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A Sobolev extension domain that is not uniform

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Abstract. In this paper we construct a Sobolev extension domain which, together with its complement, is topologically as nice as possible and yet not uniform. This shows that the well known implication that Uniform ⇒ Sobolev extension cannot be reversed under strongest possible topological conditions.

1. Introduction

In this paper we investigate whether, under certain topological conditions, the following implications of analytic/geometric properties of domains can be reversed.

$$\text{Uniform} \Rightarrow \text{Sobolev extension} \Rightarrow \text{QED} \Rightarrow \text{Loewner} \Rightarrow \text{LLC}. \tag{1}$$

1.1. Uniform domains

Recall that a domain D in the Euclidean space $\mathbb{R}^n$ ($n \geq 2$) is said to be *uniform* if there exists a constant $c = c(D)$, $1 \leq c < \infty$, such that each pair of points $x_1, x_2 \in D$ can be joined by a continuum (or, equivalently, a curve) β in D for which

$$\text{dia}(\beta) \leq c|x_1 - x_2| \quad \text{and} \quad \min_{j=1,2} |x_j - x| \leq c\, d(x, \partial D) \tag{2}$$

for each $x \in \beta$. Here dia(β) denotes the diameter of β and $d(x, \partial D)$ the distance from x to the boundary ∂D.

1.2. Sobolev extension domains

As usual, let $L^p(D)$ ($p \geq 1$) denote the Banach space of L^p-integrable functions on a domain D. We consider the Sobolev classes $W^1_p(D)$ and $L^1_p(D)$, where $W^1_p(D) = L^1_p(D) \cap L^p(D)$ and $L^1_p(D)$ is the family of measurable functions

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whose first distributional derivatives belong to $L^p(D)$. The norms on $W_p^1(D)$ and $L_p^1(D)$ are given by

$$\|u\|_{W_p^1(D)} = \|\nabla u\|_{L^p(D)} + \|u\|_{L^p(D)} \quad \text{and} \quad \|u\|_{L_p^1(D)} = \|\nabla u\|_{L^p(D)},$$

respectively. We call $D \subset \mathbb{R}^n$ an $L_p^1(D)$-*extension domain* if there is a bounded linear extension operator from $L_p^1(D)$ to $L_p^1(\mathbb{R}^n)$. Similarly one can define $W_p^1(D)$-extension domains. The $W_p^1(D)$-extension property and/or $L_p^1(D)$-extension property is usually referred as Sobolev extension property.

Uniform domains and Sobolev extension domains play important roles in analysis and geometry. These two classes of domains are closely related to each other and to several other interesting classes of domains. Jones [7] showed that uniform domains are W_p^1-extension domains for all p and L_n^1-extension domains. In this line, Herron and Koskela [5] further showed that uniform domains and bounded W_p^1-extension domains are all L_p^1-extension domains for all p. Koskela [8] established that L_n^1-extension domains are QED (quasi-extremal distance) domains while Gehring and Martio [2] proved that QED domains are LLC (linearly locally connected). For definitions and more details on QED and LLC domains we refer the reader to [2, 8] and references therein. Sobolev extension domains have also been studied by Maz'ja [10], Gol'dshtein and Vodop'yanov [3, Chapter 6].

The Loewner condition, which is similar to (but appears weaker than) the QED condition in the Euclidean space, was introduced by Heinonen and Koskela [4] in a general metric measure space setting. They showed that Loewner domains are LLC. Recently Brania and Yang [1] established that QED domains are Loewner domains and that Loewner domains enjoy many function-theoretic and geometric properties of QED domains. But the question of whether the implication QED⇒Loewner can be reversed remains open. However, all other implications in (1) cannot be reversed in general, as illustrated by numerous examples (see [2, 5–7]). Therefore, it is important to seek for conditions under which the implications in (1) can be reversed. In this regard, the following can be easily established.

1.3. Fact

Suppose D is quasiconformally equivalent to a uniform domain. Then D is uniform if and only if D is LLC.

This can be proved as follows. Let f be a quasiconformal map from a uniform domain G onto an LLC domain D. By [2, Theorem 3.1], f is quasimöbius. Therefore, it follows from [12, Theorem 4.11] that D is also uniform.

It is usually difficult to verify the condition that D is quasiconformally equivalent to a uniform domain. Thus a purely topological and intuitive condition is more desirable. In fact such a condition does exist in the plane. It is well known that a Jordan domain in the plane is LLC if and only if it is uniform (or, equivalently, a quasidisk). Thus all the conditions in (1) are equivalent for Jordan domains in the plane. In higher dimensions, however, the situation is much more complicated (as illustrated by examples mentioned above). As a matter of fact, in this paper we show that such a topological condition does not exist in higher dimensions.

1.4. Main result

Theorem 1. *There is a homeomorphism of* $\mathbb{R}^n$ $(n \geq 3)$ *such that the image of the upper half space is an* L^1_p*-extension (and a* W^1_p*-extension) domain for all* $p \geq 1$*, but not uniform.*

Corollary 1. *There is a Jordan domain in* $\mathbb{R}^n$ $(n \geq 3)$ *which is topologically equivalent to the unit ball and LLC, but not uniform.*

In Sect. 2 we give the construction of the domain and show that it is not uniform. Its Sobolev extension property will be verified in Sect. 3 as a consequence of a more general extension result. We conclude with an open discussion on what possible geometric condition can be added in order to reverse the implications of (1) in this strongest possible topological setting.

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2. Construction of the domain

For simplicity of notation, the construction will be done in $\mathbb{R}^3$. Let $\mathbf{H}$ denote the open upper half space in $\mathbb{R}^3$ and $\mathbf{B}(x, r)$ the open ball centered at x of radius $r > 0$.

2.1. The construction of domain D

For $j = 1, 2, 3, \ldots$, let F_j be the closed circular cylinder of height $h_j = \frac{1}{\sqrt{j}}$ and radius $r_j = \frac{1}{10j^3}$ with its base centered at the point $(\frac{1}{j}, 0, 0)$, that is,

$$F_j = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \left(x_1 - \frac{1}{j}\right)^2 + x_2^2 \leq r_j^2,\ 0 \leq x_3 \leq h_j\}.$$

Let $D = \mathbf{H} \setminus \bigcup_{j \geq 1} F_j$. Thus D is the complement in $\mathbf{H}$ of the union of the cylinders F_j. Note that F_j are mutually disjoint and very thin cylinders (for large j).

2.2. D is not uniform

To show that the domain D constructed above is not uniform, we consider a sequence of pairs of points y_j, z_j in D determined by

$$y_j = \left(\frac{1}{2}\left(\frac{1}{j} + \frac{1}{j+1}\right), \frac{1}{j^{3/2}}, \frac{1}{j^{3/2}}\right), \quad z_j = \left(\frac{1}{2}\left(\frac{1}{j} + \frac{1}{j+1}\right), -\frac{1}{j^{3/2}}, \frac{1}{j^{3/2}}\right)$$

for $j = 1, 2, 3, \ldots$. We shall show that the uniformity conditions are violated for the pair y_j, z_j when j is large. The idea is that in order to join y_j and z_j in D, one has to either go through the narrow passages formed by the cylinders (which will be in violation of the second condition in (2)), or go around them (in violation of the first condition in (2)).

Observe that y_j and z_j are symmetric about the x_1x_3-plane and that $|y_j - z_j| = \frac{2}{j^{3/2}}$. For a fixed j, let γ be a continuum in D joining y_j and z_j. Fix a point P in the intersection of γ and the x_1x_3-plane and let $Q = (\frac{1}{2}(\frac{1}{j} + \frac{1}{j+1}), 0, 0)$.

Assume first that $P \notin \mathbf{B}(Q, \frac{1}{2j})$. Then

$$\mathrm{dia}(\gamma) \geq |P - Q| - |Q - y_j| \geq \frac{1}{2j} - \frac{2}{j^{3/2}}.$$

This yields that $\mathrm{dia}(\gamma)/|y_j - z_j| \to \infty$ as $j \to \infty$, which violates the first condition in (2).

Assume next that $P \in \mathbf{B}(Q, \frac{1}{2j})$. Then P lies in one of the "narrow passages" between cylinders that intersect $\mathbf{B}(Q, \frac{1}{2j})$. To be more precise, let $P = (p_1, p_2, p_3)$. Then it follows that $\frac{1}{2(j+1)} < p_1 < \frac{2}{j}$, $p_2 = 0$, $0 < p_3 < \frac{1}{2j}$. Thus, when j is large enough, the third coordinate p_3 of P is smaller than the heights of the cylinders F_i for $2(j+1) < i < j/2$. Therefore, it follows that

$$d(P, \partial D) \leq \frac{1}{2} \max_{j/2 < i < 2(j+1)} d(F_{i+1}, F_i) \leq \frac{1}{2}\left(\frac{1}{j/2} - \frac{1}{1 + j/2}\right) = \frac{2}{j(j+2)},$$

which, together with the fact that $|P - y_j| = |P - z_j| \geq \frac{1}{j^{3/2}}$, yields

$$\frac{\min\{|P - y_j|, |P - z_j|\}}{d(P, \partial D)} \geq \frac{j(j+2)}{2j^{3/2}} \to \infty$$

as $j \to \infty$. This is in violation of the second condition in (2) and shows that D is not a uniform domain.

2.3. The topology of D

By the construction of D, it is routine, but tedious, to show that there is a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(\mathbf{H}) = D$. Thus the domain D is topologically as nice as possible.

3. Sobolev extension property

In this section we establish a general extension result for some domains that are not necessarily uniform (or locally uniform). As a corollary, we verify that the domain D constructed in Sect. 2 has the desired Sobolev extension properties, hence completing the proof of Theorem 1. We note that, however, the domain D is not even locally uniform (in the sense of [5, 3.1]) or uniformly collared (in the sense of [5, Sect. 7]). Therefore, the Sobolev extension property of D is not a consequence of existing extension results in this line and requires a proof.

3.1. Whitney decomposition

We employ similar extension techniques used by Jones [7] and Koskela [9], which depend heavily on Whitney cube decompositions of open sets. Recall that any open set $G \subset \mathbb{R}^n$ admits a Whitney cube decomposition $G = \cup_k Q_k$, where Q_k are closed cubes with pairwise disjoint interiors and sides parallel to the coordinate axes, and satisfy

$$1 \leq \frac{d(Q_k, \partial G)}{l(Q_k)} \leq 4\sqrt{n}, \quad \text{for all } k,$$

and

$$\frac{1}{4} \leq \frac{l(Q_j)}{l(Q_k)} \leq 4 \quad \text{if } Q_j \cap Q_k \neq \emptyset.$$

Here $l(Q_k)$ denotes the side length of a cube.

Lemma 3.2 ([7, Lemma 2.6]). *Let $G \subset \mathbb{R}^n$ be a c-uniform domain. Then there exists a constant $C_1 \geq 1$ depending only on the uniformity constant and the dimension n with the following property. If $W_1 = \{S_k\}$ and $W_2 = \{Q_i\}$ are Whitney decompositions of G and $\mathbb{R}^n \backslash \bar{G}$, respectively, then to each cube $Q_i \in W_2$ one can associate a cube $Q_i^* = S_k \in W_1$ such that*

$$1 \leq \frac{l(Q_i^*)}{l(Q_i)} \leq C_1 \quad \textit{and} \quad 1 \leq \frac{d(Q_i, Q_i^*)}{l(Q_i)} \leq C_1. \tag{3}$$

Furthermore, for each cube $S_k \in W_1$ there are at most C_1 cubes $Q_i \in W_2$ such that $Q_i^* = S_k$.

Suppose that $Q_1, Q_2, \ldots, Q_m$ are cubes (in a Whitney decomposition) such that $Q_j \cap Q_{j+1} \neq \emptyset$ for all j, $1 \leq j \leq m-1$. We say then that $\{Q_1, Q_2, \ldots, Q_m\}$ is a chain of length m connecting Q_1 to Q_m.

Lemma 3.3 ([7, Lemma 2.8]). *Let $G \subset \mathbb{R}^n$ be a c-uniform domain. Then there exists a constant $C_2 \geq 1$ depending only on the uniformity constant and the dimension n with the following property. If $W_1 = \{S_k\}$ and $W_2 = \{Q_i\}$ are Whitney decompositions of G and $\mathbb{R}^n \backslash \bar{G}$, respectively, and if $Q_j^*, Q_k^* \in W_1$ are cubes corresponding to cubes $Q_j, Q_k \in W_2$ with $Q_j \cap Q_k \neq \emptyset$, as described in Lemma 3.2, then there is a chain $F(Q_j^*, Q_k^*) = \{Q_j^* = S_1, \ldots, S_m = Q_k^*\}$ of cubes in W_1 connecting Q_j^* to Q_k^* with length $m \leq C_2$.*

Now we can formulate the extension result. It may not be in the most general form. But it does cover the example constructed in Sect. 2 as a special case.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a c-uniform domain and F_j, $j = 1, 2, \ldots$, be pairwise disjoint continua in $\bar{\Omega}$ such that $\Omega \backslash F_j$ is uniform with the uniformity constant independent of j for all j and that $\cup_j F_j$ is closed relative to Ω. Assume further that*

$$\partial F_j = \partial F_j^{\circ} \quad \textit{and} \quad d(F_j, F_k) \geq C d(x, \partial F_j) \tag{4}$$

for all $j \neq k$ and $x \in F_j^\circ$, where $C > 1$ is a constant which is to be determined in the proof and which depends only on the uniformity constant and dimension n. Then $G = \Omega \backslash \cup_j F_j$ has the W_p^1- extension property and the L_p^1- extension property for all $p \geq 1$.

3.5. Remarks Condition (4) means that the continua F_j are slim, but nowhere degenerate, "tubes" and far away from each other. It also implies that F_is cannot accumulate near any F_j. We require the constant C in (4) so large that any chain of corresponding cubes determined by a pair of cubes in F_j° will not intersect any chain of cubes determined by a pair of cubes in F_k° for $j \neq k$. Here and in what follows we let F° denote the interior of a set F and $|F|$ the n-measure of F. The idea is that under these assumptions, one can extend a Sobolev function across F_j using only the information about the function near F_j. Thus the extensions across different F_js will not interfere with each other.

Throughout the proof of Theorem 3.4, we will frequently use the properties of the Whitney cubes recorded in 3.1–3.3, without mentioning them each time they are invoked.

3.6. Proof of Theorem 3.4. We only verify the W_p^1- extension property of $\Omega \backslash \cup_j F_j$. The L_p^1- extension property can be established similarly (but more easily). Since Ω is a uniform domain, it suffices to construct a bounded extension operator

$$E: \; W_p^1(\Omega \backslash \cup_j F_j) \longrightarrow W_p^1(\Omega).$$

For each $j \geq 1$, fix a Whitney cube decomposition of $\mathbb{R}^n \backslash (\overline{\Omega \backslash F_j})$ and let $\mathcal{W}_j$ denote the collection of cubes Q in this decomposition such that $Q \cap F_j^\circ \neq \emptyset$. Since $\Omega \backslash F_j$ is a c-uniform domain, as in Lemma 3.2, we can associate to each cube $Q_{i,j} \in \mathcal{W}_j$ a cube $Q_{i,j}^* \subset \Omega \backslash F_j$ such that

$$1 \leq \frac{l(Q_{i,j}^*)}{l(Q_{i,j})} \leq C_1 \quad \text{and} \quad 1 \leq \frac{d\big(Q_{i,j}, Q_{i,j}^*\big)}{l(Q_{i,j})} \leq C_1, \tag{5}$$

where $C_1 = C_1(c, n)$ is the constant as in Lemma 3.2. In fact, according to Lemma 3.7 below, we have $Q_{i,j}^* \subset \Omega \backslash \cup_j F_j$.

Next we choose a partition of unity $\{\phi_{i,j}\}$ (see [11, Chapter 6]) corresponding to the collection $\{Q_{i,j} : Q_{i,j} \in \mathcal{W}_j\}$ such that each $\phi_{i,j}$ has support in the cube $(3/2)Q_{i,j}$ and that

$$|\nabla \phi_{i,j}(x)| \leq \frac{C}{l(Q_{i,j})} \tag{6}$$

for all x, where C is a constant depending only on n.

For any Sobolev function $u \in W_p^1(\Omega \backslash \cup_j F_j)$, we define the extension Eu on Ω by

$$Eu(x) = \begin{cases} u(x), & x \in \Omega \backslash \cup_j F_j, \\ \sum_{i,j} a_{i,j} \phi_{i,j}(x), & x \in \cup_j F_j, \end{cases} \tag{7}$$

where $a_{i,j}$ is the average of u on $Q^*_{i,j}$. We will show that E is the desired extension operator by establishing several lemmas. We note that $\phi_{i,j}(x) = 0$ on F_l for $l \neq j$. The first lemma spells out the requirement for the constant C in Theorem 3.4.

Lemma 3.7. *There is a constant $C = C(c, n) > 1$ such that if (4) is satisfied, then the collections $\mathcal{W}_j$ of Whitney cubes described above have the following property. If $Q_{i,j}, Q_{l,j} \in \mathcal{W}_j$, $Q_{s,k}, Q_{t,k} \in \mathcal{W}_k$, and if $Q^*_{i,j}, Q^*_{l,j}, Q^*_{s,k}, Q^*_{t,k}$ are the corresponding cubes described above, then the two chains $F(Q^*_{i,j}, Q^*_{l,j})$ and $F(Q^*_{s,k}, Q^*_{t,k})$ of cubes, defined in Lemma 3.3, are disjoint and are subsets in $\Omega \setminus \cup_j F_j$.*

Proof. We need to find a constant $C = C(n, c)$ such that if condition (4) is satisfied, then

$$d(S_j, S_k) > 0 \quad \text{and} \quad d(S_j, F_k) > 0$$

for all cubes $S_j \in F(Q^*_{i,j}, Q^*_{l,j})$ and $S_k \in F(Q^*_{s,k}, Q^*_{t,k})$, and all $j \neq k$. To this end, we fix such cubes S_j and S_k, and fix points $x_j \in S_j$, $y_k \in S_k$, $q_{i,j} \in Q_{i,j} \cap F^\circ_j$ and $q_{s,k} \in Q_{s,k} \cap F^\circ_k$. By 3.1–3.3, there is a constant $\lambda = \lambda(C_1, C_2)$ depending only on the constants in Lemmas 3.2 and 3.3 such that

$$d(x_j, q_{i,j}) \leq \lambda l(Q_{i,j}) \leq \lambda d(q_{i,j}, \partial F_j).$$

Thus, if we choose constant $C = C(n, c) > 2\lambda$, then (4) yields

$$\begin{aligned} d(S_j, S_k) &\geq d(q_{i,j}, q_{s,k}) - d(x_j, q_{i,j}) - d(y_k, q_{s,k}) \\ &\geq d(F_j, F_k) - \lambda(d(q_{i,j}, \partial F_j) + d(q_{s,k}, \partial F_k)) > 0 \end{aligned}$$

as desired. Similarly, we have $d(S_j, F_k) > 0$. This proves Lemma 3.7.

We want to remind the reader that the forth mentioned properties of Whitney cubes and partition of unity will be used freely throughout. We will use C to denote a generic constant whose value may vary from line to line but only depends on n, c, p.

Lemma 3.8. *There is a constant $C = C(n, c, p)$ such that*

$$\|Eu\|_{L^p(\cup F^\circ_j)} \leq C \|u\|_{L^p(\Omega \setminus \cup F_j)}.$$

Proof. For simplicity of notation, for a fixed j we write $Q_{k,j}$ as Q_k and $\phi_{k,j}$ as ϕ_k. We note that, by Hölder inequality,

$$|a_k|^p = \frac{1}{|Q^*_k|^p} \left| \int_{Q^*_k} u(x) \mathrm{d}x \right|^p \leq \frac{1}{|Q^*_k|} \int_{Q^*_k} |u(x)|^p \mathrm{d}x.$$

Therefore, for each i and fixed j, we have

$$\int_{Q_i} |Eu|^p dx = \int_{Q_i} \left| \sum_{\substack{Q_k \in \mathcal{W}_j \\ Q_k \cap Q_i \neq \emptyset}} a_k \phi_k(x) \right|^p \mathrm{d}x$$
$$\leq C \sum_{\substack{Q_k \in \mathcal{W}_j \\ Q_k \cap Q_i \neq \emptyset}} |a_k|^p |Q_k| \leq C \sum_{\substack{Q_k \in \mathcal{W}_j \\ Q_k \cap Q_i \neq \emptyset}} \int_{Q_k^*} |u(x)|^p \mathrm{d}x.$$

Since each cube $Q_{i,j}$ has only a bounded number of adjacent cubes and each Q^* can be only associated to a bounded number of $Q_{i,j}$, by taking the summation over all i, j, the above inequality yields that

$$\int_{\cup_j F_j^\circ} |Eu(x)|^p \mathrm{d}x \leq C \sum_j \sum_i \sum_{\substack{Q_k \in \mathcal{W}_j \\ Q_k \cap Q_i \neq \emptyset}} \int_{Q_k^*} |u(x)|^p dx \leq C \int_{\Omega \setminus \cup F_j} |u|^p \mathrm{d}x.$$

Thus Lemma 3.13 follows as desired.

Lemma 3.9. *There is a constant $C = C(n, c, p)$ such that*

$$\|\nabla(Eu)\|_{L^p(\cup F_j^\circ)} \leq C \|\nabla u\|_{L^p(\Omega \setminus \cup F_j)}.$$

Proof. To simplify notation, as in the proof of the previous lemma, we write $Q_{k,j}$ as Q_k and $\phi_{k,j}$ as ϕ_k for a fixed j. We observe that for $x \in Q_i$ we have

$$Eu(x) = \sum_{\substack{Q_k \in \mathcal{W}_j \\ Q_k \cap Q_i \neq \emptyset}} a_k \phi_k(x) = a_i + \sum_{\substack{Q_k \in \mathcal{W}_j \\ Q_k \cap Q_i \neq \emptyset}} (a_k - a_i) \phi_k(x).$$

Thus

$$\int_{Q_i} |\nabla Eu(x)|^p dx = \int_{Q_i} \left| \sum_{\substack{Q_k \in \mathcal{W}_j \\ Q_k \cap Q_i \neq \emptyset}} (a_k - a_i) \nabla \phi_k(x) \right|^p \mathrm{d}x$$
$$\leq C l(Q_i)^{-p} |Q_i| \sum_{\substack{Q_k \in \mathcal{W}_j \\ Q_k \cap Q_i \neq \emptyset}} |a_k - a_i|^p. \tag{8}$$

Next we estimate $|a_k - a_i|$ for $Q_k \cap Q_i \neq \emptyset$. As in Lemma 3.7, there is a chain $F(Q_i^*, Q_k^*) = \{Q_i^* = S_1, \ldots, S_m = Q_k^*\}$ of cubes in $\Omega \setminus \cup F_j$ connecting Q_i^* to Q_k^*. Let u_r denote the average of u on S_r and $u_{r,r+1}$ the average of u on $S_r \cup S_{r+1}$. Then

$$|a_k - a_i|^p = |u_1 - u_m|^p \leq C \sum_{r=1}^{m-1} (|u_r - u_{r,r+1}|^p + |u_{r,r+1} - u_{r+1}|^p).$$

Using Hölder's inequality and a variant of the classical Poincare–Sobolev inequality (see [7, Lemma 2.2] or [5, Lemma4.2]), we deduce that

$$\begin{aligned}|u_r - u_{r,r+1}| &= \frac{1}{|S_r \cup S_{r+1}|}\left|\int\limits_{S_r\cup S_{r+1}} (u-u_r)\mathrm{d}x\right| \\ &\le \frac{1}{|S_r \cup S_{r+1}|}\|u-u_r\|_{L^p(S_r\cup S_{r+1})}|S_r\cup S_{r+1}|^{1-1/p} \\ &\le \frac{Cl(S_r)}{|S_r\cup S_{r+1}|^{1/p}}\|\nabla u\|_{L^p(S_r\cup S_{r+1})}.\end{aligned}$$

Thus

$$|u_r - u_{r,r+1}|^p \le \frac{Cl(Q_i)^p}{|Q_i|}\int\limits_{S_r\cup S_{r+1}} |\nabla u|^p \mathrm{d}x$$

for $r = 1, \dots, m-1$ and hence

$$\begin{aligned}|a_k - a_i|^p &\le C\sum_{r=1}^{m-1}(|u_r - u_{r,r+1}|^p + |u_{r,r+1} - u_{r+1}|^p) \\ &\le \frac{Cl(Q_i)^p}{|Q_i|}\int\limits_{F(Q_i^*,Q_k^*)} |\nabla u|^p \mathrm{d}x\end{aligned}$$

for all k with $Q_k \cap Q_i \neq \emptyset$, where $F(Q_i^*, Q_k^*)$ also denotes the union of the cubes in the chain. Therefore, it follows from (8) that

$$\int\limits_{Q_i} |\nabla Eu(x)|^p \mathrm{d}x \le C \sum_{\substack{Q_k\in\mathcal{W}_j \\ Q_k\cap Q_i\neq\emptyset}} \int\limits_{F(Q_i^*,Q_k^*)} |\nabla u|^p \mathrm{d}x.$$

Taking the summation over i and j, the above inequality yields

$$\int\limits_{\cup F_j^\circ} |\nabla Eu(x)|^p \mathrm{d}x \le C \int\limits_{\Omega\setminus\cup_j F_j} |\nabla u|^p \mathrm{d}x.$$

Thus Lemma 3.9 follows.

Finally, we complete the proof of Theorem 3.4 as follows. Fix any Sobolev function $u \in W_p^1(\Omega\setminus\cup_j F_j)$ and let Eu be the extension in Ω determined by the extension operator (7). Since for each j $\Omega\setminus F_j$ is c-uniform, $|\partial F_j| = 0$. To see that Eu is Sobolev in Ω, we note that being in the Sobolev class is a local property. Thus it suffices to show that Eu is in the Sobolev class in a neighborhood of each boundary point $x_0 \in \partial F_j$. This follows from the fact that $\Omega\setminus F_j$ is c-uniform and the extension result of Jones [7]. Furthermore, by Lemmas 3.8 and 3.9, the Sobolev norm of Eu can be estimated as follows:

$$\|Eu\|^p_{W^1_p(\Omega)} = \int\limits_{\cup F_j^\circ} (|\nabla Eu|^p + |Eu|^p)\mathrm{d}x + \int\limits_{\Omega\setminus\cup_j F_j} (|\nabla u|^p + |u|^p)\mathrm{d}x$$
$$\leq C \int\limits_{\Omega\setminus\cup_j F_j} (|\nabla u|^p + |u|^p)\mathrm{d}x.$$

Thus

$$\|Eu\|_{W^1_p(\Omega)} \leq C\|u\|_{W^1_p(\Omega\setminus\cup_j F_j)}.$$

This completes the proof of Theorem 3.4. □

As a corollary to Theorem 3.4, we next show that the domain constructed in Sect. 2 has the desired extension property.

Corollary 3.10. *For sufficiently large N the domain $D = \mathbf{H}\setminus\bigcup_{j\geq N} F_j$ constructed in Sect. 2 satisfies the conditions of Theorem 3.4, and hence has the Sobolev extension property.*

Proof. Recall that $\mathbf{H}$ is the open upper half space in $\mathbb{R}^3$ and F_j is the closed circular cylinder of height $h_j = \frac{1}{\sqrt{j}}$ and radius $r_j = \frac{1}{10j^3}$ with its base centered at the point $(\frac{1}{j}, 0, 0)$, $j = 1, 2, 3, \ldots$.

To see that $\mathbf{H}\setminus F_j$ is c-uniform with uniformity constant c independent of j, we observe that under a similarity map of $\mathbb{R}^3$ $\mathbf{H}\setminus F_j$ is equivalent to a domain $G_h = \mathbf{H}\setminus F$, where F is the closed circular cylinder based on the unit disk of height h. Routine (but tedious) case by case verification shows that domain G_h is c-uniform with c independent of the height h.

Finally, to verify condition (4), we let $C = C(c, n) > 1$ (with $n = 3$ here) be the constant determined in Lemma 3.7. For all $j \neq k$ and $x \in F_j^\circ$, it follows that

$$d(F_j, F_k) = \left|\frac{1}{j} - \frac{1}{k}\right| - r_j - r_k, \quad d(x, \partial F_j) \leq r_j.$$

Thus

$$\frac{d(F_j, F_k)}{d(x, \partial F_j)} \geq \frac{|\frac{1}{j} - \frac{1}{k}| - \frac{1}{10j^3} - \frac{1}{10k^3}}{\frac{1}{10j^3}} \longrightarrow \infty$$

as $j, k \to \infty$. Therefore, there is a positive integer N such that, when $j, k \geq N$ and $j \neq k$, we have

$$d(F_j, F_k) \geq C d(x, \partial F_j).$$

This completes the proof of Corollary 3.10. □

3.11. Remarks Theorem 1 shows that even under the strongest possible topological conditions the implications in (1) cannot be reversed. Therefore, certain geometric conditions (on the domain or its complement) are needed. Finally we note that the complementary domain D^* of the domain D constructed in 2.1 is not linearly locally connected (or LLC). A natural question to ask is if both D and D^* are LLC, are they also uniform?

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