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# On the spectrum of a Stokes-type operator arising from flow around a rotating body

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**Abstract.** We present the description of the spectrum of a linear perturbed Stokes-type operator which arises from equations of motion of a viscous incompressible fluid in the exterior of a rotating compact body. Considering the operator in the function space $L^2_\sigma(\Omega)$ we prove that the essential spectrum consists of a set of equally spaced half lines parallel to the negative real half line in the complex plane. Our approach is based on a reduction to invariant closed subspaces of $L^2_\sigma(\Omega)$ and on a Fourier series expansion with respect to an angular variable in a cylindrical coordinate system attached to the axis of rotation. Moreover, we show that the leading part of the operator is normal if and only if the body is axially symmetric about this axis.

## 1. Introduction

Suppose that $K$ is a compact body in $\mathbb{R}^3$, i.e., the closure of a bounded domain in $\mathbb{R}^3$, rotating about the $x_1$-axis with the angular velocity $\omega$. Put $\boldsymbol{\omega} = \omega \boldsymbol{e}_1$ where $\boldsymbol{e}_1$ is unit vector oriented in the direction of the $x_1$-axis. Denote further by $\Omega(t)$ the exterior of $K$ at time $t$. Assume that $\Omega(t)$ is a domain with boundary of class $C^{1,1}$. Put

$$O(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega t & \sin\omega t \\ 0 & -\sin\omega t & \cos\omega t \end{pmatrix}.$$

Then $\boldsymbol{x} \equiv (x_1, x_2, x_3) \in \Omega(t) \iff \boldsymbol{x}' \equiv O(t)\boldsymbol{x} \in \Omega(0)$. Thus, $\boldsymbol{x}'$ denotes the Cartesian coordinates connected with the rotating body. Our assumptions do not exclude the case when $K = \emptyset$ and consequently $\Omega(t) = \mathbb{R}^3$ for all $t \geq 0$.

Let $\boldsymbol{u}$ denote the velocity and $p$ denote the pressure of a flow of a viscous incompressible fluid in the exterior of the body $K$. Then $\boldsymbol{u}$ and $p$ satisfy the Navier–Stokes equation

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} \tag{1}$$

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and the equation of continuity

$$\nabla \cdot \boldsymbol{u} \;=\; 0 \tag{2}$$

in the space–time region $\{(\boldsymbol{x}, t) \in \mathbb{R}^3 \times I;\ t \in I,\ \boldsymbol{x} \in \Omega(t)\}$ where $I$ is an interval on the time-axis. The assumption on adherence of the fluid to the body $K$ on the surface of $K$ leads to the boundary condition

$$\boldsymbol{u}(\boldsymbol{x}, t) \;=\; \boldsymbol{\omega} \times \boldsymbol{x}, \quad \boldsymbol{x} \in \partial\Omega(t). \tag{3}$$

The disadvantage of this description is the variability of the spatial domain $\Omega(t)$ which is filled, at time $t$, by the moving fluid. This is why many authors use the transformation

$$\boldsymbol{u}(\boldsymbol{x}, t) = O^T(t)\, \boldsymbol{u}'(\boldsymbol{x}', t) \;=\; O^T(t)\, \boldsymbol{u}'\big(O(t)\boldsymbol{x}, t\big), \tag{4}$$

$$p(\boldsymbol{x}, t) = p'(\boldsymbol{x}', t) \;=\; p'\big(O(t)\boldsymbol{x}, t\big). \tag{5}$$

The functions $\boldsymbol{u}'$, $p'$ satisfy the system of equations

$$\partial_t \boldsymbol{u}' - \nu\Delta'\boldsymbol{u}' - (\boldsymbol{\omega} \times \boldsymbol{x}') \cdot \nabla'\boldsymbol{u}' + \boldsymbol{\omega} \times \boldsymbol{u}' + (\boldsymbol{u}' \cdot \nabla')\boldsymbol{u}' + \nabla' p' = \boldsymbol{f}' \tag{6}$$

$$\nabla' \cdot \boldsymbol{u}' = 0 \tag{7}$$

in $\Omega(0) \times I$, where $\nabla'$, respectively $\Delta'$, denote the operator nabla, respectively the Laplace operator, with respect to $\boldsymbol{x}'$. The boundary condition (3) is transformed to

$$\boldsymbol{u}'(\boldsymbol{x}', t) \;=\; \boldsymbol{\omega} \times \boldsymbol{x}', \quad \boldsymbol{x}' \in \partial\Omega(0). \tag{8}$$

In order to have a simple notation, we shall further omit the primes in (6–8) and we shall write only $\Omega$ instead of $\Omega(0)$.

Among a series of basic results on properties of the system (6–8) or related linear problems, let us mention Cumsille and Tucsnak [2]; Hishida [15], [16], [17]; Galdi [7], [8]; Galdi and Silvestre [10]; Farwig et al. [5]; Farwig [3], [4]; Nečasová [23], [24]; Geissert et al. [11] and Kračmar et al. [20].

We shall use the basic notation: let $R_0 = \max\{|\boldsymbol{x}|;\ \boldsymbol{x} \in K\}$ and $\Omega_R = \Omega \cap B_R(\boldsymbol{0})$ with outer normal vector $\boldsymbol{n}$ on $\partial\Omega$. Moreover, we use the following spaces and operators:

- $(.\,,\,.)_{0,2}$ and $\|\,.\,\|_{0,2}$ are the scalar product and norm in $L^2(\Omega)^3$, respectively.
- $W_0^{1,2}(\Omega)$ is the subspace of the Sobolev space $W^{1,2}(\Omega)$ consisting of functions vanishing on $\partial\Omega$ in the sense of traces. As is well-known, $W_0^{1,2}(\Omega)$ equals the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,2}(\Omega)$.
- $\|\,.\,\|_{k,2}$ denotes the norm in $W^{k,2}(\Omega)^3$, $k \in \mathbb{N}$.
- $C_{0,\sigma}^\infty(\Omega)$ denotes the space of all divergence-free functions from $C_0^\infty(\Omega)^3$.
- $L^2_\sigma(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)^3$. The space $L^2_\sigma(\Omega)$ can be characterized as the space of all divergence-free (in the sense of distributions) vector functions $\boldsymbol{u}$ from $L^2(\Omega)^3$ such that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ in the sense of traces ([6], pp. 111–115).
- $\Pi_\sigma$ denotes the orthogonal projection of $L^2(\Omega)^3$ onto $L^2_\sigma(\Omega)$.

By analogy with the classical or perturbed Stokes operators, which play a fundamental role in the analysis of the Navier–Stokes equation, we introduce the following Stokes-type operators (note the sign '+' in front of $\Pi_\sigma \nu\Delta$)

$$A^\omega \boldsymbol{u} = \Pi_\sigma \nu\Delta \boldsymbol{u} + \Pi_\sigma[(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u} - \boldsymbol{\omega}\times\boldsymbol{u}], \tag{9}$$

$$L^\omega \boldsymbol{u} = A^\omega \boldsymbol{u} + B\boldsymbol{u} \tag{10}$$

in $L^2_\sigma(\Omega)$ with the dense domains

$$D(A^\omega) = D(L^\omega) = \Big\{\boldsymbol{u}\in W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap L^2_\sigma(\Omega);\ (\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u}\in L^2(\Omega)^3\Big\}.$$

Here

$$B\boldsymbol{u} = \Pi_\sigma \mathcal{B}(\boldsymbol{x})\boldsymbol{u} + \Pi_\sigma \boldsymbol{b}(\boldsymbol{x})\cdot\nabla\boldsymbol{u}$$

where $\mathcal{B}$ is supposed to be a $3\times 3$ matrix with entries in $L^2_{loc}(\overline{\Omega})$ and $\boldsymbol{b}$ vector function in $L^q_{loc}(\overline{\Omega})^3$ for some $q>3$. Moreover, we assume that

$$\lim_{R\to+\infty}\Big(\operatorname*{ess\,sup}_{|x|>R}(|\mathcal{B}(\boldsymbol{x})| + |\boldsymbol{b}(\boldsymbol{x})|)\Big) = 0. \tag{11}$$

Now our main theorems read as follows (for definitions of several kinds of spectra see Sect. 2 below):

**Theorem 1.1.** (i) *The essential spectrum $\sigma_{ess}(A^\omega)$ of the operator $A^\omega$ has the form*

$$\sigma_{ess}(A^\omega) = \{\lambda = \alpha + \mathrm{i}k\omega;\quad k\in\mathbb{Z},\ \alpha\le 0\}. \tag{12}$$

(ii) *If $\Omega$ is axially symmetric about the $x_1$-axis, then the operator $A^\omega$ is normal, the point spectrum and the residual spectrum of $A^\omega$ are empty and the continuous spectrum coincides with $\sigma_{ess}(A^\omega)$.*

(iii) *If $\omega\neq 0$ and the domain $\Omega$ is not axially symmetric about the $x_1$-axis, then the operator $A^\omega$ is not normal.*

**Theorem 1.2.** (i) *The essential spectrum $\sigma_{ess}(L^\omega)$ has the same form* (12) *as $\sigma_{ess}(A^\omega)$.*

(ii) *The spectrum $\sigma(L^\omega)$ equals $\sigma_{ess}(L^\omega)\cup\Lambda$ where $\Lambda$ consists of an at most countable set of isolated eigenvalues of $L^\omega$ which can possibly cluster only at points of $\sigma_{ess}(L^\omega)$ and each of them has a finite algebraic multiplicity.*

## 2. Preliminaries

Since the main aim of this paper is to study the spectrum of the operators $A^\omega$ and $L^\omega$, we shall consider all function spaces needed in the following to be spaces of complex-valued functions.

**Lemma 2.1.** *There exists* $c_1 > 0$ *such that if* $\boldsymbol{u} \in D(A^\omega)$ *and* $\boldsymbol{f} = A^\omega \boldsymbol{u}$*, then*

$$\|\boldsymbol{u}\|_{2,2} + \|(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u}\|_{0,2} \le c_1 \big(\|\boldsymbol{f}\|_{0,2} + \|\boldsymbol{u}\|_{0,2}\big). \tag{13}$$

If $\Omega = \mathbb{R}^3$ then estimate (13) is a direct consequence of [5]. Otherwise (13) follows from [16].

**Lemma 2.2.** $A^\omega$ *is a closed operator in* $L^2_\sigma(\Omega)$ *and its adjoint has the form*

$$(A^\omega)^* \boldsymbol{u} = \Pi_\sigma \nu \Delta \boldsymbol{u} - \Pi_\sigma [(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u}] \tag{14}$$

*with* $D\big((A^\omega)^*\big) = D(A^\omega)$.

*Proof.* Suppose that $\boldsymbol{u}_n \in D(A^\omega)$, $\boldsymbol{u}_n \to \boldsymbol{u}$ in $L^2_\sigma(\Omega)$ and $A^\omega \boldsymbol{u}_n \equiv \boldsymbol{f}_n \to \boldsymbol{f}$ in $L^2_\sigma(\Omega)$. Then $A^\omega(\boldsymbol{u}_n - \boldsymbol{u}_m) \equiv \boldsymbol{f}_n - \boldsymbol{f}_m$ and due to the estimate (13), we have

$$\|\boldsymbol{u}_n - \boldsymbol{u}_m\|_{2,2} + \|(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla(\boldsymbol{u}_n - \boldsymbol{u}_m)\|_{0,2} \le c_1 \big(\|\boldsymbol{f}_n - \boldsymbol{f}_m\|_{0,2} + \|\boldsymbol{u}_n - \boldsymbol{u}_m\|_{0,2}\big).$$

Thus we get that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3$, and the sequence $\{(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u}_n\}$ converges to some function $\boldsymbol{h}$ in $L^2(\Omega)^3$. Since $(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u}_n \to (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u}$ in $L^2(\Omega_R)^3$ for each $R \ge R_0$, we deduce that $\boldsymbol{h} = (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u}$. This implies that $\boldsymbol{u} \in D(A^\omega)$ and $A^\omega \boldsymbol{u} = \boldsymbol{f}$ which confirms that the operator $A^\omega$ is closed.

It follows from [11], Proposition 4.3, that for $\zeta > 0$ sufficiently large the range of the operator $\zeta I - A^\omega$ covers the whole space $L^2_\sigma(\Omega)$.

Let us denote by $T^\omega$ the operator on the right hand side of (14) with $D(T^\omega) = D(A^\omega)$. By analogy with $A^\omega$, the operator $T^\omega$ is closed and $R(\zeta I - T^\omega) = L^2_\sigma(\Omega)$ if $\zeta > 0$ is sufficiently large. It is easy to verify that the operators $A^\omega$ and $T^\omega$ are adjoint to each other in the sense of Kato [19], p. 167; hence $T^\omega \subset (A^\omega)^*$. In order to show that $T^\omega = (A^\omega)^*$, we need to verify that $T^\omega$ is the maximal operator adjoint to $A^\omega$. Suppose that $\boldsymbol{v} \in D((A^\omega)^*)$ and put $\boldsymbol{f} = (\zeta I - (A^\omega)^*)\boldsymbol{v}$. Since $\boldsymbol{f} \in R(\zeta I - T^\omega)$, there exists $\boldsymbol{w} \in D(T^\omega)$ such that $\boldsymbol{f} = (\zeta I - T^\omega)\boldsymbol{w}$. Hence $(\zeta I - (A^\omega)^*)\boldsymbol{v} = (\zeta I - T^\omega)\boldsymbol{w}$. Multiplying both sides of this identity by $\boldsymbol{u} \in D(A^\omega)$, we arrive at

$$\big(\boldsymbol{v},\ (\zeta I - A^\omega)\boldsymbol{u}\big)_{0,2} = \big(\boldsymbol{w},\ (\zeta I - A^\omega)\boldsymbol{u}\big)_{0,2}.$$

As this holds for all $\boldsymbol{u} \in D(A^\omega)$, we get $\boldsymbol{v} = \boldsymbol{w} \in D(B^\omega)$; thus $(A^\omega)^* = T^\omega$. □

For the proof of the following statement see also [10], Lemma 3.

**Lemma 2.3.** *If* $\boldsymbol{u} \in D(A^\omega)$*, then* $(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u}$ *belongs to* $L^2_\sigma(\Omega)$.

*Proof.* Since the space $C^\infty_{0,\sigma}(\Omega)$ is dense in $D(A^\omega)$ in the topology of $W^{1,2}(\Omega)^3$, there exists a sequence of functions $\boldsymbol{u}_n \in C^\infty_{0,\sigma}(\Omega)$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $W^{1,2}(\Omega)^3$. Let $\psi$ be a function from $W^{1,2}_{loc}(\Omega)$ such that $\nabla\psi \in L^2(\Omega)^3$. Then we have

$$\begin{aligned}
&\int_\Omega \big[(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u}\big] \cdot \nabla\psi \, \mathrm{d}\boldsymbol{x} \\
&= \lim_{n \to +\infty} \int_\Omega \big[(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u}_n - \boldsymbol{\omega} \times \boldsymbol{u}_n\big] \cdot \nabla\psi \, \mathrm{d}\boldsymbol{x} \\
&= -\lim_{n \to +\infty} \int_\Omega \mathrm{div}\big[(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u}_n - \boldsymbol{\omega} \times \boldsymbol{u}_n\big]\, \psi \, \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

We simply verify that $\mathrm{div}\big[(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u}_n - \boldsymbol{\omega} \times \boldsymbol{u}_n\big] = 0$ in $\Omega$. Thus the function $(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u}$ is orthogonal to the subspace of all gradients in $L^2(\Omega)^3$, which further implies that it belongs to $L^2_\sigma(\Omega)$, see e.g., Galdi [6], p. 103. □

Lemma 2.3 enables us to omit the projection $\Pi_\sigma$ in front of the second and the third term on the right hand side of (9). Therefore, the operator $A^\omega$ can be simplified to

$$A^\omega \boldsymbol{u} \;=\; A^0 \boldsymbol{u} + (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u} \tag{15}$$

where $A^0 \equiv \nu \Pi_\sigma \Delta$ is the usual Stokes operator in $L^2_\sigma(\Omega)$ with domain $D(A^0) = W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap L^2_\sigma(\Omega)$. The adjoint operator $(A^\omega)^*$ can similarly be simplified. It is well known that the operator $A^0$ is selfadjoint and generates an analytic semigroup $\mathrm{e}^{A^0 t},\ t \geq 0$, in $L^2_\sigma(\Omega)$ (Giga [12]; Giga and Sohr [13]).

**Lemma 2.4.** *The operator $B$ is $A^\omega$-compact.*

*Proof.* Let $\{\boldsymbol{u}_n\}$ be a bounded sequence in $L^2_\sigma(\Omega)$ such that the sequence $\{A^\omega \boldsymbol{u}_n\}$ is also bounded in $L^2_\sigma(\Omega)$. Then, due to Lemma 2.1, the sequence $\{\boldsymbol{u}_n\}$ is bounded in $W^{2,2}(\Omega)^3$. Hence there exists a subsequence of $\{\boldsymbol{u}_n\}$ (we preserve the same notation $\{\boldsymbol{u}_n\}$ for the subsequence) which converges weakly in $W^{2,2}(\Omega)^3$ to a limit function $\boldsymbol{v}$. Recall that $\boldsymbol{b} \in L^q_{loc}\big(\overline{\Omega}\big)^3$ for some $q > 3$. Put $q' = 2q/(q-2)$. Since $q' < 6$ and consequently, $W^{2,2}(\Omega_{R_0})^3 \hookrightarrow\hookrightarrow W^{1,q'}(\Omega_{R_0})^3$, there exists a subsequence $\big\{\boldsymbol{u}_n^{R_0}\big\}$ of $\{\boldsymbol{u}_n\}$ which converges in $W^{1,q'}(\Omega_{R_0})^3$. By analogy, there exists a subsequence $\big\{\boldsymbol{u}_n^{R_0+1}\big\}$ of $\big\{\boldsymbol{u}_n^{R_0}\big\}$ which converges in $W^{1,q'}(\Omega_{R_0+1})^3$. Proceeding in this way, we get a subsequence $\big\{\boldsymbol{u}_n^{R_0+2}\big\}$ of $\big\{\boldsymbol{u}_n^{R_0+1}\big\}$, etc. If we put $\boldsymbol{v}_n = \boldsymbol{u}_n^{R_0+n}$, we obtain a subsequence of $\{\boldsymbol{u}_n\}$ which converges in $W^{1,q'}(\Omega_R)^3$ for every $R \geq R_0$ to the function $\boldsymbol{v}$.

We claim that the sequence $\{B\boldsymbol{v}_n\}$ converges to $B\boldsymbol{v}$ in $L^2_\sigma(\Omega)$. For every $m \in \mathbb{N}$ and $R \geq R_0$, we have

$$\begin{aligned} \|B\boldsymbol{v}_m - B\boldsymbol{v}\|_0^2 &\leq 2 \int_\Omega \Big( \big|\mathcal{B}(\boldsymbol{v}_m - \boldsymbol{v})\big|^2 + \big|\boldsymbol{b} \cdot \nabla(\boldsymbol{v}_m - \boldsymbol{v})\big|^2 \Big)\, \mathrm{d}\boldsymbol{x} \\ &= 2 \int_{\Omega_R} \cdots + 2 \int_{\Omega - \Omega_R} \cdots \;\leq\; 2(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \end{aligned}$$

where

$$\gamma_1 = \int_{\Omega_R} |\mathcal{B}|^2 \,\mathrm{d}\boldsymbol{x} \,\Big( \operatorname*{ess\,sup}_{\Omega_R} |\boldsymbol{v}_m - \boldsymbol{v}|^2 \Big),$$

$$\gamma_2 = \Big( \operatorname*{ess\,sup}_{\Omega - \Omega_R} |\mathcal{B}|^2 \Big) \int_{\Omega - \Omega_R} |\boldsymbol{v}_m - \boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x},$$

$$\gamma_3 = \Big( \int_{\Omega_R} |\boldsymbol{b}|^q \,\mathrm{d}\boldsymbol{x} \Big)^{2/q} \Big( \int_{\Omega_R} |\nabla(\boldsymbol{v}_m - \boldsymbol{v})|^{q'} \,\mathrm{d}\boldsymbol{x} \Big)^{2/q'},$$

$$\gamma_4 = \Big(\operatorname*{ess\,sup}_{\Omega-\Omega_R} |\boldsymbol{b}|^2\Big) \int\limits_{\Omega-\Omega_R} |\nabla(\boldsymbol{v}_m - \boldsymbol{v})|^2 \, \mathrm{d}\boldsymbol{x}.$$

Here $\gamma_2$, $\gamma_4$ can be made arbitrarily small by choosing $R$ sufficiently large. Then $\gamma_1$, $\gamma_3$ can be made arbitrarily small by choosing $m$ sufficiently large. □

Lemma 2.2 and Lemma 2.4 imply that the operator $L^{\omega}$ is closed in $L^2_{\sigma}(\Omega)$. Note that under slightly different conditions on $\mathcal{B}$ and $\boldsymbol{b}$, it is proved in [11] that the operator $L^{\omega}$ generates a $C^0$-semigroup in $L^2_{\sigma}(\Omega)$, which also directly implies the closedness of $L^{\omega}$.

It will be further advantageous to work in cylindrical coordinates. We shall denote by $x_1$, $r$ and $\varphi$ the cylindrical coordinate system whose axis is the $x_1$-axis and angle $\varphi$ is measured from the positive part of the $x_2$-axis towards the positive part of the $x_3$-axis. The corresponding cylindrical components of vector functions will be denoted by the indices 1, $r$ and $\varphi$, e.g., $u_1$, $u_r$ and $u_{\varphi}$. In order to distinguish between the Cartesian and the cylindrical components of vectors, we shall write the Cartesian components in parentheses and the cylindrical components in brackets. Thus, we have $(u_1, u_2, u_3) \triangleq [u_1, u_r, u_{\varphi}]$. Using the transformations

$$\begin{aligned} u_r &= u_2 \cos\varphi + u_3 \sin\varphi, & u_{\varphi} &= -u_2 \sin\varphi + u_3 \cos\varphi, \\ u_2 &= u_r \cos\varphi - u_{\varphi} \sin\varphi, & u_3 &= u_r \sin\varphi + u_{\varphi} \cos\varphi, \end{aligned}$$

we calculate that

$$\begin{aligned} &(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u} = \omega\, \partial_{\varphi} \boldsymbol{u} - (\boldsymbol{\omega} \times \boldsymbol{u}) = \omega\, \partial_{\varphi}(u_1, u_2, u_3) - \omega\,(0, -u_3, u_2) \\ &= \omega\, \partial_{\varphi} \begin{pmatrix} u_1 \\ u_r \cos\varphi - u_{\varphi} \sin\varphi \\ u_r \sin\varphi + u_{\varphi} \cos\varphi \end{pmatrix}^T - \omega \begin{pmatrix} 0 \\ -u_r \sin\varphi - u_{\varphi} \cos\varphi \\ u_r \cos\varphi - u_{\varphi} \sin\varphi \end{pmatrix}^T \\ &= \omega \begin{pmatrix} \partial_{\varphi} u_1 \\ (\partial_{\varphi} u_r) \cos\varphi - (\partial_{\varphi} u_{\varphi}) \sin\varphi \\ (\partial_{\varphi} u_r) \sin\varphi + (\partial_{\varphi} u_{\varphi}) \cos\varphi \end{pmatrix}^T \triangleq \omega \begin{bmatrix} \partial_{\varphi} u_1 \\ \partial_{\varphi} u_r \\ \partial_{\varphi} u_{\varphi} \end{bmatrix}^T = \omega\, \partial_{\varphi} \big[u_1, u_r, u_{\varphi}\big]. \end{aligned}$$

We shall further consistently identify $\boldsymbol{u}$ with $[u_1, u_r, u_{\varphi}]$; the same holds for other vectors. Thus, we can write the relation (15) between the operator $A^{\omega}$ and the Stokes operator $A^0$ in the form

$$A^{\omega}\boldsymbol{u} \;=\; A^0\boldsymbol{u} + \omega\, \partial_{\varphi}\boldsymbol{u} \tag{16}$$

where $A^0$ naturally denotes the Stokes operator acting in cylindrical coordinates.

As there is no conformity in the names of various types of spectra in the literature, we recall some general notions. Suppose that $H$ is a Hilbert space with norm $\|\,.\,\|$ and $T$ is a closed linear operator in $H$ with a dense domain $D(T)$. Then $N(T)$ denotes the null space of $T$, $R(T)$ its range, and $T^*$ the adjoint operator to $T$. Moreover, we shall use the following notation:

- $\operatorname{nul}(T)$ (the nullity of $T$) $= \dim\ N(T)$
- $\operatorname{def}(T)$ (the deficiency of $T$) $= \dim\ H/R(T)$
- $\operatorname{ind}(T)$ (the index of $T$) $= \operatorname{nul}(T) - \operatorname{def}(T)$

- $\mathrm{nul}'(T)$ (the approximate nullity of $T$) – the greatest number $m \in \mathbb{N} \cup \{+\infty\}$ such that to any $\epsilon > 0$ there exists an $m$-dimensional closed linear manifold $M_\epsilon \subset D(T)$ with the property that $\|Tu\| \le \epsilon \|u\|$ for all $u \in M_\epsilon$
- $\mathrm{def}'(T)$ (the approximate deficiency of $T$) $= \mathrm{nul}'(T^*)$.

These numbers satisfy the inequalities, see Kato [19], pp. 230–233:

$$\mathrm{nul}'(T) \ge \mathrm{nul}(T), \quad \mathrm{def}'(T) \ge \mathrm{def}(T),$$

and, if $R(T)$ is closed, which is automatic if $\mathrm{def}(T) < +\infty$, then $\mathrm{nul}'(T) = \mathrm{nul}(T)$, $\mathrm{def}'(T) = \mathrm{def}(T)$. On the other hand, if $R(T)$ is not closed, then $\mathrm{nul}'(T) = \mathrm{def}'(T) = +\infty$.

- $\rho(T)$ (the resolvent set of $T$) is the open set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ has a bounded inverse operator defined in the whole space $H$. It is the set of $\lambda \in \mathbb{C}$ such that

$$\mathrm{nul}(T - \lambda I) = \mathrm{def}(T - \lambda I) = \mathrm{nul}'(T - \lambda I) = \mathrm{def}'(T - \lambda I) = 0.$$

- $\sigma_p(T)$ (the point spectrum of $T$) consists of eigenvalues of $T$. It is the set of $\lambda \in \mathbb{C}$ such that $\mathrm{nul}(T - \lambda I) > 0$. It can also be defined as the set of all $\lambda \in \mathbb{C}$ such that the operator $T - \lambda I$ is not injective.
- $\sigma_c(T)$ (the continuous spectrum of $T$) is the set of such $\lambda \in \mathbb{C}$ that $\mathrm{nul}(T - \lambda I) = 0$, $R(T - \lambda I)$ is dense in $H$, but $R(T - \lambda I) \neq H$. In this case,

$$\mathrm{nul}'(T - \lambda I) = \mathrm{def}(T - \lambda I) = \mathrm{def}'(T - \lambda I) = +\infty.$$

- $\sigma_r(T)$ (the residual spectrum of $T$) is the set of such $\lambda \in \mathbb{C}$ that $\mathrm{nul}(T - \lambda I) = 0$ and $R(T - \lambda I)$ is not dense in $H$. In this case, $\mathrm{def}(T - \lambda I) > 0$.
- $\sigma(T)$ (the spectrum of $T$) $= \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$. It follows from the previous definitions that $\sigma(T)$ is the complement of $\rho(T)$ in $\mathbb{C}$.
- $\sigma_{ess}(T)$ (the essential spectrum of $T$) is the set of all $\lambda \in \mathbb{C}$ such that $\mathrm{nul}'(T - \lambda I) = \mathrm{def}'(T - \lambda I) = +\infty$.
- $\widetilde{\sigma}_c(T)$ denotes the set of those $\lambda \in \mathbb{C}$ for which there exists a non-compact sequence $\{\boldsymbol{u}_n\}$ in the unit sphere in $H$ such that $(T - \lambda I)\boldsymbol{u}_n \to \mathbf{0}$ for $n \to +\infty$. It is equivalent with the equality $\mathrm{nul}'(T - \lambda I) = +\infty$ ([19], Theorem IV.5.11).

The three parts $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ of $\sigma(T)$ are mutually disjoint. The residual spectrum $\sigma_r(T)$ can be characterized as the set of $\lambda \in \mathbb{C}$ such that $\bar{\lambda} \in \sigma_p(T^*)$ and $\lambda \notin \sigma_p(T)$.

The essential spectrum $\sigma_{ess}(T)$ is defined e.g. in Kato [19]. Calling the operator $T$ semi-Fredholm if at least one of the numbers $\mathrm{nul}'(T)$, $\mathrm{def}'(T)$ is finite, $\sigma_{ess}(T)$ is the set of those $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is not semi-Fredholm. It is shown that $\sigma_{ess}(T)$ is a closed subset of $\mathbb{C}$ and $\mathrm{ind}(T - \lambda I)$ is constant in each component $G$ of $\mathbb{C} - \sigma_{ess}(T)$. Moreover, $\mathrm{nul}(T - \lambda I)$ and $\mathrm{def}(T - \lambda I)$ are constant in $G$ with the possible exception of an at most countable set of isolated eigenvalues of finite algebraic multiplicities which can cluster only at points of $\sigma_{ess}(T)$ ([19], p. 243).

The definition of $\widetilde{\sigma}_c(T)$ is due to Glazman [14] calling $\widetilde{\sigma}_c(T)$ the continuous spectrum; however, we shall not use this name for $\widetilde{\sigma}_c(T)$ because it would contradict the previous definition. It is known ([14], p. 20) that the set $\widetilde{\sigma}_c(T)$ is closed in $\mathbb{C}$. Obviously $\sigma_c(T) \subset \sigma_{ess}(T) \subset \widetilde{\sigma}_c(T) \subset \sigma(T)$.

The equality $\mathrm{nul}'(T - \lambda I) = +\infty$ for the points $\lambda \in \widetilde{\sigma}_c(T)$ enables us to construct, by mathematical induction, an orthonormal sequence $\{\boldsymbol{v}_n\}$ in the unit sphere in $H$ such that $(T - \lambda I)\boldsymbol{v}_n \to \mathbf{0}$ for $n \to +\infty$. Suppose that we have already constructed $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ so that $\|(T - \lambda I)\boldsymbol{v}_j\| \le 1/j$ for $j = 1, \dots, n$. Denote by $N_n$ the linear hull of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$. To $\epsilon_{n+1} = 1/(n+1)$ there exists an infinite dimensional linear manifold $M_{n+1}$ such that $\|(T - \lambda I)\boldsymbol{u}\| \le 1/(n+1)$ for all $\boldsymbol{u} \in M_{n+1}$. Due to Lemma IV.2.3 in [19], there exists $\boldsymbol{v}_{n+1} \in M_{n+1}$ such that $\|\boldsymbol{v}_{n+1}\| = 1$ and the distance between $\boldsymbol{v}_{n+1}$ and $N_n$ also equals 1. It can be simply shown that $\boldsymbol{v}_{n+1}$ is orthogonal to $N_n$.

An operator $T$ is said to be normal if $T^*T = TT^*$. If $T$ is normal then $T$ and $T^*$ have the same null space ([19], p. 277). It is well known that the residual spectrum of a normal operator is empty, see e.g. [25], Problem XII.9.13. [It is an easy consequence of the identities $R(T - \lambda I)^{\perp} = N(T^* - \bar{\lambda} I) = N(T - \lambda I)$].

**Lemma 2.5.** *If the operator $T$ is normal, then $\sigma_{ess}(T) = \widetilde{\sigma}_c(T)$.*

*Proof.* If $\lambda \in \widetilde{\sigma}_c(T) - \sigma_{ess}(T)$ then $R(T - \lambda I)$ is closed and consequently, also $R(T^* - \bar{\lambda} I)$ is closed. So we get

$$+\infty = \mathrm{nul}'(T - \lambda I) = \mathrm{nul}(T - \lambda I) = \mathrm{nul}(T^* - \bar{\lambda} I) = \mathrm{nul}'(T^* - \bar{\lambda} I).$$

Since $N(T^* - \bar{\lambda} I) = R(T - \lambda I)^{\perp}$ and $N(T - \lambda I) = R(T^* - \bar{\lambda} I)^{\perp}$, we have $R(T - \lambda I) = R(T^* - \bar{\lambda} I)$ and consequently,

$$+\infty > \mathrm{def}'(T - \lambda I) = \mathrm{def}(T - \lambda I) = \mathrm{def}(T^* - \bar{\lambda} I) = \mathrm{def}'(T^* - \bar{\lambda} I).$$

This implies that $\mathrm{ind}(T - \lambda I) = \mathrm{ind}(T^* - \bar{\lambda} I) = +\infty$. However, this is a contradiction to the equality $\mathrm{ind}(T - \lambda I) = -\mathrm{ind}(T^* - \bar{\lambda} I)$ which holds if $T - \lambda I$ is a semi-Fredholm operator, see [19], p. 234. We have proved that $\widetilde{\sigma}_c(T) \subset \sigma_{ess}(T)$. The opposite inclusion is obvious. □

Let us conclude this section by recalling known results on the spectrum of the Stokes operator $A^0$.

**Lemma 2.6.** $\sigma_p(A^0) = \sigma_r(A^0) = \emptyset$ *and* $\sigma(A^0) = \sigma_c(A^0) = (-\infty, 0]$.

The residual spectrum of $A^0$ is empty because $A^0$ is normal. The identity $\sigma(A^0) = (-\infty, 0]$ is well known and can be deduced from Glazman [14] and Ladyzhenskaya [21]. The non-existence of an eigenvalue is only rarely mentioned in the literature. However, it can be shown by means of results on the growth of a strong solution of the equation $\Delta w + q(\boldsymbol{x})\, w = 0$ for $|\boldsymbol{x}| \to +\infty$ proved by Kato [18]. If $\lambda$ is an eigenvalue of $A^0$ and $\boldsymbol{u} \not\equiv \mathbf{0}$ is an associated eigenfunction then $\lambda \in \mathbb{R}$. Multiplying the equation $A^0\boldsymbol{u} = \lambda\boldsymbol{u}$ by $\bar{\boldsymbol{u}}$, we can show that $\lambda < 0$. The vector field $\boldsymbol{w} = \mathbf{curl}\, \boldsymbol{u}$ satisfies $\Delta\boldsymbol{w} - \lambda\boldsymbol{w} = \mathbf{0}$ in $\Omega$. Then Theorem 1 from [18] implies that $\boldsymbol{w} = \mathbf{0}$ for all $\boldsymbol{x}$ such that $|\boldsymbol{x}| > R_0$ (here $\boldsymbol{x}$ denotes the Cartesian variables). Due to the unique continuation principle, see Leis [22], we have $\boldsymbol{w} = \mathbf{0}$ in $\Omega$. This implies, together with the boundary condition $\boldsymbol{u} = \mathbf{0}$ on $\partial\Omega$, that the circulation of $\boldsymbol{u}$ on each closed piecewise smooth curve in $\Omega$ equals zero. Thus, $\boldsymbol{u}$ has the form $\nabla\phi$ where $\phi$ is an appropriate scalar function in $\Omega$. Using now the equation of continuity $\nabla\cdot\boldsymbol{u}=0$ in $\Omega$ and the boundary condition, we derive that $\boldsymbol{u}=\boldsymbol{0}$ in $\Omega$. This is a contradiction with the assumption that $\boldsymbol{u}\not\equiv\boldsymbol{0}$.

## 3. Axially symmetric domains: decomposition of $L^2_\sigma(\Omega)$ and of $A^0$

We shall assume that the domain $\Omega\subset\mathbb{R}^3$ is axially symmetric with respect to the $x_1$-axis in Sect. 3 and 4. Clearly, this assumption is satisfied if the considered body $K$ is rotationally symmetric about the axis of rotation $x_1$.

Let $k$ be an integer. Then we introduce the following spaces:

- $L^2(\Omega)^3_k=\{\boldsymbol{v}\in L^2(\Omega)^3;\quad \boldsymbol{v}=\boldsymbol{V}(x_1,r)\,\mathrm{e}^{\mathrm{i}k\varphi}\}$
- $C_0^\infty(\Omega)^3_k=C_0^\infty(\Omega)^3\cap L^2(\Omega)^3_k$
- $C_{0,\sigma}^\infty(\Omega)_k=C_0^\infty(\Omega)^3_k\cap C_{0,\sigma}^\infty(\Omega)$
- $L^2_\sigma(\Omega)_k=$ the closure of $C_{0,\sigma}^\infty(\Omega)_k$ in $L^2(\Omega)^3_k$.

Obviously, $L^2(\Omega)^3_k$, $k\in\mathbb{Z}$, is a closed subspace of $L^2(\Omega)^3$, and $L^2_\sigma(\Omega)_k$ is a closed subspace of $L^2_\sigma(\Omega)$. The spaces $L^2(\Omega)^3_k$ and $L^2_\sigma(\Omega)_k$ are infinite dimensional. We further define the operators

- $P_k$—the orthogonal projection of $L^2(\Omega)^3$ onto $L^2(\Omega)^3_k$
- $A^0_k$—the restriction of the operator $A^0$ to the space $L^2_\sigma(\Omega)_k$.

Hence the domain of $A^0_k$ equals $D(A^0)\cap L^2_\sigma(\Omega)_k$.

Each function from $L^2(\Omega)^3$ can uniquely be written in the form of a convergent Fourier series—with respect to the variable $\varphi$—of terms from $L^2(\Omega)^3_k$, $k\in\mathbb{Z}$. To be more precise, if $\boldsymbol{v}\in L^2(\Omega)^3$, then

$$\boldsymbol{v}(x_1,r,\varphi)=\sum_{k=-\infty}^{+\infty}\boldsymbol{V}^k(x_1,r)\,\mathrm{e}^{\mathrm{i}k\varphi};\quad \boldsymbol{V}^k(x_1,r)=\frac{1}{2\pi}\int_0^{2\pi}\boldsymbol{v}(x_1,r,\varphi)\,\mathrm{e}^{-\mathrm{i}k\varphi}\,\mathrm{d}\varphi. \tag{17}$$

Thus, we have $L^2(\Omega)^3=\cdots\oplus L^2(\Omega)^3_{-2}\oplus L^2(\Omega)^3_{-1}\oplus L^2(\Omega)^3_0\oplus L^2(\Omega)^3_1\oplus L^2(\Omega)^3_2\oplus\cdots$.

**Lemma 3.1.** *Let $k\in\mathbb{Z}$. Then $\Pi_\sigma\,L^2(\Omega)^3_k=L^2_\sigma(\Omega)\cap L^2(\Omega)^3_k=L^2_\sigma(\Omega)_k=P_kL^2_\sigma(\Omega)$.*

*Proof.* Suppose that $\boldsymbol{v}\in W^{1,2}(\Omega)^3\cap L^2(\Omega)^3_k$. The analysis of the Neumann problem

$$\Delta\phi=\operatorname{div}\boldsymbol{v}\quad\text{in }\Omega,\qquad \frac{\partial\phi}{\partial\boldsymbol{n}}=\boldsymbol{v}\cdot\boldsymbol{n}\quad\text{on }\partial\Omega,$$

shows that the solution $\phi$ can be found in the form $\phi=\Phi(x_1,r)\,\mathrm{e}^{\mathrm{i}k\varphi}$. Then $\Pi_\sigma\boldsymbol{v}=\boldsymbol{v}-\nabla\phi\in L^2(\Omega)_k$. Using the density of $W^{1,2}(\Omega)^3\cap L^2(\Omega)^3_k$ in $L^2(\Omega)^3_k$, we can show that this is true for all $\boldsymbol{v} \in L^2(\Omega)^3_k$, i.e., $\Pi_\sigma L^2(\Omega)^3_k \subset L^2_\sigma(\Omega) \cap L^2(\Omega)^3_k$. The opposite inclusion is obvious.

Since $C^\infty_{0,\sigma}(\Omega)_k$ is dense in $L^2_\sigma(\Omega) \cap L^2(\Omega)^3_k$, its closure $L^2_\sigma(\Omega)_k$ equals $L^2_\sigma(\Omega) \cap L^2(\Omega)^3_k$.

Let us finally verify the last equality. Consider $\boldsymbol{v} \in C^\infty_{0,\sigma}(\Omega)^3$, let (17) be its Fourier expansion in the variable $\varphi$ and $V^k = \big[V^k_1, V^k_r, V^k_\varphi\big]$. Since

$$0 = \sum_{k=-\infty}^{+\infty} \operatorname{div}\Big[\boldsymbol{V}^k(x_1, r)\,\mathrm{e}^{\mathrm{i}k\varphi}\Big] = \sum_{k=-\infty}^{+\infty} \Big[\big(\partial_1 V^k_1\big) + \frac{1}{r}\,\partial_r\big(r V^k_r\big) + \frac{1}{r}\,\mathrm{i}\,k\,V^k_\varphi\Big]\,\mathrm{e}^{\mathrm{i}k\varphi},$$

we get $\operatorname{div}\big[\boldsymbol{V}^k(x_1, r)\,\mathrm{e}^{\mathrm{i}k\varphi}\big] = 0$ for each $k \in \mathbb{Z}$. Hence $P_k\boldsymbol{v} = \boldsymbol{V}^k(x_1, r)\,\mathrm{e}^{\mathrm{i}k\varphi} \in C^\infty_{0,\sigma}(\Omega)_k$ which is a subset of $L^2_\sigma(\Omega) \cap L^2(\Omega)^3_k$. Since $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^2_\sigma(\Omega)$, we obtain the inclusion $P_k L^2_\sigma(\Omega) \subset L^2_\sigma(\Omega) \cap L^2(\Omega)^3_k$. On the other hand, if $\boldsymbol{v} \in L^2_\sigma(\Omega) \cap L^2(\Omega)^3_k$, then $P_k\boldsymbol{v} = \boldsymbol{v}$, hence it also belongs to $P_k L^2_\sigma(\Omega)$. Thus, the opposite inclusion $L^2_\sigma(\Omega) \cap L^2(\Omega)^3_k \subset P_k L^2_\sigma(\Omega)$ is also true. □

**Lemma 3.2.** (i) $D(A^0_k) = P_k\big[D(A^0)\big]$.

(ii) $R(A^0_k) \subset L^2_\sigma(\Omega)_k$.

(iii) *The operator* $A^0_k$ *is selfadjoint in* $L^2_\sigma(\Omega)_k$.

*Proof.* Let $\boldsymbol{v} \in D(A^0) \equiv W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap L^2_\sigma(\Omega)$ and let (17) be its Fourier expansion in the variable $\varphi$. Then $\boldsymbol{V}^k(x_1, r)\,\mathrm{e}^{\mathrm{i}k\varphi} \equiv P_k\boldsymbol{v} \in W^{2,2}(\Omega)^3$, and, due to the axial symmetry of $\Omega$ and the boundary condition satisfied by $\boldsymbol{v}$ on $\partial\Omega$, $\boldsymbol{V}^k(x_1, r)\,\mathrm{e}^{\mathrm{i}k\varphi}$ also belongs to $W^{1,2}_0(\Omega)^3$. We have already seen in the proof of Lemma 3.1 that $\boldsymbol{V}^k(x_1, r)\,\mathrm{e}^{\mathrm{i}k\varphi} \in L^2_\sigma(\Omega)_k$. Hence $P_k[D(A^0)] \subset D(A^0_k)$.

On the other hand, if $\boldsymbol{v} \in D(A^0_k)$, then it belongs to $D(A^0)$, and since $P_k\boldsymbol{v} = \boldsymbol{v}$, it also belongs to $L^2(\Omega)^3_k$. Hence $\boldsymbol{v} \in D(A^0) \cap L^2(\Omega)^3_k = D(A^0) \cap L^2_\sigma(\Omega)_k = P_k[D(A^0)]$.

If $\boldsymbol{v} \in D(A^0_k)$, then $\Delta\boldsymbol{v} \in L^2(\Omega)^3_k$, and due to Lemma 3.1, $A^0\boldsymbol{v} = \nu\Pi_\sigma\Delta\boldsymbol{v} \in L^2_\sigma(\Omega)_k$. Hence $A^0$ is reduced onto $L^2_\sigma(\Omega)_k$.

The domain $D(A^0_k)$ is dense in $L^2_\sigma(\Omega)_k$ because it contains $C^\infty_{0,\sigma}(\Omega)_k$. Moreover, the operator $A^0_k$ is symmetric because it is the part of the symmetric operator $A^0$ in $L^2_\sigma(\Omega)_k$, and $A^0_k$ is closed because it is the restriction of the closed operator $A^0$ to a closed subspace of $L^2_\sigma(\Omega)$. Thus, in order to show that $A^0_k$ is selfadjoint, it is sufficient to show that $\rho(A^0_k)$ contains at least one real number ([19], p. 271). Indeed, if $\zeta \in \mathbb{R}$, $\zeta > 0$, and $\boldsymbol{f} \in L^2_\sigma(\Omega)_k$, then it can be verified that $\boldsymbol{u} = (A^0 - \zeta I)^{-1}\boldsymbol{f}$ represents the unique solution of the equation $(A^0_k - \zeta I)\boldsymbol{u} = \boldsymbol{f}$ in $L^2_\sigma(\Omega)_k$. Thus, $\zeta \in \rho(A^0_k)$. □

**Lemma 3.3.** $\sigma(A^0_k) = \sigma_c(A^0_k) = (-\infty, 0]$.

*Proof—part 1.* Since the operator $A^0_k$ is a part of $A^0$, Lemma 2.6 yields $\widetilde{\sigma}_c(A^0_k) \subset \widetilde{\sigma}_c(A^0) = (-\infty, 0]$.

Let us prove that $\widetilde{\sigma}_c(A_k^0)$ covers the whole interval $(-\infty, 0]$. Since $\widetilde{\sigma}_c(A_k^0)$ is a closed set, it is sufficient to show that it is also open in $(-\infty, 0]$ and non-empty. The last property is clear because $A_k^0$ is selfadjoint: if $\widetilde{\sigma}_c(A_k^0) = \emptyset$ then $\sigma(A_k^0) = \sigma_p(A_k^0)$, which is impossible because each eigenvalue of $A_k^0$ is also an eigenvalue of $A^0$ (with the same eigenfunction) and $\sigma_p(A^0) = \emptyset$.

Let us show that $\widetilde{\sigma}_c(A_k^0)$ is open in $(-\infty, 0]$. Suppose that $\lambda \in \widetilde{\sigma}_c(A_k^0)$. Then $\mathrm{nul}'(A_k^0 - \lambda I) = +\infty$. This enables us, cf. Sect. 2, to choose an orthonormal sequence $\{\boldsymbol{v}_n\} \subset D(A_k^0)$ in $L^2_\sigma(\Omega)_k$ such that

$$(A_k^0 - \lambda I)\, \boldsymbol{v}_n \;=\; \boldsymbol{\epsilon}_n \;\longrightarrow\; 0 \quad \text{in } L^2_\sigma(\Omega)_k \quad \text{for } n \to +\infty. \tag{18}$$

We shall further use the next lemma.

**Lemma 3.4.** *Let* $\{\boldsymbol{v}_n\} \subset D(A_k^0)$ *be an orthonormal sequence satisfying* (18). *Then there exists* $R > R_0$ *and a non-compact sequence* $\{\boldsymbol{u}_n\}$ *in* $D(A_k^0)$ *such that* $\|\boldsymbol{u}_n\|_{0,2} = 1$, $\boldsymbol{u}_n = \mathbf{0}$ *in* $\Omega_R$ *and*

$$(A_k^0 - \lambda I)\, \boldsymbol{u}_n \;\longrightarrow\; 0 \quad \textit{in } L^2_\sigma(\Omega)_k \quad \textit{for } n \to +\infty. \tag{19}$$

*Proof.* Obviously $\{\boldsymbol{v}_n\}$ converges to the zero function weakly in $L^2_\sigma(\Omega)_k$. The estimate

$$\|\nabla \boldsymbol{v}\|_{0,2} + \|\nabla^2 \boldsymbol{v}\|_{0,2} \;\leq\; c_2 \left( \|A^0 \boldsymbol{v}\|_{0,2} + \|\boldsymbol{v}\|_{0,2} \right) \tag{20}$$

(Galdi and Padula [9], pp. 205, 279) shows that the sequence $\{\boldsymbol{v}_n\}$ is bounded in $W_0^{1,2}(\Omega)^3 \cap W^{2,2}(\Omega)^3$. Then there exists a subsequence, again denoted by $\{\boldsymbol{v}_n\}$, which is weakly convergent to $\mathbf{0}$ in $W_0^{1,2}(\Omega)^3 \cap W^{2,2}(\Omega)^3$. Suppose that $R \geq R_0 + 3$ is a fixed number. The compact imbedding $W^{2,2}(\Omega_R)^3 \hookrightarrow\hookrightarrow W^{1,2}(\Omega_R)^3$ yields

$$\boldsymbol{v}_n \;\longrightarrow\; \mathbf{0} \qquad \text{strongly in } W^{1,2}(\Omega_R)^3. \tag{21}$$

The first part of (18) can be written in the form

$$\nu \Delta \boldsymbol{v}_n - \lambda \boldsymbol{v}_n + \nabla q_n \;=\; \boldsymbol{\epsilon}_n \tag{22}$$

where $q_n$ is an appropriate scalar function. It follows from (22) that $\nabla q_n \to \mathbf{0}$ weakly in $L^2(\Omega)^3$. Thus, the functions $q_n$ [which are given uniquely up to an additive constant by (22)] can be chosen so that $q_n \to q \equiv const.$ strongly in $L^2(\Omega_R)$. The constant can be chosen so that $q = 0$.

Denote by $\eta$ an infinitely differentiable cut-off function in $\Omega$ such that

$$\eta(\boldsymbol{x}) \;=\; \begin{cases} 0 & \text{if } |\boldsymbol{x}| < R - 2, \\ 1 & \text{if } |\boldsymbol{x}| > R - 1, \end{cases}$$

$0 \leq \eta(\boldsymbol{x}) \leq 1$ if $R - 2 \leq |\boldsymbol{x}| \leq R - 1$, and that $\eta$ is independent of $\varphi$. Put $\boldsymbol{u}_n = \eta \boldsymbol{v}_n - \boldsymbol{V}_n$ where $\operatorname{div} \boldsymbol{V}_n = \nabla\eta \cdot \boldsymbol{v}_n$. Although $\boldsymbol{V}_n$ is not given uniquely, the results on solutions of the equation $\operatorname{div} \boldsymbol{V} = \boldsymbol{f}$ (see e.g., [1]) show that the function $\boldsymbol{V}_n$ can be chosen such that $\operatorname{supp} \boldsymbol{V}_n \subset \{\boldsymbol{x} \in \Omega;\ R - 3 < |\boldsymbol{x}| < R\}$ and there exist $c_3, c_4 > 0$ such that

$$\|\boldsymbol{V}_n\|_{2,2} \;\leq\; c_3\, \|\nabla\eta \cdot \boldsymbol{v}_n\|_{1,2} \;\leq\; c_4 \tag{23}$$

for all $n \in \mathbb{N}$. Moreover, since $\nabla\eta$ is independent of $\varphi$ and $\boldsymbol{v}_n \in L^2_\sigma(\Omega)_k$, the function $\boldsymbol{V}_n$ can be constructed so that it belongs to $L^2(\Omega)^3_k$.

The function $\boldsymbol{u}_n$ is divergence-free, equals $\boldsymbol{0}$ in $\Omega_{R-3}$, equals $\boldsymbol{v}_n$ in $\Omega - \Omega_R$ and belongs to $L^2(\Omega)^3_k$. Due to the properties of the functions $\eta$ and $\boldsymbol{V}_n$ we get $\boldsymbol{u}_n \in D(A^0_k)$. Obviously $\boldsymbol{u}_n$ satisfies

$$\begin{aligned}\nu\Delta\boldsymbol{u}_n &- \lambda\boldsymbol{u}_n + \nabla(\eta q_n)\\ &= \eta[\nu\Delta\boldsymbol{v}_n - \lambda\boldsymbol{v}_n] + 2\nu\nabla\eta\cdot\nabla\boldsymbol{v}_n + \nu(\Delta\eta)\boldsymbol{v}_n - \nu\Delta\boldsymbol{V}_n + \lambda\boldsymbol{V}_n + \nabla(\eta q_n)\\ &= \eta\boldsymbol{\epsilon}_n + 2\nu\nabla\eta\cdot\nabla\boldsymbol{v}_n + \nu(\Delta\eta)\boldsymbol{v}_n - \nu\Delta\boldsymbol{V}_n + \lambda\boldsymbol{V}_n + (\nabla\eta)q_n \qquad (24)\end{aligned}$$

where $\eta\boldsymbol{\epsilon}_n \to \boldsymbol{0}$ in $L^2(\Omega)^3$ due to (18), and $\nu[2\nabla\eta\cdot\nabla\boldsymbol{v}_n + (\Delta\eta)\boldsymbol{v}_n] \to \boldsymbol{0}$ in $L^2(\Omega)^3$ because $\nabla\eta$ and $\Delta\eta$ are supported in $\Omega_R$ and due to (21). Furthermore, $(\nu\Delta\boldsymbol{V}_n - \lambda\boldsymbol{V}_n) \to \boldsymbol{0}$ in $L^2(\Omega)^3$ due to (23), (21). Finally, $(\nabla\eta)q_n \to \boldsymbol{0}$ in $L^2(\Omega)^3$ because $q_n \to 0$ in $L^2(\Omega_R)$ and $\nabla\eta$ is supported in $\Omega_R$. Thus,

$$\nu\Delta\boldsymbol{u}_n - \lambda\boldsymbol{u}_n + \nabla(\eta q_n) \longrightarrow \boldsymbol{0} \quad \text{in } L^2(\Omega)^3 \quad \text{for } n \to +\infty,$$

and therefore $\{\boldsymbol{u}_n\}$ satisfies (19). We have

$$\|\boldsymbol{u}_n\|^2_{0,2} \geq \int_{|x|>R} |\boldsymbol{u}_n(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} = \int_{|x|>R} |\boldsymbol{v}_n(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} \longrightarrow 1 \quad \text{for } n \to +\infty$$

because $\|\boldsymbol{v}_n\|_{0,2} = 1$ and due to (21). If we divide each of the functions $\boldsymbol{u}_n$ by its norm $\|\boldsymbol{u}_n\|_{0,2}$ and denote the new function again by $\boldsymbol{u}_n$, we obtain the sequence $\{\boldsymbol{u}_n\}$ with all the properties stated in Lemma 3.4. Finally, the orthonormality of $\{\boldsymbol{v}_n\}$ and (21) imply the non-compactness of the sequence $\{\boldsymbol{u}_n\}$. □

*Proof of Lemma 3.3—part 2.* Consider the sequence $\{\boldsymbol{u}_n\}$ constructed in Lemma 3.4. There exists $0 < \zeta_0 < 1$ such that for any $\zeta > \zeta_0$ the functions

$$\boldsymbol{u}^\zeta_n(\boldsymbol{x}) = \begin{cases} \dfrac{1}{\zeta^{3/2}}\,\boldsymbol{u}_n\Big(\dfrac{\boldsymbol{x}}{\zeta}\Big) & \text{for } \boldsymbol{x}/\zeta \in \Omega,\\ 0 & \text{for } \boldsymbol{x}/\zeta \notin \Omega\end{cases}$$

have their supports outside $\overline{\Omega}_{R_0}$. Thus $\boldsymbol{u}^\zeta_n \in D(A^0_k)$, $\{\boldsymbol{u}^\zeta_n\}$ is a non-compact sequence in $L^2_\sigma(\Omega)_k$ and

$$\|\boldsymbol{u}^\zeta_n\|^2_{0,2} = \int_\Omega |\boldsymbol{u}^\zeta_n(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} = \frac{1}{\zeta^3}\int_{x/\zeta\in\Omega}\Big|\boldsymbol{u}_n\Big(\frac{\boldsymbol{x}}{\zeta}\Big)\Big|^2\,\mathrm{d}\boldsymbol{x} = \int_\Omega |\boldsymbol{u}_n(\boldsymbol{y})|^2\,\mathrm{d}\boldsymbol{y} = 1.$$

Since $\nu\Delta_x\boldsymbol{u}^\zeta_n(\boldsymbol{x}) - \frac{\lambda}{\zeta^2}\,\boldsymbol{u}^\zeta_n(\boldsymbol{x}) = \frac{1}{\zeta^{7/2}}\left(\Delta_y\boldsymbol{u}_n(\boldsymbol{y}) - \lambda\,\boldsymbol{u}_n(\boldsymbol{y})\right)$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$ from $\Omega$ such that $\boldsymbol{y} = \boldsymbol{x}/\zeta$,

$$\Big(A^0_k\boldsymbol{u}^\zeta_n - \frac{\lambda}{\zeta^2}\,I\Big)\,\boldsymbol{u}^\zeta_n \longrightarrow \boldsymbol{0} \quad \text{for } n \to +\infty. \qquad (25)$$

This shows that $\lambda/\zeta^2 \in \widetilde{\sigma}_c(A^0_k)$. Since $\zeta$ can be chosen arbitrarily in the interval $(\zeta_0, +\infty)$, some neighborhood of $\lambda$ in $(-\infty, 0]$ is contained in $\widetilde{\sigma}_c(A^0_k)$.

We have proved that $\widetilde{\sigma}_c(A_k^0) = (-\infty, 0]$. Lemma 2.5 shows that $\widetilde{\sigma}_c(A_k^0) = \sigma_{ess}(A_k^0)$, and both sets are also equal to $\sigma_c(A_k^0)$ because $\sigma_p(A_k^0) = \sigma_r(A_k^0) = \emptyset$. □

The restriction of $e^{A^0 t}$, $t \geq 0$, to $L^2_\sigma(\Omega)_k$ defines an analytic semigroup in $L^2_\sigma(\Omega)_k$. It can be verified that its generator is the operator $A_k^0$.

## 4. Axially symmetric domains Ω: operator $A^\omega$ and its decomposition

Let $k \in \mathbb{Z}$. We shall denote by $A_k^\omega$ the restriction of $A^\omega$ to $L^2_\sigma(\Omega)_k$. The domain of $A_k^\omega$ is the same as the domain of $A_k^0$, i.e.,

$$D(A_k^\omega) = D(A_k^0) \equiv W^{2,2}(\Omega)^3 \cap W_0^{1,2}(\Omega)^3 \cap L^2_\sigma(\Omega)_k.$$

If $\boldsymbol{u} \in L^2_\sigma(\Omega)_k$, then it has the form $\boldsymbol{U}(x_1, r)\, e^{ik\varphi}$ and $\partial_\varphi \boldsymbol{u} = i\, k\, \boldsymbol{U}\, e^{ik\varphi} = i\, k\, \boldsymbol{u}$. Therefore, $A_k^\omega$ can be rewritten as

$$A_k^\omega \boldsymbol{u} \;=\; A_k^0 \boldsymbol{u} + \omega\, \partial_\varphi \boldsymbol{u} \;=\; A_k^0 \boldsymbol{u} + i\, k\omega\, \boldsymbol{u}. \tag{26}$$

**Lemma 4.1.** *$A_k^\omega$ is a normal operator in $L^2_\sigma(\Omega)_k$ and*

$$\sigma(A_k^\omega) = \sigma_c(A_k^\omega) = \{\lambda = \alpha + i\, k\omega;\ \alpha \leq 0\}.$$

*Proof.* Since $A_k^0$ is reduced by $L^2_\sigma(\Omega)_k$, the operator $A_k^\omega$ is an operator in $L^2_\sigma(\Omega)_k$ due to (26). Moreover, as $A_k^0$ is selfadjoint, the operator $A_k^\omega$ is densely defined and closed. The adjoint operator to $A_k^\omega$ has the form

$$(A_k^\omega)^* \boldsymbol{u} \;=\; A_k^0 \boldsymbol{u} - \omega\, \partial_\varphi \boldsymbol{u} \;=\; A_k^0 \boldsymbol{u} - i\, k\omega\, \boldsymbol{u}. \tag{27}$$

This operator commutes with $A_k^\omega$, hence $A_k^\omega$ is normal. The characterization of $\sigma(A_k^\omega)$ follows from the representation (26) of $A_k^\omega$ and from Lemma 3.3. □

Since $A_k^0$ generates an analytic semigroup in $L^2_\sigma(\Omega)_k$ and $A_k^\omega$ equals $A_k^0$ plus a bounded operator in $L^2_\sigma(\Omega)_k$, $A_k^\omega$ also generates an analytic semigroup in $L^2_\sigma(\Omega)_k$.

**Lemma 4.2.** *$A^\omega$ is a normal operator in the space $L^2_\sigma(\Omega)$.*

*Proof.* Equality (16) implies that $(A^\omega)^* \boldsymbol{u} = A^0 \boldsymbol{u} - \omega\, \partial_\varphi \boldsymbol{u} = \Pi_\sigma \Delta \boldsymbol{u} - \omega\, \partial_\varphi \boldsymbol{u}$.

Suppose that $\boldsymbol{u} \in D(A^\omega (A^\omega)^*)$, i.e., $\boldsymbol{u} \in D((A^\omega)^*)$ and $(A^\omega)^* \boldsymbol{u} \in D(A^\omega) = D((A^\omega)^*)$. The latter means that $A^0 \boldsymbol{u} - \omega \partial_\varphi \boldsymbol{u} \in D(A^0)$ and $\partial_\varphi (A^0 \boldsymbol{u} - \omega \partial_\varphi \boldsymbol{u}) \in L^2(\Omega)^3$, implying that $\boldsymbol{u} \in D((A^\omega)^{*2})$. Put $\boldsymbol{w} = A^\omega (A^\omega)^* \boldsymbol{u}$.

In order to show that $\boldsymbol{u} \in D((A^\omega)^* A^\omega)$, we treat the scalar product $(A^\omega \boldsymbol{u}, A^\omega \boldsymbol{v})_{0,2}$ for $\boldsymbol{v} \in D(A^\omega)$ as follows:

$$\begin{aligned}(A^\omega \boldsymbol{u}, A^\omega \boldsymbol{v})_{0,2} &= \big((A^\omega)^* \boldsymbol{u}, A^\omega \boldsymbol{v}\big)_{0,2} + 2\, (\omega \partial_\varphi \boldsymbol{u}, A^\omega \boldsymbol{v})_{0,2} \\ &= \big((A^\omega)^* \boldsymbol{u}, A^\omega \boldsymbol{v}\big)_{0,2} + 2\, (\omega \partial_\varphi \boldsymbol{u}, A^0 \boldsymbol{v})_{0,2} + 2\, (\omega \partial_\varphi \boldsymbol{u}, \omega \partial_\varphi \boldsymbol{v})_{0,2}.\end{aligned} \tag{28}$$

Let us first assume that $\boldsymbol{v}$ has a compact support in $\overline{\Omega}$. Then

$$(\partial_\varphi \boldsymbol{u}, A^0\boldsymbol{v})_{0,2} = \nu \int_\Omega \partial_\varphi \boldsymbol{u} \cdot \Delta \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \;=\; -\nu \int_\Omega \nabla\partial_\varphi \boldsymbol{u} \cdot \nabla \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}$$
$$= \nu \int_\Omega \nabla \boldsymbol{u} \cdot \partial_\varphi \nabla \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = -\nu \int_\Omega \Delta \boldsymbol{u} \cdot \partial_\varphi \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \;=\; -(A^0\boldsymbol{u}, \partial_\varphi \boldsymbol{v})_{0,2}.$$

Substituting this identity into (28), we obtain

$$\begin{aligned}(A^\omega \boldsymbol{u}, A^\omega \boldsymbol{v})_{0,2} &= \big((A^\omega)^*\boldsymbol{u}, A^\omega \boldsymbol{v}\big)_{0,2} - 2\,(A^0\boldsymbol{u}, \omega\partial_\varphi \boldsymbol{v})_{0,2} + 2\,(\omega\partial_\varphi \boldsymbol{u}, \omega\partial_\varphi \boldsymbol{v})_{0,2} \\ &= \big((A^\omega)^*\boldsymbol{u}, A^\omega \boldsymbol{v}\big)_{0,2} - 2\,\big((A^\omega)^*\boldsymbol{u}, \omega\partial_\varphi \boldsymbol{v}\big)_{0,2} = \big((A^\omega)^*\boldsymbol{u}, (A^\omega)^*\boldsymbol{v}\big)_{0,2} \\ &= \big(A^\omega(A^\omega)^*\boldsymbol{u}, \boldsymbol{v}\big)_{0,2} \;=\; (\boldsymbol{w}, \boldsymbol{v})_{0,2}. \end{aligned} \tag{29}$$

In fact, (29) holds for all $\boldsymbol{v} \in D(A^\omega)$ because the set $\{\boldsymbol{v} \in D(A^\omega);\ \boldsymbol{v}$ has a compact support in $\overline{\Omega}\}$ is a core of $A^\omega$. In order to verify it, we use a cut-off function procedure analogous to that one used in the proof of Lemma 3.4, with a sequence of cut-off functions $\eta_n(\boldsymbol{x}) = \eta(\boldsymbol{x}/n)$ where $\eta$ is independent of $\varphi$. Moreover, using the technique from [1], we can show that the sequence of correction terms $\boldsymbol{V}_n$ (satisfying $\operatorname{div} \boldsymbol{V}_n = \nabla\eta_n \cdot \boldsymbol{v}$) can be constructed so that $n\|\nabla \boldsymbol{V}_n\|_{0,2}$ and $\|\partial_\varphi \boldsymbol{V}_n\|_{0,2}$ tend to zero as $n \to +\infty$. Now, (29) shows that for fixed $\boldsymbol{u}$, $(A^\omega \boldsymbol{u}, A^\omega \boldsymbol{v})_{0,2}$ can be extended to a continuous linear functional of $\boldsymbol{v} \in L^2_\sigma(\Omega)$. Thus, $\boldsymbol{u} \in D((A^\omega)^* A^\omega)$.

We have proved the inclusion $D(A^\omega(A^\omega)^*) \subset D((A^\omega)^* A^\omega)$. The opposite inclusion can be proved in the same way. Moreover, (29) implies that

$$\big((A^\omega)^* A^\omega \boldsymbol{u}, \boldsymbol{v}\big)_{0,2} \;=\; (A^\omega (A^\omega)^* \boldsymbol{u}, \boldsymbol{v})_{0,2}$$

for all $\boldsymbol{v} \in D(A^\omega)$ and even for all $\boldsymbol{v} \in L^2_\sigma(\Omega)$, which confirms that the operators $A^\omega$ and $(A^\omega)^*$ commute. □

**Lemma 4.3.** $\sigma_p(A^\omega) = \sigma_r(A^\omega) = \emptyset$.

*Proof.* Note that $\sigma_r(A^\omega) = \emptyset$ because $A^\omega$ is normal. Suppose that $\lambda$ is an eigenvalue of $A^\omega$ and $\boldsymbol{u}$ is a corresponding eigenfunction. The equation $A^\omega \boldsymbol{u} - \lambda \boldsymbol{u} = \boldsymbol{0}$ means that there exists a scalar function $p$ such that $\nu\Delta \boldsymbol{u} + \omega\partial_\varphi \boldsymbol{u} + \nabla p - \lambda \boldsymbol{u} = \boldsymbol{0}$ in $\Omega$. Multiplying this equation by $\bar{\boldsymbol{u}}$ and integrating on $\Omega$, we can verify that $Re\,\lambda < 0$. Furthermore, expanding $\boldsymbol{u}$ and $p$ to the Fourier series in the variable $\varphi$ and denoting the coefficients by $\boldsymbol{U}^k(x_1, r)$ and $P^k(x_1, r)$ (for $k \in \mathbb{Z}$), we can deduce that

$$\nu\Delta\big(\boldsymbol{U}^k\, \mathrm{e}^{\mathrm{i}k\varphi}\big) + \omega\,\mathrm{i}\,k\,\boldsymbol{U}^k\, \mathrm{e}^{\mathrm{i}k\varphi} + \nabla\big(P^k\, \mathrm{e}^{\mathrm{i}k\varphi}\big) - \lambda\,\boldsymbol{U}^k\, \mathrm{e}^{\mathrm{i}k\varphi} \;=\; \boldsymbol{0} \tag{30}$$

in $\Omega$. Moreover, $\operatorname{div}\big(\boldsymbol{U}^k \mathrm{e}^{\mathrm{i}k\varphi}\big) = 0$ and $\boldsymbol{U}^k = \boldsymbol{0}$ on $\partial\Omega$. This implies that $\big[A^0 + (\omega\,\mathrm{i}\,k - \lambda)\big]\big(\boldsymbol{U}^k \mathrm{e}^{\mathrm{i}k\varphi}\big) = \boldsymbol{0}$. Since the Stokes operator $A^0$ has no eigenvalues, we obtain $\boldsymbol{U}^k = \boldsymbol{0}$. This identity holds for all $k \in \mathbb{Z}$, hence $\boldsymbol{u} = \boldsymbol{0}$. This is a contradiction with the assumption that $\boldsymbol{u}$ is an eigenfunction. □

**Lemma 4.4.** $\sigma_c(A^\omega) = \{z = \alpha + \mathrm{i}\,k\omega;\ k \in \mathbb{Z},\ \alpha \le 0\}$.

*Proof.* Lemma 4.3 and the inclusion $\sigma_c(A^\omega) \subset \widetilde{\sigma}_c(A^\omega)$ imply that $\sigma_c(A^\omega) = \widetilde{\sigma}_c(A^\omega)$.

Suppose that $\lambda = \alpha + \mathrm{i}\beta \in \widetilde{\sigma}_c(A^\omega)$. Then there exists a non-compact sequence $\{\boldsymbol{u}_n\}$ in the unit sphere in $L^2_\sigma(\Omega)$ such that

$$(A^\omega - \lambda I)\boldsymbol{u}_n \;=\; \boldsymbol{\epsilon}_n \;\longrightarrow\; 0 \quad \text{in } L^2_\sigma(\Omega) \quad \text{as } n \to +\infty. \tag{31}$$

Let us write $\boldsymbol{u}_n$ in the form $\boldsymbol{u}_n^{-\infty,K_1} + \boldsymbol{u}_n^{K_1,K_2} + \boldsymbol{u}_n^{K_2,+\infty}$ where $K_1,\ K_2 \in \mathbb{Z}$, $K_1 \le K_2$,

$$\boldsymbol{u}_n^{-\infty,K_1}(x_1,r,\varphi) = \sum_{m=-\infty}^{K_1-1} \boldsymbol{U}_n^m(x_1,r)\,\mathrm{e}^{\mathrm{i}m\varphi}$$

and $\boldsymbol{u}_n^{K_1,K_2}$, $\boldsymbol{u}_n^{K_2,+\infty}$ are defined by similar sums where $m$ runs from $K_1$ to $K_2$ or from $K_2+1$ to $+\infty$. Obviously, $\boldsymbol{U}_n^m\,\mathrm{e}^{\mathrm{i}m\varphi} = P_m\boldsymbol{u}_n$. Since $\boldsymbol{u}_n \in D(A^\omega) \subset D(A^0)$, part (a) of Lemma 3.2 implies that $\boldsymbol{U}_n^m\,\mathrm{e}^{\mathrm{i}m\varphi} \in D\big(A^0_m\big) \equiv D\big(A^\omega_m\big)$. The identity

$$1 \;=\; \|\boldsymbol{u}_n\|^2_{0,2} \;=\; \|\boldsymbol{u}_n^{-\infty,K_1}\|^2_{0,2} + \|\boldsymbol{u}_n^{-K_1,K_2}\|^2_{0,2} + \|\boldsymbol{u}_n^{K_2,+\infty}\|^2_{0,2} \tag{32}$$

implies that there exists a subsequence of $\{\boldsymbol{u}_n\}$ (we shall preserve the same notation for the subsequence) such that at least one of the following three statements is true:

(A) There exists an increasing sequence $\{K_2^n\}$ of integer numbers which tends to $+\infty$ as $n \to +\infty$ and $\|\boldsymbol{u}_n^{K_2^n,+\infty}\|_{0,2} > 1/\sqrt{3}$ for all $n \in \mathbb{N}$.

(B) There exists a decreasing sequence $\{K_1^n\}$ of integer numbers which tends to $-\infty$ as $n \to +\infty$ and $\|\boldsymbol{u}_n^{-\infty,K_1^n}\|_{0,2} > 1/\sqrt{3}$ for all $n \in \mathbb{N}$.

(C) There exist fixed $K_1,\ K_2 \in \mathbb{Z}$, $K_1 < K_2$, such that $\|\boldsymbol{u}_n^{K_1,K_2}\|_{0,2} > 1/\sqrt{3}$ for all $n \in \mathbb{N}$.

Suppose that statement (A) is true. Let us multiply (31) by $\overline{\boldsymbol{U}_n^m}\,\mathrm{e}^{-\mathrm{i}m\varphi}$, integrate on $\Omega$ and sum over $m$ from $K_2^n$ to $+\infty$. We obtain

$$-\|\nabla\boldsymbol{u}_n^{K_2^n,+\infty}\|^2_{0,2} - \lambda\,\|\boldsymbol{u}_n^{K_2^n,+\infty}\|^2_{0,2} + \mathrm{i}\,\omega \sum_{m=K_2^n}^{+\infty} m\|\boldsymbol{U}_n^m\,\mathrm{e}^{\mathrm{i}m\varphi}\|^2_{0,2} = (\boldsymbol{\epsilon}_n, \boldsymbol{u}_n^{K_2^n,+\infty})_{0,2}. \tag{33}$$

Note that the right hand side tends to zero as $n \to +\infty$. However, the imaginary part of the left hand side is

$$\begin{aligned}
&-\beta\,\|\boldsymbol{u}_n^{K_2^n,+\infty}\|^2_{0,2} + \omega \sum_{m=K_2^n}^{+\infty} m\,\|\boldsymbol{U}_n^m\,\mathrm{e}^{\mathrm{i}m\varphi}\|^2_{0,2} \\
&> -\beta\,\|\boldsymbol{u}_n^{K_2^n,+\infty}\|^2_{0,2} + \omega K_2^n \sum_{m=K_2^n}^{+\infty} \|\boldsymbol{U}_n^m\,\mathrm{e}^{\mathrm{i}m\varphi}\|^2_{0,2} \\
&= (-\beta + \omega K_2^n)\,\|\boldsymbol{u}_n^{K_2^n,+\infty}\|^2_{0,2} \;\ge\; \frac{1}{3}\,(-\beta + \omega K_2^n)
\end{aligned}$$

and tends to $+\infty$. This is a contradiction. If statement (B) is true, then we arrive at a similar contradiction.

Let us finally assume that (C) is true. Then there exists an integer $k$ in the interval $[K_1, K_2]$, a $\delta > 0$ and a subsequence of $\{\boldsymbol{u}_n\}$ (again denoted by $\{\boldsymbol{u}_n\}$) such that $\|\boldsymbol{u}_n^{k,k}\|_{0,2} \equiv \|\boldsymbol{U}_n^k\, \mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2} > \delta$. The first part of (31) can be written in the form

$$\sum_{m=-\infty}^{+\infty} (A^\omega - \lambda I)\left[\boldsymbol{U}_n^m\, \mathrm{e}^{\mathrm{i}m\varphi}\right] = \sum_{m=-\infty}^{+\infty} \left(A^0 - \lambda I + \mathrm{i}\, m\omega I\right)\left[\boldsymbol{U}_n^m\, \mathrm{e}^{\mathrm{i}m\varphi}\right]$$
$$= \sum_{m=-\infty}^{+\infty} \mathcal{E}_n^m\, \mathrm{e}^{\mathrm{i}m\varphi}$$

where $\mathcal{E}_n^m$ are the coefficients in the Fourier expansion of $\boldsymbol{\epsilon}_n$ in the variable $\varphi$. Hence

$$\left(A^0 - \lambda I + \mathrm{i}\, k\omega I\right)\left[\boldsymbol{U}_n^k\, \mathrm{e}^{\mathrm{i}k\varphi}\right] = \mathcal{E}_n^k\, \mathrm{e}^{\mathrm{i}k\varphi} \tag{34}$$

for all $n \in \mathbb{N}$. Multiplying (34) by $\overline{\boldsymbol{U}_n^k}\, \mathrm{e}^{-\mathrm{i}k\varphi}$ and integrating on $\Omega$, we obtain

$$-\|\nabla\left(\boldsymbol{U}_n^k\, \mathrm{e}^{\mathrm{i}k\varphi}\right)\|_{0,2}^2 - (\lambda - \mathrm{i}\, k\omega)\, \|\boldsymbol{U}_n^k\, \mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2}^2 = \left(\mathcal{E}_n^k\, \mathrm{e}^{\mathrm{i}k\varphi}, \boldsymbol{U}_n^k\, \mathrm{e}^{\mathrm{i}k\varphi}\right)_{0,2}. \tag{35}$$

The imaginary part of the left hand side is $-(\beta - k\omega)\, \|\boldsymbol{U}_n^k\, \mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2}^2$, whereas the right hand side tends to zero for $n \to +\infty$, due to (31). Hence $\beta = k\omega$. Finally, (34) shows that $\lambda - \mathrm{i}\, k\omega = \alpha \in \sigma(A^0)$. However, since $\sigma(A^0) = (-\infty, 0]$, $\alpha$ is non-positive.

We have proved the inclusion $\widetilde{\sigma}_c(A^\omega) \subset \{z = \alpha + \mathrm{i}\, k\omega;\ k \in \mathbb{Z},\ \alpha \le 0\}$. The opposite inclusion follows from the fact that each of the operators $A_k^\omega$ is a part of $A^\omega$ and so $\widetilde{\sigma}_c(A_k^\omega) \equiv \{z = \alpha + \mathrm{i}\, k\omega;\ \alpha \le 0\} \subset \widetilde{\sigma}_c(A^\omega)$ for all $k \in \mathbb{Z}$. □

Using Lemmas 4.2, 4.3 and 4.4, we proved Theorem 1.1, part (ii).

## 5. General exterior domains—operators $A^\omega$ and $L^\omega$

We denote by $\widehat{A}^\omega$ the operator which is defined in the same way as $A^\omega$, however on the whole space $\mathbb{R}^3$ instead of the exterior $\Omega \subset \mathbb{R}^3$. Obviously, the operator $\widehat{A}^\omega$ has all properties derived in Sects. 3 and 4.

**Lemma 5.1.** $\widetilde{\sigma}_c(A^\omega) = \widetilde{\sigma}_c(\widehat{A}^\omega)$.

*Proof.* Suppose that $\lambda \in \widetilde{\sigma}_c(A^\omega)$. Then there exists an orthonormal sequence $\{\boldsymbol{v}_n\} \subset D(A^\omega)$ in $L^2_\sigma(\Omega)$ such that $\|\boldsymbol{v}_n\|_{0,2} = 1$ and $\{\boldsymbol{v}_n\}$ satisfies

$$(A^\omega - \lambda I)\, \boldsymbol{v}_n \longrightarrow 0 \quad \text{in } L^2_\sigma(\Omega) \quad \text{for } n \to +\infty. \tag{36}$$

Using exactly the same procedure as in the proof of Lemma 3.4, we can prove that there exists a non-compact sequence $\{\boldsymbol{u}_n\}$ in $D(A^\omega)$ such that $\|\boldsymbol{u}_n\|_{0,2} = 1$, $\boldsymbol{u}_n = \boldsymbol{0}$ in $\Omega_R$ and

$$(A^\omega - \lambda I)\, \boldsymbol{u}_n \longrightarrow 0 \quad \text{in } L^2_\sigma(\Omega) \quad \text{for } n \to +\infty. \tag{37}$$

All functions $\boldsymbol{u}_n$, extended by zero from $\Omega$ to the whole $\mathbb{R}^3$, belong to the domain of operator $\widehat{A}^\omega$. Thus, (37) shows that $\lambda \in \widetilde{\sigma}_c(\widehat{A}^\omega)$.

On the other hand, if $\lambda \in \widetilde{\sigma}_c(\widehat{A}^\omega)$ then we can use analogous arguments and prove that $\lambda$ also belongs to $\widetilde{\sigma}_c(A^\omega)$. □

Note that $\widetilde{\sigma}_c(\widehat{A}^\omega)$ equals $\sigma_c(\widehat{A}^\omega)$ and is described by Lemma 4.4. Since $\sigma_{ess}(A^\omega)$ is closed and a subset of $\widetilde{\sigma}_c(A^\omega)$, the open set $G = \mathbb{C} - \sigma_{ess}(A^\omega)$ has just one component and $\operatorname{ind}(A^\omega - \lambda I)$ is constant in $G$. Using $\rho(A^\omega) \subset G$, we get $\operatorname{ind}(A^\omega - \lambda I) = 0$ in $G$. This shows that $G \cap \widetilde{\sigma}_c(A^\omega) = \emptyset$ and consequently,

$$\sigma_{ess}(A^\omega) = \widetilde{\sigma}_c(A^\omega) = \{z = \alpha + \mathrm{i}k\omega;\ k \in \mathbb{Z},\ \alpha \le 0\} \tag{38}$$

proving Theorem 1.1 (i).

**Lemma 5.2.** *If $\omega \neq 0$ and if the domain $\Omega$ is not axially symmetric about the $x_1$-axis, then the operator $A^\omega$ is not normal.*

*Proof.* By proving the existence of a function $\boldsymbol{z} \in D((A^\omega)^* A^\omega)$ which is not in $D(A^\omega (A^\omega)^*)$, we show that the domains $D((A^\omega)^* A^\omega)$ and $D(A^\omega (A^\omega)^*)$ do not coincide.

Let $R > R_0$ and let us denote by $A^0_R$ the Stokes operator in the space $L^2_\sigma(\Omega_R)$ with the dense domain $D(A^0_R) = W^{2,2}(\Omega_R)^3 \cap W^{1,2}_0(\Omega_R)^3 \cap L^2_\sigma(\Omega_R)$. The spectrum of $A^0_R$ (as well as the spectrum of $A^0_R + \omega\partial_\varphi$) consists of a countable number of isolated eigenvalues with finite multiplicities and negative real parts. Choose an eigenvalue $\zeta$ of $A^0_R$ and denote by $\boldsymbol{v}$ an associated eigenfunction. The equation

$$A^0_R \boldsymbol{u} + \omega\partial_\varphi \boldsymbol{u} = \boldsymbol{v} \tag{39}$$

has a unique solution $\boldsymbol{u} \in D(A^0_R)$, $\boldsymbol{u} \not\equiv \boldsymbol{0}$. Let us show, by contradiction, that $\partial_\varphi \boldsymbol{u} \not\equiv \boldsymbol{0}$ on $\partial\Omega$. Assume the opposite, i.e., $\partial_\varphi \boldsymbol{u} \equiv \boldsymbol{0}$ on $\partial\Omega$. Then $\partial_\varphi \boldsymbol{u} \in V_R$ where $V_R := W^{1,2}_0(\Omega_R)^3 \cap L^2_\sigma(\Omega_R)$. The operator $A^0_R$ can be extended to the one-to-one continuous linear operator mapping $V_R$ onto the dual space $V'_R$. Moreover, $\partial_\varphi$ maps $V_R$ into $L^2_\sigma(\Omega)$ and $A^0_R + \omega\partial_\varphi$ is an injection from $V_R$ into $V'_R$, because 0 is not an eigenvalue of $A^0_R + \omega\partial_\varphi$. The equation (39) shows that $A^0_R \boldsymbol{u}$ also belongs to $V_R$. Now, $A^0_R \partial_\varphi \boldsymbol{u} \in V'_R$ and it can simply be shown that it equals $\partial_\varphi A^0_R \boldsymbol{u}$ ($\in L^2_\sigma(\Omega)$). Indeed, if $\boldsymbol{\phi} \in C^\infty_{0,\sigma}(\Omega_R)$, the duality between the spaces $V'_R$ and $V_R$ yields

$$\begin{aligned}
\langle A^0_R \partial_\varphi \boldsymbol{u}, \boldsymbol{\phi} \rangle &= -\int_{\Omega_R} \nabla\partial_\varphi \boldsymbol{u} \cdot \nabla\boldsymbol{\phi}\, \mathrm{d}\boldsymbol{x} = \int_{\Omega_R} \partial_\varphi \boldsymbol{u} \cdot \Delta\boldsymbol{\phi}\, \mathrm{d}\boldsymbol{x} = -\int_{\Omega_R} \boldsymbol{u} \cdot \partial_\varphi \Delta\boldsymbol{\phi}\, \mathrm{d}\boldsymbol{x} \\
&= -\int_{\Omega_R} \boldsymbol{u} \cdot \Delta\partial_\varphi \boldsymbol{\phi}\, \mathrm{d}\boldsymbol{x} = -\int_{\Omega_R} \boldsymbol{u} \cdot A^0_R \partial_\varphi \boldsymbol{\phi}\, \mathrm{d}\boldsymbol{x} = -\int_{\Omega_R} A^0_R \boldsymbol{u} \cdot \partial_\varphi \boldsymbol{\phi}\, \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega_R} \partial_\varphi A^0_R \boldsymbol{u} \cdot \boldsymbol{\phi}\, \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Hence, as identities in $V_R'$, we have

$$\begin{aligned}\mathbf{0} &= \big(A_R^0 - \zeta I\big)\boldsymbol{v} \;=\; \big(A_R^0 - \zeta I\big)\big(A_R^0 + \omega\partial_\varphi\big)\boldsymbol{u}\\ &= \big(A_R^0\big)^2\boldsymbol{u} + \omega A_R^0\partial_\varphi\boldsymbol{u} - \zeta A_R^0\boldsymbol{u} - \zeta\omega\partial_\varphi\boldsymbol{u}\\ &= \big(A_R^0\big)^2\boldsymbol{u} + \omega\partial_\varphi A_R^0\boldsymbol{u} - A_R^0\zeta\boldsymbol{u} - \omega\partial_\varphi\zeta\boldsymbol{u} \;=\; \big(A_R^0 + \omega\partial_\varphi\big)\big(A_R^0 - \zeta I\big)\boldsymbol{u}.\end{aligned}$$

This implies that $\big(A_R^0 - \zeta I\big)\boldsymbol{u} = \mathbf{0}$ which means that $\boldsymbol{u}$ is an eigenfunction of $A_R^0$ associated with the eigenvalue $\zeta$, too. Since the space generated by such eigenfunctions is finite-dimensional, $\boldsymbol{v}$ can be chosen so that $\mu\boldsymbol{u} = \boldsymbol{v}$ for some $\mu \in \mathbb{C}$. Then equation (39) implies

$$\omega\partial_\varphi\boldsymbol{u} \;=\; (\mu - \zeta)\boldsymbol{u} \tag{40}$$

in $\Omega_R$. Since $\Omega_R$ is not axially symmetric, we find a point $\boldsymbol{x}_0 \in \partial\Omega$ such that in a neighborhood $U \subset \partial\Omega$ of this point $\partial_\varphi$ is *not* the tangential derivative at $\boldsymbol{x} \in U$. Consider (40) as a first order linear differential equation in $\varphi$ with initial values related to points in $U$. The boundary condition $\boldsymbol{u} = \mathbf{0}$ on $\partial\Omega$ enables us to conclude that $\boldsymbol{u}$ vanishes identically in an open subset of $\Omega_R$. Now the unique continuation principle applied to $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u}$, cf. the proof of Lemma 2.6, shows that $\boldsymbol{\omega} \equiv \mathbf{0}$ and consequently that also $\boldsymbol{u} \equiv \mathbf{0}$ in $\Omega_R$ which is impossible because $\boldsymbol{u} \neq \mathbf{0}$ in $\Omega_R$. The assumption $\partial_\varphi\boldsymbol{u} \equiv \mathbf{0}$ on $\partial\Omega$ thus leads to the contradiction, hence $\partial_\varphi\boldsymbol{u} \not\equiv \mathbf{0}$ on $\partial\Omega$.

Using an appropriate cut-off function procedure, cf. the proof of Lemma 3.4, we can construct a function $\boldsymbol{z}$ in $D((A^\omega)^*A^\omega)$ which coincides with the function $\boldsymbol{u}$ constructed just before in the neighborhood of $\partial\Omega$ and equals $\mathbf{0}$ outside $\Omega_R$. Hence $\partial_\varphi\boldsymbol{z} \not\equiv \mathbf{0}$ on $\partial\Omega$. However, then $\boldsymbol{z}$ cannot belong to $D(A^\omega(A^\omega)^*)$ because all functions $\boldsymbol{z} \in D((A^\omega)^*A^\omega) \cap D(A^\omega(A^\omega)^*)$ satisfy $\boldsymbol{z} = A^0\boldsymbol{z} + \omega\,\partial_\varphi\boldsymbol{z} = A^0\boldsymbol{z} - \omega\,\partial_\varphi\boldsymbol{z} = \mathbf{0}$ on $\partial\Omega$, which implies that $\partial_\varphi\boldsymbol{z} \equiv \mathbf{0}$ on $\partial\Omega$. □

Now Lemma 5.2 yields item (iii) of Theorem 1.1. Theorem IV.5.35 in [19] and Lemma 2.4 imply that the essential spectrum of the operator $L^\omega$ is also given by (38). Moreover, since $\operatorname{ind}(L^\omega - \lambda I) = 0$ in $G = \mathbb{C} - \sigma_{ess}(L^\omega)$ and due to Theorem IV.5.31 in [19], $G$ can contain at most countably many eigenvalues $\lambda$ of $L^\omega$, which can cluster only on the boundary of $G$ and $0 < \operatorname{nul}(L^\omega - \lambda I) = \operatorname{def}(L^\omega - \lambda I) < +\infty$ at each of them. This implies Theorem 1.2.

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