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Generalizations of some results concerning microperiodic mappings

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Abstract. It is well known that a microperiodic function that maps the set of reals into itself and is continuous at a point (Lebesgue measurable, respectively) must be constant (constant almost everywhere, resp.). We generalize those results in several directions. As a consequence we obtain conclusions concerning some systems of functional inequalities.

1. Introduction

In what follows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote, as usual, the sets of positive integers, integers, rationals and reals, respectively. Moreover $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We say a function $f : \mathbb{R} \to \mathbb{R}$ is microperiodic provided it has an arbitrarily small positive period, i.e. the set $\{a \in \mathbb{R} : f(x+a) = f(x) \text{ for } x \in \mathbb{R}\}$ is dense in $\mathbb{R}$. (For instance, if $f : \mathbb{R} \to \mathbb{R}$ is biperiodic with periods $a, b \in \mathbb{R} \setminus \{0\}$ (i.e. $f(x+a) = f(x) = f(x+b)$ for $x \in \mathbb{R}$) and $ab^{-1} \notin \mathbb{Q}$, then f is microperiodic). It is very easy to prove that a microperiodic function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at a point must be constant. The case where the function is Lebesgue measurable is more involved; it seems that it was Łomnicki [10] who first proved that such a function must be constant almost everywhere. A short proof of it was given by Semadeni [21] (see also [4]). An analogous result for functions with the Baire property has been obtained by Xenikakis [22]. Generalizations of all these are given in Kuczma [6], where a very abstract approach is assumed, and in Brzdęk [3].

Similar results were obtained by Montel [13] (cf. [17] and [7, p. 228]; see also [11] and [12]), who considered functions $f : \mathbb{R} \to \mathbb{R}$, continuous at a point and satisfying the following system of simultaneous inequalities

$$f(x+a) \le f(x), \quad f(x+b) \le f(x) \quad \text{for } x \in \mathbb{R} \tag{1}$$

with some $a, b \in \mathbb{R} \setminus \{0\}$, $ab^{-1} \notin \mathbb{Q}$, $ab < 0$. It is easily seen that the set $P := \{na + mb : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}$ and (1) implies

$$f(x+p) \le f(x) \quad \text{for } x \in \mathbb{R},\ p \in P. \tag{2}$$

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Therefore the study of system (1) leads us to the problem of description of solutions of (2). For measurable functions and in more general settings inequalities (1) and (2) have been studied in Brzdęk [3].

We present generalizations of all the mentioned results. In particular, we show how to unify most of these outcomes into a more general one, from which the particular cases can be easily derived by specifications of some terms. Thus we supply an answer to a question of Volkmann, who asked (private communication) about a unification of the both mentioned above results, concerning microperiodic measurable functions and microperiodic functions that are continuous at a point. (The question was motivated by well known such a unification for the Cauchy equations (see e.g. [1])). We also obtain a unification of the results in Brzdęk [3] (see Remark 3.1 and Theorem 3.1). We achieve that using an abstract property of measurability with respect to some families of sets satisfying a hypothesis being an abstract analogue of the Steinhaus Theorem (cf. [6]).

As a consequence we get generalizations of some results of Krassowska [8] and Matkowski [9], concerning some systems of simultaneous functional inequalities (see Corollaries 3.6 and 3.7), arising from a characterization of the L^p norm (see [9]).

2. Preliminaries

Let us start with some definitions (some of them are quite well known).

Definition 2.1. *A group* $(G, \cdot)$, *endowed with a topology, is a semitopological group provided the mappings* $G \ni x \to x \cdot y \in G$ *and* $G \ni x \to y \cdot x \in G$ *are continuous for every* $y \in G$ (*cf. e.g.* [5], [6] *or* [18]).

Definition 2.2. *Let* X *be a nonempty set. Then* $\mathcal{I} \subset 2^X$ *is an ideal (in* X*) provided* $A \cup B \in \mathcal{I}$ *and* $2^A \subset \mathcal{I}$ *for every* $A, B \in \mathcal{I}$. *If, moreover,* $\mathcal{I} \neq 2^X$, *then we say* $\mathcal{I}$ *is proper. Next, we say* $\mathcal{I}$ *is nontrivial provided* $\mathcal{I} \neq \{\emptyset\}$.

Definition 2.3. *Let* $\mathcal{I}$ *be an ideal in a set* $X \neq \emptyset$. *Then* $\mathcal{I}$ *is a* σ*-ideal provided* $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{I}$ *for every* $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{I}$.

Definition 2.4. *Let* $(X, \cdot)$ *be a group and* $\mathcal{I} \subset 2^X$. *Then* $\mathcal{I}$ *is left-translation invariant provided* $x \cdot A := \{x \cdot a : a \in A\} \in \mathcal{I}$ *for every* $A \in \mathcal{I}$, $x \in X$.

Definition 2.5. *Let* X *be a topological space,* $\mathcal{I} \subset 2^X$ *be an ideal and* $A \subset X$. *If for every* $x \in A$ *there exists a neighbourhood* $W_x \subset X$ *of* x *such that* $W_x \cap A \in \mathcal{I}$, *then we say* A *is locally in* $\mathcal{I}$. *We write* $l(\mathcal{I}) := \{A \subset X : A \text{ is locally in } \mathcal{I}\}$.

We say $\mathcal{I}$ *is local provided* $l(\mathcal{I}) \subset \mathcal{I}$.

Remark 2.1. Clearly, for every ideal $\mathcal{I} \subset 2^X$, $\mathcal{I} \subset l(\mathcal{I})$, whence $\mathcal{I} = l(\mathcal{I})$ if and only if $\mathcal{I}$ is local. There exist left-translation invariant ideals that are not local. For instance let $\mathcal{I}$ be the σ-ideal of the left-Haar measure zero subsets of a locally compact topological group. Then $l(\mathcal{I})$ is the σ-ideal of subsets of the group that are locally of the left-Haar measure zero and, in some locally compact topological groups, $\mathcal{I} \neq l(\mathcal{I})$.

It is also easily seen that in a topological group $(G, \cdot)$, with the discrete topology, for every nontrivial left-translation invariant ideal $\mathcal{I} \subset 2^G$ we have $G \in l(\mathcal{I})$; but on the other hand $G \in \mathcal{I}$ if and only if $\mathcal{I} = 2^G$.

Proposition 2.1. *Let (G, d) be a separable metric space. Then every σ-ideal $\mathcal{I} \subset 2^G$ is local.*

Proof. Let P be a dense countable subset of G and $\mathcal{I} \subset 2^G$ be a σ-ideal. Write $B(x, r) := \{y \in G : d(x, y) < r\}$ for $x \in G$, $r > 0$. Take $A \in l(\mathcal{I})$. Then, for every $x \in A$, there is $r_x \in \mathbb{Q}^+$ (positive rationals) with $B(x, 2r_x) \cap A \in \mathcal{I}$. Let $P_x := B(x, r_x) \cap P$ for $x \in A$. It is easily seen that $B(x, r_x) \subset \bigcup_{y \in P_x} B(y, r_x)$ and $B(y, r_x) \cap A \subset B(x, 2r_x) \cap A \in \mathcal{I}$ for every $x \in A$, $y \in P_x$. Next $L := \{B(y, r_x) : x \in A, y \in P_x\} \subset \{B(y, r) : y \in P, r \in \mathbb{Q}^+\}$, whence L is countable. Since $A \subset \bigcup_{x \in A} B(x, r_x) \subset \bigcup_{B \in L} B$ and $D \cap A \in \mathcal{I}$ for $D \in L$, we obtain $A \in \mathcal{I}$. □

Remark 2.2. It seems to be an open question whether the converse is true, i.e. whether in a metric space that is not separable there exist σ-ideals that are not local.

Let $(G, \cdot)$ be a group endowed with a topology. For $D, E \subset G$ and $a \in G$ we write $a \cdot E := \{a \cdot y : y \in E\}$, $D^{-1} := \{x^{-1} : x \in D\}$ and $D \cdot E := \{x \cdot y : x \in D, y \in E\}$. In the sequel we need the following hypothesis.

(M) $\mathcal{M} \subset 2^G$ and there exist σ-ideals $\mathcal{I}, \mathcal{S} \subset 2^G$ such that

$$\operatorname{int}(D \cdot C^{-1}) \neq \emptyset \quad \text{and} \quad \operatorname{int}(C \cdot D^{-1}) \neq \emptyset \quad \text{for } D \in \mathcal{M} \setminus \mathcal{I}, C \in 2^G \setminus \mathcal{S}.$$

We have the following two well known examples of families $\mathcal{M}$ satisfying hypothesis (M).

Example 2.1. $\mathcal{M}$ is the family of Haar measurable subsets of a locally compact topological group and $\mathcal{S} = \mathcal{I}$ is the σ-ideal of all locally of Haar measure zero subsets of the group (see [2]).

Example 2.2. $(G, \cdot)$ is a semitopological group such that the mapping $G \ni x \to x^{-1} \in G$ is continuous, $\mathcal{M}$ is the family of all subsets of G with the Baire property and $\mathcal{S} = \mathcal{I}$ is the σ-ideal of all subsets of G of the first category (see e.g. [5, 18, 19]).

Next two examples of families $\mathcal{M}$ satisfying hypothesis (M) are supplied in Remark 3.2 and in the subsequent Proposition 2.2. (Proposition 2.2 and Lemma 2.1 correspond to the abstract generalization of the Baire property introduced in [14] and [15] and to the results generalizing the Piccard Theorem in Sander [18–20] and Kominek and Kuczma [5]). For the proof of Proposition 2.2 we need the following lemma.

Lemma 2.1. *Let $(G, \cdot)$ be a group endowed with a topology such that the mapping $G \in y \to z \cdot y$ is continuous for every $z \in G$. Let $\mathcal{I}_0 \subset 2^G$ be a left-translation invariant local ideal, $B \in 2^G \setminus \mathcal{I}_0$, $T \in \mathcal{I}_0$, and $U \subset G$ be open and nonempty. Then there is $y \in B$ with*

$$U \cdot y^{-1} \subset (U \setminus T) \cdot B^{-1} \quad \text{and} \quad y \cdot U^{-1} \subset B \cdot (U \setminus T)^{-1}. \tag{3}$$

Proof. $B \notin \mathcal{I}_0$ and $\mathcal{I}_0$ is local, whence there is $y \in B$ with $(y \cdot W) \cap B \notin \mathcal{I}_0$ for every neighbourhood W of the neutral element in G. Take $x \in U$. Then $U_0 := x^{-1} \cdot U$ is a neighbourhood of the neutral element and consequently $(y \cdot U_0) \cap B \notin \mathcal{I}_0$. Hence $(x^{-1} \cdot U) \cap (y^{-1} \cdot B) \notin \mathcal{I}_0$. Since $x^{-1} \cdot T \in \mathcal{I}_0$ and

$$D := (x^{-1} \cdot (U \setminus T)) \cap (y^{-1} \cdot B) = (x^{-1} \cdot U) \cap (y^{-1} \cdot B) \setminus x^{-1} \cdot T,$$

we have $D \neq \emptyset$. Take $t \in D$. Clearly $xt \in U \setminus T$ and $yt \in B$, whence $x \cdot y^{-1} = x \cdot t \cdot (y \cdot t)^{-1} \in (U \setminus T) \cdot B^{-1}$ and $y \cdot x^{-1} = y \cdot t \cdot (x \cdot t)^{-1} \in B \cdot (U \setminus T)^{-1}$. Thus we have shown (3). □

Proposition 2.2. *Let $(G, \cdot)$ be a semitopological group such that the mapping $G \in y \to y^{-1} \in G$ is continuous and $\mathcal{I}_0 \subset 2^G$ be a left-translation invariant local σ-ideal. Then the family*

$$\mathcal{M} := \{(U \setminus C) \cup D : U \subset G \text{ is open}, C, D \in \mathcal{I}_0\}$$

satisfies hypothesis (M) *with $\mathcal{S} = \mathcal{I} := \mathcal{I}_0$.*

Proof. Let $U \subset G$ be open and nonempty, $T \in \mathcal{I}_0$ and $B \in 2^G \setminus \mathcal{I}_0$. Then, according to Lemma 2.1, condition (3) holds with some $y \in G$, which means $\operatorname{int}((U \setminus T) \cdot B^{-1}) \neq \emptyset$ and $\operatorname{int}(B \cdot (U \setminus T)^{-1}) \neq \emptyset$. □

3. The main results

We need the following three definitions.

Definition 3.1. *Let (X, d) be a metric space. We say $\mathcal{R} \subset X^2$ has intersection property provided $B_1 \cap B_2 \neq \emptyset$ for every two balls $B_1, B_2 \subset X$ such that $\mathcal{R} \cap (B_1 \times B_2) \neq \emptyset$ and $\mathcal{R} \cap (B_2 \times B_1) \neq \emptyset$.*

Remark 3.1. Clearly $\mathcal{R} = \{(x, x) : x \in X\}$ has intersection property for every metric space (X, d). This is also the case if $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : x \leq y\}$ (with the usual metric in $X = \mathbb{R}$).

Definition 3.2. *Let Y be a nonempty set, $Z \subset Y$ and $\mathcal{I} \subset 2^Y$. We say a property $p(x) (x \in Z)$ holds $\mathcal{I}$-almost everywhere in a set $E \subset Z$ (abbreviated in the sequel to $\mathcal{I}$-a.e. in E) provided there exists a set $A \in \mathcal{I}$ such that $p(x)$ holds for every $x \in E \setminus A$.*

Definition 3.3. *Let G be a nonempty set, $\mathcal{M} \subset 2^G$, and X be a topological space. We say $f : G \to X$ is $\mathcal{M}$-measurable on a set $D \subset G$ provided $f^{-1}(U) \cap D \in \mathcal{M}$ for every open set $U \subset X$.*

Remark 3.2. Let X be a topological space, $(G, \cdot)$ be a semitopological group with the mapping $G \ni x \to x^{-1} \in G$ continuous, $f : G \to X$ be continuous at a point $x_0 \in G$, and $\mathcal{M} := \{U \subset G : x_0 \in \operatorname{int} U \text{ or } x_0 \notin U\}$. Then f is $\mathcal{M}$-measurable on G and hypothesis (M) holds with $\mathcal{S} = \{\emptyset\}$ and $\mathcal{I} = \{U \subset G : x_0 \notin U\}$.

Now we are in a position to prove the following theorem.

Theorem 3.1. *Let (X, d) be a separable metric space, $(G, \cdot)$ be a semitopological group, $P \subset G$ be dense in G, $E \subset G$,* (M) *hold, and $\preceq \subset X^2$ have intersection property. Suppose $w : E \to X$ is $\mathcal{M}$-measurable on a set $D \in 2^E \setminus \mathcal{I}$ and satisfies*

$$w(p \cdot x) \preceq w(x) \quad \text{for } x \in E, p \in P \quad \text{with } p \cdot x \in E. \tag{4}$$

Then w is constant $\mathcal{S}$-a.e. in E.

Proof. Let $Q = \{q_i : i \in \mathbb{N}\}$ be a dense subset of X and $B_i^n := \{x \in X : d(q_i, x) < 1/n\}$ for $i, n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we have

$$D \subset w^{-1}(X) = \bigcup_{i \in \mathbb{N}} w^{-1}(B_i^n).$$

Thus, for each $n \in \mathbb{N}$, there exists $i(n) \in \mathbb{N}$ with

$$D_n := w^{-1}\left(B_{i(n)}^n\right) \cap D \in \mathcal{M} \setminus \mathcal{I},$$

because w is $\mathcal{M}$-measurable on D and $D \notin \mathcal{I}$.

Since X is separable, for every $n \in \mathbb{N}$ there exists a countable family $\mathcal{B}(n)$ of balls in X such that

$$X \setminus \mathrm{cl}\, B_{i(n)}^n = \bigcup_{B \in \mathcal{B}(n)} B.$$

Suppose there exist $k \in \mathbb{N}$ and $B_0 \in \mathcal{B}(k)$ with $B_k := w^{-1}(B_0) \notin \mathcal{S}$. Then, on account of (M), there are $p_1, p_2 \in P$ such that $p_1 \in \mathrm{int}\,(B_k \cdot D_k^{-1})$ and $p_2 \in \mathrm{int}\,(D_k \cdot B_k^{-1})$, which means that $p_1 \cdot d_1 = b_1 \in B_k \subset E$ and $p_2 \cdot b_2 = d_2 \in D_k \subset D$ with some $b_1, b_2 \in B_k$ and $d_1, d_2 \in D_k$. Hence

$$B_0 \ni w(b_1) = w(p_1 \cdot d_1) \preceq w(d_1) \in B_{i(k)}^k, \tag{5}$$

and

$$B_{i(k)}^k \ni w(d_2) = w(p_2 \cdot b_2) \preceq w(b_2) \in B_0,$$

whence $\preceq \cap (B_0 \times B_{i(k)}^k) \neq \emptyset$ and $\preceq \cap (B_{i(k)}^k \times B_0) \neq \emptyset$. This is a contradiction, because $\preceq$ has intersection property and $B_0 \cap B_{i(k)}^k = \emptyset$.

Thus we have proved that $w^{-1}(B) \in \mathcal{S}$ for every $k \in \mathbb{N}$, $B \in \mathcal{B}(k)$. Let

$$L := \bigcap_{k \in \mathbb{N}} \mathrm{cl}\, B_{i(k)}^k.$$

Clearly L has at most one element,

$$A := w^{-1}(X \setminus L) = w^{-1}\left(\bigcup_{k \in \mathbb{N}} \left(X \setminus \mathrm{cl}\, B_{i(k)}^k\right)\right)$$
$$= \bigcup_{k \in \mathbb{N}} w^{-1}\left(X \setminus \mathrm{cl}\, B_{i(k)}^k\right) = \bigcup_{k \in \mathbb{N}} \left(\bigcup_{B \in \mathcal{B}(k)} w^{-1}(B)\right) \in \mathcal{S},$$

and $w(x) \in L$ for $x \in E \setminus A$. □

Remark 3.3. Note that if $\preceq = \{(x, x) : x \in X\}$, then condition (5) is not necessary to obtain a contradiction in the proof of Theorem 3.1. Hence, in that case, (4) may be replaced by the following weaker condition

$$w(p \cdot x) = w(x) \quad \text{for } x \in E, p \in P \quad \text{with } p \cdot x \in D, \tag{6}$$

and hypothesis (M) by the weaker one

(M′) $\mathcal{M} \subset 2^G$ and there exist σ-ideals $\mathcal{I}, \mathcal{S} \subset 2^G$ such that

$$\text{int}\,(D \cdot C^{-1}) \neq \emptyset \quad \text{for } D \in \mathcal{M} \setminus \mathcal{I}, C \in 2^G \setminus \mathcal{S}.$$

Thus we have the subsequent two generalizations of some well known results concerning microperiodic functions (cf. [3, 4, 6, 10, 21, 22]).

Corollary 3.1. *Let X, G, E, P be as in Theorem 3.1 and* (M′) *hold. Suppose $w : E \to X$ is $\mathcal{M}$-measurable on a set $D \in 2^E \setminus \mathcal{I}$ and satisfies* (6)*. Then w is constant $\mathcal{S}$-a.e. in E.*

Proof. It is an immediate consequence of Theorem 3.1 and Remark 3.3. □

Corollary 3.2. *Let X, G, E, P be as in Theorem 3.1. Suppose $w : E \to X$ is continuous at a point $x_0 \in D :=$ int E and* (6) *holds. Then w is constant.*

Proof. Let $\mathcal{M} := \{U \subset G : x_0 \in \text{int}\, U \text{ or } x_0 \notin U\}$. Then w is $\mathcal{M}$-measurable on D and hypothesis (M') holds with $\mathcal{S} = \{\emptyset\}$ and $\mathcal{I} = \{U \subset G : x_0 \notin U\}$. Hence Corollary 3.1 yields the statement. □

For the next corollary we need the following definition.

Definition 3.4. *Let Y be a real linear space and $v \in Y$. Then we write*

$$\ll_v := \{(x, y) \in Y^2 : y - x = av \quad \text{with some } a \in \mathbb{R}, a \geq 0\}.$$

Corollary 3.3. *Let G, P, E be as in Theorem 3.1, X be a real linear separable normed space, $v \in X$, $h : G \to X$, and $w : E \to X$. Suppose $h(x \cdot y) = h(x) + h(y)$ for $x, y \in G$ and*

$$w(p \cdot x) \ll_v w(x) + h(p) \quad \text{for } x \in E, p \in P \quad \text{with } p \cdot x \in E. \tag{7}$$

Then the following two conditions hold.

1. *If* (M) *holds and the function $w - h : E \to X$, $(w - h)(x) := w(x) - h(x)$ for $x \in E$, is $\mathcal{M}$-measurable on a set $D \in 2^E \setminus \mathcal{I}$, then there exists $z_0 \in X$ such that $w(x) = z_0 + h(x)$ $\mathcal{S}$-a.e. in E.*
2. *If the mapping $G \ni y \to y^{-1} \in G$ is continuous and the function $w - h$ is continuous at a point $x_0 \in$ int E, then there exists $z_0 \in X$ such that $w(x) = z_0 + h(x)$ for $x \in E$.*

Proof. Take balls $B_1, B_2 \subset X$ such that $(B_1 \times B_2) \cap \ll_v \neq \emptyset$ and $(B_2 \times B_1) \cap \ll_v \neq \emptyset$. Then there exist $x_i, y_i \in B_i$ for $i = 1, 2$ such that $x_1 \ll_v x_2$ and $y_2 \ll_v y_1$. This means that $a_1 v, a_2 v \in B_1 - B_2$ for some $a_1, a_2 \in \mathbb{R}$, $a_1 a_2 \leq 0$. Since the balls are convex sets, so is the set $B_1 - B_2$ and consequently $0 \in B_1 - B_2$, which means $B_1 \cap B_2 \neq \emptyset$.

Thus we have shown $\ll_v$ has the intersection property. Next, by (7),

$$(w - h)(p \cdot x) \ll_v w(x) + h(p) - h(p) - h(x) = (w - h)(x)$$

for every $x \in E$, $p \in P$ with $p \cdot x \in E$. Hence Theorem 3.1 yields statement (1). Next, arguing as in the proof of Corollary 3.2, from (1) we derive statement (2). □

Remark 3.4. Note that, in the case $X = \mathbb{R}$ and $v = 1$, we have $\ll_v = \{(x, y) \in \mathbb{R}^2 : x \leq y\}$. Thus Corollary 3.3 generalizes the results in Brzdęk [3].

In the case where (4) is postulated only almost everywhere we have for instance the subsequent two corollaries.

Corollary 3.4. *Let $G, P, X, \preceq$ be as in Theorem* 3.1, (M) *be valid, $\mathcal{J} \in \{\mathcal{S}, \mathcal{I}\}$,*

$$B \setminus T \in \mathcal{M} \quad \text{for } B \in \mathcal{M}, T \in \mathcal{J}, \tag{8}$$

$E_0 \subset G$, P be countable, $w : E_0 \to X$ be $\mathcal{M}$-measurable on a set $D_0 \subset E_0$ and, for each $p \in P$, the condition

$$\text{if } p \cdot x \in E_0, \quad \text{then } w(p \cdot x) \preceq w(x) \tag{9}$$

hold $\mathcal{J}$-a.e. in E_0. Then the following two conditions are satisfied.

1. *If $\mathcal{J} = \mathcal{S}$ and $D_0 \notin \mathcal{J}_0 := \{T \cup V : T \in \mathcal{I}, V \in \mathcal{S}\}$, then w is constant $\mathcal{S}$-a.e. in E_0.*
2. *If $\mathcal{J} = \mathcal{I}$ and $D_0 \notin \mathcal{I}$, then w is constant $\mathcal{J}_0$-a.e. in E_0.*

Proof. For every $p \in P$ there exists a set $A_p \in \mathcal{J}$ such that (9) holds for every $x \in E_0 \setminus A_p$. Write

$$A := \bigcup_{p \in P} A_p.$$

Then $A \in \mathcal{J}$ and (4) is valid for $E := E_0 \setminus A$. Moreover, according to (8), w is $\mathcal{M}$-measurable on $D := D_0 \setminus A \notin \mathcal{I}$. Hence, by Theorem 3.1, w is constant $\mathcal{S}$-a.e. in E, which yields the statement. □

Corollary 3.5. *Let $G, P, X, \preceq$ be as in Theorem 3.1, $E_0 \subset G$, and $\mathcal{J} \subset 2^G$ be a left-translation invariant σ-ideal. Suppose the topology in G is metrizable, the mapping $G \ni y \to y^{-1} \in G$ is continuous, $P \subset G$ is countable, $w : E_0 \to X$ is continuous at every point $x \in$* int $E_0 \neq \emptyset$ *and, for each $p \in P$, condition (9) holds $\mathcal{J}$-a.e. in E_0. Then w is constant $\mathcal{J}$-a.e. in E_0.*

Proof. The case $\mathcal{J} = 2^G$ is trivial, so suppose $\mathcal{J}$ is proper. This means int $T \neq \emptyset$ for every $T \in \mathcal{J}$ (because P is dense and countable). Next, from Proposition 2.1 we derive $\mathcal{J}$ is local. Define A as in the proof of Corollary 3.4. Then $A \in \mathcal{J}$ and (4) is valid for $E := E_0 \setminus A$. Clearly $(\text{int } E_0) \setminus A \notin \mathcal{J}$. Take $x_0 \in \text{int } E_0 \setminus A$. It is easily seen w is $\mathcal{M}$-measurable on E with

$$\mathcal{M} := \{U \setminus A : U \subset G \quad \text{and either } x_0 \in \text{int } U \quad \text{or } x_0 \notin U\}$$

and, on account of Lemma 2.1, hypothesis (M) holds with $\mathcal{S} = \mathcal{J}$ and $\mathcal{I} := \{U \subset G : x_0 \notin U\}$. Consequently Theorem 3.1 implies the statement. □

It seems that, with the same method as in the proof of Corollary 3.5, we cannot derive from Theorem 3.1 an analogous corollary under the weaker assumption of continuity of w at least at one point, because it may happen that the only point of continuity of w, say x_0, belongs to A and then $E \in \mathcal{I}$, which means we cannot apply Theorem 3.1. However we can show such a result using a more direct method of proof. Namely we have the following theorem.

Theorem 3.2. *Let (X, d) be a metric space, $\preceq \subset X^2$ have the intersection property, $(G, \cdot)$ be a semitopological group, $E_0 \subset G$, $\mathcal{J} \subset 2^G$ be a left-translation invariant local σ-ideal, and $H \in \mathcal{J}$. Suppose $P \subset G$ is dense and countable, $w : E_0 \to X$ is continuous at a point $x_0 \in$ int E_0 and, for each $p \in P$, the condition*

$$\text{if } p \cdot x \in E_0 \setminus H, \quad \text{then } w(p \cdot x) \preceq w(x) \tag{10}$$

holds $\mathcal{J}$-a.e. in E_0. Moreover assume one of the following two conditions:

1. $\preceq = \{(x, x) : x \in X\}$*;*
2. *X is separable and the mapping $G \ni y \to y^{-1} \in G$ is continuous.*

Then $w(x) = w(x_0)$ $\mathcal{J}$-a.e. in E_0.

Proof. Assume $\mathcal{J}$ is proper (otherwise the statement trivially holds). Then int $T = \emptyset$ for $T \in \mathcal{J}$. For each $p \in P$ there is $A_p \in \mathcal{J}$ such that (10) holds for every $x \in E_0 \setminus A_p$. Let

$$A := H \cup \bigcup_{p \in P} A_p \in \mathcal{J}.$$

Clearly $E := E_0 \setminus A \notin \mathcal{J}$ and (4) is valid. For each $n \in \mathbb{N}$ write $B_n := \{x \in X : d(w(x_0), x) < 1/n\}$, $X_n := X \setminus B_n$, $D_n := w^{-1}(B_n) \setminus A$ and $E_n := w^{-1}(X_n) \setminus A$. Note

$$D_n \in \mathcal{M} := \{U \setminus A : U \subset G \quad \text{and } x_0 \in \text{int } U\} \quad \text{for } n \in \mathbb{N}.$$

In the remaining part we repeat in many fragments some arguments from the proof of Theorem 3.1. However, for convenience of a reader we present them here as well.

First consider the case of (1). Suppose $E_k \notin \mathcal{J}$ for some $k \in \mathbb{N}$. Then, by Lemma 2.1, int $(D_k \cdot E_k^{-1}) \neq \emptyset$, whence $p \cdot e = d \in D_k \subset E$ with some $p \in P$,

$e \in E_k$ and $d \in D_k$. Hence $B_k \ni w(d) = w(p \cdot e) = w(e) \in X_k$. This is a contradiction, because $B_k \cap X_k = \emptyset$.

Thus we have proved that $w^{-1}(X_n) \in \mathcal{J}$ for every $n \in \mathbb{N}$. Let $L := \bigcap_{n\in\mathbb{N}} B_n$. Clearly $L = \{w(x_0)\}$, $T := w^{-1}(X \setminus L) = \bigcup_{n\in\mathbb{N}} w^{-1}(X_n) \in \mathcal{J}$ and $w(x) \in L$ for $x \in E_0 \setminus T$. This completes the proof in the case where (1) holds.

Now assume (2). For each $n \in \mathbb{N}$ there is a countable family $\mathcal{B}(n) \subset 2^X$ of balls with $X \setminus \mathrm{cl}\, B_n = \bigcup_{B\in\mathcal{B}(n)} B$. Suppose there are $k \in \mathbb{N}$ and $B_0 \in \mathcal{B}(k)$ with $C_k := w^{-1}(B_0) \setminus A \notin \mathcal{J}$. Then, on account of Lemma 2.1, there are $p_1, p_2 \in P$ such that $p_1 \in \mathrm{int}\,(C_k \cdot D_k^{-1})$ and $p_2 \in \mathrm{int}\,(D_k \cdot C_k^{-1})$, whence $p_1 \cdot d_1 = c_1 \in C_k \subset E$ and $p_2 \cdot c_2 = d_2 \in D_k \subset E$ with some $c_1, c_2 \in C_k$ and $d_1, d_2 \in D_k$. Hence $B_0 \ni w(c_1) = w(p_1 \cdot d_1) \preceq w(d_1) \in B_k$ and $B_k \ni w(d_2) = w(p_2 \cdot c_2) \preceq w(c_2) \in B_0$, which means $\preceq \cap (B_0 \times B_k) \neq \emptyset$ and $\preceq \cap (B_k \times B_0) \neq \emptyset$. This is a contradiction, because $\preceq$ has intersection property and $B_0 \cap B_k = \emptyset$.

In this way we have shown $w^{-1}(B) \in \mathcal{J}$ for every $k \in \mathbb{N}$, $B \in \mathcal{B}(k)$. Let $L := \bigcap_{k\in\mathbb{N}} \mathrm{cl}\, B_k$ and $V := w^{-1}(X \setminus L)$. Clearly $L = \{w(x_0)\}$, $V = w^{-1}\left(\bigcup_{k\in\mathbb{N}} (X \setminus \mathrm{cl}\, B_k)\right) = \bigcup_{k\in\mathbb{N}} \left(\bigcup_{B\in\mathcal{B}(k)} w^{-1}(B)\right) \in \mathcal{J}$, and $w(x) \in L$ for $x \in E_0 \setminus V$. $\square$

Remark 3.5. In the case $\mathcal{J} = \{\emptyset\}$, the assumption of countability of P in Theorem 3.2 is superfluous; then in the proof we simply take $A = H$.

The next corollary is a generalization of the result of Montel [13] (cf. [7], pp. 227–229, and [3]).

Corollary 3.6. *Let $a_1, a_2 \in \mathbb{R}$, $a_1 < 0 < a_2$, $a_1 a_2^{-1} \notin \mathbb{Q}$, and I be a real infinite interval. Then the following two conditions are valid.*

1. *Suppose* (M) *holds with $(G, \cdot) = (\mathbb{R}, +)$, $T \in \mathcal{I}$,*

$$T + na_1 + ma_2 \in \mathcal{I} \quad \text{for } n, m \in \mathbb{N}_0, \tag{11}$$

$$A \Bigg\backslash \left(\bigcup_{m,n\in\mathbb{N}_0} (T + na_1 + ma_2) \right) \in \mathcal{M} \quad \text{for } A \in \mathcal{M}, \tag{12}$$

$E := I \setminus T$, $w : E \to \mathbb{R}$ is $\mathcal{M}$-measurable on a set $D \in 2^E \setminus \mathcal{I}$ and satisfies the subsequent two conditional inequalities

$$\text{if } a_1 + x \in E, \quad \text{then } w(a_1 + x) \leq w(x), \tag{13}$$

$$\text{if } a_2 + x \in E, \quad \text{then } w(a_2 + x) \leq w(x). \tag{14}$$

Then w is constant $\mathcal{S}$-a.e. in E.

2. *Suppose $\mathcal{J} \subset 2^{\mathbb{R}}$ is a proper σ-ideal with*

$$y + A \in \mathcal{J} \quad \text{for } A \in \mathcal{J}, y \in \mathbb{R},$$

$V \in \mathcal{J}$, $E := I \setminus V$, $w : I \to \mathbb{R}$ is continuous at a point $x_0 \in \mathrm{int}\, I$ and satisfies conditions (13) and (14) $\mathcal{J}$-a.e. in I. Then $w(x) = w(x_0)$ $\mathcal{J}$-a.e. in I.

Proof. 1. Let

$$H := \bigcup_{m,n\in\mathbb{N}_0} (T + na_1 + ma_2) \in \mathcal{I}, \tag{15}$$

and $F := I \setminus H$. Then w is $\mathcal{M}$-measurable on $D \setminus H \notin \mathcal{I}$ and, for every $x \in F$,

$$w(x + na_1 + ma_2) \le w(x) \quad \text{for } m, n \in \mathbb{N} \text{ with } x + na_1 + ma_2 \in F. \tag{16}$$

Since the set $P := \{na_1 + ma_2 : m, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$, Corollary 3.3 (1) (with $h \equiv 0$) completes the proof.

2. There is a set $T \in \mathcal{J}$ such that $V \subset T$ and conditions (13), (14) hold for every $x \in I \setminus T$. Analogously as in the proof of (1) (with $\mathcal{I} = \mathcal{J}$) we obtain (16) for every $x \in F := I \setminus H \subset E$, where H is given by (15). Consequently Theorem 3.2 (with $E_0 = I$) and Proposition 2.1 complete the proof.

□

Our last corollary corresponds to some recent results of Krassowska and Matkowski (cf. [8] and [9]).

Corollary 3.7. *Let $a_1, a_2, \alpha_1, \alpha_2 \in \mathbb{R}$, $a_1 < 0 < a_2$, $a_1 a_2^{-1} \notin \mathbb{Q}$,*

$$c_1 := \frac{\alpha_1}{a_1} \ge \frac{\alpha_2}{a_2} =: c_2$$

and I be a real infinite interval. Then the subsequent two conditions are valid.

1. *Suppose* (M) *holds with* $(G, \cdot) = (\mathbb{R}, +)$, $T \in \mathcal{I}$, $E := I \setminus T$, *conditions* (11), (12) *are valid, and*

$$\text{card}\,(E \setminus A) > 1 \quad \text{for } A \in \mathcal{S}. \tag{17}$$

If $w : E \to \mathbb{R}$ satisfies the following two conditional inequalities

$$\textit{if } a_1 + x \in E, \quad \text{then } w(a_1 + x) \le w(x) + \alpha_1, \tag{18}$$

$$\textit{if } a_2 + x \in E, \quad \text{then } w(a_2 + x) \le w(x) + \alpha_2, \tag{19}$$

and the functions $g_1, g_2 : E \to \mathbb{R}$, $g_i(x) = w(x) - c_i x$ for i=1,2, are $\mathcal{M}$-measurable on a set $D \in 2^E \setminus \mathcal{I}$, then $\alpha_1 a_2 = \alpha_2 a_1$ and there is $d \in \mathbb{R}$ with

$$w(x) = \frac{\alpha_1}{a_1} x + d \quad \mathcal{S}\text{-a.e. in } E. \tag{20}$$

Moreover, if $\alpha_1 a_2 = \alpha_2 a_1$, $w : E \to \mathbb{R}$ and (20) *holds with some $d \in \mathbb{R}$, then w satisfies* (18) *and* (19) *$\mathcal{S}$-a.e. in E.*

2. *Let $\mathcal{J}$ and E be as in Corollary 3.6 (2). Then a function $w : I \to \mathbb{R}$, continuous at a point $x_0 \in \text{int } I$, satisfies conditions* (18) *and* (19) *$\mathcal{J}$-a.e. in I if and only if $\alpha_1 a_2 = \alpha_2 a_1$ and*

$$w(x) = \frac{\alpha_1}{a_1}(x - x_0) + w(x_0) \quad \mathcal{J}\text{-a.e. in } I. \tag{21}$$

Proof. 1. For every $i, j \in \{1, 2\}$, we have $\alpha_j \le c_i a_j$ and consequently

$$g_i(x+a_j) = w(x+a_j) - c_i(x+a_j) \le w(x) + \alpha_j - c_i x - c_i a_j \le g_i(x)$$

for $x \in E$ with $x+a_j \in E$. Next g_1, g_2 are $\mathcal{M}$-measurable on D. Hence, according to Corollary 3.6(1), there exist $d_i \in \mathbb{R}$ and $A_i \in \mathcal{S}$ such that $g_i(x) = d_i$ for $x \in E \setminus A_i$, $i = 1, 2$. Since $c_1 x + d_1 = w(x) = c_2 x + d_2$ for $x \in E \setminus (A_1 \cup A_2)$, by (17) we have $c_1 = c_2$ and $d_1 = d_2 =: d$.

The converse statement is easy to check. This ends the proof of (1).

2. Assume $w : I \to \mathbb{R}$ is continuous at a point $x_0 \in \operatorname{int} I$ and satisfies (18) and (19) $\mathcal{J}$-a.e. in I. Arguing analogously as in the proof of (1) and using Corollary 3.6 (2) we obtain $c_1 x - c_1 x_0 + w(x_0) = w(x) = c_2 x - c_2 x_0 + w(x_0)$ for $x \in I \setminus (A_1 \cup A_2)$, with some $A_1, A_2 \in \mathcal{J}$. Since $\mathcal{J}$ is proper and $x + A \in \mathcal{J}$ for $A \in \mathcal{J}, x \in \mathbb{R}$, we have int $A = \emptyset$ for $A \in \mathcal{J}$ and consequently card $(I \setminus (A_1 \cup A_2)) \ge 2$, which yields $\alpha_1 a_2 = \alpha_2 a_1$ and (21).

   Since it is easy to check that, in the case $\alpha_1 a_2 = \alpha_2 a_1$, the function $w : E \to \mathbb{R}$, given by: $w(x) = \alpha_1/a_1 x + d$ for $x \in I$ satisfies (18) and (19) with every $E \subset I$ and $d \in \mathbb{R}$, this completes the proof.

□

Remark 3.6. Let $\mathcal{I} \subset 2^{\mathbb{R}}$ be the σ-ideal of sets of first category and $\mathcal{S} \subset 2^{\mathbb{R}}$ be the σ-ideal of sets of the Lebesgue measure zero. Then for every set $E \subset \mathbb{R}$ there are $A \in \mathcal{S}$, $B \in \mathcal{I}$ with $E = A \cup B$ (see e.g. [16], Corollary 1.7). Therefore it seems that without condition (17) in Corollary 3.7 (1) we cannot obtain the equality $\alpha_1/a_1 = \alpha_2/a_2$.

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