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Irreducibility of Hurwitz spaces of coverings with one special fiber and monodromy group a Weyl group of type D_d

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Abstract. Let Y be a smooth, connected, projective complex curve. In this paper, we study the Hurwitz spaces which parameterize branched coverings of Y whose monodromy group is a Weyl group of type D_d and whose local monodromies are all reflections except one. We prove the irreducibility of these spaces when $Y \simeq \mathbb{P}^1$ and successively we extend the result to curves of genus $g \geq 1$.

0. Introduction

Let X, X' and Y be smooth, connected, projective complex curves of genus ≥ 0 and let $H_{d,n}(Y)$ be the Hurwitz spaces which parameterize degree d simple coverings of Y with n branch points. The irreducibility of the Hurwitz spaces $H_{d,n}(\mathbb{P}^1)$ was proved by Hurwitz in [9] using a result of Clebsch and Lüroth. Severi's proof of the irreducibility of the moduli of curves of genus g was obtained by combining the Hurwitz's result with the fact that if $d \geq g+1$ each curve of genus g may be represented as a degree d simple covering of $\mathbb{P}^1$ (see [18]). Coverings of $\mathbb{P}^1$ simply branched in all but one point of the discriminant were studied by Natanzon and Kluitmann, who proved the irreducibility of the corresponding Hurwitz spaces (see [16,14]). Natanzon's work was inspired by applications to the theory of completely integrable Hamiltonian systems. The Hurwitz spaces $H^o_{d,n}(Y)$ parameterizing coverings with full monodromy group S_d of curves of genus ≥ 1 were studied by Graber et al. [8]. They proved in [8] the irreducibility of these spaces for $n \geq 2d$. Kanev in [12] sharpened this result and moreover he extended it to coverings which are simply branched in all but one point of the discriminant. Fixing the branching data of special point, i.e, a partition $\underline{e} = (e_1, \ldots, e_r)$ of d where $e_1 \geq \cdots \geq e_r$, he obtained the Hurwitz spaces $H^o_{d,n,\underline{e}}(Y)$ parametrizing coverings with monodromy group S_d, simply branched in n points and ramified with multiplicities $e_1, \ldots, e_r$ over one addition point. Kanev proved that they are irreducible if $n \geq 2d-2$. The author sharpened the latter result proving in [19] the irreducibility of $H^o_{d,n,\underline{e}}(Y)$ for $n + |\underline{e}| \geq 2d$ where $|\underline{e}| = \sum_{i=1}^{r}(e_i - 1)$.

We notice that the results above concern Hurwitz spaces of coverings whose monodromy group is the symmetric group S_d. It is natural to ask what happens

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if one replaces S_d with another Weyl group. Coverings with monodromy group a Weyl group are interesting because they appear in the study of spectral curves and of Prym–Tyurin varieties (see [5,10,11]). Branched coverings with monodromy group a Weyl group were studied by Biggers and Fried in [1], by Kanev in [13] and by the author in [20,21]. Biggers and Fried proved the irreducibility of Hurwitz spaces that parameterize coverings of $\mathbb{P}^1$, with monodromy group a Weyl group of type D_d, whose local monodromies are all reflections. Kanev generalized the result to Hurwitz spaces parameterizing Galois coverings of $\mathbb{P}^1$ whose Galois group is an arbitrary Weyl group. We in [20] were interesting in coverings of Y whose monodromy group is a Weyl group of type B_d and whose local monodromies are all reflections except one. We verified the irreducibility of the corresponding Hurwitz spaces when $Y \simeq \mathbb{P}^1$. We proved that, in the case in which among the local monodromies there are both reflections with respect to long roots and reflections with respect to short roots, the result can be generalized to curves of genus ≥ 1 under the hypothesis $\tilde{n} + |\underline{e}| \geq 2d$, where $\tilde{n}$ is the number of the local monodromies that are reflections with respect to long roots. We completed the study of coverings with monodromy group $W(B_d)$ in [21] showing that, in the case of one special fiber and all other local monodromies being reflections with respect to long roots, the Hurwitz spaces are not irreducible and furthermore we showed that if $\tilde{n} + |\underline{e}| \geq 2d$ they have $2^{2g} - 1$ connected components where $g = g(Y)$.

In this paper, we study branched coverings $X \xrightarrow{\pi} X' \xrightarrow{f} Y$, whose monodromy group is a Weyl group of type D_d, satisfying the following: π is a degree 2 étale covering and f is a degree $d \geq 3$ covering, with monodromy group S_d, branched in $n + 1$ points, $n > 0$ of which are points of simple branching while one is a special point whose local monodromy has cycle type $\underline{e}$. We first prove the irreducibility of corresponding Hurwitz spaces when $Y \simeq \mathbb{P}^1$ (cf. Theorem 1) and then we extended the result to curves of genus ≥ 1 (cf. Theorem 2). In order to generalize the result obtained for $\mathbb{P}^1$ to curves of genus ≥ 1, we make use of the Theorem 1 of [19] that states the irreducibility of the Hurwitz spaces $H^o_{d,n,\underline{e}}(Y)$ under the hypothesis $n + |\underline{e}| \geq 2d$. Because of this the Theorem 2 is verified under the same hypothesis. Our result generalizes the one of Biggers and Fried, namely we allow one special fiber.

Conventions. Here the natural action of S_d on $\{1, \ldots, d\}$ is on the *right* and we write i^σ for $i \in \{1, \ldots, d\}$ and $\sigma \in S_d$.

1. Preliminaries

In this section, we recall some notions on the Weyl groups of type B_d and D_d and on the braid moves. Moreover, we introduce the Hurwitz spaces that will be object of our study. The references for the material on the Weyl groups are [3,4]. One can look [2,6,9,17] and [12] for the material on the braid moves.

1.1. Weyl groups of type B_d and D_d

Let $\{\varepsilon_1, \ldots, \varepsilon_d\}$ be the standard base of $\mathbb{R}^d$ and let R be the root system $\{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j : 1 \leq i, j \leq d\}$. The generators of the Weyl group of type B_d, that we denote by $W(B_d)$, are the reflections with respect to the short roots $\varepsilon_i, i = 1, \dots, d,$ and to long roots $\varepsilon_i - \varepsilon_j, 1 \le i < j \le d$. The Weyl group of type D_d is the subgroup of $W(B_d)$ generated by the reflections with respect to the long roots $\varepsilon_1 - \varepsilon_i$ and $\varepsilon_1 + \varepsilon_i,\ 2 \le i \le d$. We denote it by $W(D_d)$. The reflection s_{ε_i} interchanges ε_i and $-\varepsilon_i$ while unchanging each ε_h with $h \neq i$. The reflection $s_{\varepsilon_i - \varepsilon_j}$ interchanges ε_i and ε_j, $-\varepsilon_i$ and $-\varepsilon_j$, leaving unchanged ε_h for each $h \neq i,\ j$. Consequently, the elements of the Weyl group $W(B_d)$ operate on $\{\pm\, \varepsilon_1, \dots, \pm\, \varepsilon_d\}$ permuting the elements. If we identify $\{\pm\, \varepsilon_1, \dots, \pm\, \varepsilon_d\}$ with $\{\pm\, 1, \dots, \pm\, d\}$ by the map that transforms $\pm\, \varepsilon_i$ into $\pm\, i$ and if we ignore the sign-changes, we see that each element $w \in W(B_d)$ determines a permutation of the indexes $1, \dots, d$. This permutation can be expressed in the usual way as a product of disjoint cycles. Let $(i_1 i_2 \dots i_e)$ be a such cycle. Then w sends $\{+\varepsilon_{i_j}, -\varepsilon_{i_j}\}$ to $\{+\varepsilon_{i_{j+1}}, -\varepsilon_{i_{j+1}}\}$, $j = 1, \dots, e-1$, and $\{+\varepsilon_{i_e}, -\varepsilon_{i_e}\}$ to $\{+\varepsilon_{i_1}, -\varepsilon_{i_1}\}$. The cycle $(i_1 \dots i_e)$ is called positive if $w^e(\varepsilon_{i_1}) = \varepsilon_{i_1}$ and negative if $w^e(\varepsilon_{i_1}) = -\varepsilon_{i_1}$. The lengths of these disjoint cycles together with their signs give a set of positive or negative integers called the signed cycle-type of w. It is well known that two elements of $W(B_d)$ are conjugate if and only if they have the same signed cycle-type.

Definition 1. We call positive cycle of the form $(i_1 \dots i_e)$ each element $w \in W(B_d)$ satisfying the following: w sends $\{+\, \varepsilon_{i_j}, -\varepsilon_{i_j}\}$ to $\{+\, \varepsilon_{i_{j+1}}, -\varepsilon_{i_{j+1}}\}$, $j = 1, \dots, e$ where $i_{e+1} := i_1$, leaving unchanged ε_h for each $h \notin \{i_1, \dots, i_e\}$, $1 \le h \le d$, and moreover $w^e(\varepsilon_{i_1}) = \varepsilon_{i_1}$. The integer e is called the length of the cycle w. Two positive cycles in $W(B_d)$ of the form $(i_1 \dots i_e)$ and $(h_1 \dots h_l)$ are disjoint if $(i_1 \dots i_e)$ and $(h_1 \dots h_l)$ are disjoint cycles of S_d.

The action of $W(B_d)$ on $\{\pm\, \varepsilon_i : i = 1, \dots, d\}$ allows us also to define an injective homomorphism τ from $W(B_d)$ into S_{2d} that sends $s_{\varepsilon_i - \varepsilon_j}$ to $(i\ j)(-i\ -j)$, s_{ε_i} to $(i\ -i)$ and $s_{\varepsilon_i + \varepsilon_j} = s_{\varepsilon_i} s_{\varepsilon_j} s_{\varepsilon_i - \varepsilon_j}$ to $(i\ -j)(-i\ j)$. A positive cycle of the form $(i_1 \dots i_e)$ corresponds in S_{2d} to a product of two disjoint e-cycles, ss', which move the indexes $\{\pm i_1, \dots, \pm i_e\}$ and are such that if s sends i_j to i_{j+1} (i_j to $-i_{j+1}$) then s' sends $-i_j$ to $-i_{j+1}$ (resp. $-i_j$ to i_{j+1}), where $\pm i_{e+1} := \pm i_1$.

Let $(\mathbf{Z}_2)^d$ be the set of the functions from $\{1, \dots, d\}$ to $\mathbf{Z}_2$ equipped with the sum operation. We write z_{ij} for the function of $(\mathbf{Z}_2)^d$ defined as

$$z_{ij}(i) = z_{ij}(j) = z \quad \text{and} \quad z_{ij}(h) = \bar{0} \quad \text{for each } h \neq i, j \text{ and } z \in \mathbf{Z}_2$$

and write $\bar{1}_{i\dots k}$ for the function of $(\mathbf{Z}_2)^d$ which sends to $\bar{1}$ only the indexes $i \dots k$. Let Φ be the homomorphism from S_d in $Aut((\mathbf{Z}_2)^d)$ which assigns to $t \in S_d$ $\Phi(t) \in Aut((\mathbf{Z}_2)^d)$ where

$$[\Phi(t)\, z']\,(j) := z'(j^t) \quad \text{for each } z' \in (\mathbf{Z}_2)^d.$$

Let $(\mathbf{Z}_2)^d \times^s S_d$ be the semidirect product of $(\mathbf{Z}_2)^d$ and S_d through the homomorphism Φ. Given $(z'; t_1), (z''; t_2) \in (\mathbf{Z}_2)^d \times^s S_d$ we let

$$(z'; t_1)\,(z''; t_2) := (z' + \Phi(t_1) z'';\ t_1 t_2).$$

It is easy to check that it is possible to define an isomorphism Ψ from $W(B_d)$ to $(\mathbf{Z}_2)^d \times^s S_d$ which sends $s_{\varepsilon_i - \varepsilon_j}$ to $(0; (i\ j))$, s_{ε_i} to $(\bar{1}_i; id)$ and $s_{\varepsilon_i + \varepsilon_j}$ to

$(\bar{1}_{ij}; (i\ j))$. *In particular, the isomorphism* Ψ *sends a positive cycle of the form* $(i_1 i_2 \dots i_e)$ *to an element of type* $(\bar{1}_{i_h \dots i_k}; (i_1 i_2 \dots i_e))$ *where* $\{i_h, \dots, i_k\} \subseteq \{i_1, i_2, \dots, i_e\}$ *and* $\sharp\{i_h, \dots, i_k\}$ *is either even or equal to* 0.

It follows from what we have said that $W(D_d)$ is isomorphic to the subgroup of $(\mathbf{Z}_2)^d \times^s S_d$ generated by the elements $(0; (1\ i))$ and $(\bar{1}_{1i}; (1\ i))$, $2 \le i \le d$. Therefore elements of type $(\bar{1}_{i \dots h}; id)$, where $\sharp\{i, \dots, h\}$ is odd, do not belong to $W(D_d)$, while positive cycles and products of positive cycles belong to $W(D_d)$.

Definition 2. Let $\underline{e} = (e_1, \dots, e_r)$ be a partition of d where $e_1 \ge \cdots \ge e_r \ge 1$. We denote by C the conjugate class of $(\mathbf{Z}_2)^d \times^s S_d$ containing elements of type $(z_{ij}; (ij))$ and by $C_{\underline{e}}$ the conjugate class of $(\mathbf{Z}_2)^d \times^s S_d \cong W(B_d)$ containing elements that are product of r disjoint positive cycles whose lengths are given by the elements of the partition $\underline{e}$.

From now on we will use $(a; \xi)$ to denote an element in $C_{\underline{e}}$. Note that ξ is the permutation that $(a; \xi)$ determines on the indexes $1, \dots, d$ and ξ has cycle type $\underline{e}$.

Observation 1. Let $(a; \xi) \in C_{\underline{e}}$ and let $\xi = \xi_1 \cdots \xi_r$ where the ξ_i are disjoint cycles and ξ_i is a e_i-cycle. Let $i^{k_1}, i^{k_2}, \dots, i^{k_{s_i}}$ be the indexes moved by ξ_i that a sends to $\bar{1}$ and let

$$\xi_i = \left(\dots i^{k_1}\ i^{k_1+1} \dots i^{k_2-1}\ i^{k_2} \dots i^{k_{(s_i-1)}}\ i^{k_{(s_i-1)}+1} \dots i^{k_{s_i}-1}\ i^{k_{s_i}} \dots\right).$$

If we conjugate $(a; \xi)$ with $(\bar{1}_{i^{k_{(s_i-1)}+1} \dots i^{k_{s_i}-1}\ i^{k_{s_i}}};\ id)$ we obtain

$$\left(\bar{1}_{i^{k_{(s_i-1)}+1} \dots i^{k_{s_i}-1}\ i^{k_{s_i}}} + a + \bar{1}_{i^{k_{(s_i-1)}} \dots i^{k_{s_i}-1}};\ \xi\right)$$

where $a' = \bar{1}_{i^{k_{(s_i-1)}+1} \dots i^{k_{s_i}-1}\ i^{k_{s_i}}} + a + \bar{1}_{i^{k_{(s_i-1)}} \dots i^{k_{s_i}-1}}$ is a function which sends to $\bar{1}$ the same indexes sent to $\bar{1}$ by a except $i^{k_{s_i}}$ and $i^{k_{(s_i-1)}}$. Hence, if we conjugate $(a'; \xi)$ with

$$\left(\bar{1}_{i^{k_{(s_i-3)}+1} \dots i^{k_{(s_i-2)}-1}\ i^{k_{(s_i-2)}}};\ id\right) \left(\bar{1}_{i^{k_{(s_i-5)}+1} \dots i^{k_{(s_i-4)}-1} i^{k_{(s_i-4)}}};\ id\right) \cdots \left(\bar{1}_{i^{k_1+1} \dots i^{k_2-1}\ i^{k_2}};\ id\right)$$

we obtain a new element $(b;\ \xi)$ belonging to $C_{\underline{e}}$ such that b is a function which sends to $\bar{0}$ all indexes of ξ_i. If we proceed in this way for each $i = 1, \dots, r$ we obtain the element $(0; \xi)$.

Note that the preceding argument is a check of the fact that $(a; \xi)$ and $(0; \xi)$ belong to the same conjugate class and moreover it shows that if s is a permutation of S_d with cycle type $\underline{e}$ then $(0; s)$ belong to $C_{\underline{e}}$.

1.2. The Hurwitz spaces $H_{W(D_d),\ n,\ \underline{e}}\ (Y)$

We assume throughout what follows that d is an integer greater or equal to 3 and n is a positive integer.

Definition 3. An ordered sequence $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ of elements of $(\mathbf{Z}_2)^d \times^s S_d \simeq W(B_d)$ such that $t_i \neq (0; id)$ for each $i = 1, \dots, n$ and $t_1 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g]$ is called a Hurwitz system with values in $(\mathbf{Z}_2)^d \times^s S_d$. The subgroup of $(\mathbf{Z}_2)^d \times^s S_d$ generated by t_i, λ_k, μ_k with $i = 1, \dots, n$ and $k = 1, \dots, g$ is called the monodromy group of the Hurwitz system. Note that if $g = 0$ the Hurwitz systems $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ are of the form $(t_1, \dots, t_n)$ and $t_1 \cdots t_n = (0; id)$.

Definition 4. Two Hurwitz systems with values in $(\mathbf{Z}_2)^d \times^s S_d \simeq W(B_d)$, $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ and $(\tilde{t}_1, \dots, \tilde{t}_n; \tilde{\lambda}_1, \tilde{\mu}_1, \dots, \tilde{\lambda}_g, \tilde{\mu}_g)$, are called equivalent if there exists $s \in (\mathbf{Z}_2)^d \times^s S_d$ such that $\tilde{t}_i = s^{-1}\, t_i\, s$, $\tilde{\lambda}_k = s^{-1}\, \lambda_k\, s$ and $\tilde{\mu}_k = s^{-1}\, \mu_k\, s$ for each $i = 1, \dots, n$, $k = 1, \dots, g$. The equivalence class containing $(t_1, \dots, \mu_g)$ is denoted by $[t_1, \dots, \mu_g]$.

Let X, X' and Y be smooth, connected, projective complex curves of genus ≥ 0. In this paper, we work with branched coverings $\widetilde{f}$ of Y such that $\widetilde{f} = f \circ \pi$ where $\pi : X \to X'$ is a degree 2 unramified covering and $f : X' \to Y$ is a degree d branched covering.

Definition 5. Two coverings $X_1 \xrightarrow{\pi_1} X_1' \xrightarrow{f_1} Y$ and $X_2 \xrightarrow{\pi_2} X_2' \xrightarrow{f_2} Y$ are called equivalent if there exist two biholomorphic maps $p : X_1 \to X_2$ and $p' : X_1' \to X_2'$ such that $p' \circ \pi_1 = \pi_2 \circ p$ and $f_2 \circ p' = f_1$. The equivalence class containing the covering $X \xrightarrow{\pi} X' \xrightarrow{f} Y$ is denoted by $[X \xrightarrow{\pi} X' \xrightarrow{f} Y]$.

Let $\underline{e} = (e_1, \dots, e_r)$ be a partition of d where $e_1 \geq \cdots \geq e_r \geq 1$. We write $H_{W(D_d),\, n,\, \underline{e}}\, (Y)$ for the Hurwitz space that parameterizes equivalence classes of coverings $X \xrightarrow{\pi} X' \xrightarrow{f} Y$, whose monodromy group is $W(D_d)$, satisfying the following:
π is a degree 2 unramified covering and f is a degree d covering, with monodromy group S_d, branched in $n+1$ points, n of which are points of simple branching while one is a special point whose local monodromy has cycle type $\underline{e}$.

Let $b_0 \in Y$ and let g be the genus of Y. From now on we will denote by D and by $m : \pi_1(Y - D, b_0) \to S_{2d}$ respectively the branch locus and the monodromy homomorphism associated to the covering $f \circ \pi$. The image via the monodromy homomorphism m of a standard generating system for $\pi_1(Y - D, b_0)$ determines an equivalence class $[t_1, \dots, t_{n+1};\ \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$ of Hurwitz systems *with values in $(\mathbf{Z}_2)^d \times^s S_d$ and monodromy group $W(D_d)$ such that n among the t_j belong to C and one belongs to $C_{\underline{e}}$.* We denote by $A_{W(D_d),\, n,\, \underline{e},\, g}$ the set of all the equivalence classes, $[t_1, \dots, t_{n+1}; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$, of Hurwitz systems as above. Note that when $g = 0$ we write $A_{W(D_d),n,\, \underline{e}}$ for $A_{W(D_d),n,\, \underline{e},\, 0}$.

Let $Y^{(n+1)}$ be the $(n+1)$-fold symmetric product of Y and let Δ be the codimension 1 locus of $Y^{(n+1)}$ consisting of non simple divisors. We write δ to denote the map $H_{W(D_d),\, n,\, \underline{e}}\, (Y) \to Y^{(n+1)} - \Delta$ which assigns to each equivalence class $[X \xrightarrow{\pi} X' \xrightarrow{f} Y]$ the branch locus D of $X \xrightarrow{\pi} X' \xrightarrow{f} Y$. It is well known, there is a unique topology on $H_{W(D_d),\, n,\, \underline{e}}\, (Y)$ such that δ is a topological covering map (see [7]). By Riemann's existence theorem we can identify the fiber of

δ over D with $A_{W(D_d),\ n,\ \underline{e},\ g}$. Therefore the braid group $\pi_1(Y^{(n+1)} - \Delta, D)$ acts on $A_{W(D_d),\ n,\ \underline{e},\ g}$. If this action is transitive the Hurwitz space $H_{W(D_d),\ n,\ \underline{e}}(Y)$ is connected.

1.3. Braid moves

Let Y be a smooth, projective complex curve of genus ≥ 1. The generators of the braid group $\pi_1(Y^{(n+1)} - \Delta, D)$ are the elementary braids σ_j with $j = 1, \ldots, n$ and the braids $\rho_{ik},\ \tau_{ik}$ with $1 \leq i \leq n+1$ and $1 \leq k \leq g$ (see [2,6,17]). To each generator $\sigma_j,\ \rho_{ik},\ \tau_{ik}$ is associated a pair of braid moves: σ'_j and $\sigma''_j = (\sigma'_j)^{-1}$, ρ'_{ik} and $\rho''_{ik} = (\rho'_{ik})^{-1}$, τ'_{ik} and $\tau''_{ik} = (\tau'_{ik})^{-1}$, respectively (see [9,12]). The moves σ'_j and σ''_j are called *elementary moves* while $\rho'_{ik},\ \rho''_{ik},\ \tau'_{ik},\ \tau''_{ik}$ are simply called *braid moves*.

The elementary move σ'_j transforms (see [9]) $[t_1, \ldots, t_{j-1}, t_j, t_{j+1}, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g]$ to

$$\left[t_1, \ldots, t_{j-1}, t_j t_{j+1} t_j^{-1}, t_j, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g\right]$$

and then σ''_j transforms $[t_1, \ldots, t_{j-1}, t_j, t_{j+1}, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g]$ to $[t_1, \ldots, t_{j-1}, t_{j+1}, t_{j+1}^{-1} t_j t_{j+1}, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g]$.

The action of braid moves ρ'_{1k} and τ''_{1k} is described by the following proposition.

Proposition 1. ([12], Corollary 1.9) *Let* $(t_1, \ldots, t_n; \lambda_1, \mu_1, \ldots, \lambda_g, \mu_g)$ *be a Hurwitz system. Let* $u_k = [\lambda_1, \mu_1] \cdots [\lambda_k, \mu_k]$ *for* $k = 1, \ldots, g$ *and let* $u_0 = id$. *The following formulae hold*:

(i) *For* ρ'_{1k}:

$$\rho'_{1k} : \mu_k \to \mu'_k = (b_1^{-1} t_1^{-1} b_1)\mu_k,$$

where $b_1 = u_{k-1}\lambda_k$

(ii) *For* τ''_{1k}:

$$\tau''_{1k} : \lambda_k \to \lambda''_k = (u_{k-1}^{-1} t_1^{-1} u_{k-1})\lambda_k.$$

In particular,

$$\tau''_{11} : \lambda_1 \to t_1^{-1}\lambda_1.$$

2. Irreducibility of $H_{W(D_d),\ n,\underline{e}}(Y)$

In this section, we prove the irreducibility of $H_{W(D_d),\ n,\ \underline{e}}(Y)$ when $Y \simeq \mathbb{P}^1$ and successively we extend the result to curves of genus ≥ 1 under the hypothesis $n + |\underline{e}| \geq 2d$ where $|\underline{e}| = \sum_{i=1}^{r}(e_i - 1)$.

Definition 6. We call two Hurwitz systems with values in $(\mathbf{Z}_2)^d \times^s S_d \simeq W(B_d)$ braid-equivalent if one is obtained from the other by a finite sequence of braid moves σ'_j, ρ'_{ik}, τ'_{ik}, σ''_j, ρ''_{ik}, τ''_{ik} where $1 \leq j \leq n-1$, $1 \leq i \leq n$ and $1 \leq k \leq g$. We say braid-equivalent two ordered n-tuples (or sequences) of elements in $(\mathbf{Z}_2)^d \times^s S_d$, $(t_1, \dots, t_n)$ and $(\tilde{t}_1, \dots, \tilde{t}_n)$, if $(\tilde{t}_1, \dots, \tilde{t}_n)$ is obtained from $(t_1, \dots, t_n)$ by a finite sequence of braid moves of type σ'_j, σ''_j. We denote the braid equivalence by $\sim$.

Lemma 1. *Let $(t_1, \dots, t_i, t_{i+1}, \dots, t_n)$ be a sequence of elements in $(\mathbf{Z}_2)^d \times^s S_d$ such that $t_{i+1} = t_i^{-1}$. Then, acting with elementary moves σ'_j and their inverses, we can move to the left and to the right the pair (t_i, t_{i+1}) leaving unchanged the other elements of the sequence.*

Proof. The lemma follows from the braid equivalences $(t, t_i, t_{i+1}) \sim (t_i, t_i^{-1} t\, t_i, t_{i+1}) \sim (t_i, t_{i+1}, t)$ and $(t_i, t_{i+1}, t) \sim (t_i, t_{i+1} t\, t_{i+1}^{-1}, t_{i+1}) \sim (t, t_i, t_{i+1})$. □

We now enunciate two results that we will use late on.

Lemma 2. ([12], Main Lemma 2.1) *Let $(t_1, ..., t_n; \lambda_1, \mu_1, ..., \lambda_g, \mu_g)$ be a Hurwitz system with values in $(\mathbf{Z}_2)^d \times^s S_d \simeq W(B_d)$. Suppose that $t_i t_{i+1} = (0; id)$. Let H be the subgroup of $(\mathbf{Z}_2)^d \times^s S_d$ generated by $\{t_1, ..., t_{i-1}, t_{i+2}, ..., t_n, \lambda_1, \mu_1, ..., \lambda_g, \mu_g\}$. Then for every $h \in H$ the given Hurwitz system is braid equivalent to*

$$\left(t_1, ..., t_{i-1}, h^{-1}\, t_i\, h, h^{-1}\, t_{i+1}\, h, t_{i+2}, ..., t_n; \lambda_1, \mu_1, ..., \lambda_g, \mu_g\right).$$

From now on we associate to the partition $\underline{e} = (e_1, \dots, e_r)$, where $e_1 \geq \cdots \geq e_r \geq 1$, the following element in S_d

$$(12 \dots e_1)(e_1 + 1 \dots e_1 + e_2) \cdots ((e_1 + \cdots + e_{r-1}) + 1 \dots d). \qquad (1)$$

Following [15] we also denote the permutation (1) by

$$\epsilon = (1_1\, 2_1 \dots (e_1)_1)(1_2\, 2_2 \dots (e_2)_2) \cdots (1_r\, 2_r \dots (e_r)_r).$$

We write q_i for the cycle $(1_i 2_i \dots (e_i)_i)$, Z_i for the sequence $((1_i 2_i), (1_i 3_i), \dots, (1_i (e_i)_i))$ and Z for the concatenation $Z_1 Z_2 \dots Z_r$. Moreover, we use $|\underline{e}|$ to denote $\sum_{i=1}^r (e_i - 1)$.

Proposition 2. ([14] or [15] pp. 369–370) *Let $(t'_1, \dots, t'_n)$ be a sequence of transpositions such that $t'_1 \cdots t'_n = \epsilon$ and $< t'_1, \dots, t'_n >$ is transitive. Then $(t'_1, \dots, t'_n)$ is braid equivalent to*

$$(Z, t''_{N+1}, \dots, t''_n)$$

where $n - N \equiv 0\ (mod\ 2)$ and

(i) *If $r = 1$ $\quad t''_i = (1_1\, 2_1)$ $\quad$ for each $\quad i \geq N + 1$,*
(ii) *If $r > 1$ $\quad$ then*

$$(t''_{N+1}, \dots, t''_n) = ((1_1 1_2), (1_1 1_2), (1_1\, 1_3), (1_1\, 1_3), \dots, (1_1 1_r), \dots, (1_1\, 1_r))$$

where each $(1_1\ 1_i)$ *appears twice if* $2 \le i \le r-1$ *and* $(1_1\ 1_r)$ *appears an even number of times.*

In what follows, we use $\widetilde{Z}_i,\ i = 1, \dots, r,$ to denote the sequence $((0;(1_i 2_i)), (0;(1_i 3_i)), \dots, (0;(1_i (e_i)_i)))$ and $\widetilde{Z}$ to denote the concatenation $\widetilde{Z}_1 \widetilde{Z}_2 \dots \widetilde{Z}_r$.

Proposition 3. *Let* $[\,\underline{t}\,] = [t_1, \dots, t_{n+1}]$ *be an equivalence class of Hurwitz systems with values in* $(\mathbf{Z}_2)^d \times^s S_d \simeq W(B_d)$ *such that n among the* t_j *belong to C, one belongs to* $C_{\underline{e}}$ *and moreover if* $t_j = (z';\ t'_j),\ j = 1, \dots, n+1,$ *the group generated by the permutations* t'_j *is all* S_d*. Then* $[\,\underline{t}\,]$ *is braid-equivalent to a class of the form*:

(i) *if* $r > 1$

$$[\underline{t}_1] = \Big[\widetilde{Z},\ (0;(1_1 1_2)),\ (0;(1_1 1_2)), \dots, (0;(1_1 1_{r-1})), (0;(1_1 1_{r-1}))\ ,\ (z^1_{1_1 1_r};(1_1 1_r)), \dots, (z^s_{1_1 1_r};(1_1 1_r)), (0;\epsilon^{-1})\Big]$$

where each $(0;(1_1\ 1_i)),\ 2 \le i \le r-1,$ *appears twice, the* z^h *are elements of* $\mathbb{Z}_2$ *and* s *is an even positive integer,*

(ii) *if* $r = 1$

$$[\,\underline{t}_2\,] = \Big[\widetilde{Z}_1,\ (z^1_{1_1 2_1};(1_1 2_1)), \dots, (z^s_{1_1 2_1};(1_1 2_1)), (0;\epsilon^{-1})\Big]$$

where the z^h *are elements of* $\mathbb{Z}_2$ *and* s *is an even positive integer.*

Proof. Let $(a;\xi)$ be the element of $\underline{t}$ that belongs to $C_{\underline{e}}$. With elementary moves σ'_i we move $(a;\xi)$ to the place $n+1$. Because $(0;\epsilon^{-1})$ and $(a;\xi)$ belong to the same conjugate class of $(\mathbf{Z}_2)^d \times^s S_d$ (see Observation 1), there exists one element $(\tilde{z};s) \in (\mathbf{Z}_2)^d \times^s S_d$ so that $(\tilde{z};s)^{-1}(a,\xi)(\tilde{z};s) = (0;\epsilon^{-1})$. Hence conjugating each element of our Hurwitz system by $(\tilde{z};s)$ we obtain a class braid-equivalent to $[\,\underline{t}\,]$ of the form $[\hat{t}_1, \dots, \hat{t}_n, (0;\epsilon^{-1})]$. Let $\hat{t}_j = (*;t''_j)$. From the equality $\hat{t}_1 \cdots \hat{t}_n = (0;\epsilon)$ it follows that $t''_1 \cdots t''_n = \epsilon$ and so $\langle t''_1, \dots, t''_n \rangle = S_d$.

At this point we discuss one at a time the cases: $r > 1$ and $r = 1$.

Case $r > 1$. By Proposition 2 $[\hat{t}_1, \dots, \hat{t}_n, (0;\epsilon^{-1})]$ is braid-equivalent to a class of the form

$$[\,\tilde{\underline{t}}_1\,] = \Big[\left(a^1_{1_1 2_1};(1_1 2_1)\right),\ \left(b^1_{1_1 3_1};(1_1 3_1)\right), \dots, \left(e^1_{1_1 (e_1)_1};(1_1 (e_1)_1)\right), \dots,\ \left(a^r_{1_r 2_r};(1_r 2_r)\right),\ \left(b^r_{1_r 3_r};(1_r 3_r)\right), \dots, \left(e^r_{1_r (e_r)_r};(1_r (e_r)_r)\right), \left(z^2_{1_1 1_2};(1_1 1_2)\right),\ \left((z^2)'_{1_1 1_2};(1_1 1_2)\right), \dots, \left(z^{r-1}_{1_1 1_{r-1}};(1_1 1_{r-1})\right), \left((z^{r-1})'_{1_1 1_{r-1}};(1_1 1_{r-1})\right),\ \left((z^r)^1_{1_1 1_r};(1_1 1_r)\right), \dots, \left((z^r)^s_{1_1 1_r};(1_1 1_r)\right),\ (0;\epsilon^{-1})\Big]$$

where $a^i,\ b^i, \dots, e^i,\ z^j,\ (z^j)',\ (z^r)^h$ belong to $\mathbb{Z}_2$ and s is an even positive integer.

From the equality

$$(a^1_{1_1 2_1};(1_1 2_1)) \cdots \left(e^r_{1_r (e_r)_r};(1_r (e_r)_r)\right)(z^2_{1_1 1_2};(1_1 1_2)) \cdots \left((z^r)^s_{1_1 1_r};(1_1 1_r)\right) = (0;\epsilon)$$

one deduces that: $a^i = b^i = \cdots = e^i = \bar{0}$, for each $i = 1, \ldots, r$, while $z^j + (z^j)' \equiv \bar{0} \pmod 2$, for each $j = 2, \ldots, r-1$ and thus $z^j = (z^j)'$.

Now we show that $[\, \tilde{\underline{t}}_1 \,]$ is the required class. This is obvious if $z^j = \bar{0}$ for each $j = 2, \ldots, r-1$. If instead $z^j = \bar{1}$ for some $j \in \{2, \ldots, r-1\}$, we observe that in $\tilde{\underline{t}}_1$ in addition to the pair $((\bar{1}_{1_1 1_j}; (1_1 1_j)),\ (\bar{1}_{1_1 1_j}; (1_1 1_j)))$ and to $(0; \epsilon^{-1})$ there are elements of type $(z_{\alpha\ \beta}; (\alpha\ \beta))$ where the indexes $\alpha,\ \beta$ are moved both either by q_j or by a cycle different from q_j. So it is sufficient to conjugate any element of $\tilde{\underline{t}}_1$ with $(\bar{1}_{1_j \ldots (e_j)_j}; id)$, where $\bar{1}_{1_j \ldots (e_j)_j}$ is the function which sends to $\bar{1}$ only the indexes moved by q_j, to replace the pair $((\bar{1}_{1_1 1_j}; (1_1 1_j)), (\bar{1}_{1_1 1_j}; (1_1 1_j)))$ by $((0; (1_1 1_j)), (0; (1_1 1_j)))$ leaving unchanged each other element of $\tilde{\underline{t}}_1$. In fact

$$\begin{aligned}
&(\bar{1}_{1_j \ldots (e_j)_j}; id)\ (\bar{1}_{1_1 1_j}; (1_1 1_j))\ (\bar{1}_{1_j \ldots (e_j)_j}; id) \\
&\quad = (\bar{1}_{1_j \ldots (e_j)_j} + \bar{1}_{1_1 1_j} + \bar{1}_{1_1 2_j \ldots (e_j)_j}; (1_1 1_j)) \\
&\quad = (0; (1_1 1_j)),
\end{aligned}$$

while

$$\begin{aligned}
&(\bar{1}_{1_j \ldots (e_j)_j}; id)\ (0; \epsilon^{-1})\ (\bar{1}_{1_j \ldots (e_j)_j}; id) \\
&\quad = (\bar{1}_{1_j \ldots (e_j)_j} + \Phi(\epsilon^{-1})(\bar{1}_{1_j \ldots (e_j)_j});\ \epsilon^{-1}) \\
&\quad = (\bar{1}_{1_j \ldots (e_j)_j} + \bar{1}_{(e_j)_j \ldots 2_j 1_j}; \epsilon^{-1}) = (0; \epsilon^{-1}).
\end{aligned}$$

Analogously one checks that

$$(\bar{1}_{1_j \ldots (e_j)_j}; id)\ (z_{\alpha\ \beta}; (\alpha\ \beta))\ (\bar{1}_{1_j \ldots (e_j)_j}; id) = (z_{\alpha\ \beta}; (\alpha\ \beta)).$$

So reasoning for each $j \in \{2, \ldots, r-1\}$ such that $z^j = \bar{1}$ we obtain a Hurwitz system belonging to $[\, \tilde{\underline{t}}_1 \,]$ that is of same type of $\underline{t}_1$.

Case $r = 1$. By Proposition 2 $[\hat{t}_1, \ldots, \hat{t}_n, (0; \epsilon^{-1})]$ is braid-equivalent to

$$\begin{aligned}
[\, \tilde{\underline{t}}_2 \,] = [&(a^1_{1_1 2_1}; (1_1 2_1)), (b^1_{1_1 3_1}; (1_1 3_1)), \ldots, (e^1_{1_1 (e_1)_1}; (1_1 (e_1)_1)), \\
&(z^1_{1_1 2_1}; (1_1 2_1)), \ldots, (z^s_{1_1 2_1}; (1_1 2_1)), (0; \epsilon^{-1})]
\end{aligned}$$

where $a^1,\ b^1, \ldots, e^1,\ z^h$ belong to $\mathbb{Z}_2$ and s is an even positive integer.

From the equality

$$(a^1_{1_1 2_1}; (1_1 2_1)) \cdots (e^1_{1_1 (e_1)_1}; (1_1 (e_1)_1))\ (z^1_{1_1 2_1}; (1_1 2_1)) \cdots (z^s_{1_1 2_1}; (1_1 2_1)) = (0; \epsilon)$$

one deduces that: $a^1 = b^1 = \cdots = e^1$ and $a^1 + z^1 + \cdots + z^s \equiv \bar{0} \pmod 2$.

If $a^1 = b^1 = \cdots = e^1 = \bar{0}$ the equivalence class of Hurwitz systems so obtained is one required. Then we suppose that $a^1 = b^1 = \cdots = e^1 = \bar{1}$. The relation $a^1 + z^1 + \cdots + z^s \equiv \bar{0} \pmod 2$ assures that the z^h equal to $\bar{1}$ are odd in number. Because of this and since s is even, we know that in $\tilde{\underline{t}}_2$ among the elements of type $(z^h_{1_1 2_1}; (1_1 2_1))$ there is at least one pair of the form $((\bar{1}_{1_1 2_1}; (1_1 2_1)),\ (0, (1_1 2_1)))$. Because it is not restrictive suppose that the elements of this pair occupy the places e_1 and $e_1 + 1$, to obtain a class as required it is sufficient to use the elementary moves $\sigma''_{(e_1)_1 - 1},\ \sigma''_{(e_1)_1},\ \sigma''_{(e_1)_1 - 2},\ \sigma''_{(e_1)_1 - 1}, \ldots, \sigma''_2,\ \sigma''_3$ and then Lemma 1. □

Theorem 1. *The Hurwitz space* $H_{W(D_d),\ n,\ \underline{e}}(\mathbb{P}^1)$ *is irreducible.*

Proof. Since the Hurwitz space $H_{W(D_d),\ n,\ \underline{e}}\ (\mathbb{P}^1)$ is smooth in order to prove its irreducibility it suffices to show that it is connected and then that the braid group $\pi_1(Y^{(n+1)} - \Delta, D)$ acts transitively on $A_{W(D_d),n,\ \underline{e}}$. To do this it is enough to show that, acting by elementary moves σ'_j and their inverses, it is possible to replace any equivalence class in $A_{W(D_d),n,\ \underline{e}}$ with the normal form:

(i) if $r > 1$

$$[T_1] = [\widetilde{Z},\ (0; (1_1 1_2)), (0; (1_1 1_2)),\ldots, (0; (1_1 1_{r-1})),\ (0; (1_1 1_{r-1})), (\bar{1}_{1_1 1_r}; (1_1 1_r)), (0; (1_1 1_r)), (\bar{1}_{1_1 1_r}; (1_1 1_r)), (0; (1_1 1_r)), (0; (1_1 1_r)), \ldots, (0; (1_1 1_r)),\ (0; \epsilon^{-1})]$$

where each $(0; (1_1\ 1_i))$, $2 \le i\ \le r-1$, and $(\bar{1}_{1_1 1_r}; (1_1 1_r))$ appear twice while $(0; (1_1 1_r))$ appears an even number of times,

(ii) if $r = 1$

$$[T_2]=[\widetilde{Z}_1, (\bar{1}_{1_1 2_1}; (1_1 2_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)), (0; (1_1 2_1)),\ldots, (0; (1_1 2_1)), (0; \epsilon^{-1})]$$

where $(\bar{1}_{1_1 2_1}; (1_1 2_1))$ appears twice while $(0; (1_1 2_1))$ appears an even number of times.

The equivalence classes belonging to $A_{W(D_d),n,\ \underline{e}}$ satisfy all the hypothesis of Proposition 3 and therefore each class in $A_{W(D_d),n,\ \underline{e}}$ is braid-equivalent to a class of the form either $[\ \underline{t}_1\ ]$ or $[\ \underline{t}_2\ ]$ depending on whether $r > 1$ or $r = 1$. Recall that in $A_{W(D_d),n,\ \underline{e}}$ there are equivalence classes of Hurwitz systems whose monodromy group is $W(D_d)$ and moreover the conjugation with elements of $W(B_d)$ and the action of elementary moves leave unchanged the monodromy group. Then we can affirm that each class in $A_{W(D_d),n,\ \underline{e}}$ is braid-equivalent, depending on whether $r > 1$ or $r = 1$, to a class of the form either $[\ \underline{t}_1\ ]$ where among the elements of $\underline{t}_1$ there is a pair of type $((\bar{1}_{1_1 1_r}; (1_1 1_r)), (0; (1_1 1_r)))$ or $[\ \underline{t}_2\ ]$ where certainly one z^h is equal to $\bar{1}$.

In fact if $z^1 = \cdots = z^s = \bar{0}$ the monodromy group of $\underline{t}_1$ and $\underline{t}_2$ is contained properly in $W(D_d)$. The same thing one can say on the monodromy group of $\underline{t}_1$ if $z^1 = \cdots = z^s = \bar{1}$. In fact it is enough conjugate each element of $\underline{t}_1$ by $(\bar{1}_{1_r \ldots (e_r)_r}; id)$ (see proof of Proposition 3) to reduce us to the case $z^1 = \cdots = z^s = \bar{0}$.

At this point we analyze separately the cases: $r > 1$ and $r = 1$.

Case $r > 1$. By the preceding argument we know that each class of $A_{W(D_d),n,\ \underline{e}}$ is braid-equivalent to a class of type $[\widetilde{Z},\ (0; (1_1 1_2)),\ (0; (1_1 1_2)), \ldots, (0; (1_1 1_{r-1})), (0; (1_1 1_{r-1})),\ (z^1_{1_1 1_r}; (1_1 1_r)), \ldots, (z^s_{1_1 1_r}; (1_1 1_r)),\ (0; \epsilon^{-1})]$ where there are both $(\bar{1}_{1_1 1_r}; (1_1\ 1_r))$ and $(0; (1_1\ 1_r))$. From the equality

$$\widetilde{Z}\ (0; (1_1 1_2)) \cdots (0; (1_1 1_{r-1}))\ (z^1_{1_1 1_r}; (1_1 1_r)) \cdots (z^s_{1_1 1_r}; (1_1 1_r)) = (0; \epsilon)$$

we deduce that $z^1 + \cdots + z^s \equiv \bar{0}$ (mod 2) and so the number of $z^h = \bar{1}$ is even and greater or equal to 2. Let $2m + 2$ be the number of the elements of type $(\bar{1}_{1_1 1_r}; (1_1\ 1_r))$ in $\underline{t}_1$. With elementary moves we can replace these elements as following

$$[\ldots, (0; (1_1\ 1_{r-1})),\ (\bar{1}_{1_1 1_r}; (1_1\ 1_r)), \ldots, (\bar{1}_{1_1 1_r}; (1_1 1_r)),\ (\bar{1}_{1_1 1_r}; (1_1\ 1_r)), (0; (1_1 1_r)), (\bar{1}_{1_1 1_r}; (1_1 1_r)),\ (0; (1_1\ 1_r)), (0; (1_1 1_r)),\ldots, (0; (1_1 1_r)), (0; \epsilon^{-1})].$$

Hence using the moves $\sigma'_{\sum_i e_i+r-4},\ \sigma'_{(\sum_i e_i+r-4)+1},\ \ldots,\ \sigma'_{(\sum_i e_i+r-4)+2m-1}$ and Lemma 1 we can replace the sequence $((0;\,(1_1\ 1_{r-1})),\,(\bar{1}_{1_1 1_r};\,(1_1\ 1_r)),\ldots,$ $(\bar{1}_{1_1 1_r};\,(1_1\ 1_r)),\ (\bar{1}_{1_1 1_r};\,(1_1\ 1_r)),\ (0;\,(1_1\ 1_r)))$ by

$$((0;\,(1_1\ 1_{r-1})),\ (\bar{1}_{1_{r-1}1_r};\,(1_{r-1}\ 1_r)),\ldots,(\bar{1}_{1_{r-1}1_r};\,(1_{r-1}\ 1_r)),$$
$$(\bar{1}_{1_1 1_r};\,(1_1\ 1_r)),(0;\,(1_1\ 1_r))).$$

Now applying $\sigma''_{(\sum_i e_i+r-4)+2m},\ \sigma''_{(\sum_i e_i+r-4)+2m-1},\ldots,\sigma''_{(\sum_i e_i+r-4)+1}$ we obtain that the sequence above is braid-equivalent to

$$((0;\,(1_1 1_{r-1})),\ (\bar{1}_{1_1 1_r};\,(1_1 1_r)),\ (0;\,(1_1 1_{r-1})),\ldots,(0;\,(1_1 1_{r-1})),\ (0;\,(1_1 1_r))).$$

Using the braid moves $\sigma''_{(\sum_i e_i+r-4)+2m+1},\ldots,\sigma''_{(\sum_i e_i+r-4)+2}$ and Lemma 1 we replace the our sequence by

$$((0;\,(1_1 1_{r-1})),\ (0;\,(1_{r-1} 1_r)),\ldots,(0;\,(1_{r-1} 1_r)),(\bar{1}_{1_1 1_r};\,(1_1 1_r)),(0;\,(1_1 1_r))).$$

At this point, to complete the proof in the case $r>1$, we make use of the braid moves $\sigma'_{\sum_i e_i+r-4},\ \ldots,\sigma'_{(\sum_i e_i+r-4)+2m-1}$ and of Lemma 1.

Case $r=1$. We have already observed that each equivalence class in $A_{W(D_d),n,\,\underline{e}}$ is braid-equivalent to a class of the form $[\widetilde{Z}_1,\ (z^1_{1_1\,2_1};\,(1_1\,2_1)),\ldots,(z^s_{1_1 2_1};\,(1_1 2_1)),$ $(0;\,\epsilon^{-1})]$ where there is certainly one $(\bar{1}_{1_1 2_1};\,(1_1 2_1))$. It follows from the relation

$$\widetilde{Z}_1\ (z^1_{1_1 2_1};\,(1_1 2_1))\cdots(z^s_{1_1 2_1};\,(1_1 2_1))=(0;\,\epsilon)$$

that the number of elements of type $(\bar{1}_{1_1 2_1};\,(1_1 2_1))$ is even and greater or equal to 2. We write $2m+2$ for the number of these elements. With suitable elementary moves we can replace to the right of $(0;\,(1_1(e_1)_1))$ the elements of type $(\bar{1}_{1_1 2_1};\,(1_1 2_1))$ and then we use $\sigma'_{e_1-1},\ \sigma'_{e_1},\ldots,\sigma'_{e_1+2m-2}$ and Lemma 1 so that results

$$((0;\,(1_1(e_1)_1)),\ (\bar{1}_{1_1 2_1};\,(1_1 2_1)),\ldots,(\bar{1}_{1_1 2_1};\,(1_1 2_1)))$$
$$\sim((0;\,(1_1(e_1)_1)),(\bar{1}_{(e_1)_1 2_1};\,((e_1)_1 2_1)),\ldots,(\bar{1}_{(e_1)_1 2_1};\,((e_1)_1 2_1)),$$
$$(\bar{1}_{1_1 2_1};\,(1_1 2_1)),(\bar{1}_{1_1 2_1};\,(1_1 2_1))).$$

Applying the elementary moves $\sigma''_{e_1+2m-1},\ \sigma''_{e_1+2m-2},\ldots,\sigma''_{e_1}$ and using Lemma 1 we obtain that the sequence above is braid-equivalence to

$$((0;\,(1_1(e_1)_1)),(0;\,(1_1(e_1)_1)),\ldots,(0;\,(1_1(e_1)_1)),(\bar{1}_{1_1 2_1};\,(1_1 2_1)),(\bar{1}_{1_1 2_1};\,(1_1 2_1))).$$

By Lemma 1 we can move $2m$ elements of type $(0;\,(1_1(e_1)_1))$ to the places $2,\ldots,2m+1$ leaving unchanged the other elements of the Hurwitz system. Now using the elementary moves $\sigma'_1,\ldots,\sigma'_{2m}$ and Lemma 1 we obtain a class braid-equivalent to ours of type

$[\widetilde{Z}_1, (0; (2_1(e_1)_1)), \dots, (0; (2_1(e_1)_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)), (0; \epsilon^{-1})]$.

Hence to complete the proof it is sufficient to apply the moves $\sigma'_{e_1-1}, \sigma'_{e_1}, \dots, \sigma'_{e_1+2m-2}$ and Lemma 1. □

Theorem 2. *If $n + |\underline{e}| \geq 2d$ the Hurwitz space $H_{W(D_d),\, n,\, \underline{e}}\,(Y)$ is irreducible.*

Proof. Since $H_{W(D_d),\, n,\, \underline{e}}\,(Y)$ is smooth to prove the theorem it is enough to show that it is connected. To do this it is sufficient to check that $\pi_1(Y^{(n+1)} - \Delta, D)$ acts transitively on $A_{W(D_d),n,\, \underline{e},\, g}$ and then it suffices to prove that each equivalence class $[t_1, \dots, t_{n+1}; \lambda_1,\ \mu_1, \dots, \lambda_g, \mu_g]$ belonging to $A_{W(D_d),n,\, \underline{e},g}$ is braid-equivalent to the normal form:

(i) If $r > 1$

$$[T_1;\ (0; id), \dots, (0; id)],$$

(ii) If $r = 1$

$$[T_2;\ (0; id), \dots, (0; id)],$$

where T_1 and T_2 are the Hurwitz systems which give the normal forms in Theorem 1.
Step 1. Let $t_j = (*; t'_j)$, $\lambda_k = (*; \lambda'_k)$ and $\mu_k = (*; \mu'_k)$, $j = 1, \dots, n+1$, $k = 1, \dots, g$. By Riemann's existence theorem the equivalence class of Hurwitz systems $[t'_1, \dots, t'_{n+1}; \lambda'_1, \dots, \mu'_g]$ corresponds to an equivalence class of coverings belonging to $H^o_{d,n,\, \underline{e}}(Y)$. Since $n + |\underline{e}| \geq 2d$ the Hurwitz space $H^o_{d,n,\, \underline{e}}(Y)$ is irreducible (see [19], Theorem 1). Therefore it is possible, acting by braid moves σ'_j, ρ'_{ik}, τ'_{ik} and their inverses, to replace $[t'_1, \dots, \mu'_g]$ with $[t''_1, \dots, t''_n, \epsilon^{-1}; id, \dots, id]$. In this way $[t_1, \dots, t_{n+1}; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g]$ results braid-equivalent to a class of the form $[\tilde{t}_1, \dots, \tilde{t}_n, (b'; \epsilon^{-1}); (a_1; id), (b_1; id), \dots, (a_g; id), (b_g; id)]$. Because $(b'; \epsilon^{-1})$ and $(0; \epsilon^{-1})$ belong to the same conjugate class of $(\mathbf{Z}_2)^d \times^s S_d$, there exists one element $(z'; id) \in (\mathbf{Z}_2)^d \times^s S_d$ such that $(z'; id)(b'; \epsilon^{-1})(z'; id) = (0; \epsilon^{-1})$ (see Observation 1). Conjugating each element of our Hurwitz system with $(z'; id)$ we obtain a new system belonging to our class of type $(\widehat{t}_1, \dots, \widehat{t}_n, (0; \epsilon^{-1});\ (a_1; id),\ (b_1; id), \dots, (a_g; id), (b_g; id))$.
Step 2. In step 1 we showed that $[t_1, \dots, \mu_g]$ is braid-equivalent to $[\widehat{t}_1, \dots, \widehat{t}_n, (0; \epsilon^{-1});\ (a_1; id),\ (b_1; id), \dots, (a_g; id), (b_g; id)]$. At this point we claim that it is braid-equivalent to a class of type $[\tilde{t}_1, \dots, \tilde{t}_n,\ (0; \epsilon^{-1}); (0; id), \dots, (0; id)]$. Once proved this one observes that $[\tilde{t}_1, \dots, \tilde{t}_n, (0; \epsilon^{-1})]$ is the equivalence class of Hurwitz systems associated to a class of coverings in $H_{W(D_d),\, n,\, \underline{e}}\,(\mathbb{P}^1)$ and so the proof follows by Theorem 1.

Recall that $(a_k; id)$ and $(b_k; id)$ are elements of $W(D_d)$. Therefore if a_k and b_k are functions different from 0, they send to $\bar{1}$ an even number of indexes. Suppose that a_1 is a function different from 0. Let i and j be two indexes sent to $\bar{1}$ by a_1. Observe that if, acting by braid moves of type σ'_l, σ''_l, $1 \leq l \leq n-1$, we can obtain a class braid-equivalent to ours in which there are both $(\bar{1}_{ij}; (ij))$ and $(0; (ij))$ then our class is braid-equivalent to a class of the form

$[\hat{t}_1, \ldots, \hat{t}_n, (0;\ \epsilon^{-1}); (\hat{a}_1; id), (b_1; id), \ldots, (a_g; id), (b_g; id)]$ where $\hat{a}_1$ is a function which sends to $\bar{1}$ the same indexes sent to $\bar{1}$ by a_1 except i and j. In fact, using elementary moves σ_l'' we can bring to the first place one of two elements of type $(z_{ij}; (ij))$ and then we apply the move τ_{11}'' that transforms $(a_1; id)$ in $(z_{ij}; (ij))(a_1; id)$. Now we move to the first place the other element of type $(z_{ij}'; (ij))$, where $z' = \bar{1}$ if $z = \bar{0}$ and $z' = \bar{0}$ if $z = \bar{1}$ and we again act by τ_{11}''. In this way we replace $(z_{ij}; (ij))(a_1; id)$ with $(z_{ij}'; (ij))\ (z_{ij}; (ij))\ (a_1; id) = (\bar{1}_{ij}; id)\ (a_1; id) = (\bar{1}_{ij} + a_1; id)$ where $\hat{a}_1 = \bar{1}_{ij} + a_1$ is a function which sends i and j to $\bar{0}$.

We start showing that $[\widehat{t_1}, \ldots, \widehat{t_n}, (0; \epsilon^{-1}); (a_1; id), (b_1; id), \ldots, (a_g; id), (b_g; id)]$ is braid-equivalent to a class of the form $[\hat{t}_1, \ldots, \hat{t}_n, (0; \epsilon^{-1});\ (\hat{a}_1; id), (b_1; id), \ldots, (a_g; id), (b_g; id)]$.

The relation

$$[(a_1; id), (b_1; id)] \cdots [(a_g; id), (b_g; id)] = (0; id),$$

implies that $\widehat{t_1} \cdots \widehat{t_n} = (0; \epsilon)$ and then the group generated by the transpositions corresponding to the $\widehat{t_j}$ is all S_d. Hence $[\widehat{t_1}, \ldots, \widehat{t_n}, (0; \epsilon^{-1})]$ satisfies all the hypothesis of Proposition 3 and thus it is braid-equivalent to a class of the form $[\ \underline{t}_1\ ]$ or $[\ \underline{t}_2\ ]$ depending wether $r > 1$ or $r = 1$. Note that by transforming $(\widehat{t_1}, \ldots, \widehat{t_n}, (0; \epsilon^{-1}))$ to $\underline{t}_1$ or $\underline{t}_2$ we leave unchanged the elements $(a_k; id)$ and $(b_k; id)$ (see proof of Proposition 3). Because of this we can affirm that our class is braid-equivalent to a class of the form $[\underline{t}_1; (a_1; id), (b_1; id), \ldots, (a_g; id), (b_g; id)]$ or $[\underline{t}_2; (a_1; id), (b_1; id), \ldots, (a_g; id), (b_g; id)]$ depending wether $r > 1$ or $r = 1$. We discuss separately the cases: $r > 1$ and $r = 1$.

Case $r > 1$. Note that is not restrictive to suppose that in $\underline{t}_1$ there are at least two $(\bar{1}_{1_1 1_r}; (1_1 1_r))$ and two $(0; (1_1 1_r))$. In fact the hypothesis $n + |\underline{e}| \geq 2d$ assures that in $\underline{t}_1$ there are at least four elements of type $(*; (1_1 1_r))$. Hence if $z^1 = \cdots = z^s$ and we cancel two among the $(*; (1_1 1_r))$ the group generated by the remaining elements in $(\underline{t}_1; (a_1; id), (b_1; id), \ldots, (a_g; id), (b_g; id))$ is still $W(D_d)$. Because of this we can by Lemma 2 to replace the pair $((z^1_{1_1 1_r}; (1_1 1_r)), (z^1_{1_1 1_r}; (1_1 1_r)))$ with $((z_{1_1 1_r}; (1_1 1_r)),\ (z_{1_1 1_r}; (1_1 1_r)))$ where $z + z^1 \equiv \bar{1}$ (mod 2), it is sufficient to choose $h = (\bar{1}_{1_1 2_1}; id)$. We discuss at first the case $i = 1_1$ and $j \neq 1_r$ (in a similar manner one affronts the case $i = 1_r$ and $j \neq 1_1$). If j is an index moved by the cycle q_r in $\underline{t}_1$ there is the element $(0; (1_r j))$. We move $(0; (1_r j))$ to the left of one pair of type $((\bar{1}_{1_1 1_r}; (1_1 1_r)),\ (0; (1_1 1_r)))$. If the elements of this pair occupy the places $h,\ h + 1$, we use $\sigma_{h-1}',\ \sigma_h'$ to obtain a new class in which there is the pair $((\bar{1}_{1_1 j}; (1_1 j)),\ (0; (1_1 j)))$. If j is an index moved by q_1 in $\underline{t}_1$ there is already $(0; (1_1 j))$. We move it to the left of one sequence of type $((0; (1_1 1_r)),\ (0; (1_1 1_r)),\ (\bar{1}_{1_1 1_r}; (1_1 1_r)))$. If the elements of these sequence occupy the places $h,\ h + 1,\ h + 2$ we use $\sigma_{h-1}',\ \sigma_h'$ and Lemma 1 to have

$$\begin{aligned} &((0; (1_1 j)),\ (0; (1_1 1_r)),\ (0; (1_1 1_r)),\ (\bar{1}_{1_1 1_r}; (1_1 1_r))) \\ &\quad \sim ((0; (1_1 j)),\ (0; (1_r j)),\ (0; (1_r j)), (\bar{1}_{1_1 1_r}; (1_1 1_r))) \end{aligned}$$

and then it is sufficient to apply σ_{h+1}' to obtain a sequence in which there is the pair $((0; (1_1 j)),\ (\bar{1}_{1_1 j}; (1_1 j)))$. In the end if j is an index moved by a cycle q_a, with $a \neq 1, r$, in $\underline{t}_1$ there is the element $(0; (1_a j))$ and there are both the pair $((0; (1_a 1_1)),\ (0; (1_a 1_1)))$ and the pair $((\bar{1}_{1_1 1_r}; (1_1 1_r)),\ (0; (1_1 1_r)))$. By Lemma 1, we can move the pair $((0; (1_a 1_1)),\ (0; (1_a 1_1)))$ to the left of the pair $((\bar{1}_{1_1 1_r}; (1_1 1_r)),\ (0; (1_1 1_r)))$ and then with suitable elementary moves we bring $(0; (1_a j))$ to the left of $((0; (1_a 1_1)),\ (0; (1_a 1_1)))$. If now $(0; (1_a j))$ is at the place h, we apply $\sigma'_h,\ \sigma'_{h+2},\ \sigma''_{h+3},\ \sigma''_{h+2}$ to replace $((0; (1_a j)),\ (0; (1_a 1_1)),\ (0; (1_a 1_1)),$ $(\bar{1}_{1_1 1_r}; (1_1 1_r)),\ (0; (1_1 1_r)))$ by

$$((0; (1_1 j)),\ (0; (1_a j)),\ (0; (1_1 1_r)),\ (\bar{1}_{1_1 1_a}; (1_1 1_a)),\ (0; (1_r 1_a))).$$

At this point to do in way that among the elements of our Hurwitz system there is the required pair we use $\sigma''_{h+2},\ \sigma'_{h+1}$. In the end we analyze the case in which i and j are indexes different from 1_1 and 1_r. We distinguish the case in which i and j are indexes moved by a same cycle q_a from one in which i and j are indexes moved by two different cycles q_a and q_b. If i and j are indexes moved by a same cycle q_a in $\underline{t}_1$ there are the elements $(0; (1_a i))$, $(0; (1_a j))$ and the pairs $((0; (1_a 1_1)), (0; (1_a 1_1)))$, $((0; (1_1 1_r)), (0; (1_1 1_r)))$, $((\bar{1}_{1_1 1_r}; (1_1 1_r)), (\bar{1}_{1_1 1_r}; (1_1 1_r)))$. Suppose $i < j$. Using suitable elementary moves and Lemma 1 we can replace them as following $[\ldots, (0; (1_a i)),\ (0; (1_a j)),\ (0; (1_a 1_1)),\ (\bar{1}_{1_1 1_r}; (1_1 1_r)),\ (\bar{1}_{1_1 1_r}; (1_1 1_r)), (0; (1_a 1_1)), ((0; (1_1 1_r)), (0; (1_1 1_r)), \ldots]$. If now $(0; (1_a i))$ is at the place h we act by $\sigma''_{h+1},\ \sigma''_h,\ \sigma''_{h+5}$ to replace the sequence above with

$$((0; (1_1 1_a)), (0; (1_1 i)), (0; (1_1 j)), (\bar{1}_{1_1 1_r}; (1_1 1_r)), (\bar{1}_{1_1 1_r}; (1_1 1_r)), (0; (1_1 1_r)),$$
$$(0; (1_a 1_r)), (0; (1_1 1_r))). \qquad (\star)$$

Note that when $a = r$ the pair $((0; (1_a 1_1)), (0; (1_a 1_1)))$ coincides with the pair $((0; (1_1 1_r)), (0; (1_1 1_r)))$ and so to obtain the sequence ($\star$) one only uses σ''_{h+1} and σ''_h. When $a = 1$ in $\underline{t}_1$ there are already both $(0; (1_1 i))$ and $(0; (1_1 j))$ and thus to obtain the sequence ($\star$) it is sufficient to move $(0; (1_1 i))$ to the left of $(0; (1_1 j))$ and then to use Lemma 1.

By Lemma 1 we move the pair $((\bar{1}_{1_1 1_r}; (1_1 1_r)),\ (\bar{1}_{1_1 1_r}; (1_1 1_r)))$ to the right of $(0; (1_1 i))$, after we act by $\sigma'_{h+1},\ \sigma'_{h+2},\ \sigma'_{h+4}$ and again we use Lemma 1 to obtain that ($\star$) is braid-equivalent to $((0; (1_a 1_1)),\ (0; (1_1 i)),\ (\bar{1}_{i 1_r}; (i 1_r)),\ (\bar{1}_{i 1_r}; (i 1_r)),$ $(0; (j 1_r)), (0; (1_1 j)),\ (0; (1_a 1_r)), (0; (1_1 1_r)))$. Acting by $\sigma''_{h+3},\ \sigma''_{h+2},\ \sigma'_{h+1}$ and using Lemma 1 we can replace the sequence above with

$$((0; (1_a 1_1)),\ (0; (1_r j)),\ (\bar{1}_{ij}; (ij)),\ (\bar{1}_{ij}; (ij)),\ (0; (i 1_1)),\ (0; (1_1 j)),$$
$$(0; (1_a 1_r)), (0; (1_1 1_r))).$$

Now one obtains a sequence braid-equivalent contained the pair $((0; (ij)),$ $(\bar{1}_{ij}; (ij)))$ acting with σ'_{h+4}. Observe that the case in which i and j are indexes moved by two different cycles q_a and q_b can be reduced at the case just analyzed. In fact if i is an index moved by q_a and j by q_b in $\underline{t}_1$ there are the elements $(0; (1_a i)),\ (0; (1_b j))$ and the pairs $((0; (1_1 1_a)),\ (0; (1_1 1_a)))$, $((0; (1_1 1_b)),$ $(0; (1_1 1_b)))$. By Lemma 1 we can move to the right of $(0; (1_a i))$ the pair $((0; (1_1 1_a)),$

$(0; (1_1 1_a)))$ and to the right of $(0; (1_b j))$ the pair $((0; (1_1 1_b)), (0; (1_1 1_b)))$. If $(0; (1_a i))$ and $(0; (1_b j))$ occupy respectively the places h and k, we use σ'_h and σ'_k and so we return to the case in which i and j are indexes moved by a same cycle q_a with $a = 1$.
Case $r = 1$. We already observed that our class is braid-equivalent to a class of type $[\underline{t}_2; (a_1; id), (b_1; id), \dots, (a_g; id), (b_g; id)]$. Note that is not restrictive to suppose that in $\underline{t}_2$ there is the pair $((\bar{1}_{1_1 2_1}; (1_1 2_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)))$. In fact the hypothesis $n + |\underline{e}| \geq 2d$ assures that in $\underline{t}_2$ there are at least two elements of type $(z^k; (1_1 2_1))$. If $z^1 = \dots = z^s = \bar{0}$ in $\underline{t}_2$ there are three elements of type $(0; (1_1 2_1))$, so if we cancel two of these the group generated by the remaining elements of $(\underline{t}_2; (a_1; id), (b_1; id), \dots, (a_g; id), (b_g; id))$ is still $W(D_d)$. Then by Lemma 2 we can replace $((0; (1_1 2_1)), (0; (1_1 2_1)))$ with $((\bar{1}_{1_1 2_1}; (1_1 2_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)))$, it is sufficient to choose $h = (\bar{1}_{1_1 3_1}; id)$ (recall that $d \geq 3$). Now we check that acting by elementary moves it is possible to obtain a sequence braid-equivalent to $\underline{t}_2$ in which there is the pair $((\bar{1}_{ij}; (ij)), (0; (ij)))$. If i is equal either to 1_1 or to 2_1 while $j \notin \{1_1, 2_1\}$ in $\underline{t}_2$ there is $(0; (1_1 j))$ and there is $((\bar{1}_{1_1 2_1}; (1_1 2_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)))$. We move $(0; (1_1 j))$ to the second place and then use Lemma 1 to move the pair $((\bar{1}_{1_1 2_1}; (1_1 2_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)))$ to its right. Now to obtain the required pair it is sufficient to act either with σ'_2, σ'_1 or with $\sigma'_2, \sigma'_1, \sigma''_3, \sigma''_2, \sigma'_1$ depending if i is equal to 1_1 or to 2_1. If instead the indexes $i, j \notin \{1_1, 2_1\}$, in $\underline{t}_2$ there are $(0; (1_1 i))$, $(0; (1_1 j))$ and the pair $((\bar{1}_{1_1 2_1}; (1_1 2_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)))$. Suppose $i < j$. We move $(0; (1_1 i))$ and $(0; (1_1 j))$ respectively to the second and to the third place and after we use Lemma 1 to bring the pair $((\bar{1}_{1_1 2_1}; (1_1 2_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)))$ to the right of $(0; (1_1 j))$. Applying $\sigma''_1, \sigma'_3, \sigma'_4$ and using Lemma 1 we have that the sequence $((0; (1_1 2_1)), (0; (1_1 i)), (0; (1_1 j)), (\bar{1}_{1_1 2_1}; (1_1 2_1)), (\bar{1}_{1_1 2_1}; (1_1 2_1)))$ is braid-equivalent to

$$((0; (1_1 i)), (0; (2_1 i)), (0; (1_1 j)), (\bar{1}_{j 2_1}; (j 2_1)), (\bar{1}_{j 2_1}; (j 2_1))).$$

Now we obtain the required pair using the elementary moves $\sigma'_2, \sigma'_3, \sigma'_1$.

Till now we proved that both $r > 1$ and $r = 1$ the class $[\widehat{t_1}, \dots, \widehat{t_n}, (0; \epsilon^{-1}); (a_1; id), (b_1; id), \dots, (a_g; id), (b_g; id)]$ is braid-equivalent to a class of the form $[\hat{t}_1, \dots, \hat{t}_n, (0; \epsilon^{-1}); (\hat{a}_1; id), (b_1; id), \dots, (a_g; id), (b_g; id)]$ where $\hat{a}_1$ is a function which sends to $\bar{1}$ the same indexes sent to $\bar{1}$ by a_1 except i and j.

We note that $[\hat{t}_1, \dots, \hat{t}_n, (0; \epsilon^{-1})]$ is still a class that satisfies the hypothesis of Proposition 3, so one can proceed for each pair of indexes which $\hat{a}_1$ sends to $\bar{1}$ as one made by the pair (i, j). In this way, after a finite number of steps, we are able to replace $[\hat{t}_1, \dots, \hat{t}_n, (0; \epsilon^{-1}); (\hat{a}_1; id), (b_1; id), \dots, (a_g; id), (b_g; id)]$ with a class of the form $[\check{t}_1, \dots, \check{t}_{n_2}, (0; \epsilon^{-1}); (0; id), (b_1; id), \dots, (a_g; id), (b_g; id)]$.

Now if b_1 is a function different from 0 to replace $(b_1; id)$ with $(0; id)$ one proceeds in the same way but using the braid move ρ'_{11}. Analogously one reasons when a_k is different from 0 and a_l, b_l are equal to 0 for each $l \leq k - 1$, but one uses the braid move τ''_{1k}. In the end if b_k is different from 0 and a_l, b_l, a_k, $l \leq k - 1$, are equal to 0, to replace $(b_k; id)$ with $(0; id)$ one applies the braid moves ρ'_{1k}.

This completes the proof of the theorem. □

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