i}(M, R))$ for all $0 \leq i \leq d$. Note that $\mathrm{pd}_R(M) \leq d$ and $\mathrm{Ext}_R^j(M, R) = 0$ for all $j > \mathrm{pd}_R(M)$. Therefore $\dim(\mathrm{Ext}_R^{d-i}(M, R)) \leq i$ for all $0 \leq i \leq d$ by Lemma 2.4. It follows by Brodmann and Sharp [2, Theorem 7.1.6] that $H_I^i(\mathrm{Ext}_R^{d-i}(M, R))$ is Artinian for all $0 \leq i \leq d$ and the claim is proved. Next, there exists an exact sequence $0 \to L \to R^n \to N \to 0$, where n is an integer and L is a finitely generated R-module. Hence we get by Theorem 3.1 an exact sequence

$$H_I^d(M, R^n) \to H_I^d(M, N) \to 0.$$

Since $H_I^d(M, R^n) \cong H_I^d(M, R)^n$, we get by the claim that $H_I^d(M, R^n)$ is Artinian. Therefore $H_I^d(M, N)$ is Artinian.

(ii) Firstly, we conclude that $\mathrm{Supp}(H_I^{d-1}(M, R))$ is finite. By similar arguments as in the proof of (i), we need only to show that $\mathrm{Supp}(E_2^{i,d-1-i})$ is finite for all $0 \leq i \leq d-1$, where $E_2^{i,d-1-i} = H_I^i(\mathrm{Ext}_R^{d-1-i}(M, R))$. We consider two cases.

The first case: $0 \leq \mathrm{pd}_R(M) < d-1$. If $0 \leq i < d-1-\mathrm{pd}_R(M)$ then $\mathrm{pd}_R(M) < d-1-i \leq d-1$. It implies $\mathrm{Ext}_R^{d-1-i}(M, R) = 0$, so that $E_2^{i,d-1-i} = 0$ for all $0 \leq i < d-1-\mathrm{pd}_R(M)$. If $d-1-\mathrm{pd}_R(M) \leq i \leq d-1$ then $0 \leq d-1-i \leq \mathrm{pd}_R(M)$. Thus, by Lemma 2.4, $\dim(\mathrm{Ext}_R^{d-1-i}(M, R)) \leq i+1$. It implies by Lemma 2.6 that $\mathrm{Supp}(E_2^{i,d-1-i})$ is finite for all $d-1-\mathrm{pd}_R(M) \leq i \leq d-1$. Therefore $\mathrm{Supp}(E_2^{i,d-1-i})$ is finite for all $0 \leq i \leq d-1$.

The second case: $\mathrm{pd}_R(M) \geq d-1$. Similar as in the first case, we get by Lemma 2.4 that $\dim(\mathrm{Ext}_R^{d-1-i}(M, R)) \leq i+1$ for all $0 \leq i \leq d-1$. Hence, by Lemma 2.6 again, $\mathrm{Supp}(E_2^{i,d-1-i})$ is finite for all $0 \leq i \leq d-1$ and the conclusion follows.

Next, with the same method as in the proof of (i) we get an exact sequence

$$H_I^{d-1}(M, R^n) \to H_I^{d-1}(M, N) \to H_I^d(M, L).$$

Since $\mathrm{Supp}(H_I^{d-1}(M, R^n))$ is a finite set by the conclusion above and $H_I^d(M, L)$ is Artinian by (i), it follows that $\mathrm{Supp}(H_I^{d-1}(M, N))$ is finite as required. □

Proof of Theorem 5.1(b). Let $d = \dim(R)$. Let s be the least integer satisfying one of three equivalent conditions in Theorem 5.1(a). Then, by Lemma 5.4, we have $s < d-1$. It follows from Theorem 4.1 that in order to prove $\max\{r, \gamma\} \leq s$, where $r = \mathrm{gdepth}(I_M, N)$ and

$$\gamma = \sup\{\mathrm{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}(N/I_M N), \dim(R/\mathfrak{p}) \geq 2\},$$

we have only to show that $\gamma \leq s$. Indeed, assume that $\gamma > s$. Then there exists $\mathfrak{p} \in \mathrm{Supp}(N/I_M N)$ such that $\dim(R/\mathfrak{p}) \geq 2$ and $\mathrm{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) > s$. Therefore we get by

Eisenbud [5, Corollary 19.5] that $\operatorname{Tor}^{R_{\mathfrak{p}}}_{s+1}(M_{\mathfrak{p}}, k(\mathfrak{p})) \neq 0$. So $\operatorname{Ext}^{s+1}_R(M, R/\mathfrak{p})_{\mathfrak{p}} \neq 0$ by Bruns and Herzog [3, Proposition 1.3.1]. It follows by Kaplansky [10, Theorem 144] that $\operatorname{Supp}(\operatorname{Ext}^{s+1}_R(M, R/\mathfrak{p}))$ is an infinite set. On the other hand, since $I_M \subseteq \mathfrak{p}$, we get by Lemma 2.1 that $H^{s+1}_I(M, R/\mathfrak{p}) = \operatorname{Ext}^{s+1}_R(M, R/\mathfrak{p})$. It follows that $\operatorname{Supp}(H^{s+1}_I(M, R/\mathfrak{p}))$ is infinite. This contradicts with the choice of s. Thus $\gamma \leq s$ as required. □

Remark 5.5. (i) In general, there does not exist the integer s in Theorem 5.1(a) if $\operatorname{pd}_R(M) = +\infty$. For example, let k be a field and $R = k[[x, y, u, v]]/\mathfrak{a}$, where $\mathfrak{a} = (x^2, y^2)$. Set $\mathfrak{p} = (x, y)/\mathfrak{a}$. It is clear that $R_{\mathfrak{p}}$ is not a regular local ring. Hence $\operatorname{pd}_{R_{\mathfrak{p}}}(k(\mathfrak{p})) = \operatorname{injd}_{R_{\mathfrak{p}}}(k(\mathfrak{p})) = +\infty$, where $\operatorname{injd}_{R_{\mathfrak{p}}}(k(\mathfrak{p}))$ is the injective dimension of $R_{\mathfrak{p}}$-module $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Now, let $M = N = R/\mathfrak{p}$ and $I = \mathfrak{p}$. Hence $\operatorname{pd}_R(M) = \operatorname{injd}_R(N) = +\infty$ and $\operatorname{Ext}^j_R(M, N)_{\mathfrak{p}} = \operatorname{Ext}^j_{R_{\mathfrak{p}}}(k(\mathfrak{p}), k(\mathfrak{p})) \neq 0$ for all $j \geq 0$. Thus $\operatorname{Supp}(H^j_I(M, N)) = \operatorname{Supp}(\operatorname{Ext}^j_R(M, N))$ is an infinite set for all $j \geq 0$.
(ii) We cannot replace the condition that $\mathfrak{p}$ runs through the set $\operatorname{Supp}(N)$ in statement (iii) of Theorem 5.1(a) by the condition that $\mathfrak{p}$ runs through the set $\operatorname{Ass}(N)$. Indeed, let k be a field and $R = k[[x, y, u, v]]$. Set $I = (x, y)R$ and $M = R/(y)$, then $I_M = I$. Since R is a regular local ring, $\operatorname{pd}_R(M) < \infty$. Let $N = R/((x) \cap (x^2, u) \cap (x^2, y, u^2))$. Then $\dim(N) = 3$ and $\operatorname{Ass}(N) = \{xR, (x, u)R, (x, y, u)R\}$. It is clear that $y \in I_M$ is a generalized regular element of N, and $\operatorname{ann}(N/yN) \supseteq (x^2, y)$. Thus, for any $\mathfrak{q} \in \operatorname{Ass}(N/yN)$ with $\dim(R/\mathfrak{q}) \geq 2$, we have $\mathfrak{q} \supseteq I_M$, so that $\operatorname{gdepth}(I_M, N/yN) = 0$. This implies $\operatorname{gdepth}(I_M, N) = 1$. Moreover, it is easy to see that $H^0_I(M, R/(x)) = 0$, $H^0_I(M, R/(x, u)) = 0$ and $\dim(H^0_I(M, R/(x, y, u))) = 1$. Hence $\operatorname{Supp}(H^0_I(M, R/\mathfrak{p}))$ is finite for all $\mathfrak{p} \in \operatorname{Ass}(N)$. However, we get by Theorem 4.1 that $\operatorname{Supp}(H^1_I(M, N))$ is not finite.

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