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Lower order eigenvalues of Dirichlet Laplacian

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Abstract. In this paper, we investigate an eigenvalue problem for the Dirichlet Laplacian on a domain in an n-dimensional compact Riemannian manifold. First we give a general inequality for eigenvalues. As one of its applications, we study eigenvalues of the Laplacian on a domain in an n-dimensional complex projective space, on a compact complex submanifold in complex projective space and on the unit sphere. By making use of the orthogonalization of Gram–Schmidt (QR-factorization theorem), we construct trial functions. By means of these trial functions, estimates for lower order eigenvalues are obtained.

1. Introduction

Let M be an n-dimensional compact C^∞ Riemannian manifold with or without boundary, where the boundary ∂M of M is assumed to be C^∞. It is known that a large amount of information about the manifold is carried by the spectrum of its Laplacian. The spectrum of the Laplacian on M is an important analytic invariant and has important geometric meanings (cf. Chavel [7] and Protter [23]).

For $M = \Omega$ a bounded domain in $\mathbf{R}^n$, let $\{\lambda_i\}$ be the set of eigenvalues and $\{u_i\}$ an orthonormal basis of eigenfunctions of the following Dirichlet eigenvalue problem:

$$\begin{cases} \triangle u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\triangle$ denotes the Laplacian on $\mathbf{R}^n$. It is well known that the spectrum of this eigenvalue problem (1.1) is real and discrete:

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty,$$

where each eigenvalue is repeated with its multiplicity. When $\Omega = \mathbf{B}^n$ is the n-dimensional unit ball in $\mathbf{R}^n$, we write $\lambda_i(\mathbf{B}^n)$ for these eigenvalues. It is well known that $\lambda_i(\mathbf{B}^n)$ are given by squares of the positive zeros of Bessel functions, e.g. $\lambda_1(\mathbf{B}^n) = j^2_{n/2-1,1}$ and $\lambda_2(\mathbf{B}^n) = \cdots = \lambda_{n+1}(\mathbf{B}^n) = j^2_{n/2,1}$, where $j_{p,k}$ denotes the kth positive zero of the Bessel function $J_p(x)$ of the first kind of order p. The following conjecture of Payne, Pólya and Weinberger is well known:

Conjecture of Payne, Pólya and Weinberger: *For a bounded domain Ω in $\mathbf{R}^n$, the eigenvalues of* (1.1) *satisfy*

(1) $\dfrac{\lambda_2}{\lambda_1} \le \dfrac{\lambda_2(\mathbf{B}^n)}{\lambda_1(\mathbf{B}^n)}$,

(2) $\dfrac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \le \dfrac{\lambda_2(\mathbf{B}^n) + \lambda_3(\mathbf{B}^n) + \cdots + \lambda_{n+1}(\mathbf{B}^n)}{\lambda_1(\mathbf{B}^n)} = n\dfrac{\lambda_2(\mathbf{B}^n)}{\lambda_1(\mathbf{B}^n)}$.

The conjecture (1) of Payne, Pólya and Weinberger was studied by many mathematicians, for examples, Payne, Pólya and Weinberger [21, 22], Brands [6], de Vries [12], Chiti [11], Hile and Protter [16]. Finally, Ashbaugh and Benguria [3] (cf. [1, 2] and [4]) proved this conjecture.

With regard to the conjecture (2) of Payne, Pólya and Weinberger, in the case $n = 2$, the bound $\dfrac{\lambda_2 + \lambda_3}{\lambda_1} \le 6$ of Payne, Pólya and Weinberger [22] was improved to $\dfrac{\lambda_2 + \lambda_3}{\lambda_1} \le 3 + \sqrt{7}$ by Brands [6]. Furthermore, Hile and Protter [16] obtained $\dfrac{\lambda_2 + \lambda_3}{\lambda_1} \le 5.622$. In [20], Marcellini proved $\dfrac{\lambda_2 + \lambda_3}{\lambda_1} \le (15 + \sqrt{345})/6$. In 1993, for general dimensions $n \ge 2$, Ashbaugh and Benguria [5] proved

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \le n\left(1 + \frac{4}{n}\right). \tag{1.2}$$

In this paper, we consider an eigenvalue problem for the Dirichlet Laplacian on a domain Ω in an n-dimensional compact Riemannian manifold without boundary. In the sequel, we will always assume that boundary $\partial\Omega$ of the domain Ω is C^∞. First we will give a general inequality for eigenvalues of the Dirichlet Laplacian. As an application, we study lower order eigenvalues of the Laplacian on a domain in an n-dimensional complex projective space $\mathbf{CP}^n(4)$, on a compact complex submanifold in complex projective space and on the unit sphere, that is, we will give an upper bound for $\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}$, where n is the dimension of the Riemannian manifold. We use the notation $\mathbf{CP}^n(4)$ in this paper to denote the n-dimensional complex projective space equipped with the Fubini-Study metric of the holomorphic sectional curvature 1 (whereas $\mathbf{CP}^n$ carries the Fubini-Study metric with holomorphic sectional curvature $\frac{1}{4}$). We emphasize that in the sequence of eigenvalues $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ each eigenvalue is always repeated with its multiplicity.

Theorem 1.1. *For a domain Ω in $\mathbf{CP}^n(4)$, we consider the eigenvalue problem*:

$$\begin{cases} \triangle u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\triangle$ denotes the Laplacian on $\mathbf{CP}^n(4)$. Let λ_k be the k^{th} eigenvalue of the eigenvalue problem (1.3)*. Then we have*

$$\frac{1}{2n}\sum_{i=1}^{2n}\lambda_{i+1} \leq 4(n+1) + \left(1+\frac{2}{n}\right)\lambda_1.$$

Theorem 1.2. *For a domain Ω in an n-dimensional compact complex submanifold M of $\mathbf{CP}^{n+m}(4)$, we consider the eigenvalue problem*:

$$\begin{cases} \triangle u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\triangle$ is the Laplacian on M. Then, the eigenvalues λ_k $(k = 1, 2, \ldots, 2n+1)$ of the eigenvalue problem (1.4) *satisfy*

$$\frac{1}{2n}\sum_{i=1}^{2n}\lambda_{i+1} \leq 4(n+1) + \left(1+\frac{2}{n}\right)\lambda_1.$$

Theorem 1.3. *For a domain Ω in the n-dimensional unit sphere $S^n(1)$, let λ_k be the k^{th} eigenvalue of the eigenvalue problem*:

$$\begin{cases} \triangle u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\triangle$ is the Laplacian on $S^n(1)$. Then we have

$$\frac{1}{n}\sum_{i=1}^{n}\lambda_{i+1} \leq n + \left(1+\frac{4}{n}\right)\lambda_1. \tag{1.6}$$

Remark 1.1. When $\Omega = S^n(1)$, we know that $\lambda_1 = 0$ and $\lambda_2 = \cdots = \lambda_{n+1} = n$. Hence, inequality (1.3) in the Theorem 1.3 becomes an equality. Thus, the inequality (1.3) is optimal.

On the other hand, it seems to be an interesting and difficult problem to discuss the sharpness of the inequalities in Theorems 1.1 and 1.2.

Remark 1.2. Estimates for higher order eigenvalues of the Laplacian have been obtained by many mathematicians (cf. [8–10, 13–19, 22, 24, 25] and [26]). For instance, when Ω is a bounded domain in $\mathbf{R}^n$, the sharpest estimate for higher order eigenvalues is due to Yang [25] (cf. Payne, Pólya and Weinberger [22], Hile and Protter [16]), that is

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_{k+1}-\left(1+\frac{4}{n}\lambda_i\right)\right) \leq 0, \text{ for } k = 1, 2, \ldots$$

In particular, we should remark that, in [19], Levitin and Parnovski have used commutator identities to obtain universal estimates for eigenvalues. They have given abstract generalizations of the Payne, Pólya and Weinberger formula and of the Yang's formula. It seems difficult, however, to make the estimates in [19] explicit for the situation treated in this paper so that the relation of our present results to the general results of Levitin and Parnovski would be clarified. We believe that it is not possible to derive our present results from [19], at least if the ambient Riemannian manifold has non-constant curvature.

When Ω is a domain in the unit sphere $S^n(1)$, Cheng and Yang [8] have proved

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(n+\frac{4}{n}\lambda_i\right), \quad \text{for } k=1,2,\dots$$

When Ω is a domain in the n-dimensional complex projective space $\mathbf{CP}^n(4)$, in [10], they have derived

$$\lambda_{k+1} \leq \left(1+\frac{1}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i + 2(n+1)$$
$$+\left\{\left[\frac{1}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i + 2(n+1)\right]^2 - \left(1+\frac{2}{n}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j - \frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2\right\}^{1/2}.$$

This paper is organized as follows. In Section 2 we consider an eigenvalue problem for the Laplacian on a domain in an n-dimensional compact Riemannian manifold. A general inequality for eigenvalues λ_{i+1} will be given. As applications, in Sections 3, 4 and 5, we shall prove our Theorems 1.1, 1.2 and 1.3, respectively. In order to prove our theorems, we must find good trial functions. In this paper, we make use of the orthogonalization of Gram–Schmidt (QR-factorization theorem) to construct trial functions. By means of these trial functions we obtain our estimates for eigenvalues.

2. An estimate for the eigenvalues of the Laplacian

In this section, we shall consider an eigenvalue problem for the Laplacian on a domain Ω in an n-dimensional Riemannian manifold M. We shall obtain a general inequality for the eigenvalues which plays an important role in proofs of the Theorems 1.1, 1.2 and 1.3.

Theorem 2.1. *For a domain Ω in an n-dimensional compact Riemannian manifold M without boundary, we consider the eigenvalue problem*:

$$\begin{cases} \triangle u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\triangle$ denotes the Laplacian on M. Assume that λ_i is the ith eigenvalue and $\{u_i\}$ be an orthonormal system of eigenfunctions corresponding to $\{\lambda_i\}$. If $g_i \in C^2(\bar{\Omega})$ satisfies $\int_\Omega g_i u_1 u_j = 0$ for $j = 2, \ldots, i$, then, the following holds:

$$(\lambda_{i+1} - \lambda_1)\|(\nabla g_i)u_1\|^2 \leq \|(\triangle g_i)u_1 + 2\nabla g_i \cdot \nabla u_1\|^2,$$

where ∇ denotes the gradient operator on M and $\|f\|^2 = \int_\Omega f^2$.

Proof. From the assumptions of the Theorem 2.1, we have

$$\int_\Omega g_i u_1 u_j = 0, \quad for\ i \geq j > 1. \tag{2.1}$$

We define a function φ_i by

$$\varphi_i = g_i u_1 - u_1 \int_\Omega g_i u_1^2. \tag{2.2}$$

It is easy to see

$$\int_\Omega \varphi_i u_1 = 0.$$

Combining with (2.1) φ_i satisfies

$$\int_\Omega \varphi_i u_j = 0, \quad \text{for any } j \text{ with } j \leq i.$$

Thus, φ_i is a trial function. According to the Rayleigh–Ritz inequality, we have

$$\lambda_{i+1} \leq \frac{\int_\Omega |\nabla \varphi_i|^2}{\int_\Omega \varphi_i^2}. \tag{2.3}$$

From the definition of φ_i, we have

$$\int_\Omega \varphi_i^2 = \int_\Omega \varphi_i \left(g_i u_1 - u_1 \int_\Omega g_i u_1^2 \right) = \int_\Omega \varphi_i g_i u_1, \tag{2.4}$$

and

$$\triangle \varphi_i = (\triangle g_i)u_1 + 2\nabla g_i \cdot \nabla u_1 - \lambda_1 g_i u_1 + \lambda_1 u_1 \int_\Omega g_i u_1^2. \tag{2.5}$$

From (2.2), (2.4) and (2.5), we infer

$$\begin{aligned}
\int_\Omega |\nabla \varphi_i|^2 &= -\int_\Omega \varphi_i \triangle \varphi_i \\
&= -\int_\Omega \varphi_i \{(\triangle g_i)u_1 + 2\nabla g_i \cdot \nabla u_1 - \lambda_1 g_i u_1\} \\
&= \lambda_1 \int_\Omega \varphi_i^2 - \int_\Omega \varphi_i \{(\triangle g_i)u_1 + 2\nabla g_i \cdot \nabla u_1\}.
\end{aligned}$$

From (2.3) and the above inequality, we obtain

$$(\lambda_{i+1} - \lambda_1) \int_\Omega \varphi_i^2 \leq - \int_\Omega \varphi_i \{(\triangle g_i) u_1 + 2\nabla g_i \cdot \nabla u_1\}.$$

Letting $\omega_i = - \int_\Omega \varphi_i \{(\triangle g_i) u_1 + 2\nabla g_i \cdot \nabla u_1\}$, we have

$$(\lambda_{i+1} - \lambda_1) \|\varphi_i\|^2 \leq \omega_i. \tag{2.6}$$

From the Cauchy–Schwarz inequality, we derive

$$\omega_i^2 \leq \|\varphi_i\|^2 \|(\triangle g_i) u_1 + 2\nabla g_i \cdot \nabla u_1\|^2. \tag{2.7}$$

Multiplying (2.7) by $(\lambda_{i+1} - \lambda_1)$, we get

$$(\lambda_{i+1} - \lambda_1) \omega_i^2 \leq (\lambda_{i+1} - \lambda_1) \|\varphi_i\|^2 \|(\triangle g_i) u_1 + 2\nabla g_i \cdot \nabla u_1\|^2. \tag{2.8}$$

Combining this with (2.6) we obtain

$$(\lambda_{i+1} - \lambda_1) \omega_i \leq \|(\triangle g_i) u_1 + 2\nabla g_i \cdot \nabla u_1\|^2. \tag{2.9}$$

On the other hand, we have

$$\begin{aligned} \omega_i &= - \int_\Omega \varphi_i \{(\triangle g_i) u_1 + 2\nabla g_i \cdot \nabla u_1\} \\ &= - \int_\Omega g_i (\triangle g_i) u_1^2 - \frac{1}{2} \int_\Omega \nabla g_i^2 \cdot \nabla u_1^2 \\ &\quad + \int_\Omega (\triangle g_i) u_1^2 \int_\Omega g_i u_1^2 + \int_\Omega \nabla g_i \cdot \nabla u_1^2 \int_\Omega g_i u_1^2. \end{aligned} \tag{2.10}$$

By making use of Stokes' formula, it is easy to obtain

$$- \int_\Omega g_i (\triangle g_i) u_1^2 = \int_\Omega |u_1 \nabla g_i|^2 + \frac{1}{2} \int_\Omega \nabla g_i^2 \cdot \nabla u_1^2 \tag{2.11}$$

and

$$\int_\Omega (\triangle g_i) u_1^2 = - \int_\Omega \nabla g_i \cdot \nabla u_1^2. \tag{2.12}$$

Substituting (2.11) and (2.12) into (2.10), we have

$$\omega_i = \int_\Omega |u_1 \nabla g_i|^2 = \|(\nabla g_i) u_1\|^2. \tag{2.13}$$

According to (2.13) and (2.9), we infer

$$(\lambda_{i+1} - \lambda_1) \|(\nabla g_i) u_1\|^2 \leq \|(\triangle g_i) u_1 + 2\nabla g_i \cdot \nabla u_1\|^2.$$

It completes the proof of the Theorem 2.1. □

3. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. First we state two simple algebraic lemmas, which will be used in proof of the Theorem.

Let A^* denote the adjoint matrix of a matrix $A = (a_{ij})$, $U(n)$ and $O(n)$ be the set of all $n \times n$ unitary matrices and the set of all $n \times n$ orthogonal matrices, respectively.

Lemma 3.1. *For a matrix $C = (C_{pq}) \in U(n)$, we have $A = (A_{\alpha\beta}) = (C_{ps}\overline{C_{qt}}) \in U(n^2)$ and $B = (B_{\alpha\beta}) = (\overline{C_{ps}}C_{qt}) \in U(n^2)$, where $\alpha = (p, q)$, $\beta = (s, t)$.*

Lemma 3.2. *For a complex matrix $A + iB \in U(n)$, where A and B are $n \times n$ real matrices, we have $D = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in O(2n)$.*

Let $Z = (Z^0, Z^1, \dots, Z^n)$ be a homogeneous coordinate system on $\mathbf{CP}^n(4)$. Defining functions $f_{p\bar{q}}$ by

$$f_{p\bar{q}} = \frac{Z^p\overline{Z^q}}{\sum_{r=0}^{n} Z^r\overline{Z^r}}, \tag{3.1}$$

we have

$$f_{p\bar{q}} = \overline{f_{q\bar{p}}}, \quad \sum_{p,q=0}^{n} f_{p\bar{q}}\overline{f_{p\bar{q}}} = 1. \tag{3.2}$$

Let Ω be as in Theorem 1.1. For any fixed point $P \in \Omega$, we can choose a new homogeneous coordinate system on $\mathbf{CP}^n(4)$ such that, at P,

$$\widetilde{Z}^0 \neq 0, \ \widetilde{Z}^1 = \cdots = \widetilde{Z}^n = 0 \tag{3.3}$$

and

$$Z^p = \sum_{r=0}^{n} C_{pr}\widetilde{Z}^r, \tag{3.4}$$

where the $(n+1) \times (n+1)$-matrix $C = (C_{pr}) \in U(n+1)$. Therefore, if we denote $z^p = \widetilde{Z}^p/\widetilde{Z}^0$, then $z = (z^1, \dots, z^n)$ is a local holomorphic coordinate system on $\mathbf{CP}^n(4)$ in a neighborhood U of $P \in \Omega$ and

$$z^0 = 1, \ z^1 = \cdots = z^n = 0 \tag{3.5}$$

at P. Define functions $\widetilde{f}_{p\overline{q}}$ by

$$\widetilde{f}_{p\overline{q}} = \frac{\widetilde{Z}^p\overline{\widetilde{Z}^q}}{\sum_{r=0}^{n}\widetilde{Z}^r\overline{\widetilde{Z}^r}} = \frac{z^p\overline{z^q}}{1 + \sum_{r=1}^{n} z^r\overline{z}^r}. \tag{3.6}$$

It is easy to check that $\widetilde{f}_{p\overline{q}}$ and $f_{p\overline{q}}$ satisfy

$$f_{p\overline{q}} = \sum_{r,s=0}^{n} C_{pr}\overline{C_{qs}}\widetilde{f}_{r\overline{s}}, \quad p, q = 0, 1, \dots, n. \tag{3.7}$$

Now we consider the $2(n+1)^2$ functions $\mathrm{Re}(f_{p\overline{q}})$ and $\mathrm{Im}(f_{p\overline{q}})$, denoted by g_α, where $p, q = 0, 1, \ldots, n$. Then, we have

$$\sum_{\alpha=1}^{2(n+1)^2} g_\alpha^2 = \sum_{p,q=0}^{n} f_{p\overline{q}}\overline{f_{p\overline{q}}} = \sum_{p,q=0}^{n} \widetilde{f}_{p\overline{q}}\overline{\widetilde{f}_{p\overline{q}}} = 1, \tag{3.8}$$

and

$$\sum_{\alpha=1}^{2(n+1)^2} g_\alpha \nabla g_\alpha = 0. \tag{3.9}$$

In the local coordinate system we have

$$\Delta f = \sum_{p,q=1}^{n} 4g^{p\overline{q}} \frac{\partial^2 f}{\partial z^p \partial \overline{z}^q},$$

where $ds^2 = \sum_{p,q=1}^{n} g_{p\overline{q}} dz^p d\overline{z}^q$ is the Fubini-Study metric of $\mathbf{CP}^n(4)$, and

$$g_{p\overline{q}} = \frac{\delta_{p\overline{q}}}{1 + \sum_{r=1}^{n} |z^r|^2} - \frac{z^q \overline{z^p}}{\left(1 + \sum_{r=1}^{n} |z^r|^2\right)^2},$$

$$(g_{p\overline{q}})^{-1} = (g^{p\overline{q}}),$$

$$g^{p\overline{q}} = \left(1 + \sum_{r=1}^{n} |z^r|^2\right)(\delta^{p\overline{q}} + z^q \overline{z^p}).$$

Let $\widetilde{g}_\alpha$ denote the $2(n+1)^2$ functions $\mathrm{Re}(\widetilde{f}_{p\overline{q}})$ and $\mathrm{Im}(\widetilde{f}_{p\overline{q}})$, where $p, q = 0, 1, \ldots, n$. From (3.5) and (3.6), it is not difficult to check that, at P,

$$\Delta = 4\sum_{r=1}^{n} \frac{\partial^2}{\partial z^r \partial \overline{z^r}}, \tag{3.10}$$

$$\begin{cases} \nabla \widetilde{f}_{p\overline{q}} = 0, & \text{when } pq \neq 0 \text{ or } p = q = 0, \\ \mathrm{Re}\nabla_p \widetilde{f}_{q\overline{0}} = \delta_{pq}, & \mathrm{Im}\nabla_p \widetilde{f}_{q\overline{0}} = \delta_{pq}, \\ \mathrm{Re}\nabla_p \widetilde{f}_{0\overline{q}} = \delta_{pq}, & \mathrm{Im}\nabla_p \widetilde{f}_{0\overline{q}} = -\delta_{pq}, \end{cases} \tag{3.11}$$

$$\Delta \widetilde{f}_{p\overline{q}} = \begin{cases} 0, & \text{when } p \neq q, \\ -4n, & \text{when } p = q = 0, \\ 4, & \text{when } p = q = r \neq 0. \end{cases} \tag{3.12}$$

Lemma 3.3. *At any point $P \in \Omega$, the functions g_α satisfy*

$$\begin{cases} \sum_{\alpha=1}^{2(n+1)^2} |\nabla g_\alpha|^2 = 4n, \\ \sum_{\alpha=1}^{2(n+1)^2} |\Delta g_\alpha|^2 = 16n(n+1), \\ \sum_{\alpha=1}^{2(n+1)^2} \nabla g_\alpha \Delta g_\alpha = 0, \\ \sum_{\alpha=1}^{2(n+1)^2} |\nabla g_\alpha \cdot \nabla u_1|^2 = 2|\nabla u_1|^2. \end{cases}$$

Proof. By making use of the same notation as above, because of $C = (C_{pq}) \in U(n+1)$, from the Lemma 3.1 we infer $A = (A_{\alpha\beta}) = (C_{ps}\overline{C_{qt}}) \in U((n+1)^2)$. Put $A = A_1 + iA_2$. From (3.7), we know

$$(g_\alpha) = \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix} (\widetilde{g}_\beta).$$

From the Lemma 3.2, we see

$$\begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix}$$

is a $2(n+1)^2 \times 2(n+1)^2$ orthogonal matrix. We denote it by $O = (O_{\alpha\beta})$. Thus, we have, for any α,

$$g_\alpha = \sum_\beta O_{\alpha\beta}\widetilde{g}_\beta. \tag{3.13}$$

Without loss of generality, we rearrange the $2(n+1)^2$ functions $\widetilde{g}_\alpha$ such that the first $4n$ functions are

$$\mathrm{Re}\widetilde{f}_{1\bar{0}}, \ldots, \mathrm{Re}\widetilde{f}_{n\bar{0}}, \mathrm{Im}\widetilde{f}_{1\bar{0}}, \ldots, \mathrm{Im}\widetilde{f}_{n\bar{0}}, \mathrm{Re}\widetilde{f}_{0\bar{1}}, \ldots, \mathrm{Re}\widetilde{f}_{0\bar{n}}, \mathrm{Im}\widetilde{f}_{0\bar{1}}, \ldots, \mathrm{Im}\widetilde{f}_{0\bar{n}},$$

denoted by $\widetilde{g}_{s0}$ and $\widetilde{g}_{0t}$, where $s, t = 1, \ldots, n$. And we still denote the other $2(n+1)^2 - 4n$ functions by $\widetilde{g}_\alpha$. Therefore, from (3.11), we have

$$\begin{cases} \nabla_p \widetilde{g}_{p0} = 1, & p = 1, \ldots, 2n, \\ \nabla_p \widetilde{g}_{0p} = 1, & p = 1, \ldots, n, \\ \nabla_p \widetilde{g}_{0p} = -1, & p = n+1, \ldots, 2n, \\ \nabla_p \widetilde{g}_\alpha = 0, & \alpha = 4n+1, \ldots, 2(n+1)^2. \end{cases} \tag{3.14}$$

Since O is an orthogonal matrix, from (3.13) and (3.14), we have

$$\begin{aligned} \sum_{\alpha=1}^{2(n+1)^2} |\nabla g_\alpha|^2 &= \sum_{\alpha=1}^{2(n+1)^2} \sum_{\beta=1}^{2(n+1)^2} O_{\alpha\beta}\nabla\widetilde{g}_\beta \cdot \sum_{\gamma=1}^{2(n+1)^2} O_{\alpha\gamma}\nabla\widetilde{g}_\gamma \\ &= \sum_{\alpha=1}^{2(n+1)^2} |\nabla\widetilde{g}_\alpha|^2 = \sum_{p=1}^{2n} [(\nabla_p\widetilde{g}_{p0})^2 + (\nabla_p\widetilde{g}_{0p})^2] \\ &= 4n. \end{aligned}$$

Similarly, we have

$$\sum_{\alpha=1}^{2(n+1)^2} \nabla g_\alpha \Delta g_\alpha = \sum_{p,q=0}^{n} \left(\nabla \mathrm{Re}\widetilde{f}_{p\overline{q}} \Delta \mathrm{Re}\widetilde{f}_{p\overline{q}} + \nabla \mathrm{Im}\widetilde{f}_{p\overline{q}} \Delta \mathrm{Im}\widetilde{f}_{p\overline{q}}\right) = 0,$$

$$\begin{aligned} \sum_{\alpha=1}^{2(n+1)^2} |\Delta g_\alpha|^2 &= \sum_{\alpha=1}^{2(n+1)^2} |\Delta \widetilde{g}_\alpha|^2 = \sum_{p,q=0}^{n} \overline{\Delta \widetilde{f}_{p\overline{q}}} \Delta \widetilde{f}_{p\overline{q}} \\ &= 4n \cdot 4n + 4 \cdot 4 \cdot n = 16n(n+1), \end{aligned}$$

$$\begin{aligned} \sum_{\alpha=1}^{2(n+1)^2} (\nabla g_\alpha \cdot \nabla u_1)^2 &= \sum_{\alpha=1}^{2(n+1)^2} \sum_{\beta=1}^{2(n+1)^2} O_{\alpha\beta} \nabla \widetilde{g}_\beta \cdot \nabla u_1 \sum_{\gamma=1}^{2(n+1)^2} O_{\alpha\gamma} \nabla \widetilde{g}_\gamma \cdot \nabla u_1 \\ &= \sum_{\beta=1}^{2(n+1)^2} (\nabla \widetilde{g}_\beta \cdot \nabla u_1)^2 = \sum_{\beta=1}^{2(n+1)^2} \left(\sum_{p=1}^{2n} \nabla_p \widetilde{g}_\beta \nabla_p u_1 \right)^2 \\ &= \sum_{p=1}^{2n} \left[\left(\nabla_p \widetilde{g}_{p0} \nabla_p u_1\right)^2 + \left(\nabla_p \widetilde{g}_{0p} \nabla_p u_1\right)^2 \right] \\ &= 2|\nabla u_1|^2. \end{aligned}$$

This finishes the proof of the Lemma 3.3. □

Lemma 3.4. *Let* $(h_\alpha) = Q(g_\beta)$*, where* $Q = (q_{\alpha\beta})$ *is a constant orthogonal* $2(n+1)^2 \times 2(n+1)^2$ *matrix. At any point* $P \in \Omega$*, we then have*

$$|\nabla h_\alpha|^2 \leq 2, \quad \alpha = 1, \ldots, 2(n+1)^2.$$

Proof. From (3.13), we have

$$(h_\alpha) = Q(g_\beta) = QO(\widetilde{g}_\beta).$$

Without loss of generality, we still denote the orthogonal $2(n+1)^2 \times 2(n+1)^2$ matrix QO by $O = (O_{\alpha\beta})$. Thus, we have

$$(h_\alpha) = O(\widetilde{g}_\beta).$$

By rearranging the $2(n+1)^2$ functions $\widetilde{g}_\alpha$ as in the proof of the Lemma 3.3, from (3.13) and (3.14) we obtain

$$\begin{aligned} |\nabla h_\alpha|^2 &= \sum_{p=1}^{2n} \sum_{\beta=1}^{2(n+1)^2} O_{\alpha\beta} \nabla_p \widetilde{g}_\beta \sum_{\gamma=1}^{2(n+1)^2} O_{\alpha\gamma} \nabla_p \widetilde{g}_\gamma \\ &= \sum_{p=1}^{2n} \left(O_{\alpha(p,0)} \nabla_p \widetilde{g}_{p0} + O_{\alpha(0,p)} \nabla_p \widetilde{g}_{0p} \right)^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{p=1}^{n} \left(O_{\alpha(p,0)} + O_{\alpha(0,p)}\right)^2 + \sum_{p=n+1}^{2n} \left(O_{\alpha(p,0)} - O_{\alpha(0,p)}\right)^2 \\
&\leq \sum_{p=1}^{2n} \left[\left(O_{\alpha(p,0)}\right)^2 + \left(O_{\alpha(0,p)}\right)^2 + 2|O_{\alpha(p,0)} O_{\alpha(0,p)}|\right] \\
&\leq 2 \sum_{p=1}^{2n} \left[\left(O_{\alpha(p,0)}\right)^2 + \left(O_{\alpha(0,p)}\right)^2\right] \\
&\leq 2 \sum_{\beta=1}^{2(n+1)^2} (O_{\alpha\beta})^2 = 2.
\end{aligned}
$$

Hence, the Lemma 3.4 is proved. □

Proof of Theorem 1.1. Let $Z = (Z^0, Z^1, \ldots, Z^n)$ be a homogeneous coordinate system on $\mathbf{CP}^n(4)$. We consider the functions f_{pq} defined by (3.1). Let g_α denote the $2(n+1)^2$ functions $\operatorname{Re} f_{pq}$ and $\operatorname{Im} f_{pq}$ as above. We consider the $2(n+1)^2 \times 2(n+1)^2$ matrix A defined by

$$
A = \begin{pmatrix}
\int_\Omega g_1 u_1 u_2 & \int_\Omega g_1 u_1 u_3 & \cdots & \int_\Omega g_1 u_1 u_{2(n+1)^2+1} \\
\int_\Omega g_2 u_1 u_2 & \int_\Omega g_2 u_1 u_3 & \cdots & \int_\Omega g_2 u_1 u_{2(n+1)^2+1} \\
\cdots & \cdots & \cdots & \cdots \\
\int_\Omega g_{2(n+1)^2} u_1 u_2 & \int_\Omega g_{2(n+1)^2} u_1 u_3 & \cdots & \int_\Omega g_{2(n+1)^2} u_1 u_{2(n+1)^2+1}
\end{pmatrix}.
$$

From the orthogonalization of Gram–Schmidt (QR-factorization theorem), we know that A can be written by

$$
T = OA,
$$

where $O = (O_{kl})$ is an orthogonal $2(n+1)^2 \times 2(n+1)^2$ matrix and T is an upper triangular matrix. Hence, we have, for any k and j with $k > j$,

$$
\sum_{l=1}^{2(n+1)^2} O_{kl} \int_\Omega g_l u_1 u_{j+1} = 0.
$$

Defining functions h_k by $(h_k) = O(g_j)$, i.e. $h_k = \sum_{j=1}^{2(n+1)^2} O_{kj} g_j$, we infer, for any $i, j = 1, 2, \ldots, 2(n+1)^2$ satisfying $i > j$,

$$
\int_\Omega h_i u_1 u_{j+1} = 0. \tag{3.15}
$$

Hence, these functions h_α, $\alpha = 1, 2, \ldots, 2(n+1)^2$, satisfy the conditions in the Theorem 2.1. Applying the theorem we obtain

$$
(\lambda_{\alpha+1} - \lambda_1)\|(\nabla h_\alpha) u_1\|^2 \leq \|(\triangle h_\alpha) u_1 + 2\nabla h_\alpha \cdot \nabla u_1\|^2.
$$

Summing on α from 1 to $2(n+1)^2$, we have

$$\sum_{\alpha=1}^{2(n+1)^2} \lambda_{\alpha+1}\|(\nabla h_\alpha)u_1\|^2 \leq \sum_{\alpha=1}^{2(n+1)^2} \|(\triangle h_\alpha)u_1 + 2\nabla h_\alpha \cdot \nabla u_1\|^2. \tag{3.16}$$

Since $h_\alpha = \sum_{\beta=1}^{2(n+1)^2} O_{\alpha\beta} g_\beta$ holds, from the Lemma 3.3 we obtain

$$\begin{cases} \sum_{\alpha=1}^{2(n+1)^2} |\nabla h_\alpha|^2 = 4n, \\ \sum_{\alpha=1}^{2(n+1)^2} |\Delta h_\alpha|^2 = 16n(n+1), \\ \sum_{\alpha=1}^{2(n+1)^2} \nabla h_\alpha \Delta h_\alpha = 0, \\ \sum_{\alpha=1}^{2(n+1)^2} |\nabla h_\alpha \cdot \nabla u_1|^2 = 2|\nabla u_1|^2. \end{cases} \tag{3.17}$$

Hence, we infer, from (3.16) and (3.17),

$$\sum_{\alpha=1}^{2(n+1)^2} \lambda_{\alpha+1}\|\nabla h_\alpha u_1\|^2 \leq 16n(n+1) + 4(n+2)\lambda_1.$$

On the other hand, from (3.17) and Lemma 3.4, we have

$$\begin{aligned} \sum_{\alpha=1}^{2(n+1)^2} \lambda_{\alpha+1}|\nabla h_\alpha|^2 &\geq \sum_{\alpha=1}^{2n} \lambda_{\alpha+1}|\nabla h_\alpha|^2 + \lambda_{2n+1} \sum_{\alpha=2n+1}^{2(n+1)^2} |\nabla h_\alpha|^2 \\ &= \sum_{\alpha=1}^{2n} \lambda_{\alpha+1}|\nabla h_\alpha|^2 + \lambda_{2n+1}\left(4n - \sum_{\alpha=1}^{2n} |\nabla h_\alpha|^2\right) \\ &= \sum_{\alpha=1}^{2n} \lambda_{\alpha+1}|\nabla h_\alpha|^2 + \lambda_{2n+1} \sum_{\alpha=1}^{2n}(2 - |\nabla h_\alpha|^2) \\ &\geq \sum_{\alpha=1}^{2n} \lambda_{\alpha+1}|\nabla h_\alpha|^2 + \sum_{\alpha=1}^{2n}(2 - |\nabla h_\alpha|^2)\lambda_{\alpha+1} \\ &= 2\sum_{\alpha=1}^{2n} \lambda_{\alpha+1}. \end{aligned}$$

Therefore, we have

$$\int_\Omega 2\sum_{\alpha=1}^{2n}\lambda_{\alpha+1}u_1^2 \le \int_\Omega \sum_{\alpha=1}^{2(n+1)^2}\lambda_{\alpha+1}|\nabla h_\alpha|^2u_1^2 = \sum_{\alpha=1}^{2(n+1)^2}\lambda_{\alpha+1}\|\nabla h_\alpha u_1\|^2.$$

Thus, we finally infer

$$\frac{1}{2n}\sum_{i=1}^{2n}\lambda_{i+1} \le 4(n+1) + \left(1+\frac{2}{n}\right)\lambda_1,$$

which is the claim made in Theorem 1.1. □

4. Proof of Theorem 1.2

In this section, we shall give the proof of the Theorem 1.2. Let Ω be a domain in an n-dimensional compact complex submanifold M of $\mathbf{CP}^{n+m}(4)$. Let $Z = (Z^0, Z^1, \ldots, Z^{n+m})$ be a homogeneous coordinate system on $\mathbf{CP}^{n+m}(4)$. The functions $f_{p\bar{q}}$ defined by

$$f_{p\bar{q}} = \frac{Z^p\overline{Z^q}}{\sum_{r=0}^{n+m} Z^r\overline{Z^r}} \tag{4.1}$$

satisfy

$$f_{p\bar{q}} = \overline{f_{q\bar{p}}}\,, \quad \sum_{p,q=0}^{n+m} f_{p\bar{q}}\overline{f_{p\bar{q}}} = 1. \tag{4.2}$$

By making use of the same assertion as in the Sect. 3, for any point $P \in \Omega$, we can choose a new homogeneous coordinate system on $\mathbf{CP}^{n+m}(4)$ such that, at P,

$$\widetilde{Z}^0 \ne 0,\ \widetilde{Z}^1 = \cdots = \widetilde{Z}^{n+m} = 0 \tag{4.3}$$

and

$$Z^p = \sum_{r=0}^{n+m} C_{pr}\widetilde{Z}^r, \tag{4.4}$$

where $C = (C_{pr}) \in U(n+m+1)$. Therefore, if we denote $z^p = \widetilde{Z}^p/\widetilde{Z}^0$, then $z = (z^1, \ldots, z^{n+m})$ is a local holomorphic coordinate system of $\mathbf{CP}^{n+m}(4)$ in a neighborhood U of $P \in \Omega$ and

$$z^0 = 1,\ z^1 = \cdots = z^{n+m} = 0 \tag{4.5}$$

at P, and $z^{n+i} = l_i(z^1, \ldots, z^n)\quad (i = 1, \ldots, m)$ are holomorphic functions of $z^1, \ldots, z^n$ which satisfy

$$\frac{\partial l_i}{\partial z^p}(P) = 0, \quad p = 1, \ldots, n. \tag{4.6}$$

Then, we can easily compute

$$\widetilde{f}_{p\overline{q}} = \frac{\widetilde{Z}^p \overline{\widetilde{Z}^q}}{\sum_{r=0}^{n+m} \widetilde{Z}^r \overline{\widetilde{Z}^r}} = \frac{z^p \overline{z^q}}{1 + \sum_{r=1}^{n+m} z^r \overline{z}^r}, \tag{4.7}$$

and

$$f_{p\overline{q}} = \sum_{r,s=0}^{n+m} C_{pr} \overline{C_{qs}} \widetilde{f}_{r\overline{s}}, \quad p, q = 0, 1, \ldots, n+m. \tag{4.8}$$

Now we consider the $2(n+m+1)^2$ functions $\mathrm{Re}(f_{p\overline{q}})$ and $\mathrm{Im}(f_{p\overline{q}})$, denoted by g_α, where $p, q = 0, 1, \ldots, n+m$. Then, we have

$$\sum_{\alpha=1}^{2(n+m+1)^2} g_\alpha^2 = \sum_{p,q=0}^{n+m} f_{p\overline{q}} \overline{f_{p\overline{q}}} = \sum_{p,q=0}^{n+m} \widetilde{f}_{p\overline{q}} \overline{\widetilde{f}_{p\overline{q}}} = 1, \tag{4.9}$$

and

$$\sum_{\alpha=1}^{2(n+m+1)^2} g_\alpha \nabla g_\alpha = 0. \tag{4.10}$$

In the local coordinate system on U, we have $ds^2{}_M = \sum_{p=1}^n dz^p \overline{dz^p} + O(z^2)$. For the Laplacian $\triangle$ on the n-dimensional complex submanifold M in $\mathbf{CP}^{n+m}(4)$ we have $\triangle_{\mathbf{CP}^{n+m}(4)} f = \triangle f + \sum_{i=1}^m f_{n+i\ n+i}$. We denote the $2(n+m+1)^2$ functions $Re(\widetilde{f}_{p\overline{q}})$ and $Im(\widetilde{f}_{p\overline{q}})$ by $\widetilde{g}_\alpha$, where $p, q = 0, 1, \ldots, n+m$. From (4.6), (4.7) and (4.8) we have, at P,

$$\Delta = 4 \sum_{r=1}^n \frac{\partial^2}{\partial z^r \partial \overline{z^r}}, \tag{4.11}$$

$$\begin{cases} \nabla \widetilde{f}_{p\overline{q}} = 0, \text{ when } pq \neq 0 \text{ or } p = q = 0, \\ Re \nabla_p \widetilde{f}_{q\overline{0}} = \delta_{pq}, \quad Im \nabla_p \widetilde{f}_{q\overline{0}} = \delta_{pq}, \\ Re \nabla_p \widetilde{f}_{0\overline{q}} = \delta_{pq}, \quad Im \nabla_p \widetilde{f}_{0\overline{q}} = -\delta_{pq}, \end{cases} \tag{4.12}$$

$$\Delta \widetilde{f}_{p\overline{q}} = \begin{cases} 0, & \text{when } p \neq q,\ or\ p = q = n+1, \ldots, n+m, \\ -4n, & \text{when } p = q = 0, \\ 4, & \text{when } p = q = 1, \ldots, n. \end{cases} \tag{4.13}$$

By making use of the same calculations as in the Lemmas 3.3 and 3.4, we now obtain the following:

Lemma 4.1. *For any point $P \in \Omega$, we have*

$$\begin{cases} \sum_{\alpha=1}^{2(n+m+1)^2} |\nabla g_\alpha|^2 = 4n, \\ \sum_{\alpha=1}^{2(n+m+1)^2} |\Delta g_\alpha|^2 = 16n(n+1), \\ \sum_{\alpha=1}^{2(n+m+1)^2} \nabla g_\alpha \Delta g_\alpha = 0, \\ \sum_{\alpha=1}^{2(n+m+1)^2} |\nabla g_\alpha \cdot \nabla u_1|^2 = 2|\nabla u_1|^2. \end{cases} \tag{4.14}$$

Lemma 4.2. *Let* $(h_\alpha) = Q(g_\beta)$*, where* $Q = (q_{\alpha\beta})$ *is a constant orthogonal* $2(n+m+1)^2 \times 2(n+m+1)^2$ *matrix. At any point* $P \in \Omega$*, we have*

$$|\nabla h_\alpha|^2 \leq 2, \quad \alpha = 1, \ldots, 2(n+m+1)^2. \tag{4.15}$$

Proof of Theorem 1.2. From Lemma 4.1 and Lemma 4.2, we can derive Theorem by employing the same arguments as in the proof of Theorem 1.1. □

5. Proof of Theorem 1.3

In this section, we shall give the proof of the Theorem 1.3.

Let $\Omega \subset S^n(1)$ be a domain in the n-dimensional unit sphere $S^n(1)$. Let $x^1, x^2, \ldots, x^{n+1}$ be the standard coordinate functions on $\mathbf{R}^{n+1}$ so that $S^n(1) = \{(x^1, x^2, \ldots, x^{n+1}) \in \mathbf{R}^{n+1}; \sum_{j=1}^{n+1}(x^j)^2 = 1\}$. It is well known that x^p (for $p = 1, \ldots, n+1$) satisfy

$$\Delta x^p = -n x^p.$$

Lemma 5.1. *Let* $(h_\alpha) = Q(x^\beta)$*, where* $Q = (q_{\alpha\beta})$ *is a constant orthogonal* $(n+1) \times (n+1)$ *matrix. For any point* P *in* Ω*, we have*

$$\begin{aligned}
&|\nabla h_p|^2 \leq 1, \text{ for } p = 1, 2, \ldots, n,\\
&\sum_{p=1}^{n+1} |\nabla h_p|^2 = n,\\
&\sum_{p=1}^{n+1} (\nabla h_p \cdot \nabla u_i)^2 = |\nabla u_i|^2.
\end{aligned}$$

Proof. For any fixed point $P \in \Omega$, we can find a coordinate system $(\bar{x}^1, \ldots, \bar{x}^{n+1})$ on $\mathbf{R}^{n+1}$ such that, at P,

$$\begin{aligned}
&\bar{x}^1 = \cdots = \bar{x}^n = 0, \quad \bar{x}^{n+1} = 1,\\
&\nabla \bar{x}^{n+1} = 0; \quad \nabla_p \bar{x}^q = \delta_{pq}. \quad (p, q = 1, \ldots, n).
\end{aligned} \tag{5.1}$$

In fact, we can choose a constant $(n+1) \times (n+1)$ orthonormal matrix $A = (a_{ij})$ such that

$$x^p = \sum_{\alpha=1}^{n+1} a_{p\alpha} \bar{x}^\alpha,$$

and (5.1), (5.2) is satisfied at P. Hence, we have

$$(h_\alpha) = QA(\bar{x}^\beta),$$

where QA is also a constant orthogonal $(n+1)\times(n+1)$ matrix. We still denote it by $A=(a_{ij})$ without loss of generality. Thus, at P, we have

$$
\begin{aligned}
|\nabla h_p|^2 &= \sum_{\alpha=1}^{n+1} a_{p\alpha}\nabla \bar{x}^{\alpha}\cdot\sum_{\beta=1}^{n+1} a_{p\beta}\nabla\bar{x}^{\beta}\\
&= \sum_{j=1}^{n}\sum_{\alpha,\beta=1}^{n+1} a_{p\alpha}a_{p\beta}\nabla_j\bar{x}^{\alpha}\cdot\nabla_j\bar{x}^{\beta}\\
&= \sum_{j=1}^{n} a_{pj}a_{pj}\le 1,
\end{aligned}
$$

$$
\sum_{p=1}^{n+1}|\nabla h_p|^2 = n,
$$

and

$$
\begin{aligned}
\sum_{p=1}^{n+1}(\nabla h_p\cdot\nabla u_i)^2 &= \sum_{p,q,\alpha=1}^{n+1} a_p^{\alpha}a_q^{\alpha}(\nabla\bar{x}^p\cdot\nabla u_i)(\nabla\bar{x}^q\cdot\nabla u_i)\\
&= \sum_{p=1}^{n+1}(\nabla\bar{x}^p\cdot\nabla u_i)^2 = \sum_{p=1}^{n}(\nabla_p u_i)^2\\
&= |\nabla u_i|^2.
\end{aligned}
$$

Since P is arbitrary the Lemma is proved. □

Proof of Theorem 1.3. For the functions $g_i = x^i$, we consider the $(n+1)\times(n+1)$ matrix A defined by

$$
A=\begin{pmatrix}
\int_{\Omega} g_1u_1u_2 & \int_{\Omega} g_1u_1u_3 & \cdots & \int_{\Omega} g_1u_1u_{n+2}\\
\int_{\Omega} g_2u_1u_2 & \int_{\Omega} g_2u_1u_3 & \cdots & \int_{\Omega} g_2u_1u_{n+2}\\
\cdots & \cdots & \cdots & \cdots\\
\int_{\Omega} g_{n+1}u_1u_2 & \int_{\Omega} g_{n+1}u_1u_3 & \cdots & \int_{\Omega} g_{n+1}u_1u_{n+2}
\end{pmatrix}.
$$

From the same arguments as in the proof of Theorem 1.1 in the Section 3, we infer that there exists an orthogonal matrix $O=(O_{kj})$ such that $h_k=\sum_{j=1}^{n+1}O_{kj}g_j$ satisfies, for any $i, j = 1, 2, \ldots, n+1$ with $i > j$,

$$
\int_{\Omega} h_i u_1 u_{j+1} = 0. \tag{5.2}
$$

Applying Theorem 2.1 to the functions h_i and summing on i from 1 to $n+1$, we get

$$
\sum_{i=1}^{n+1}(\lambda_{i+1}-\lambda_1)\|(\nabla h_i)u_1\|^2 \le \sum_{i=1}^{n+1}\|(\Delta h_i)u_1 + 2\nabla h_i\cdot\nabla u_1\|^2.
$$

Since $\sum_{p=1}^{n+1}(x^p)^2 = 1$, $\triangle x^p = -nx^p$, we have

$$\sum_{p=1}^{n+1} \nabla (x^p)^2 = 0,$$
$$\sum_{p=1}^{n+1} |\nabla x^p|^2 = -\sum_{p=1}^{n+1} x^p \Delta x^p = n.$$

Hence, from Lemma 5.1, we infer

$$\sum_{i=1}^{n+1} \lambda_{i+1} \|\nabla h_i u_1\|^2 \leq n^2 + (4+n)\lambda_1$$

and

$$\begin{aligned}\sum_{i=1}^{n+1} \lambda_{i+1} |\nabla h_i|^2 &\geq \sum_{i=1}^{n} \lambda_{i+1} |\nabla h_i|^2 + \lambda_{n+1} |\nabla h_{n+1}|^2 \\ &= \sum_{i=1}^{n} \lambda_{i+1} |\nabla h_i|^2 + \lambda_{n+1} \left(n - \sum_{i=1}^{n} |\nabla h_i|^2 \right) \\ &\geq \sum_{i=1}^{n} \lambda_{i+1}.\end{aligned}$$

Thus we have proved the claim

$$\frac{1}{n} \sum_{i=1}^{n} \lambda_{i+1} \leq n + \left(1 + \frac{4}{n}\right) \lambda_1.$$

□

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