manuscripta math. 120, 131–150 (2006)

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Rational Lagrangian fibrations on punctual Hilbert schemes of K3 surfaces

Received: 29 September 2005 / Revised version: 31 January 2006
Published online: 21 March 2006

Abstract. A rational Lagrangian fibration f on an irreducible symplectic variety V is a rational map which is birationally equivalent to a regular surjective morphism with Lagrangian fibers. By analogy with K3 surfaces, it is natural to expect that a rational Lagrangian fibration exists if and only if V has a divisor D with Bogomolov–Beauville square 0. This conjecture is proved in the case when V is the Hilbert scheme of d points on a generic K3 surface S of genus g under the hypothesis that its degree $2g-2$ is a square times $2d-2$. The construction of f uses a twisted Fourier–Mukai transform which induces a birational isomorphism of V with a certain moduli space of twisted sheaves on another K3 surface M, obtained from S as its Fourier–Mukai partner.

0. Introduction

According to Beauville [Beau-1], [Beau-2], the d-th symmetric power $S^{(d)}$ of a K3 surface S has a natural resolution of singularities, the punctual Hilbert scheme $S^{[d]} = \mathrm{Hilb}^d S$, which is a $2d$-dimensional irreducible symplectic variety. Here a holomorphically symplectic manifold is called *an irreducible symplectic variety* if it is projective and simply connected and has a unique symplectic structure up to proportionality. In dimension 2, the irreducible symplectic varieties are exactly K3 surfaces. Here and throughout the paper, a *K3 surface* means a *projective K3 surface*, and all the *varieties* are assumed to be projective.

The 2-dimensional cohomology $H^2(V, \mathbb{Z})$ of an irreducible symplectic variety V has an integral quadratic form q_V with remarkable properties, called the Bogomolov–Beauville form. This form together with the Hodge structure on $H^2(X)$ gives rise to many striking analogies between K3 surfaces, where q_V is just the intersection form, and higher-dimensional irreducible symplectic varieties. They have very similar deformation theories, descriptions of period maps, Torelli theorems, structures of the ample (or Kähler) cone (see [Bou], [Hu-1], [Hu-2], [O'G-1], [O'G-2], [S-1], [S-2] and references therein). There are also differences. For example, there is only one deformation class of K3 surfaces, but at least two deformation classes of irreducible symplectic varieties of any even dimension > 2; one of them is the class of the variety $S^{[d]}$. Another difference is the existence of nontrivial birational isomorphisms (flops) between irreducible symplectic varieties. Huybrechts

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Mathematics Subject Classification (2000): 14J60, 14J40

DOI: 10.1007/s00229-006-0631-4

shows in [Hu-0] that two birationally equivalent irreducible symplectic varieties not only are deformation equvalent and have the same period, but also represent nonseparated points of the moduli space.

It is natural to expect that the analogy between the K3 surfaces and irreducible symplectic varieties has many more manifestations that are still to be proved. For example, O'Grady conjectures in [O'G-2] that an irreducible symplectic variety V, deformation equivalent to $S^{[d]}$, has an antisymplectic birational involution if it has a divisor class with Bogomolov–Beauville square 2. This is of course true for K3 surfaces. He also raises the problem of finding explicit geometric constructions for the irreducible symplectic varieties having a divisor class with small square.

A more specific question in this direction is the characterization of varieties V that have a structure of a fibration, that is a regular surjective map $f : V \to B$ to some other variety B, different from a point, with connected fibers of positive dimension. If V is K3, then such a map exists if and only if V has a divisor D with square 0, and then the divisor D' obtained from $\pm D$ by a number of reflections in (-2)-curves defines the structure of an elliptic pencil $\varphi_{|D'|} : V \to \mathbb{P}^1$.

One cannot generalize this straightforwardly by saying that a higher dimensional irreducible symplectic variety admits a fibration if and only if the Bogomolov–Beauville form on $\mathrm{Pic}(V)$ represents zero. Indeed, it follows from the description of $\mathrm{Pic}(S^{[d]})$ in Theorem 1.2 that if S is a generic K3 surface of degree $2d-2$, then $\mathrm{Pic}(S^{[d]})$ has exactly two primitive effective divisor classes $h \pm e$ of square zero and none of them defines a fibration structure. Nevertheless, $V = S^{[d]}$ admits a structure of a rational fibration, that is a rational map $f : V \dashrightarrow B$ which can be birationally transformed to a genuine fibration $g : W \to B$ on a symplectic variety W birational to V. This map is introduced by formula (2) and it coincides with the rational map $\varphi_{|h-e|}$ given by the complete linear system $|h-e|$.

Thus the expected generalization is the following: an irreducible symplectic variety V has a structure of a *rational* fibration if and only if it has a divisor D with square 0. Matsushita [Mat-1], [Mat-2] has proved important results on the structure of regular fibrations $f : V \to B$ on an irreducible symplectic variety V: $\dim B = \frac{1}{2}\dim V$ and the generic fiber of f is an abelian variety which is a Lagrangian subvariety of V with respect to the symplectic structure. If, moreover, B is nonsingular, then B has the Hodge numbers of a projective space. Remark that no examples are known of fibrations on irreducible symplectic varieties with base different from a projective space. So one might complete the conjecture in saying that the base B of any rational fibration f on an irreducible symplectic variety V of dimension $2n$ is the projective space $\mathbb{P}^n$ and that f is given by the complete linear system $|D|$ of a divisor D with $q_V(D) = 0$.

In the present article, we prove this conjecture for the varieties $S^{[d]}$ constructed from *generic* primitively polarized K3 surfaces S of degree $(2d-2)m^2$, $m \geq 2$ (Corollary 4.4). In the next few lines we describe briefly our construction. We define another $K3$ surface M, which is a moduli space of sheaves on S, and a birational map μ from $S^{[d]}$ to another irreducible symplectic variety V. The latter is a moduli space of α-twisted sheaves on M for some element α of the Brauer group $\mathrm{Br}(M)$, and μ is induced by the twisted Fourier–Mukai transform defined by a twisted universal sheaf on $S \times M$. Further, V is a torsor under the relative Jacobian of a linear system $|C|$ on M, and hence has a natural morphism $f : V \to |C| \simeq \mathbb{P}^d$ which sends each twisted sheaf to its support. This map is a *regular* Lagrangian fibration, and the *rational* one on $S^{[d]}$ is $\pi = f \circ \mu$.

The same result was obtained independently and almost simultaneously in [S-2]. Later Yoshioka proved the regularity of this rational Lagrangian fibration; Yoshioka's proof is included in the version 3 of loc. cit. The paper [Gu] contains a similar result for the other series of Beauville's examples, generalized Kummer varieties; in this case the proof does not necessitate a use of twisted sheaves, and it seems very likely that the rational Lagrangian fibration constructed by the author is indeed not regular.

By Proposition 1.2, $S^{[d]}$ has a divisor class with Bogomolov–Beauville square zero for a generic primitively polarized K3 surface S of degree n if and only if $k^2 n = (2d-2)m^2$ for some integers $k \geq 1$, $m \geq 1$, $d \geq 2$ with relatively prime k, m. Regular Lagrangian fibrations on $S^{[d]}$ have been known before in some particular cases when $k = 1$. Hassett and Tschinkel [HasTsch-1], [HasTsch-2] have proved the existence of a regular Lagrangian fibration on $S^{[d]}$ for a generic K3 surface S of degree $2m^2$, which corresponds to $d = 2$, $m \geq 2$. The authors of [IR] provided an explicit construction of such a fibration in the case $d = 3$, $m = 2$. No examples are known with $k > 1$.

In Section 1, we gather generalities on irreducible symplectic varieties and fibrations on them. In Section 2, we cite necessary notions and results on Fourier–Mukai transforms and on twisted sheaves following [Cal-2], [HS], [Y-2]. In Section 3 we define the moduli K3 surface $M = M(m, H, (d-1)m)$ and study the properties of the sheaves on the initial surface S represented by points of M. In Section 4, the main result (Theorem 4.3, Corollary 4.4) is proved. In conclusion, we show that the same construction applied to nongeneric K3 surfaces of degree $(2d-2)m^2$ yields nonregular rational fibration maps f (Proposition 4.6).

1. Preliminaries

A *symplectic variety* is a nonsingular projective variety V over $\mathbb{C}$ having a nondegenerate holomorphic 2-form $\alpha \in H^0(\Omega^2_V)$. It is called *irreducible symplectic* if it is, moreover, simply connected and $h^{0,2}(V) = 1$. By the Bogomolov–Beauville decomposition theorem [Beau-2], every symplectic variety becomes, after a finite étale covering, a product of a complex torus and a number of irreducible symplectic varieties.

Theorem 1.1 (Beauville, [Beau-2]). *Let V be an irreducible symplectic variety of dimension $2d$. Then there exists a constant c_V and an integral idivisible quadratic form q_V of signature $(3, b_2(V)-3)$ on the cohomology $H^2(V, \mathbb{Z})$ such that $\gamma^{2d} = c_V q_V(\gamma)^d$ for all $\gamma \in H^2(V, \mathbb{Z})$, where $\gamma^{2d} \in H^{4d}(V, \mathbb{Z})$ denotes the $2d$-th power of γ with respect to the cup product in $H^*(V, \mathbb{Z})$, and $H^{4d}(V, \mathbb{Z})$ is naturally identified with $\mathbb{Z}$.*

The form q_V was first introduced by Bogomolov in [Bo], so we will call it Bogomolov–Beauville form, and c_V is called Fujiki's constant.

In dimension 2, the irreducible symplectic varieties are just K3 surfaces. Historically, the first constructions of higher-dimensional irreducible symplectic varieties belong to Fujiki [F] (one example of dimension 4) and Beauville [Beau-1], [Beau-2] (two infinite series of examples in all even dimensions ≥ 4). The Beauville's examples are: 1) $X^{[d]} = \mathrm{Hilb}^d(X)$, the Hilbert scheme of 0-dimensional subschemes of length d in a K3 surface X, and 2) $K_n(A)$, the generalized Kummer variety associated to an abelian surface A. The latter is defined as the fiber of the summation map $A^{[n+1]} \to A$. The punctual Hilbert scheme $X^{[d]}$ has a natural Hilbert–Chow map $X^{[d]} \to X^{(d)}$ sending each 0-dimensional subscheme to the associated 0-dimensional cycle of degree d, considered as a point of the d-th symmetric power $X^{(d)}$ of X. The Hilbert–Chow map is a resolution of singularities whose exceptional locus is a single irreducible divisor E, the inverse image of the big diagonal of $X^{(d)}$. By a *lattice* we mean a free $\mathbb{Z}$-module of finite rank endowed with a nondegenerate integer quadratic form.

Proposition 1.2. *Let X be a K3 surface. Then $c_{X^{[d]}} = \frac{(2d)!}{d!2^d}$ and there is a natural isomorphism of lattices $H^2(X^{[d]}, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) \overset{\perp}{\oplus} \mathbb{Z}e,\ e^2 = -2(d-1)$, where e is the class of the exceptional divisor E of the Hilbert–Chow resolution, and e^2 stands for the square of e with respect to the Bogomolov–Beauville form $q_{X^{[d]}}$.*

Proof. See [HL], 6.2.14. □

Using the isomorphism of Proposition 1.2, we will denote a class in Pic X or $H^2(X, \mathbb{Z})$ and its image in Pic $X^{[d]}$ or $H^2(X^{[d]}, \mathbb{Z})$ by the same symbol.

Other examples of irreducible symplectic varieties are given by moduli spaces of sheaves on a K3 or abelian surface Y. Mukai [Mu-1] has endowed the integer cohomology $H^*(Y)$ with the following bilinear form:

$$\langle (v_0, v_1, v_2), (w_0, w_1, w_2) \rangle = v_1 \cup w_1 - v_0 \cup w_2 - v_2 \cup w_0, \tag{1}$$

where $v_i, w_i \in H^{2i}(Y)$. We will denote $\langle v, v \rangle$ simply by v^2. For a sheaf $\mathcal{F}$ on Y, the *Mukai vector* is

$$\begin{aligned} v(\mathcal{F}) = \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{Td}(Y)} &= (\mathrm{rk}\,\mathcal{F}, c_1(\mathcal{F}), \chi(\mathcal{F}) - \epsilon\,\mathrm{rk}\,\mathcal{F}) \\ &\in H^0(Y) \oplus H^2(Y) \oplus H^4(Y) = H^*(Y), \end{aligned}$$

where $\mathrm{Td}(Y)$ is the Todd genus, $H^4(Y)$ is naturally identified with $\mathbb{Z}$ and $\epsilon = 0$, resp. 1 if Y is abelian, resp. K3. We refer to [Sim] or to [HL] for the definition and the basic properties of the Simpson (semi-)stable sheaves. Let $M_Y^{H,s}(v)$ (resp. $M_Y^{H,ss}(v)$) denote the moduli space of Simpson stable (resp. semistable) sheaves $\mathcal{F}$ on Y with respect to an ample class H with Mukai vector $v(\mathcal{F}) = v$. According to Mukai ([Mu-1], [Mu-2], see also [HL]), $M_Y^{H,s}(v)$, if nonempty, is smooth of dimension $v^2 + 2$ and carries a holomorphic symplectic structure.

Theorem 1.3. *If Y is a $K3$ surface, then a nonempty moduli space $M_Y^{H,s}(v)$ is an irreducible symplectic variety whenever it is compact or, equivalently, projective. Moreover, $M_Y^{H,s}(v)$ is compact if v is primitive and H is a sufficiently generic ample* *class in the Kähler cone of Y. The last condition means that H does not lie on a certain discrete family of walls in the Kähler cone, and it is automatically verified if* $\mathrm{Pic}\, Y \simeq \mathbb{Z}$.

Proof. This summarizes the results of several papers: [Mu-1], [Mu-2], [Hu-1], [Hu-2], [O'G-1] and [Y-1]. See also [HL], 6.2.5 and historical comments to Chapter 6. □

In particular, the following statement holds:

Corollary 1.4. *Let Y be a K3 surface with* $\mathrm{Pic}\, Y \simeq \mathbb{Z}$ *and v a primitive Mukai vector. Assume that $M = M_Y^{H,s}(v)$ is nonempty. Then M is an irreducible symplectic variety of dimension $v^2 + 2$. If v is, moreover, isotropic, then M is a K3 surface.*

There are similar results for the case when Y is abelian [Y-1], however in this case not $M_Y^{H,s}(v)$ itself is irreducible symplectic, but the fiber of its Albanese map $M_Y^{H,s}(v) \to Y \times \hat{Y}$. The papers cited above prove also that all the irreducible symplectic varieties obtained in this way are deformation equivalent to Beauville's examples.

Let X be a K3 surface containing a nef curve C of degree $2d-2$, $d \geq 2$. Then the d-th punctual Hilbert scheme admits a dominant rational map $\theta : X^{[d]} \dashrightarrow |C| \simeq \mathbb{P}^d$ sending $\xi \in X^{[d]}$ to the generically unique curve $C_\xi \in |C|$ containing ξ. If $|C|$ embeds X into $\mathbb{P}^d$, then θ can be described as follows:

$$\theta : X^{[d]} \dashrightarrow \mathbb{P}^{d\vee}, \quad \xi \mapsto \langle \xi \rangle_{\mathbb{P}^d}. \tag{2}$$

Here $\langle \xi \rangle_{\mathbb{P}^d}$ denotes the linear span of a subscheme $\xi \subset X$ in the embedding into $\mathbb{P}^d$, which is generically a hyperplane in $\mathbb{P}^d$, that is a point of the dual projective space $\mathbb{P}^{d\vee}$.

For a $2d$-dimensional symplectic variety V, we will call a morphism $\pi : V \to B$ a *Lagrangian fibration* if it is surjective and its generic fiber is a connected Lagrangian subvariety of V, that is a d-dimensional subvariety such that the restriction of the symplectic form to it is zero. By the classical Liouville's theorem, the generic fiber is then an abelian variety.

Theorem 1.5 (Matsushita, [Mat-1], [Mat-2]). *Let V be an irreducible $2d$-dimensional symplectic variety, and $\pi : V \to B$ a surjective morphisme with connected fibers. Then* $\dim B = d = \frac{1}{2} \dim V$ *and f is a Lagrangian fibration. If, moreover, B is nonsingular, then $R^i f_* \mathcal{O}_V \simeq \Omega_B^i$ for all $i \geq 0$, and B has the Hodge numbers of a projective space.*

We will call a rational map $\pi : V \dashrightarrow B$ a *rational Lagrangian fibration*, if there exists a birational map $\mu : W \dashrightarrow V$ of $2d$-dimensional symplectic varieties such that $\pi\mu$ is a regular Lagrangian fibration. Such a π is dominant and its generic fiber is a connected Lagrangian subvariety of V, birational to an abelian variety. The above map θ is a rational Lagrangian fibration. Its fibers are birational to symmetric powers $C_\xi^{(d)}$, and hence to Jacobians of the genus-d curve C_ξ. This birationality is globalized as follows.

Let $J = J^d X$ be the relative compactified Jacobian of the linear system $|C|$. It can be defined as the moduli space $M_X(0, [C], 1)$ of Simpson-semistable torsion sheaves on X with Mukai vector $(0, [C], 1)$ [Sim], [LeP-2]. To speak about semistable sheaves, one has to fix a polarization H on X, so it is better to write $J^{d,H} X$ to explicitize the dependence on the polarization. If all the curves in $|C|$ are reduced and irreducible, then every semistable sheaf is stable, hence J is smooth and its definition does not depend on polarization. In this case J can be equally understood as the moduli space of simple sheaves [LeP-1], [Mu-1] with given Mukai vector. Let $\psi : J \longrightarrow |C| \simeq \mathbb{P}^d$ be the natural map sending each sheaf to its support. It is known that J is an irreducible symplectic variety and ψ is a Lagrangian fibration [Beau-3]. The varieties $X^{[d]}$ and J are related by a generalized Mukai flop $\mu : X^{[d]} \dashrightarrow J$ introduced by Markman in [Mar]. It sends $\xi \in X^{[d]}$ to the same subscheme ξ considered as a degree-d divisor on the curve C_ξ, and we have $\theta = \psi \circ \mu$.

According to Huybrechts (Lemma (2.6) of [Hu-2]), any birational map between irreducible symplectic varieties induces a Hodge isometry of their integer second cohomology lattices $H^2(\cdot, \mathbb{Z})$ equiped with the Bogomolov–Beauville form. Thus we have the isomorphisms of the Bogomolov–Beauville lattices $H^2(J, \mathbb{Z}) \simeq H^2(X^{[d]}, \mathbb{Z})$ and $\mathrm{Pic}(J) \simeq \mathrm{Pic}(X^{[d]})$.

Lemma 1.6. *Let X be a $K3$ surface with an effective divisor class f_{2d-2} such that all the curves in the linear system $|f_{2d-2}|$ are reduced and irreducible. Let D be a divisor on $X^{[d]}$ with class $f_{2d-2} - e$. Then $h^0(\mathcal{O}(D)) = d + 1$ and θ is given by the complete linear system $|D|$.*

Proof. Let $\mathcal{L} = \psi^* \mathcal{O}_{\mathbb{P}^d}(1)$. Then the self-intersection $(\mathcal{L})^{2d}$ is 0, hence $q_J(\mathcal{L}) = 0$, where q_J denotes the Bogomolov–Beauville form on $H^2(J, \mathbb{Z})$. By the Riemann–Roch Theorem for hyperkähler manifolds [Hu-1], if V is a hyperkähler manifold V of dimension $2d$ with Bogomolov–Beauville form q_V, then $\chi(\mathcal{O}_V(D)) = \chi(\mathcal{O}_V) = d+1$ for any divisor D such that $q_V(D) = 0$. Hence $\chi(\mathcal{L}) = d+1$. By Matsushita's theorem [Mat-2], $R^i \psi_* \mathcal{O}_J \simeq \Omega^i_{\mathbb{P}^d}$. Applying the Leray spectral sequence and the Bott formula, we obtain for $\mathcal{L} = \psi^*(\mathcal{O}(1))$: $h^0(\mathcal{L}) = d + 1$, $h^i(\mathcal{L}) = 0$ for $i > 0$.

The map $\theta = \psi \circ \mu$ is given by a subsystem of $|\mu^* \mathcal{L}|$. As μ^* is an isometry of Bogomolov–Beauville lattices $(H^2(J, \mathbb{Z}), q_J)$ and $(H^2(X^{[d]}, \mathbb{Z}), q)$, $q(\mu^* \mathcal{L}) = q_J(\mathcal{L}) = 0$. For X as in the hypothesis, it is possible that there are many primitive divisors D with $q(D) = 0$ on $X^{[d]}$. Let us consider a deformation of X to a surface with $\mathrm{Pic}\, X \simeq \mathbb{Z}$, polarized by a divisor f_{2d-2} of degree $2d - 2$. Then the map θ deforms with X, $\mathrm{rk}\,\mathrm{Pic}\, X^{[d]} = 2$ and the only two primitive effective classes with square 0 on $X^{[d]}$ are $f_{2d-2} - e$, $f_{2d-2} + e$. The latter has negative intersection with the generic fiber $\mathbb{P}^1$ of the Hilbert–Chow map $X^{[d]} \longrightarrow X^{(d)}$, hence has the whole exceptional divisor E in its base locus, which is not the case for θ, so the linear system defining θ is a subsystem of $|f_{2d-2} - e|$.

As μ is an isomorphism in codimension 1, $h^0(\mu^* \mathcal{L}) = h^0(\mathcal{L}) = d + 1$, and this ends the proof. $\square$

As follows from Proposition 1.2, if X is a K3 surface with a curve class f_{2d-2} of degree $2d - 2$, then the Bogomolov–Beauville form on the Picard lattice of $X^{[d]}$

represents zero. The classes $\pm f_{2d-2} \pm e \in \operatorname{Pic}(X^{[d]})$ are primitive with square 0, and one of them, namely $f_{2d-2} - e$, defines a rational Lagrangian fibration. Thus Lemma 1.6 provides an example illustrating the following conjecture:

Conjecture 1.7. *Let X be a K3 surface. Then $X^{[d]}$ admits a rational Lagrangian fibration with base $\mathbb{P}^d$ if and only if the Bogomolov–Beauville form of the Picard lattice of $X^{[d]}$ represents zero. In this case, there exists $m \geq 1$ and an effective curve class $f_{(2d-2)m^2}$ of degree $(2d-2)m^2$ on X such that the linear system defining the rational Lagrangian fibration map on $X^{[d]}$ is of the form $f_{(2d-2)m^2} - me$.*

In Section 4, we will prove the conjecture for K3 surfaces with Picard group $\mathbb{Z}$, generated by a curve class of degree $(2d-2)m^2$, and we will identify the Lagrangian fibration as a torsor under the relative Jacobian of a linear system of curves on some other K3 surface. The structure of the torsor is defined by twisting by an element of the Brauer group of the second K3 surface.

2. Twisted sheaves and twisted Fourier-Mukai transforms

The sheaves twisted by an element of the Brauer group were introduced by Căldăraru [Cal-1], [Cal-2].

The cohomological Brauer group $\operatorname{Br}'(Y)$ of a scheme Y is defined as $H^2_{\text{ét}}(Y, \mathcal{O}_Y^*)$. The Brauer group $\operatorname{Br}(Y)$ is the union of the images of $H^1(Y, PGL(n))$ in $\operatorname{Br}'(Y)$ for all n. For a smooth curve, $\operatorname{Br}(Y) = \operatorname{Br}'(Y) = 0$. For a smooth surface, $\operatorname{Br}(Y) = \operatorname{Br}'(Y)$. If Y is a K3 surface, then $\operatorname{Br}(Y) \simeq \operatorname{Hom}_{\mathbb{Z}}(T_Y, \mathbb{Q}/\mathbb{Z})$, where T_Y is the transcendental lattice of Y, defined as the orthogonal complement of $\operatorname{Pic} Y$ in $H^2(Y, \mathbb{Z})$.

Let $\alpha \in \operatorname{Br}(Y)$ be represented by a Čech 2-cocycle $(\alpha_{ijk})_{i,j,k \in I}, \alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_Y^*)$ on some open covering $\{U_i\}_{i \in I}$. An α-twisted sheaf $\mathcal{F}$ on Y is a pair $(\{\mathcal{F}_i\}_{i \in I}, \{\varphi_{ij}\}_{i,j \in I})$, where $\mathcal{F}_i$ is a sheaf on U_i $(i \in I)$ and

$$\varphi_{ij} : \mathcal{F}_j|_{U_i \cap U_j} \longrightarrow \mathcal{F}_i|_{U_i \cap U_j} \qquad (i,\ j \in I)$$

are isomorphisms of sheaves with the following three properties:

$$\varphi_{ii} = \mathrm{id}, \quad \varphi_{ji} = \varphi_{ij}^{-1}, \quad \varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}\,.$$

The α-twisted sheaves form an abelian category $\mathfrak{Mod}(Y, \alpha)$. It has enough injectives and enough $\mathcal{O}_Y$-flat sheaves. An α-twisted sheaf is coherent if all the sheaves $\mathcal{F}_i$ are the α-twisted coherent sheaves form an abelian category $\mathfrak{Coh}(Y, \alpha)$. If $\mathcal{F}$ is an α-twisted sheaf, and $\mathcal{G}$ an α'-twisted sheaf, then $\mathcal{F} \otimes \mathcal{G}$ is an $\alpha\alpha'$-twisted sheaf, and $\mathcal{H}om\ (\mathcal{F}, \mathcal{G})$ is an $\alpha^{-1}\alpha'$-twisted sheaf. If $f : X \to Y$ is a morphism, then $f^*\mathcal{F} \in \mathfrak{Mod}(X, f^*\alpha)$, and for any $\mathcal{G} \in \mathfrak{Mod}(X, f^*\alpha)$, $\quad f_*\mathcal{G} \in \mathfrak{Mod}(Y, \alpha)$.

We denote by $\mathbf{D}(Y, \alpha)$ the derived category $\mathbf{D}^b_{\mathrm{coh}}(\mathfrak{Mod}(Y, \alpha))$ of bounded complexes of α-twisted sheaves with coherent cohomology. By a standard machinery one defines the derived functors $\mathbf{R}f_*, \overset{\mathbf{L}}{\otimes}, \mathbf{L}f^*$.

To define the Chern character on $\mathfrak{Coh}(Y, \alpha)$, Căldăraru fixes some α^{-1}-twisted locally free sheaf $\mathcal{E}$ on Y. Then $\mathcal{F} \otimes \mathcal{E}$ is a usual (untwisted) sheaf, so one can define the modified Chern character $\mathrm{ch}_{\mathcal{E}}(\mathcal{F}) := \frac{1}{\mathrm{rk}\,\mathcal{E}}\,\mathrm{ch}(\mathcal{F}\otimes\mathcal{E})$ and the associated Mukai vector $v_{\mathcal{E}}(\mathcal{F}) = \mathrm{ch}_{\mathcal{E}}(\mathcal{F})\sqrt{\mathrm{Td}(Y)}$. Remark that such an $\mathcal{E}$ exists only with rank which is a multiple of the order of α in the Brauer group. These definitions depending on $\mathcal{E}$ are not the ones best suited for application to the Fourier–Mukai transform. Huybrechts and Stellari [HS] tensor $\mathcal{F}$ by an α^{-1}-twisted C^∞ line bundle in place of an α^{-1}-twisted holomorphic locally free sheaf $\mathcal{E}$ of higher rank. Their construction depends on a so called B-field $B \in H^2(Y,\mathbb{Q})$. The latter can be defined for any Y for which $H^3(Y,\mathbb{Z}) = 0$ as a lift of α via the surjection in the exact triple

$$0 \to \mathrm{Pic}\, Y \otimes \mathbb{Q}/\mathbb{Z} \to H^2(Y,\mathbb{Q}/\mathbb{Z}) \to H^2(Y,\mathcal{O}_Y^*)_{\mathrm{tor}} \to 0$$

composed with the natural map $H^2(Y,\mathbb{Q}) \to H^2(Y,\mathbb{Q}/\mathbb{Z})$. We will write $\alpha = e^{2\pi i B}$.

Let L_B denote the C^∞ line bundle on Y with transition functions $e^{2\pi i\beta_{ij}}$, where (β_{ij}) is a C^∞ 1-cochain whose coboundary is some 2-cocycle $B_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathbb{Q})$ representing B. Then the twisted Chern character is $\mathrm{ch}_B(\mathcal{F}) = \mathrm{ch}(\mathcal{F}\otimes L_B^{-1})$ and the twisted Mukai vector is $v_B(\mathcal{F}) = \mathrm{ch}_B(\mathcal{F})\sqrt{\mathrm{Td}(Y)}$.

Let now X and Y be smooth projective varieties, $\alpha \in \mathrm{Br}(Y)$, and $\mathcal{P}^\bullet \in \mathbf{D}(X \times Y, \pi_Y^*\alpha^{-1})$, where π_X, π_Y denote the projections of $X \times Y$ to the two factors. The twisted Fourier–Mukai transform

$$\Phi_{Y\to X}^{\mathcal{P}^\cdot} : \mathbf{D}(Y,\alpha) \longrightarrow \mathbf{D}(X)$$

is defined by

$$\Phi_{Y\to X}^{\mathcal{P}^\cdot}(K^\bullet) = \pi_{X*}(\pi_Y^*(K^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{P}^\bullet).$$

It can be pushed down to the cohomology level in a natural way to give a map $\varphi : H^*(Y,\mathbb{Q}) \to H^*(X,\mathbb{Q})$ so that the following diagram is commutative:

$$\begin{tikzcd} \mathbf{D}(Y,\alpha) \arrow[r, "\Phi_{Y\to X}^{\mathcal{P}^\cdot}"] \arrow[d, "v_B(\cdot)"'] & \mathbf{D}(X) \arrow[d, "v(\cdot)"] \\ H^*(Y,\mathbb{Q}) \arrow[r, "\varphi"] & H^*(X,\mathbb{Q}) \end{tikzcd}$$

The Grothendieck–Riemann–Roch Theorem implies the following expression for $\varphi = \varphi_{Y\to X}^{\mathcal{P}^\cdot,\,B}$:

$$\varphi_{Y\to X}^{\mathcal{P}^\cdot,\,B}(\gamma) = \pi_{X*}\left(\pi_Y^*(\gamma)\cdot \mathrm{ch}_{\pi_Y^* B}(\mathcal{P}^\bullet)\cdot\sqrt{\mathrm{Td}(X\times Y)}\right).$$

If $\alpha = 1$, $B = 0$, we get the usual Fourier–Mukai transform introduced by Mukai, and $\varphi_{Y\to X}^{\mathcal{P}^\cdot,\,0}$ is denoted by $\varphi_{Y\to X}^{\mathcal{P}^\cdot}$. Remark that the cohomological Fourier–Mukai transform φ does not respect the grading of cohomology, neither the ring structure given by the cup product.

Let now X be a K3 surface and $v = (r, L, s) \in H^*(X, \mathbb{Z})$ a primitive Mukai vector with algebraic $L \in H^2(X, \mathbb{Z})$ such that $v^2 := (v, v) = 0$. Let H be a sufficiently generic ample class, so that $M = M_X^{H,s}(v)$ is compact. Then, by Corollary 1.4, M is a K3 surface. Denote $m = \text{g.c.d.}\,(r, L\cdot\gamma, s)_{\gamma\in\operatorname{Pic} X}$. If $m = 1$, then, according to the Appendix to [Mu-2], M is a fine moduli space, that is, there exists a universal sheaf $\mathcal{P}$ on $X \times M$. It is defined by the condition that for any $t \in M$, the isomorphism class of stable sheaves corresponding to t is represented by the restriction $\mathcal{P}_t = \mathcal{P}|_{X\times t}$. The same Mukai's argument shows that if $m > 1$, then there exists an element $\alpha \in \operatorname{Br}(M)$ of order m and $\pi_M^*\alpha^{-1}$-twisted universal sheaf $\mathcal{P}$ on $X \times M$. We will not consider separately the case $m = 1$. It is a particular case of the next theorem corresponding to $\alpha = 1$, $B = 0$, though historically, it was proved by Mukai in [Mu-2], [Mu-3] before Căldăraru's work on twisted sheaves. To state the theorem, we need to introduce a new weight-2 integral Hodge structure on the total cohomology $H^*(M, \mathbb{Z})$ of M. This Hodge structure is determined by a B-field lifting α, which we will fix once and forever, and will be denoted by $\tilde{H}_B(M)$. The integer structure is defined by

$$\tilde{H}_B(M, \mathbb{Z}) = (\exp B \cdot H^*(M, \mathbb{Q})) \cap H^*(M, \mathbb{Z}),\ \exp B := 1 + B + \frac{B^2}{2},$$

and the Hodge decomposition over $\mathbb{C}$ by

$$\tilde{H}_B^{1,1}(M) = \exp B \cdot (H^0(M) \oplus H^{1,1}(M) \oplus H^4(M)),$$

$$\tilde{H}_B^{2,0}(M) = \exp B \cdot H^{2,0}(M),\quad \tilde{H}_B^{0,2}(M) = \exp B \cdot H^{0,2}(M).$$

The lattice $\tilde{H}_B(M, \mathbb{Z})$ is endowed with Mukai's form (1). We will also consider the total cohomology of X with a weight-2 Hodge structure $\tilde{H}(X, \mathbb{Z})$ constructed in the same way, but with B-field equal to zero.

The following theorem holds.

Theorem 2.1 (Mukai, Căldăraru, Huybrechts–Stellari). *Under the hypotheses and with the notation of the previous paragraph, let $\mathcal{P}$ be a $\pi_M^*\alpha^{-1}$-twisted universal sheaf on $X \times M$ and $\mathcal{P}^\vee := \mathbf{R}\mathcal{H}om\,(\mathcal{P}, \mathcal{O}_{X\times M})$. The following assertions are verified:*

(i) $\Phi^{\mathcal{P}}_{M\to X} : \mathbf{D}(M, \alpha) \longrightarrow \mathbf{D}(X)$ *is an equivalence of categories with inverse* $\Phi^{\mathcal{P}^\vee}_{X\to M} : \mathbf{D}(X) \longrightarrow \mathbf{D}(M, \alpha)$.

(ii) $\varphi = \varphi^{\mathcal{P}^\vee, B}_{X\to M}$ *is defined over* $\mathbb{Z}$ *and is a Hodge isometry* $\tilde{H}(X, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}_B(M, \mathbb{Z})$.

(iii) We have $\varphi(v) = (0, 0, 1)$, $\varphi(v^\perp) \subset (0, *, *)$, *and the induced map*

$$\overline{\varphi} : v^\perp/v \to H^2(M, \mathbb{Z}),\quad w \mapsto [\varphi(w)]_{H^2(M)}$$

is a Hodge isometry. Here $[\cdot]_{H^2(M)}$ *denotes the* $H^2(M)$*-component of an element of* $H^*(M)$*, and the orthogonal complement* $v^\perp$ *is taken in* $\tilde{H}(X, \mathbb{Z})$.

Yoshioka in [Y-2] defined the notion of (semi)-stability of α-twisted sheaves and constructed the moduli spaces of α-twisted sheaves. Let Y be a K3 surface polarized by an ample class H, $B \in H^2(Y,\mathbb{Q})$ and $\alpha = e^{2\pi i B} \in \mathrm{Br}(Y)$. Let $v \in \tilde{H}_B^{1,1}(M,\mathbb{Z})$. We will denote by $M_{Y,B}^{H,s}(v)$ the moduli space of stable α-twisted sheaves $\mathcal{F}$ on Y with $v_B(\mathcal{F}) = v$. The next theorem is a generalization of Theorem 1.3 to twisted sheaves.

Theorem 2.2 (Yoshioka). *In the hypotheses of the previous paragraph, assume that H is sufficiently generic, that v is primitive, and that $M = M_{Y,B}^{H,s}(v)$ is nonempty. Then M is an irreducible symplectic manifold of dimension v^2+2.*

3. The K3 moduli space $M(m, H, (d-1)m)$

Let X be a K3 surface such that $\operatorname{Pic} X = \mathbb{Z}\cdot H$, where H is ample and $H^2 = (2d-2)m^2$ for some $m \geq 2$, $d \geq 2$. Denote by η_X the positive generator of $H^4(X,\mathbb{Z})$. Let $M = M_X(m, H, (d-1)m)$ be the moduli space of semistable sheaves on X with Mukai vector $v = (m, H, (d-1)m) = m + H + (d-1)m\eta_X$.

Lemma 3.1. *The following statements hold:*

(i) M is a K3 surface with $\operatorname{Pic} M \simeq \mathbb{Z}$.
(ii) Every semistable sheaf with Mukai vector v is μ-stable and locally free.

Proof. As v is primitive and $v^2 = 0$, (i) follows from Theorem 2.1, (iii). Further, $\operatorname{Pic} X = \mathbb{Z}H$, hence every semistable sheaf $\mathcal{E}$ with $c_1(\mathcal{E}) = H$ is μ-stable.

The local freeness of $\mathcal{E}$ follows from [Mu-2], Proposition 3.16. □

Now we will describe Serre's construction, which permits to obtain all the vector bundles from M as some sheaf extensions. For a 0-dimensional subscheme $Z \subset X$, the number $\delta(Z) = h^1(\mathcal{I}_Z(1))$ is called the index of speciality (or $\mathcal{O}(1)$-speciality) of Z; it is equal to the number of independent linear relations between the points of Z. In a more formal way, $\delta(Z) = l(Z) - 1 - \dim\langle Z\rangle$, where $l(Z)$ stands for the length of Z. Following Tyurin [Tyu-1], [Tyu-2], we will call Z stable if $\delta(Z') < \delta(Z)$ for any $Z' \subset Z$, $Z' \neq Z$.

Lemma 3.2. *Let $Z \subset X$ be a stable* 0*-dimensional subscheme of degree $c = (d-1)(m^2-m)+m$ with $\delta(Z) = m-1$. Define a sheaf $\mathcal{E} = \mathcal{E}_Z$ as the middle term of the exact triple*

$$0 \longrightarrow H^1(\mathcal{I}_Z(1)) \otimes \mathcal{O}_X \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{I}_Z(1) \longrightarrow 0 \tag{3}$$

whose extension class is the identity map on $H^1(\mathcal{I}_Z(1))$ under the canonical isomorphism $\mathrm{Ext}^1(\mathcal{I}_Z(1), H^1(\mathcal{I}_Z(1)) \otimes \mathcal{O}_X) = \mathrm{End}(H^1(\mathcal{I}_Z(1)))$*. Then $\mathcal{E}$ is a stable locally free sheaf with Mukai vector $v = (m, H, (d-1)m)$ and $h^0(\mathcal{E}) = dm$, $h^1(\mathcal{E}) = h^2(\mathcal{E}) = 0$.*

Proof. The local freeness follows from [Tyu-2], Lemma 1.2. The assertions on the cohomology and the Mukai vector of $\mathcal{E}$ are obvious. As $\mathrm{Pic}(X) = \mathbb{Z}H$ and $c_1(\mathcal{E}) = H$, it is enough to prove that $\mathcal{E}$ is semistable.

Assume that $\mathcal{E}$ is unstable and $\mathcal{M}$ is the maximal destabilizing subsheaf of $\mathcal{E}$. Then $c_1(\mathcal{M}) = nH$ with $n \geq 1$. Let i be the inclusion $\mathcal{M} \longrightarrow \mathcal{E}$. If $\beta \circ i = 0$, then α^{-1} maps $\mathcal{M}$ into the trivial vector bundle $H^1(\mathcal{I}_Z(1)) \otimes \mathcal{O}_X \simeq \mathcal{O}_X^{m-1}$, which is impossible. Hence $\beta \circ i(\mathcal{M})$ is a rank-1 subsheaf of $\mathcal{I}_Z(1)$, which we can represent as $\mathcal{I}_W(1)$ for some subscheme $W \subset X$ containing Z. Thus we have the exact triple

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{I}_W(1) \longrightarrow 0.$$

If we assume that W has one-dimensional components, then $c_1(\mathcal{I}_W(1)) \leq 0 < \mu(\mathcal{M})$, which contradicts the semistability of $\mathcal{M}$. Hence W is 0-dimensional. We have $\mathcal{M}' \subset \mathcal{O}_X^{m-1} \subset \mathcal{E}$, so $c_1(\mathcal{M}') = n - 1 \leq 0$ and $n = 1$. As $\mathcal{M}$ is maximal, it is saturated in $\mathcal{E}$, hence so is $\mathcal{M}'$, and then $\mathcal{M}' \simeq \mathcal{O}_X^{k-1}$ is a trivial subbundle of $\mathcal{O}_X^{m-1}$. We obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X^{k-1} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{I}_W(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X^{m-1} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{I}_Z(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X^{m-k} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{I}_{W,Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4}$$

For any Artinian $\mathcal{O}_X$-module M, we have $\mathrm{Ext}^1(M, \mathcal{O}_X) \simeq \mathrm{Ext}^1(\mathcal{O}_X, M)^\vee = H^1(X, M)^\vee = 0$. Applying this to $M = \mathcal{I}_{W,Z}$, we see that $\mathrm{Ext}^1(\mathcal{I}_{W,Z}, \mathcal{O}_X^{m-k}) = 0$ and $\mathcal{N}$ has torsion whenever $\mathcal{I}_{W,Z} \neq 0$, which contradicts the fact that $\mathcal{M}$ is a saturated subsheaf of $\mathcal{E}$. Hence $\mathcal{I}_{W,Z} = 0$ and $Z = W$.

We have the following monomorphism of extensions of sheaves:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X^{k-1} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{I}_Z(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_X^{m-1} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{I}_Z(1) & \longrightarrow & 0
\end{array}$$

The next lemma implies that $k = m$, hence the monomorphism is an isomorphism, $\mathcal{M} = \mathcal{E}$, and $\mathcal{E}$ is semistable. $\square$

Lemma 3.3. *Let V be a smooth variety, F_1, F_2, G sheaves on V,*

$$0 \longrightarrow F_i \longrightarrow \mathcal{E}_i \longrightarrow G \longrightarrow 0\ , \qquad i = 1,\ 2,$$

two sheaf extensions with classes $e_i \in \mathrm{Ext}^1(G, F_i)$. Then a homomorphism of sheaves $\varphi : F_1 \longrightarrow F_2$ extends to a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \longrightarrow & E_1 & \longrightarrow & G & \longrightarrow & 0 \\ & & {\scriptstyle\varphi}\big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & F_2 & \longrightarrow & E_2 & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

if and only if $e_2 = \varphi \circ e_1$.

Proof. Standard; compare to [Mac], Proposition III.1.8. □

The vector bundle $\mathcal{E} = \mathcal{E}_Z$ defined by (3) is the result of *Serre's construction* applied to the 0-dimensional susbscheme Z.

Remark 3.4. To define $\mathcal{E}_Z$, one can replace the identity extension class by any linear automorphism e of $H^1(\mathcal{J}_Z(1))$. By the lemma, this will provide an isomorphic sheaf $\mathcal{E}_Z$. If e is not of maximal rank, then $\mathcal{E}_Z$ is not locally free. By [Tyu-2], § 1, $\mathcal{E}_Z$ will also be non-locally-free if Z is not stable.

The description of properties of $\mathcal{E}$ in terms of Z is particularly simple if Z is contained in a smooth hyperplane section $C \in |H|$ of X, in which case we can consider Z as a divisor on C. By the geometric Riemann–Roch Theorem, the index of speciality of Z according to our definition is the same as the index of speciality of the divisor Z on C. Thus $\delta(Z) = m - 1$ if and only if Z belongs to a linear series g_c^{m-1} on C. We denote by $W_c^r(C)$ the Brill–Noether locus of linear series g_c^r on C and by $G_c^r(C)$ the union the corresponding linear series as a subvariety of $\mathrm{Div}^c(C)$.

Lemma 3.5. *Let $C \in |H|$ be a smooth curve and $Z \in \mathrm{Div}^c(C)$ such that $\delta(Z) = m - 1$. Let $\mathcal{E}_Z$ be defined by the extension (3). Then the following assertions hold:*

(i) $\mathcal{E}_Z$ is locally free if and only if the linear system $|Z|$ is base point free.
(ii) $\mathcal{E}_Z$ is globally generated if and only if the linear system $|K_C - Z|$ is base point free.
(iii) $\mathcal{E}_Z$ fits into the exact triple

$$0 \longrightarrow \mathcal{O}_X^m \longrightarrow \mathcal{E}_Z \longrightarrow \mathcal{O}_C(K - Z) \longrightarrow 0\ ,$$

where $K = K_C$ is the canonical class of C, and $\mathcal{E}_Z \simeq \mathcal{E}_{Z'}$ for any $Z' \in |Z|$.

Proof. See [Tyu-1], Lemma 3.4 or [Mor], Section 5. □

For any $t \in M$ we will denote by $\mathcal{E}_t$ the rank-m sheaf on X represented by t.

Proposition 3.6. *Let $t \in M$ be generic and $\mathcal{E} = \mathcal{E}_t$. The following properties are verified:*

(i) $\mathcal{E}$ is obtained from a divisor $Z \in G_c^{m-1}(C)$ for some smooth hyperplane section C of X by the construction of Lemma 3.5.

(ii) $\mathcal{E}$ is globally generated.

Proof. As X is generic, any of its smooth hyperplane sections C is Brill–Noether generic by [L]. As the Brill–Noether number $\rho_c^{m-1} = 1$, there exists a divisor $Z \in g_c^{m-1}$ on C. Let $\mathcal{E} = \mathcal{E}_Z$ be the corresponding vector bundle. As $\rho_{c-1}^{m-1} < 0$ and $\rho_{c+1}^{m} < 0$, the linear systems $|Z|$ and $|K - Z|$ are base point free, so $\mathcal{E} = \mathcal{E}_t$ for some $t \in M$ and $\mathcal{E}$ is globally generated by Lemma 3.5. The conditions (i), (ii) are open, so the proposition is proved. □

For a future use, we will prove the following lemma.

Lemma 3.7. *Let $t \in M$ be generic and $\mathcal{E} = \mathcal{E}_t$. The following properties are verified:*

(i) Let $s_1, \dots, s_m$ be m linearly independent global sections of $\mathcal{E}$. Then $s_1, \dots, s_m$ generate $\mathcal{E}$ at generic point of X.

(ii) The vanishing at a generic point $z \in X$ imposes precisely $m = \operatorname{rk} \mathcal{E}$ independent linear conditions on a section of $\mathcal{E}$. For any $k = 1, \dots, d$ and for k generic points $z_1, \dots, z_k$ of X, we have $h^0(X, \mathcal{E} \otimes \mathcal{I}_\xi) = m(d - k)$, where $\xi = z_1 + \cdots + z_k$.

(iii) Let $z_1, \dots, z_{d-1}$ be generic. Then any nonzero section s of $\mathcal{E}$ vanishing at $z_1, \dots, z_{d-1}$ has exactly $z_1 + \cdots + z_{d-1}$ as its scheme of zeros.

Proof. (i) Let $\mathcal{F}$ be the saturate in $\mathcal{E}$ of the subsheaf generated by $s_1, \dots, s_m$. As $\mathcal{E}$ is locally free, $\mathcal{F}$ is reflexive. Any reflexive sheaf on a smooth surface is locally free, so $\mathcal{F}$ is locally free. By the stability of $\mathcal{E}$, $\mathcal{F}$ has no subsheaves $\mathcal{F}'$ with $c_1(\mathcal{F}') > 0$. If $k = \operatorname{rk} \mathcal{F} < m$, then $s_1, \dots, s_m$ are linearly dependent at generic point of $\mathcal{E}$. Let us renumber the s_i in such a way that $s_1, \dots, s_k$ are linearly independent over $\mathbb{C}(X)$. Then $s_1, \dots, s_k$ define an inclusion $\mathcal{O}_X^{\oplus k} \hookrightarrow \mathcal{F}$. An inclusion of locally free sheaves of the same rank is either isomorphic, or has a cokernel supported on a nonempty divisor. The second case is impossible, as $c_1(\mathcal{F}) \leq 0$. Hence $\mathcal{F} \simeq \mathcal{O}_X^{\oplus k}$, and s_i for $i = k + 1, \dots, m$ are linear combinations of $s_1, \dots, s_k$ with constant coefficients, which is absurd. Hence $k = m$ and $\mathcal{F} = \mathcal{E}$.

(ii) If z_1 is generic, then $H^0(\mathcal{I}_{z_1} \otimes \mathcal{E})$ has codimension m in $H^0(\mathcal{E})$ by (i). Choose $s_1, \dots, s_m \in H^0(\mathcal{I}_{z_1} \otimes \mathcal{E})$ linearly independent. Then for $z_2 \in X$ generic, $s_1, \dots, s_m$ span $\mathcal{E}_{z_2}$ by (i). Thus z_1, z_2 impose $2m$ conditions on sections of $\mathcal{E}$. Iterating one gets (ii).

(iii) Choose $d - 2$ generic points $z_1, \dots, z_{d-2}$ of X and a basis of $2m$ sections $e_1, \dots, e_m, s_1, \dots, s_m$ vanishing at $z_1, \dots, z_{d-2}$, as in (ii) with $k = d - 2$. Consider these sections on a sufficiently small open set U on which both $e_1, \dots, e_m$ and $s_1, \dots, s_m$ are bases of $\mathcal{E}$. Let $A = A(x)$ be a m by m matrix of rational functions in $x \in X$ such that $s_i = Ae_i$, that is, $s_i = \sum_j a_{ji} e_j$, where $A = (a_{ji})$ $(i, j = 1, \dots, m)$. Then A is regular and nondegenerate for $x \in U$. For any $y \in U$, the m sections $\sigma_i = s_i - A(y)e_i$ form a basis of all the sections of $\mathcal{E}$ vanishing at $z_1, \dots, z_{d-2}$ and y. We may expand them in the basis (e_j) with coefficients in $\mathbb{C}(X)$: $\sigma_i = \sigma_i(x) = (A(x) - A(y))e_i$. Here x is a variable point of U, so that $\sigma_i(x)$ denotes the value of σ_i in the fiber $\mathcal{E}_x$. Thus, any section σ of $\mathcal{E}$ vanishing at y can be written over U in the form $\sigma(x) = (A(x) - A(y))\boldsymbol{v}$, where $\boldsymbol{v} = v_1 e_1 + \ldots + v_m e_m$, $(v_1, \ldots, v_m) \in \mathbb{C}^m$. This σ depends on $\boldsymbol{v}$, y as parameters, and we will denote it by $\sigma_{\boldsymbol{v},y}$. By the stability of $\mathcal{E}$, no nonzero section of $\mathcal{E}$ vanishes along a curve, so y is an isolated zero of $\sigma_{\boldsymbol{v},y}$. Let us now fix $\boldsymbol{v} \neq 0$ and let vary both x and y. The zeros of $\sigma_{\boldsymbol{v},y}$ are the solutions of the equation $A(x)\boldsymbol{v} = A(y)\boldsymbol{v}$. The fact that y is an isolated zero of $\sigma_{\boldsymbol{v},y}$ for any $y \in U$ implies that the fibers of the map $f : U \to \mathbb{C}^m$, $y \mapsto A(y)\boldsymbol{v}$ are 0-dimensional (here we interprete $\boldsymbol{v}$ as the vector $(v_1, \ldots, v_m) \in \mathbb{C}^m$). By Chevalley's Theorem, $f(U)$ is a 2-dimensional constructible subset of $\mathbb{C}^m$ in the Zariski topology. By Bertini-Sard Theorem, $f(U)$ contains an open subset of noncritical values of f. Thus there is a smaller Zariski open subset $U_0 \subset U$ such that $f|_{U_0}$ is locally holomorphically invertible ("locally" in the classical topology). Hence for any $y_0 \in U_0$, the equation $f(x) = f(y_0)$ is locally analytically equivalent to $x = y_0$, thus y_0 is a simple zero of $f(x) - f(y_0)$ and a simple zero of $\sigma_{\boldsymbol{v},y_0}$.

We have proved that any nonzero section of $\mathcal{E}$ vanishing at $d-1$ generic points $z_1, \ldots, z_{d-2}, z_{d-1} = y_0$ of X has a simple zero at z_{d-1}. By the symmetry of the roles of the points z_i, all the z_i are simple zeros. $\square$

4. Lagrangian fibration via Fourier–Mukai transform

As in Section 3, let X be a generic K3 surface of degree $(2d-2)m^2$. Denote by H, resp. η_X the positive generator of Pic X, resp. $H^4(X, \mathbb{Z})$. Let $M = M_X(m, H, (d-1)m)$ be the moduli space of semistable sheaves on X with Mukai vector $v = (m, H, (d-1)m)$. Denote by η_M the positive generator of $H^4(M, \mathbb{Z})$.

Lemma 4.1. *Let $\mathcal{P}$ be a $\pi_M^*(\alpha)^{-1}$-twisted universal sheaf on $X \times M$ for some $\alpha \in \mathrm{Br}(M)$ of order m. Let $B \in H^2(M, \mathbb{Q})$ be such that $\alpha = e^{2\pi i B}$ and $\varphi = \varphi^{\mathcal{P}^\vee, B}_{X \to M}$ the map used in Theorem 2.1. Then* Pic $M \simeq \mathbb{Z}$*, and for the positive generator $\hat{H}$ of* Pic M *we have*

$$\hat{H} = \pm\ [\varphi(1 + (1-d)\eta_X)]_{H^2(M)},$$

where $[\cdot]_{H^2(M)}$ denotes the H^2-component of a cohomology class.

Proof. The intersection of $v^\perp \subset \tilde{H}(X, \mathbb{Z})$ with $\tilde{H}^{1,1}(X) = \mathbb{C}\cdot 1 + \mathbb{C}\cdot H + \mathbb{C}\cdot \eta_X$ is the lattice of rank 2 generated by v and $1 + (1-d)\eta_X$. Hence, by Theorem 2.1, the class of the hyperplane section $\hat{H}$ of M is equal to $\pm\ [\varphi(1 + (1-d)\eta_X)]_{H^2(M)}$. $\square$

Let ξ be a subscheme of length d in X and $\mathcal{J}_\xi \subset \mathcal{O}_X$ its ideal sheaf. Define

$$C_\xi = \{t \in M \mid h^0(X, \mathcal{E}_t \otimes \mathcal{J}_\xi) \neq 0\},$$

where $\mathcal{E}_t$ denotes a rank-m vector bundle whose isomorphism class is represented by t.

Proposition 4.2. *The sign in the formula of Lemma 4.1 for $\hat{H}$ is plus, and for generic $\xi \in X^{[d]}$, C_ξ is a curve from the linear system $|\hat{H}|$.*

Proof. Let $\Phi = \Phi^{\mathcal{P}^\vee}_{X\to M}$ be the Fourier–Mukai transform associated to $\mathcal{P}$. The Mukai vector $1 + (1-d)\eta_X$ is realized by either one of the objects $\mathcal{I}_\xi$ or $\mathcal{I}_\xi^\vee = \mathbf{R}\mathcal{H}om\,(\mathcal{I}_\xi, \mathcal{O}_X)$. Hence $\varphi(1+(1-d)\eta_X) = \mathrm{ch}_B(\Phi(\mathcal{I}_\xi^\vee))\sqrt{\mathrm{Td}(M)}$, where $\Phi(\mathcal{I}_\xi^\vee) = \mathbf{R}\pi_{M*}(\pi_X^*\mathcal{I}_\xi^\vee \overset{\mathbf{L}}{\otimes} \mathcal{P}^\vee)$. As $\mathcal{P}$ is locally free, we may replace $\overset{\mathbf{L}}{\otimes}$ by $\otimes$. By the relative duality for π_M (see [Har-1], p. 210 and [Cal-2], 2.7), and because the canonical sheaf of M is trivial, $\Phi(\mathcal{I}_\xi^\vee)[2] \simeq (\mathbf{R}\pi_{M*}(\pi_X^*\mathcal{I}_\xi \otimes \mathcal{P}))^\vee$. The Cohomology and Base Change Theorem ([Har-2], Theorem III.12.11) reduces the computation of the latter to that of the cohomology groups $H^i(\mathcal{G}_t)$, where $t \in M$, $\mathcal{G} = \pi_X^*\mathcal{I}_\xi \otimes \mathcal{P}$, and $\mathcal{G}_t := \mathcal{G}|_{X\times t} \simeq \mathcal{P}_t \otimes \mathcal{I}_\xi$. The cohomology of the sheaves $\mathcal{P}_t \otimes \mathcal{I}_\xi$ can be determined from the exact triples

$$0 \longrightarrow H^1(\mathcal{I}_Z(1)) \otimes \mathcal{I}_\xi \longrightarrow \mathcal{P}_t \otimes \mathcal{I}_\xi \longrightarrow \mathcal{I}_{Z\cup\xi}(1) \longrightarrow 0,$$

where we can assume that $\mathrm{Supp}\, Z \cap \mathrm{Supp}\, \xi = \varnothing$, and

$$0 \longrightarrow \mathcal{I}_{Z\cup\xi}(1) \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_{Z\cup\xi}(1) \longrightarrow 0.$$

By Serre duality, $H^2(\mathcal{P}_t \otimes \mathcal{I}_\xi) = \mathrm{Hom}(\mathcal{I}_\xi, \mathcal{P}_t^\vee)^\vee = H^0(\mathcal{P}_t^\vee)^\vee = 0$. Hence $R^2\pi_{M*}\mathcal{G} = 0$, and $h^0(\mathcal{P}_t \otimes \mathcal{I}_\xi) = h^1(\mathcal{P}_t \otimes \mathcal{I}_\xi)$, so both of them are different from zero if and only if $t \in C_\xi$.

Let us verify that C_ξ is a proper closed subset of M for generic ξ. It suffices to show that $h^0(X, \mathcal{P}_t \otimes \mathcal{I}_\xi) = 0$ for generic $\xi \in X^{[d]}$ and generic $t \in M$. Choose a generic $\mathcal{E} = \mathcal{P}_t$ and d generic points $z_1, \dots, z_d$ of X. By Lemma 3.7, (ii), $h^0(X, \mathcal{E} \otimes \mathcal{I}_\xi) = 0$ for $\xi = z_1 + \dots + z_d$, which implies the result.

Thus, C_ξ is a union of finitely many curves and isolated points for generic ξ. By Proposition 2.26 of [Mu-2], C_ξ is of pure dimension 1 and $R^0\pi_{M*}\mathcal{G} = 0$. As $R^i\pi_{M*}\mathcal{G} = 0$ is nonzero only in odd degree $i = 1$ and $\mathcal{E}xt^p(R^1\pi_{M*}\mathcal{G}, \mathcal{O}_M)$ is nonzero only for odd $p = 1$, we have

$$\left[\varphi(1 + (1-d)\eta_X)\right]_{H^2(M)} = c_1(R^1\pi_{M*}\mathcal{G}) = h^0(\mathcal{P}_t \otimes \mathcal{I}_\xi)[C_\xi]$$

for generic $t \in C_\xi$. But by Lemma 4.1, $\left[\varphi(1 + (1-d)\eta_X)\right]_{H^2(M)} = \pm\hat{H}$, hence $C_\xi \in |\hat{H}|$, $h^0(\mathcal{P}_t \otimes \mathcal{I}_\xi)$ is generically 1, and the sign is plus. □

Theorem 4.3. *Let X be a $K3$ surface with* $\mathrm{Pic}\, X \simeq \mathbb{Z}$, *and H the ample generator of* $\mathrm{Pic}\, X$. *Let $M = M_X^{H,s}(m, H, (d-1)m)$ be the moduli space of H-stable sheaves on X with Mukai vector $v = (m, H, (d-1)m)$ ($d \geq 2, m \geq 2$) and $\hat{H}$ the ample generator of* $\mathrm{Pic}\, M$. *Let $\mathcal{P}$ be a $\pi_M^*(\alpha)^{-1}$-twisted universal sheaf on $X \times M$ for some $\alpha \in \mathrm{Br}(M)$ of order m and B a lifting of α in $H^2(M, \mathbb{Q})$. Let $\Phi = \Phi^{\mathcal{P}^\vee}_{X\to M}$ be the associated Fourier–Mukai transform, and $\varphi = \varphi^{\mathcal{P}^\vee, B}_{X\to M}$ its cohomological descent. Denote by w the Mukai vector* $(1, 0, 1-d)$ *of the sheaves $\mathcal{I}_\xi$ for $\xi \in X^{[d]}$, so that $X^{[d]} = M_X^{H,s}(w)$. Then the following assertions hold:*

(i) For generic $\xi \in X^{[d]}$, the only nonzero cohomology of the complex $\Phi(\mathcal{I}_\xi^\vee)$ is h^2, and $h^2\Phi(\mathcal{I}_\xi^\vee)$ is a rank-1 torsion-free $\alpha|_{C_\xi}$-twisted sheaf on a curve $C_\xi \in |\hat{H}|$.

(ii) $\varphi(w) = (0, \hat{H}, k)$ for some $k \in \mathbb{Z}$, and the moduli space $V = M_{M,B}^{\hat{H},s}(0, \hat{H}, k)$ is an irreducible symplectic manifold of dimension $2d$. There is a birational isomorphism $\mu : X^{[d]} - - \to V$ defined by $\xi \mapsto [h^2\Phi(\mathcal{I}_\xi^\vee)]$, where the brackets denote the isomorphism class of a sheaf.

(iii) The support of any sheaf $\mathcal{L}_t$ on M represented by a point $t \in V$ is a curve from the linear system $|\hat{H}|$, and the map $f : V \to |\hat{H}| \simeq \mathbb{P}^d$, $t \mapsto \operatorname{Supp} \mathcal{L}_t$, is a Lagrangian fibration. If we denote by $\{C\}$ the point of the projective space $\mathbb{P}^d \simeq |\hat{H}|$ representing a curve C from the linear system $|\hat{H}|$, then the fiber $f^{-1}(\{C\})$ for generic $\{C\} \in |\hat{H}|$ is isomorphic to the Jacobian of C.

Proof. (i) was verified in the proof of Proposition 4.2.

(ii) The equality $\varphi(w) = (0, \hat{H}, k)$ follows from Proposition 4.2 and Theorem 2.1 (iii). The fact that V is irreducble symplectic will follow from Yoshioka's Theorem 2.2 as soon as we see that V is nonempty. But we have constructed stable sheaves represented by points of V in part (i). Indeed, as Pic $M = \mathbb{Z}\hat{H}$, any curve C_ξ is irreducible, and a rank-1 torsion free (twisted or usual) sheaf on an irreducible curve is stable with respect to any polarization.

To prove the birationality of μ, remark that Φ and the duality functor D are equivalences of categories, so the composite functor $\Phi \circ D$ transforms non-isomorphic sheaves $\mathcal{I}_\xi$ into the complexes $\Phi(\mathcal{I}_\xi^\vee)$ that are non-isomorphic in $\mathbf{D}(M, \alpha)$. For generic ξ, the cohomology of the complex $\Phi(\mathcal{I}_\xi^\vee)$ is concentrated in degree 2, hence the complex is quasi-isomorphic to $h^2\Phi(\mathcal{I}_\xi^\vee)[2]$. Thus for generic $\xi \neq \xi'$, the sheaves $h^2\Phi(\mathcal{I}_\xi^\vee)$, $h^2\Phi(\mathcal{I}_{\xi'}^\vee)$ are non-isomorphic. This implies that μ is a generically injective rational map between irreducible varieties of the same dimension, hence it is birational.

(iii) The fact that f is a Lagrangian fibration is an obvious consequence of the above and of Matsushita's Theorem 1.5.

As $\operatorname{Br}(C) = 0$ for a smooth curve C, $\alpha|_C = 0$ for any smooth $C \in |\hat{H}|$. Hence the fiber of the support map over C is isomorphic to the Jacobian $J(C)$. This isomorphism is not canonical, for two different Čech 1-cochains β such that $\check{d}(\beta) = \alpha|_C$ may differ by a Čech 1-cocycle defining an invertible sheaf $\mathcal{L}$ on C, and the corresponding isomorphisms of $f^{-1}(\{C\})$ with $J(C)$ will differ by a translation by the class of $\mathcal{L}$ in $J(C)$. Hence V represents a birational torsor (biregular over the smooth curves $C \in |\hat{H}|$) under the relative Jacobian J of the linear system $|\hat{H}|$, and the generic fiber of f is isomorphic to $J(C)$ with $C \in |\hat{H}|$. □

Corollary 4.4. *Let X be a generic $K3$ surface of degree $(2d-2)m^2$, H the positive generator of* Pic X*, and h the divisor class in* $\operatorname{Pic}(X^{[d]})$ *corresponding to H under the isomorphism of Proposition 1.2. Consider the rational map*

$$\pi : X^{[d]} - - \to |\hat{H}| \simeq \mathbb{P}^d, \;\; \xi \mapsto C_\xi \; .$$

Then π is defined by the complete linear system $|h-me|$ and is a rational Lagrangian fibration. The fiber $\pi^{-1}(\{C\})$ for generic $C \in |\hat{H}|$ is birational to the Jacobian of C.

Proof. In the notation of Theorem 4.3, $\pi = f \circ \mu$, where f is a Lagrangian fibration and μ is birational. Hence π is a rational Lagrangian fibration. By Proposition 1.2, $\mathrm{Pic}(X^{[d]})$ is of rank 2 and the only primitive effective classes with square zero are $h \pm me$, so $\pi^*[\mathcal{O}_{\mathbb{P}^d}(1)] = h \pm me$. By construction, $\pi^*[\mathcal{O}_{\mathbb{P}^d}(1)]$ is represented by a divisor of the form

$$D_t = \{\xi \in X^{[d]} \mid h^0(\mathcal{I}_\xi \otimes \mathcal{E}_t) \neq 0\} \subset X^{[d]}$$

for generic $t \in M$, where $\mathcal{E}_t$ denotes a stable vector bundle on X representing t. The class $h + me$ has negative intersection with the generic fiber $\mathbb{P}^1$ of the Hilbert–Chow map $X^{[d]} \longrightarrow X^{(d)}$, hence has the whole exceptional divisor E in its base locus. Hence, to see that $D_t \sim h - me$, it suffices to verify that D_t does not contain E as a fixed component. The support $\mathrm{Supp}\,\xi$ of a generic $\xi \in E$ is a set of $d-1$ generic points of X. By Lemma 3.7, (iii), the scheme of zeros of any nonzero section σ of $\mathcal{E}$ vanishing on $\mathrm{Supp}\,\xi$ is exactly $\mathrm{Supp}\,\xi$. But $\mathrm{Supp}\,\xi \subsetneq \xi$, so $h^0(\mathcal{I}_\xi \otimes \mathcal{E}_t) = 0$, and $D_t \sim h - me$. By the same argument as in Lemma 1.6, π is given by the complete linear system $|h - me|$. □

Remark 4.5. Though, as we mentioned in the introduction, π is regular for generic X, it may be nonregular for some special K3 surfaces with $\mathrm{Pic}\,X \simeq \mathbb{Z}^2$ which have a divisor class of degree $(2d-2)m^2$. The following Proposition provides such a special K3 surface. The map $\varphi_{|h-me|}$ for this K3 surface is not regular, but a small deformation of X kills its indeterminacy.

Proposition 4.6. *Let X be a generic lattice-polarized K3 surface with Picard lattice*

$$Q = \begin{pmatrix} 2d & 2d-1+m \\ 2d-1+m & 2d-2 \end{pmatrix}.$$

Let f_{2d}, f_{2d-2} be effective classes forming a basis of the Picard lattice in which the intersection form is given by the above matrix. Then the following properties are verified:

(i) If $d \geq 3$, then the linear system $|f_{2d}|$, resp. $|f_{2d-2}|$ embeds X into $\mathbb{P}^{d+1}$, resp. $\mathbb{P}^d$. If $d = 2$, then $|f_{2d}|$ embeds X as a smooth quartic in $\mathbb{P}^3$, and $|f_{2d-2}|$ defines a double covering of $\mathbb{P}^2$.

(ii) Every curve in the linear systems $|f_{2d-2}|$, $|f_{2d}|$ is reduced and irreducible.

(iii) In addition to the rational map θ introduced in (2), define the birational involution

$$\iota : X^{[d]} \dashrightarrow X^{[d]}, \quad \xi \mapsto (\langle\xi\rangle_{\mathbb{P}^{d+1}} \cap X) - \xi,$$

where $\langle\xi\rangle_{\mathbb{P}^{d+1}}$ denotes the linear span of ξ in its embedding into $\mathbb{P}^{d+1}$ by the linear system $|f_{2d}|$. The corresponding isometry of the Bogomolov–Beauville lattice is the reflection with respect to the vector $f_{2d} - e$ with square 2:

$$\iota^* : H^2(X^{[d]}) \longrightarrow H^2(X^{[d]}), \quad c \mapsto -c + (c, f_{2d} - e)(f_{2d} - e). \qquad (5)$$

Then the composite map $\pi = \theta \circ \iota$ *is given by the complete linear system* $|f_{(2d-2)m^2} - me|$, *where* $f_{(2d-2)m^2} = (m+1)f_{2d} - f_{2d-2}$ *is an effective divisor class of degree* $(2d-2)m^2$.

Proof. This is similar to the work of Hassett–Tschinkel [HasTsch-1] who produce on a K3 surface X with two polarizations of degrees 4 and 8 an infinite series of polarizations f_{2m^2} of degree $2m^2$ $(m \geq 2)$ and the abelian fibration maps on $X^{[2]}$ given by the linear system $f_{2m^2} - me$. The assertions (i), (ii) follow easily from the surjectivity of the period mapping for K3 surfaces [LP] and from the results of [SD], [Kov]. For formula (5) of part (iii), see [O'G-2], 4.1.2. A direct calculation using (5) shows that $f_{(2d-2)m^2} - me = \iota^*(f_{2d-2} - e)$. The class $f_{(2d-2)m^2}$ has positive square and positive scalar product with f_{2d}, hence is effective. As ι is an isomorphism in codimension 1, the dimensions of the linear systems $|f_{2d-2} - e|$ and $|\iota^*(f_{2d-2} - e)|$ are the same, so π is defined by a complete linear system. □

Acknowledgements. I thank A. Iliev for extensive discussions: in fact, we started a joint work on Lagrangian fibrations, but finally divided our problem into two parts and continued the work separately. I am grateful to D. Orlov for introducing me to the twisted sheaves. I acknowledge discussions with K. Ranestad and S. Popescu, as well as the hospitality of the Mathematishches Forschgungsinstitut in Oberwolfach, where a part of the work was done. I would also like to thank the referee for a thorough reading of the manuscript and for suggestions which helped to correct some imprecisions.

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