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Higher integrability of the gradient for vectorial minimizers of decomposable variational integrals

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Abstract. We consider local minimizers $u\colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ of variational integrals $I[u] := \int_\Omega F(\nabla u)\,\mathrm{d}x$, where F is of anisotropic (p,q)-growth with exponents $1 < p \le q < \infty$. If F is in a certain sense decomposable, we show that the dimensionless restriction $q \le 2p+2$ together with the local boundedness of u implies local integrability of ∇u for all exponents $t \le p+2$. More precisely, the initial exponents for the integrability of the partial derivatives can be increased by two, at least locally. If $n = 2$, then we use these facts to prove $C^{1,\alpha}$-regularity of u for any exponents $2 \le p \le q$.

In this note we discuss the higher integrability properties of the gradient of local minimizers $u\colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ of variational integrals

$$I[u,\Omega] = \int_\Omega F(\nabla u)\,\mathrm{d}x, \tag{1}$$

where Ω denotes an open set in $\mathbb{R}^n$ and where $F\colon \mathbb{R}^{nN} \to [0,\infty)$ satisfies the anisotropic growth condition

$$a|Z|^p - b \le F(Z) \le A|Z|^q + B, \qquad Z \in \mathbb{R}^{nN}, \tag{2}$$

with constants $a, A > 0$, $b, B \ge 0$ and with exponents $1 < p \le q < \infty$. Due to the growth condition (2) it is natural to call a function u from the Sobolev-space $W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^N)$ (see [1] for a definition of these spaces) a local minimizer of (1) if and only if $I[u,\Omega'] < \infty$ and $I[u,\Omega'] \le I[v,\Omega']$ for all $\Omega' \Subset \Omega$ and any $v \in W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^N)$ s.t. $\mathrm{spt}(u-v) \subset \Omega'$. In the isotropic case (i.e. (2) holds with $p = q$) the local higher integrability of ∇u for a local minimizer u is a nowadays classical result which follows from Gehring's lemma as it is outlined for example in Giaquinta's monograph [15], where it is also summarized how to get better results for the isotropic scalar case. If the anisotropic scalar case is considered, then—roughly speaking—under mild smoothness assumptions on F local minimizers are locally Lipschitz provided that p and q are not too far apart, we refer to [7, 10, 14, 21] for a detailled overview and further references.

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Our note is addressed to the local higher integrability of the gradient in the anisotropic vector case. Here we first like to mention results of Marcellini [22] formulated for integrands depending on the modulus of the gradient, whereas the question of higher integrability for more general densities has been successfully attacked by Esposito et al. [12]. To explain their contribution let us assume that F is of class C^2 satisfying the ellipticity condition

$$\lambda(1+|Z|^2)^{\frac{p-2}{2}}|Y|^2 \le D^2F(Z)(Y,Y) \le \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2 \qquad (3)$$

for all $Y, Z \in \mathbb{R}^{nN}$ with constants $\lambda, \Lambda > 0$. Note that in [12] actually the case of degenerate ellipticity is considered. Note also that obviously (3) implies (2). Now, assuming (3) they show

$$\nabla u \in L^q_{\mathrm{loc}}(\Omega; \mathbb{R}^{nN}) \qquad (4)$$

for any local I-minimizer u provided that in addition

$$q < p + 2\min\{1, p/n\} \qquad (5)$$

is true. A typical example for which (3) holds (with $p = 2$ and $q \geq 2$) is the density

$$F_q(\nabla u) := |\nabla u|^2 + (1 + |\partial_n u|^2)^{\frac{q}{2}},$$

and we get the higher integrability (4) if according to (5) $q < 2 + 4/n$ which means that the range for admissible exponents q becomes smaller if the dimension increases. On the other hand, F_q is of the special form

$$F(Z) = G(|Z_1|, \dots, |Z_n|), \quad Z = (Z_1, \dots, Z_n), \quad Z_\alpha \in \mathbb{R}^N, \qquad (6)$$

for a suitable function G increasing w.r.t. each of its arguments. The structure (6) guarantees the convex hull property or a maximum principle (see, e.g. [3, 11]) which means that global minimizers for boundary data in $L^\infty(\Omega; \mathbb{R}^N)$ are bounded functions. From this point of view it makes sense to study locally bounded local minimizers of integrals (1) with F satisfying (3) and (6) having the hope that (4) or at least "some" higher integrability up to an exponent $s > p$ *not depending on* n can now be obtained under a *dimensionless condition* relating p and q. This idea was worked out by Choe [10] for the scalar case and also assuming even stronger than (6) that $F(\nabla u) = F(|\nabla u|)$ with the result that $q < p + 1$ implies (4) (and this gives $C^{1,\alpha}$). Later we proved in [3] that (3), (6) together with $u \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ implies (4) provided that $q < \max\{p(n+2)/n, p+2/3\}$. Thus $q < p+2/3$ gives (4), and in [2], Theorem 5.12, (see also [4]) this bound was improved ending up with

$$q < p + 2 \qquad (7)$$

as a sufficient condition for (4) under the hypotheses (3), (6) and $u \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$. At this stage we like to mention that (7) also occurs in [13], where it is shown that under conditions similar to (3) and (6) the global minimizer u for a boundary datum in $L^\infty(\Omega;\mathbb{R}^N)$ satisfies

$$\nabla u \in L^r_{\mathrm{loc}}(\Omega;\mathbb{R}^{nN}) \quad \text{for all } r < \frac{pn}{n-p+q-2},$$

which for large n is weaker that (4).

The main purpose of our note is to prove for the vector-case local higher integrability of the gradient of local minimizers up to a certain degree being *independent of* n under a *dimensionless* condition on p and q for a class of integrands for which (3) does not hold but which are in some sense *decomposable* into elliptic parts of different growth rates. As a typical example let us consider the density

$$F_{p,q}(\nabla u) = (1+|\tilde{\nabla}u|^2)^{\frac{p}{2}} + (1+|\partial_n u|^2)^{\frac{q}{2}}$$

with exponents $2 \le p \le q < \infty$. Here we have set

$$\tilde{\nabla}u := (\partial_1 u, \dots, \partial_{n-1}u) \in \mathbb{R}^{(n-1)N}.$$

Obviously $F_{p,q}$ does not satisfy (3), we just have the inequality

$$c|Z|^2 \le D^2F_{p,q}(X)(Z,Z) \le C(1+|X|^2)^{\frac{q-2}{2}}|Z|^2$$

so that according to (7) we need the bound $q < 4$ in order to apply Theorem 5.12 of [2]. However, $F_{p,q}$ falls in the category of integrands studied in [7], and from Theorem 1.1 of this reference we get $\tilde{\nabla}u \in L^{p+1}_{\mathrm{loc}}(\Omega;\mathbb{R}^{(n-1)N})$, $\partial_n u \in L^{q+1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ for a locally bounded minimizer u of $\int_\Omega F_{p,q}(\nabla u)\,\mathrm{d}x$ provided $q \le 2p$. Here we are going to improve this result under the following assumptions on the data: assume that $F\colon \mathbb{R}^{nN} \to [0,\infty)$ is of *splitting-type* (or *decomposable*) which means that

$$F(Z_1,\dots,Z_n) = f(\tilde{Z}) + g(Z_n), \tag{8}$$

$Z = (\tilde{Z}, Z_n)$, $\tilde{Z} = (Z_1,\dots,Z_{n-1}) \in \mathbb{R}^{(n-1)N}$, $Z_n \in \mathbb{R}^N$, with C^2-functions $f\colon \mathbb{R}^{(n-1)N} \to [0,\infty)$, $g\colon \mathbb{R}^N \to [0,\infty)$ for which

$$\left.\begin{aligned} \lambda(1+|\tilde{X}|^2)^{\frac{p-2}{2}}|\tilde{Z}|^2 &\le D^2f(\tilde{X})(\tilde{Z},\tilde{Z}) \le \Lambda(1+|\tilde{X}|^2)^{\frac{p-2}{2}}|\tilde{Z}|^2,\\ \lambda(1+|X_n|^2)^{\frac{q-2}{2}}|Z_n|^2 &\le D^2g(X_n)(Z_n,Z_n) \le \Lambda(1+|X_n|^2)^{\frac{q-2}{2}}|Z_n|^2 \end{aligned}\right\} \tag{9}$$

holds with constants $\lambda, \Lambda > 0$ and exponents $1 < p \le q$ for all matrices $X, Z \in \mathbb{R}^{nN}$. In order to have a maximum principle we further require ($X = (X_1,\dots,X_n) \in \mathbb{R}^{nN}$)

$$f(X_1,\dots,X_{n-1}) = \hat{f}(|X_1|,\dots,|X_{n-1}|), \quad g(X_n) = \hat{g}(|X_n|), \tag{10}$$

with $\hat{g}$ increasing and $\hat{f}$ increasing w.r.t. each argument. Note that (8)–(10) exactly correspond to the hypotheses imposed on the density in [7]. Let us now state our main result.

Theorem 1. *Suppose that F satisfies* (8)–(10) *with exponents* $1 < p \le q$ *and let* $u \in W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *denote a local I-minimizer such that* $u \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$.

(a) *Then we have that* $\partial_n u \in L^t_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ *for all* $t \le q+2$.
(b) *If in addition* $q \le 2p+2$, *then* $\tilde{\nabla} u \in L^s_{\mathrm{loc}}(\Omega; \mathbb{R}^{(n-1)N})$ *for all* $s \le p+2$.
(c) *If* Ω *denotes a domain in* $\mathbb{R}^2$ *and if* $2 \le p$, *then* ∇u *is in the space* $L^t_{\mathrm{loc}}(\Omega; \mathbb{R}^{2N})$ *for any* $t < \infty$. *This can be improved to* $u \in C^{1,\mu}(\Omega; \mathbb{R}^N)$ *for any* $\mu < 1$.

Remark 1. (a) In particular, the assumption $q \le p+2$ implies $\nabla u \in L^q_{\mathrm{loc}}(\Omega; \mathbb{R}^{nN})$.
(b) It will become clear from the proof that the assumption $p \le q$ concerning the growth rates of f and g is just an inessential hypothesis. If $p > q$, then the local minimizer u has to be taken from the space $W^1_{q,\mathrm{loc}}(\Omega; \mathbb{R}^N)$, and we get $\tilde{\nabla} u \in L^{p+2}_{\mathrm{loc}}(\Omega; \mathbb{R}^{(n-1)N})$ without further condition on p and q, whereas $p \le 2q+2$ implies $\partial_n u \in L^{q+2}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$. In particular, in the 2-D case, we arrive at $\nabla u \in L^t_{\mathrm{loc}}(\Omega; \mathbb{R}^{2N})$ and we have $u \in C^{1,\mu}(\Omega; \mathbb{R}^N)$ for arbitrary choices of q and $p \ge 2$. In the same spirit we like to remark that our result is not limited to the specific decomposition (8). With minor modifications in the proof we can discuss, for instance, the integrand

$$F(Z) := \sum_{i=1}^{n} (1 + |Z_i|^2)^{\frac{p_i}{2}}$$

with the result that $\partial_i u \in L^{p_i+2}_{\mathrm{loc}}$ provided that $\max\{p_i\} \le 2\min\{p_i\} + 2$. It is also evident that we can consider a function f depending on the whole variable Z and that in addition g can be allowed to depend on more than one partial derivative, provided the corresponding variants of (9) and (10) are true. In particular, our results apply to the density F_q defined after (5).
(c) In the non-splitting case, the assumption $q < p+2$ is in some sense natural to obtain higher integrability of the gradient of a bounded solution (compare [2] Remark 5.5, (ii)). We do not know if the hypothesis $q \le 2p+2$ also leads to $\nabla u \in L^t_{\mathrm{loc}}(\Omega; \mathbb{R}^{nN})$ for all $t \le p+2$ in case that u is a locally bounded local I-minimizer but with integrand F just satisfying (3) and (6).

Remark 2. (a) The structural condition (10) just enters through the fact that we need the uniform boundedness of the solutions of some approximate problems.
(b) Even if $N = 1$, the counterexamples of Giaquinta [16] and Hong [19] show that the local boundedness of the function u cannot be dropped from the set of assumptions of Theorem 1. If we take $n = 6$ in Giaquinta's example, for which $p = 2$ and $q = 4$, then it is easy to show that the singular minimizer u_0 constructed there satisfies

$$\int_{|x|<1} |\partial_n u_0|^t \, \mathrm{d}x < \infty$$

if and only if $t < 5$: the integrability of $\partial_n u_0$ does not hold for exponents up to 6 which would be the case for locally bounded minimizers of Giaquinta's energy.

(c) Let us mention that for $p > n$ the structural conditions (6) and (10), respectively, can be dropped by Sobolev's embedding theorem. By the maximum principle this is also possible in the scalar case.

Remark 3. (a) Again with minor modifications the proof of Theorem 1, (a) and (b) given below extends to the case when (9) is replaced by its degenerate variant (at least if $p \geq 2$). For a degenerate version of Theorem 1, (c) we again need the assumption $q \leq 2p + 2$.

(b) Moreover, it is not hard to prove Theorem 1 for the non-autonomous situation which means that now we have a splitting integrand $F = F(x, Z)$ depending also on $x \in \overline{\Omega}$ and where $D_x D_Z F$ satisfies a natural growth condition. Due to the splitting structure a Lavrentiev phenomenon in the approximation process presented below can be excluded (see, e.g. [5], Sect. 6) which enables us to continue to work with the regularized problems introduced in the beginning of the proof of Theorem 1.

(c) Going through the proof of Theorem 1 one easily checks that the explicit additive structure of F formulated in (8) is not really needed. In fact, if we require in place of (9) the ellipticity condition similar to the one used in [9]

$$\lambda\Big[(1+|\tilde{X}|^2)^{\frac{p-2}{2}}|\tilde{Z}|^2 + (1+|X_n|^2)^{\frac{q-2}{2}}|Z_n|^2\Big] \leq D^2F(X)(Z,Z) \leq \Lambda\Big[(1+|\tilde{X}|^2)^{\frac{p-2}{2}}|\tilde{Z}|^2 + (1+|X_n|^2)^{\frac{q-2}{2}}|Z_n|^2\Big], \tag{11}$$

drop (8) and replace (10) by (6), then we still have the results of Theorem 1. Note that in [9] the local higher integrability is studied for the case $p = 2$ together with $q < (2n-4)/(n-4) \to 2$ as $n \to \infty$.

(d) Up to now we concentrated on the anisotropic case. Let us finally mention, that even in the isotropic case $p = q$ Theorem 1 gives a result which we could not trace in the literature.

Proof of Theorem 1. From now on assume that F satisfies (8)–(10) with exponents $1 < p \leq q < \infty$ and that u is a locally bounded local I-minimizer. For $\varepsilon > 0$ let $(u)_\varepsilon$ denote the mollification of u with small radius $\varepsilon > 0$ and consider a ball $B := B_R(x_0)$ with compact closure in Ω. We fix an exponent $\tilde{q} > \max\{2, q\}$ and let

$$\delta := \delta(\varepsilon) := \frac{1}{1 + \varepsilon^{-1} + \|(\nabla u)_\varepsilon\|_{L^{\tilde{q}}(B)}^{2\tilde{q}}},$$

moreover, with $F_\delta(Z) := \delta(1 + |Z|^2)^{\tilde{q}/2} + F(Z)$, $Z \in \mathbb{R}^{nN}$, we define $u_\delta \in W^1_{\tilde{q}}(B; \mathbb{R}^N)$ as the unique solution of the problem

$$I_\delta[w, B] := \int_B F_\delta(\nabla w)\,\mathrm{d}x \to \min \quad \text{in } (u)_\varepsilon + \overset{\circ}{W}{}^1_{\tilde{q}}(B; \mathbb{R}^N).$$

We recall the following facts about this approximation:

Lemma 1. (a) *We have as* $\varepsilon \to 0$*:* $u_\delta \rightharpoondown u$ *in* $W^1_p(B;\mathbb{R}^N)$*;*

$$\delta \int_B (1+|\nabla u_\delta|^2)^{\frac{\tilde q}{2}}\,\mathrm{d}x \to 0\,; \quad \int_B F(\nabla u_\delta)\,\mathrm{d}x \to \int_B F(\nabla u)\,\mathrm{d}x.$$

(b) $\|u_\delta\|_{L^\infty(B)}$ *is bounded independent of* ε.
(c) ∇u_δ *is in the space* $L^\infty_{\mathrm{loc}} \cap W^1_{2,\mathrm{loc}}(B;\mathbb{R}^{nN})$.

Proof of Lemma 1. For (a) and (b) we refer to [3], Lemma 2.1, (c) follows from [17] and from [8]. □

Remark 4. (a) We have to regularize with exponent $\tilde q > \max\{2, q\}$ in order to ensure the initial regularity of u_δ stated in (c) of Lemma 1 which is essential for our calculations.
(b) Part (b) of Lemma 1 is the only place where we make use of (10). Any other condition giving (b) of the lemma could replace (10). Clearly (10) is superfluous in the scalar case or if $p > n$.

Lemma 2. (*Caccioppoli-type inequality*) *For any* $\eta \in C^\infty_0(B)$ *and any* $\gamma \in \{1,\dots,n\}$ *we have*

$$\int_B \eta^2 D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\,\mathrm{d}x \le c\int_B D^2 F_\delta(\nabla u_\delta)(\nabla\eta\otimes\partial_\gamma u_\delta, \nabla\eta\otimes\partial_\gamma u_\delta)\,\mathrm{d}x. \tag{12}$$

(No summation w.r.t. γ*,* $\otimes$ *denotes the tensor product and c is independent of* ε *and* η*).*

Proof of Lemma 2. Compare, e.g. [3], proof of Lemma 3.1. Inequality (12) follows from this reference by applying the Cauchy-Schwarz inequality to the bilinear form $D^2F_\delta(\nabla u_\delta)$. □

In what follows we let

$$\Gamma_\delta := 1+|\nabla u_\delta|^2, \quad \tilde\Gamma_\delta := 1+|\tilde\nabla u_\delta|^2, \quad \Gamma_{n,\delta} := 1+|\partial_n u_\delta|^2$$

and consider $\eta \in C^\infty_0(B)$, $0 \le \eta \le 1$, such that $\eta = 1$ on $B_r := B_r(x_0)$ for some $r < R$ and $|\nabla\eta| \le c/(R-r)$, where here and in the sequel c always is a finite positive constant being independent of ε, R and r. W.l.o.g. we may assume that $R < 1$. For the proof of part (a) of Theorem 1 we proceed similar to [2], Theorem 5.12, noting that all the integrals below are well-defined by Lemma 1, (c). We have for fixed $k \in \mathbb{N}$ (to be specified later)

$$\begin{aligned}\int_B \eta^{2k}|\partial_n u_\delta|^2\Gamma_{n,\delta}^{\frac{q}{2}}\,\mathrm{d}x &= \int_B \partial_n u_\delta \cdot \Big[\eta^{2k}\partial_n u_\delta \Gamma_{n,\delta}^{\frac{q}{2}}\Big]\,\mathrm{d}x\\ &= -\int_B u_\delta \cdot \partial_n\Big[\eta^{2k}\partial_n u_\delta \Gamma_{n,\delta}^{\frac{q}{2}}\Big]\,\mathrm{d}x\\ &\le c\Big[\int_B \eta^{2k}|\partial_n\partial_n u_\delta|\Gamma_{n,\delta}^{\frac{q}{2}}\,\mathrm{d}x + \int_B \eta^{2k-1}|\nabla\eta|\Gamma_{n,\delta}^{\frac{q+1}{2}}\,\mathrm{d}x\Big]\\ &=: c[I_1+I_2],\end{aligned} \tag{13}$$

where Lemma 1, (b) has been applied. For the l.h.s. of (13) we use the lower bound

$$\begin{aligned}\int_B \eta^{2k}|\partial_n u_\delta|^2\Gamma_{n,\delta}^{\frac{q}{2}}\,\mathrm{d}x &\geq \int_{B\cap[|\partial_n u_\delta|\geq 1]} \eta^{2k}|\partial_n u_\delta|^2\Gamma_{n,\delta}^{\frac{q}{2}}\,\mathrm{d}x\\ &\geq c\int_{B\cap[|\partial_n u_\delta|\geq 1]} \eta^{2k}\Gamma_{n,\delta}^{\frac{q+2}{2}}\,\mathrm{d}x\\ &\geq c\int_B \eta^{2k}\Gamma_{n,\delta}^{\frac{q+2}{2}}\,\mathrm{d}x-\tilde{c},\end{aligned}$$

thus (13) implies

$$\int_B \eta^{2k}\Gamma_{n,\delta}^{\frac{q+2}{2}}\,\mathrm{d}x \leq c[1+I_1+I_2]. \tag{14}$$

We apply Young's inequality to the integral I_2, where $\tau\in(0,1)$ is arbitrary:

$$\begin{aligned}I_2 &= \int_B \eta^{2k-1}\Gamma_{n,\delta}^{\frac{q+2}{4}}|\nabla\eta|\Gamma_{n,\delta}^{\frac{q}{4}}\,\mathrm{d}x\\ &\leq \tau\int_B \eta^{2k}\Gamma_{n,\delta}^{\frac{q+2}{2}}\,\mathrm{d}x + c(\tau)(R-r)^{-2}\int_B \eta^{2k-2}\Gamma_{n,\delta}^{\frac{q}{2}}\,\mathrm{d}x,\end{aligned}$$

and for τ small enough the first term on the r.h.s. can be absorbed on the l.h.s. of (14), whereas by Lemma 1, (a), we have

$$\int_B \Gamma_{n,\delta}^{\frac{q}{2}}\,\mathrm{d}x \leq c\left[1+\int_B F(\nabla u)\,\mathrm{d}x\right],$$

at least for ε sufficiently small. This implies with a local constant c_{loc} (depending in particular on r and R but being independent of ε)

$$\int_B \eta^{2k}\Gamma_{n,\delta}^{\frac{q+2}{2}}\,\mathrm{d}x \leq c[1+c_{\mathrm{loc}}+I_1]. \tag{15}$$

We discuss I_1: by Young's inequality we get

$$\begin{aligned}I_1 &= \int_B \eta^k|\partial_n\partial_n u|\Gamma_{n,\delta}^{\frac{q-2}{4}}\eta^k\Gamma_{n,\delta}^{\frac{q+2}{4}}\,\mathrm{d}x\\ &\leq \tau\int_B \eta^{2k}\Gamma_{n,\delta}^{\frac{q+2}{2}}\,\mathrm{d}x + c(\tau)\int_B \eta^{2k}|\partial_n\partial_n u|^2\Gamma_{n,\delta}^{\frac{q-2}{2}}\,\mathrm{d}x\\ &=: \tau J_1 + c(\tau)J_2,\end{aligned}$$

i.e. by absorbing terms for τ sufficiently small we arrive at

$$\int_B \eta^{2k}\Gamma_{n,\delta}^{\frac{q+2}{2}}\,\mathrm{d}x \le c[1+c_{\mathrm{loc}}+J_2]. \tag{16}$$

For J_2 we observe (recall (12) and Lemma 1, (a))

$$\begin{aligned} J_2 &\le c\int_B \eta^{2k} D^2F_\delta(\nabla u_\delta)(\partial_n\nabla u_\delta,\partial_n\nabla u_\delta)\,\mathrm{d}x \\ &\le c\Bigg[\delta\int_B \eta^{2k-2}|\nabla\eta|^2\Gamma_\delta^{\frac{\tilde q}{2}}\,\mathrm{d}x + \int_B \eta^{2k-2}|\nabla\eta|^2\Gamma_{n,\delta}^{\frac{q}{2}}\,\mathrm{d}x \\ &\qquad + \int_B \eta^{2k-2}|\nabla\eta|^2\Gamma_{n,\delta}\tilde\Gamma_\delta^{\frac{p-2}{2}}\,\mathrm{d}x\Bigg] \\ &\le c_{\mathrm{loc}} + c\int_B \eta^{2k-2}|\nabla\eta|^2\Gamma_{n,\delta}\tilde\Gamma_\delta^{\frac{p-2}{2}}\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} \int_B \eta^{2k-2}|\nabla\eta|^2\Gamma_{n,\delta}\tilde\Gamma_\delta^{\frac{p-2}{2}}\,\mathrm{d}x &\le c\Bigg[\int_B |\nabla\eta|^2\tilde\Gamma_\delta^{\frac{p}{2}}\,\mathrm{d}x + \int_B |\nabla\eta|^2\Gamma_{n,\delta}^{\frac{p}{2}}\,\mathrm{d}x\Bigg] \\ &\le c_{\mathrm{loc}}, \end{aligned}$$

provided $p \ge 2$. If $1 < p < 2$, then we estimate

$$\begin{aligned} \int_B |\nabla\eta|^2\eta^{2k-2}\Gamma_{n,\delta}\tilde\Gamma_\delta^{\frac{p-2}{2}}\,\mathrm{d}x &\le \int_B |\nabla\eta|^2\eta^{2k-2}\Gamma_{n,\delta}\,\mathrm{d}x \\ &\le \tau\int_B \eta^{(2k-2)\frac{q+2}{2}}\Gamma_{n,\delta}^{\frac{q+2}{2}}\,\mathrm{d}x + c_{\mathrm{loc}}(\tau) \end{aligned}$$

and for k sufficiently large (together with $\tau \ll 1$) we can absorb the first integral on the l.h.s. of (16). Thus we derive in both cases from (16)

$$\int_B \eta^{2k}\Gamma_{n,\delta}^{\frac{q+2}{2}}\,\mathrm{d}x \le c_{\mathrm{loc}} \tag{17}$$

and (17) means $\partial_n u_\delta \in L^{q+2}_{\mathrm{loc}}(B;\mathbb{R}^N)$ uniformly w.r.t. $\delta = \delta(\varepsilon)$, i.e. part (a) of Theorem 1 follows with the help of Lemma 1 by passing to the limit $\varepsilon \to 0$.

In order to prove part (b) we consider η as before and again fix $k \in \mathbb{N}$ to be specified later. In what follows we always take the sum w.r.t. $\gamma = 1, \dots, n-1$. In place of (13) we now have

$$\begin{aligned}\int_B \eta^{2k} |\tilde{\nabla} u_\delta|^2 \tilde{\Gamma}_\delta^{\frac{p}{2}} \,\mathrm{d}x &= -\int_B u_\delta \cdot \partial_\gamma \Big[\eta^{2k} \partial_\gamma u_\delta \tilde{\Gamma}_\delta^{\frac{p}{2}} \Big] \mathrm{d}x \\ &\le c \Bigg[\int_B \eta^{2k} |\tilde{\nabla}^2 u_\delta| \tilde{\Gamma}_\delta^{\frac{p}{2}} \,\mathrm{d}x + \int_B \eta^{2k-1} |\nabla \eta| \tilde{\Gamma}_\delta^{\frac{p+1}{2}} \,\mathrm{d}x \Bigg] \\ &=: c[I_1' + I_2'] \end{aligned} \tag{18}$$

with an obvious meaning of $\tilde{\nabla}^2$. In (18) we use the estimate (once more applying Young's inequality)

$$\begin{aligned} I_2' &= \int_B \eta^{2k-1} \tilde{\Gamma}_\delta^{\frac{p+2}{4}} \tilde{\Gamma}_\delta^{\frac{p}{4}} |\nabla \eta| \,\mathrm{d}x \\ &\le \tau \int_B \eta^{2k} \tilde{\Gamma}_\delta^{\frac{p+2}{2}} \,\mathrm{d}x + c(\tau) \int_B |\nabla \eta|^2 \tilde{\Gamma}_\delta^{\frac{p}{2}} \,\mathrm{d}x \end{aligned}$$

and as above we arrive at the counterpart to (15)

$$\int_B \eta^{2k} \tilde{\Gamma}_\delta^{\frac{p+2}{2}} \,\mathrm{d}x \le c[1 + c_{\mathrm{loc}} + I_1']. \tag{19}$$

I_1' is handled via

$$\begin{aligned} I_1' &= \int_B \eta^{2k} \tilde{\Gamma}_\delta^{\frac{p-2}{4}} |\tilde{\nabla}^2 u_\delta| \tilde{\Gamma}_\delta^{\frac{p+2}{4}} \,\mathrm{d}x \\ &\le \tau \int_B \eta^{2k} \tilde{\Gamma}_\delta^{\frac{p+2}{2}} \,\mathrm{d}x + c(\tau) \int_B \eta^{2k} \tilde{\Gamma}_\delta^{\frac{p-2}{2}} |\tilde{\nabla}^2 u_\delta|^2 \,\mathrm{d}x \\ &=: \tau J_1' + c(\tau) J_2', \end{aligned}$$

and again with the same arguments as above we obtain from (19) (compare (16))

$$\int_B \eta^{2k} \tilde{\Gamma}_\delta^{\frac{p+2}{2}} \,\mathrm{d}x \le c[1 + c_{\mathrm{loc}} + J_2']. \tag{20}$$

Now we apply Lemma 2 and Lemma 1, (a), with the result

$$
\begin{aligned}
J_2' &\le c \int_B \eta^{2k} D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\, \mathrm{d}x \\
&\le c\Bigg[\delta \int_B \eta^{2k-2} |\nabla \eta|^2 \Gamma_\delta^{\frac{\tilde{q}}{2}}\, \mathrm{d}x + \int_B \eta^{2k-2} |\nabla \eta|^2 \tilde{\Gamma}_\delta^{\frac{p}{2}}\, \mathrm{d}x \\
&\qquad + \int_B |\nabla \eta|^2 \eta^{2k-2} \tilde{\Gamma}_\delta \Gamma_{n,\delta}^{\frac{q-2}{2}}\, \mathrm{d}x\Bigg] \\
&\le c_{\mathrm{loc}} + c \int_B |\nabla \eta|^2 \eta^{2k-2} \tilde{\Gamma}_\delta \Gamma_{n,\delta}^{\frac{q-2}{2}}\, \mathrm{d}x
\end{aligned} \tag{21}
$$

and for discussing the remaining integral we observe that for any $\tau > 0$

$$
\begin{aligned}
\int_B \eta^{2k-2} \tilde{\Gamma}_\delta \Gamma_{n,\delta}^{\frac{q-2}{2}} |\nabla \eta|^2\, \mathrm{d}x &\le \tau \int_B \tilde{\Gamma}_\delta^{\frac{p+2}{2}} \eta^{(2k-2)\frac{p+2}{2}}\, \mathrm{d}x \\
&\quad + c(\tau) \int_B |\nabla \eta|^{2\frac{p+2}{p}} \Gamma_{n,\delta}^{\frac{q-2}{2}\frac{p+2}{p}}\, \mathrm{d}x.
\end{aligned}
$$

If $q \ge 2$, by (17) the second item on the r.h.s. of the above inequality is bounded by a local constant on account of

$$
\frac{q-2}{2} \frac{p+2}{p} \le \frac{q+2}{2}
$$

which follows from our assumption $q \le 2p + 2$. If $q < 2$, then the boundedness of this item is immediate.

Therefore, the claim of part (b) of Theorem 1 follows from (21) by choosing k in such a way that $(2k-2)(p+2)/2 \ge 2k$, and by finally absorbing the τ-term into the l.h.s. of (20).

For proving part (c) of Theorem 1 we keep the notation introduced above. Inequality (12) implies (compare the calculations after (16))

$$
\begin{aligned}
&\int_B \eta^2 D^2 F_\delta(\nabla u_\delta)(\partial_2 \nabla u_\delta, \partial_2 \nabla u_\delta)\, \mathrm{d}x \\
&\quad \le c \int_B D^2 F_\delta(\nabla u_\delta)(\partial_2 u_\delta \otimes \nabla \eta, \partial_2 u_\delta \otimes \nabla \eta)\, \mathrm{d}x \\
&\quad \le c \|\nabla \eta\|_\infty^2 \Bigg[\delta \int_B \Gamma_\delta^{\frac{\tilde{q}}{2}}\, \mathrm{d}x + \int_B \tilde{\Gamma}_\delta^{\frac{p-2}{2}} \Gamma_{2,\delta} + \int_B \Gamma_{2,\delta}^{\frac{q}{2}}\, \mathrm{d}x\Bigg],
\end{aligned}
$$

and since

$$
\int_B \tilde{\Gamma}_\delta^{\frac{p-2}{2}} \Gamma_{2,\delta}\, \mathrm{d}x \le c\Bigg[\int_B \tilde{\Gamma}_\delta^{\frac{p}{2}}\, \mathrm{d}x + \int_B \Gamma_{2,\delta}^{\frac{p}{2}}\, \mathrm{d}x\Bigg]
$$

we have with a suitable local constant (recalling $2 \le p$)

$$\begin{aligned}\int_B \eta^2\lambda|\nabla\partial_2 u_\delta|^2\,\mathrm{d}x &= \int_B \eta^2\lambda\big(|\partial_1\partial_2 u_\delta|^2 + |\partial_2\partial_2 u_\delta|^2\big)\,\mathrm{d}x\\ &\le \int_B \eta^2 D^2 f(\partial_1 u_\delta)(\partial_1\partial_2 u_\delta, \partial_1\partial_2 u_\delta)\,\mathrm{d}x\\ &\quad + \int_B \eta^2 D^2 g(\partial_2 u_\delta)(\partial_2\partial_2 u_\delta, \partial_2\partial_2 u_\delta)\,\mathrm{d}x\\ &\le \int_B \eta^2 D^2 F_\delta(\nabla u_\delta)(\partial_2\nabla u_\delta, \partial_2\nabla u_\delta)\,\mathrm{d}x \le c_{\mathrm{loc}} \qquad (22)\end{aligned}$$

and we have shown with (22)

$$\partial_2 u_\delta \in W^1_{2,\mathrm{loc}}(B) \quad \text{uniformly in } \varepsilon,$$

thus we get from Sobolev's embedding theorem

$$\partial_2 u_\delta \in L^t_{\mathrm{loc}}(B;\mathbb{R}^N) \quad \text{for all } t < \infty \text{ and uniformly in } \varepsilon. \qquad (23)$$

With (23) we return to the last step in the proof of (b) observing that

$$\int_B |\nabla\eta|^{2\frac{p+2}{p}}\Gamma_{2,\delta}^{\frac{q-2}{2}\frac{p+2}{p}}\,\mathrm{d}x$$

stays bounded without the requirement $q \le 2p+2$, thus we get $\partial_1 u_\delta \in L^{p+2}_{\mathrm{loc}}(B;\mathbb{R}^N)$ uniformly in ε just under the hypothesis that $2 \le p \le q$. Now we agree to take the sum w.r.t. $\gamma = 1, 2$. Then inequality (12) can be rewritten as

$$\int_B \eta^2 H_\delta^2\,\mathrm{d}x \le c\int_B D^2F_\delta(\nabla u_\delta)(\nabla\eta\otimes\partial_\gamma u_\delta, \nabla\eta\otimes\partial_\gamma u_\delta)\,\mathrm{d}x, \qquad (24)$$

where

$$\begin{aligned}H_\delta^2 := {}&\delta D^2F_0(\nabla u_\delta)(\partial_\gamma\nabla u_\delta, \partial_\gamma\nabla u_\delta)\\ &+ D^2 f(\partial_1 u_\delta)(\partial_\gamma\partial_1 u_\delta, \partial_\gamma\partial_1 u_\delta)\\ &+ D^2 g(\partial_2 u_\delta)(\partial_\gamma\partial_2 u_\delta, \partial_\gamma\partial_2 u_\delta),\end{aligned}$$

and where

$$F_0(Z) := (1+|Z|^2)^{\frac{\tilde q}{2}}, \quad Z \in \mathbb{R}^{2N}.$$

Let us consider any disc $B_{2r}(z_0) \Subset B = B_R(x_0)$ and let us suppose that $\eta \equiv 1$ on $B_r(z_0)$, $0 \le \eta \le 1$, $\operatorname{spt}\eta \subset B_{2r}(z_0)$, $|\nabla\eta| \le c/r$. Then the r.h.s. of (24) can be bounded by

$$\frac{c}{r^2}\int_{B_{2r}(z_0)} \Big[\delta\Gamma_\delta^{\frac{\tilde q}{2}} + \tilde\Gamma_\delta^{\frac{p-2}{2}}|\partial_\gamma u_\delta||\partial_\gamma u_\delta| + \Gamma_{2,\delta}^{\frac{q-2}{2}}|\partial_\gamma u_\delta||\partial_\gamma u_\delta|\Big]\,\mathrm{d}x.$$

In this expression all terms are "uncritical" with the exception of

$$\int_{B_{2r}(z_0)} \Gamma_{2,\delta}^{\frac{q-2}{2}} |\partial_1 u_\delta|^2 \,\mathrm{d}x$$

but this integral can be bounded (as done after (21)) with the help of Young's inequality

$$\int_{B_{2r}(z_0)} \Gamma_{2,\delta}^{\frac{q-2}{2}} \tilde{\Gamma}_\delta \,\mathrm{d}x \le c\Bigg[\int_{B_{2r}(z_0)} \tilde{\Gamma}_\delta^{\frac{p+2}{2}} \,\mathrm{d}x + \int_{B_{2r}(z_0)} \Gamma_{2,\delta}^{\frac{p+2}{p}\frac{q-2}{2}} \,\mathrm{d}x \Bigg],$$

thus with (23) and the comments thereafter (note that for $p > 2$ we could just choose the exponents $p/2$ and $\frac{p}{p-2}\frac{q-2}{2}$ in the r.h.s. of the above inequality) it is shown that

$$H_\delta \in L^2_{\mathrm{loc}}(B) \quad \text{uniform w.r.t. } \varepsilon. \tag{25}$$

But (25) together with (9) immediately implies $\nabla u_\delta \in W^1_{2,\mathrm{loc}}(B;\mathbb{R}^{2N})$ uniform w.r.t. ε, thus

$$\nabla u_\delta \in L^t_{\mathrm{loc}}(B;\mathbb{R}^{2N}) \quad \text{uniform in } \varepsilon \tag{26}$$

for any $t < \infty$, moreover, at least for a subsequence

$$\nabla u_\delta \to \nabla u \quad \text{a.e. on } B. \tag{27}$$

Now, with (26), the first part of Theorem 1, (c), is established. Next we claim that (25) is enough to follow the calculations from [5], proof of Theorem 1.1, to get the second result stated in (c) of Theorem 1. For the readers convenience we sketch some details. First, with η as specified after (24), we can replace (24) by the inequality

$$\int_{B_{2r}(z_0)} \eta^2 H_\delta^2 \,\mathrm{d}x \le -2 \int_{B_{2r}(z_0)} \eta D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u^* \otimes \nabla\eta) \,\mathrm{d}x, \tag{28}$$

where $u^*_\delta(x) := u(x) - P(x)$ for $P \in \mathbb{R}^{2N}$ to be fixed later. Letting

$$\begin{aligned} h_{1,\delta} &:= (1 + |\partial_1 u_\delta|^2)^{\frac{p-2}{4}}, \\ h_{2,\delta} &:= (1 + |\partial_2 u_\delta|^2)^{\frac{q-2}{4}}, \\ h_{3,\delta} &:= (1 + |\nabla u_\delta|^2)^{\frac{\tilde{q}-2}{4}} \sqrt{\delta}, \\ h_\delta &:= (h_{1,\delta}^2 + h_{2,\delta}^2 + h_{3,\delta}^2)^{\frac{1}{2}} \end{aligned}$$

it follows from (9) and (28)

$$\begin{aligned} \int_{B_r(z_0)} H_\delta^2 \,\mathrm{d}x &\le \frac{c}{r} \int_{B_{2r}(z_0)} H_\delta |\nabla u_\delta - P| \big[h_{1,\delta} + h_{2,\delta} + h_{3,\delta}\big] \,\mathrm{d}x \\ &\le \frac{c}{r} \int_{B_{2r}(z_0)} H_\delta h_\delta |\nabla u_\delta - P| \,\mathrm{d}x, \end{aligned}$$

thus with $s = 4/3$, using Hölder's as well as the Sobolev-Poincarè inequality we get

$$\fint_{B_r(z_0)} H_\delta^2 \,\mathrm{d}x \le c\Big[\fint_{B_{2r}(z_0)} (H_\delta h_\delta)^s \,\mathrm{d}x\Big]^{\frac{1}{s}}\Big[\fint_{B_{2r}(z_0)} |\nabla^2 u_\delta|^s \,\mathrm{d}x\Big]^{\frac{1}{s}}. \tag{29}$$

In (29) $\fint \dots$ denotes the mean value, and we have chosen $P := \fint_{B_{2r}(z_0)} \nabla u_\delta \,\mathrm{d}x$. Since

$$|\nabla^2 u_\delta| \le c H_\delta \le c H_\delta h_\delta$$

on account of $p, q \ge 2$, (29) implies

$$\Big[\fint_{B_r(z_0)} H_\delta^2 \,\mathrm{d}x\Big]^{\frac{1}{2}} \le c\Big[\fint_{B_{2r}(z_0)} (h_\delta H_\delta)^s \,\mathrm{d}x\Big]^{\frac{1}{s}}. \tag{30}$$

Note that c is uniform in $B_r(z_0)$ and ε if for example $B_{2r}(z_0) \subset B_\rho(x_0)$ for some fixed $\rho < R$. Letting $d := 2/s$, $\bar{f} := H_\delta^s$, $\bar{g} := h_\delta^s$, $\bar{h} := 0$, then (30) reads as

$$\Big[\fint_{B_r(z_0)} \bar{f}^d \,\mathrm{d}x\Big]^{\frac{1}{d}} \le c \fint_{B_{2r}(z_0)} \bar{f}\bar{g} \,\mathrm{d}x, \tag{31}$$

which exactly corresponds to (1.3) in [6]. (25) gives $\bar{f} \in L^d_{\mathrm{loc}}(B)$, and in order to apply Lemma 1.2 of [6] it remains to check that

$$\exp(\beta h_\delta^2) \in L^1_{\mathrm{loc}}(B) \tag{32}$$

for arbitrary $\beta > 0$. (Of course everything is meant uniform in ε). To this purpose let

$$\begin{aligned}
\tilde{h}_{1,\delta} &:= (1 + |\partial_1 u_\delta|^2)^{\frac{p}{4}},\\
\tilde{h}_{2,\delta} &:= (1 + |\partial_2 u_\delta|^2)^{\frac{q}{4}},\\
\tilde{h}_{3,\delta} &:= (1 + |\nabla u_\delta|^2)^{\frac{\tilde{q}}{4}}\sqrt{\delta},\\
\tilde{h}_\delta &:= (\tilde{h}_{1,\delta}^2 + \tilde{h}_{2,\delta}^2 + \tilde{h}_{3,\delta}^2)^{\frac{1}{2}}
\end{aligned}$$

and observe that (25) implies

$$\tilde{h}_{1,\delta}, \quad \tilde{h}_{2,\delta}, \quad \tilde{h}_{3,\delta} \in W^1_{2,\mathrm{loc}}(B). \tag{33}$$

Since

$$|\nabla \tilde{h}_\delta| \le |\nabla \tilde{h}_{1,\delta}| + |\nabla \tilde{h}_{2,\delta}| + |\nabla \tilde{h}_{3,\delta}|$$

(33) also gives

$$\tilde{h}_\delta \in W^1_{2,\mathrm{loc}}(B). \tag{34}$$

But (34) enables us to apply Trudinger's inequality (see [18], Theorem 7.15) with the result

$$\int_{B_\rho} \exp(\beta_0 \tilde{h}_\delta^2)\,dx \le c(\rho) \tag{35}$$

for discs B_ρ, $\rho < R$, with β_0 depending on the $W_2^1(B_\rho)$-norm of $\tilde{h}_\delta$. Finally we observe that

$$h_\delta^2 \le c\tilde{h}_\delta^{2(1-2/q)}$$

and therefore

$$\int_{B_\rho} \exp(\beta h_\delta^2)\,dx \le \int_{B_\rho} \exp(c\beta \tilde{h}_\delta^{2(1-2/q)})\,dx \le \int_{B_\rho} \exp(\beta_0 \tilde{h}_\delta^2 + c(\beta))\,dx,$$

so that (35) gives (32). Lemma 1.2 of [6] shows the existence of a positive constant c_0 with the property

$$\int_{B_\rho} H_\delta^2 \ln^{c_0\beta}(e + H_\delta)\,dx \le c(\beta, \rho) \tag{36}$$

for any $\beta > 0$, and as outlined in [5] (36) is true if H_δ is replaced by $|\sigma_{i,\delta}|$, $i = 1, 2$, where $\sigma_{1,\delta} := Df(\partial_1 u_\delta)$, $\sigma_{2,\delta} := Dg(\partial_2 u_\delta)$. This implies by quoting for example [20], Example 5.3, the uniform continuity of $\sigma_{1,\delta}$, $\sigma_{2,\delta}$ which means that ∇u_δ is continuous uniform in ε. Using (27) and Arzela's theorem we find that $u \in C^1(B; \mathbb{R}^N)$, and the final claim follows from elliptic regularity theory for systems with continuous coefficients (applied to the equation satisfied by $\partial_\gamma u$, $\gamma = 1$, 2, compare [15]). □

Remark 5. In the 2D-case and for $p < 2$, then under the more restrictive assumption that $q \le 2p + 2$ it is possible to prove (25) starting from (24), where now the r.h.s. (24) stays bounded on account of the requirement $q \le 2p+2$ combined with the results of (a) and (b). From (25) we directly deduce (33) so that the uniform local higher integrability of $\partial_1 u_\delta$, $\partial_2 u_\delta$ follows. Thus the first part of Theorem 1 (c) is true at least for exponents $1 < p \le q \le 2p + 2$.

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