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Hermite functions and weighted spaces of generalized functions

Dedicated to Professor H.-G. Tillmann on the occasion of his 80th birthday

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Abstract. We prove that the Hermite functions are an absolute Schauder basis for many weighted spaces of (ultra)differentiable functions and ultradistributions including the space of Fourier hyperfunctions. The coefficient spaces are also determined.

1. Introduction

It is well known that the Hermite functions $H_\gamma, \gamma \in \mathbb{N}_0^n$, are an orthonormal basis in $L_2(\mathbb{R}^n)$ and that they also are a Schauder basis in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing $C^\infty-$functions (see e.g. Meise and Vogt [7, 14.9 and 29.5(2)]). In this paper we will study the natural question, to which weighted spaces of (ultra)differentiable functions the latter result can be extended.

Starting with a (multi)sequence $(M_\alpha)_{\alpha\in\mathbb{N}_0^n}$ which satisfies Komatsu's standard condition $(M2')$ (stability under differential operators) we will consider weighted $(FS)-$ spaces (and $(DFS)-$ spaces) of ultradifferentiable functions $f \in C^\infty(\mathbb{R}^n)$ such that

$$\sup_{\alpha,\beta\in\mathbb{N}_0^n} \|x^\alpha \partial^\beta f\|_\infty C^{|\alpha|+|\beta|}/M_{\alpha+\beta} < \infty \tag{1.1}$$

for any $C \geq 1$ (and for some $C > 0$, respectively).

The specific condition needed for our problem is the following: there is $H > 0$ such that for any $C > 0$ there is $B > 0$ (there are $C > 0$ and $B > 0$, respectively) such that

$$\alpha^{\alpha/2} M_\beta \leq BC^{|\alpha|} H^{|\alpha+\beta|} M_{\alpha+\beta} \text{ for any } \alpha, \beta \in \mathbb{N}_0^n. \tag{1.2}$$

Notice that we do not assume that the classes of functions are non quasi analytic. For example, (1.2) is satisfied by $M_\alpha := \alpha!^r$ for $r > 1/2$ and (1.1) then leads to spaces of entire functions if $1 > r > 1/2$ (see section 5).

We will show that the Hermite functions are a basis in the spaces of type (1.1) if (1.2) is satisfied. This result is optimal, since we can prove that (1.2) already

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holds if H_0 is contained in these spaces and if we additionally assume that there is $H > 0$ such that

$$M_\alpha M_\beta \leq C H^{\alpha+\beta} M_{\alpha+\beta} \text{ for any } \alpha, \beta \in \mathbb{N}_0^n$$

(see Remark 3.3). In the case of one variable, the latter condition is satisfied if the sequence (M_α) is logarithmically convex.

We will obtain corresponding results for the dual spaces of weighted (ultra) distributions and analytic functionals. Moreover, we will determine the coefficient spaces corresponding to this Hermite expansion.

In this way we obtain explicit sequence space representations for many classical spaces of analysis including the spaces S_r^r of Gelfand and Shilov [1] for $r \geq 1/2$ and Sato's Fourier hyperfunctions.

The paper is organized as follows: In the next section we will give the basic notions and definitions which are used in this paper. Section 3 contains the proof of the Hermite expansion, and we then discuss the connection of our results to power series spaces in section 4. Many concrete examples are finally presented in section 5.

The results of the present paper will be applied to the study of convolution operators and their continuous linear right inverses on Fourier hyperfunctions in the forthcoming papers [5] and [6].

2. Preliminaries

In this paper $(M_\alpha)_{\alpha \in \mathbb{N}_0^n}$ will always denote a (multi) sequence of positive numbers with $M_0 = 1$. We will generally assume that Komatsu's condition $(M2')$ is satisfied, i.e. that there is $A \geq 1$ such that

$$M_{\alpha+e_j} \leq A^{|\alpha|+1} M_\alpha \text{ for any } \alpha \in \mathbb{N}_0^n \text{ and any } j \leq n. \tag{2.1}$$

$(M2')$ is the standard assumption which implies that the spaces of ultradifferentiable functions defined below are stable under differentiation and multiplication with polynomials.

To define the coefficient space for the Hermite expansion we will need the associated function of (M_α) defined by

$$M(t) := \ln \sup_{\alpha \in \mathbb{N}_0^n} \frac{|t^\alpha|}{M_\alpha} \text{ for } t \in \mathbb{R}^n.$$

An easy calculation shows that there are $B_1, B_2 \geq 1$ such that for any $t \in \mathbb{R}^n$

$$e^{M(t)}(1 + |t|)^{2n+2} \leq B_1 e^{M(B_2 t)} \tag{2.2}$$

if (M_α) satisfies (2.1).

We do not assume that the sequence (M_α) satisfies Komatsu's condition $(M3')$, i.e. that the corresponding classes of ultradifferentiable functions are non quasi analytic. Instead, we will use (1.2) from the introduction.

We will consider the following two types of weighted spaces (and their dual spaces) in this paper:

$$\mathcal{S}_{\{M_\alpha\}} := \{f : \mathbb{R}^n \to \mathbb{C} \mid$$
$$\|f\|_{\infty,j} := \sup_{\alpha,\beta\in\mathbb{N}_0^n} \|x^\alpha \partial^\beta f\|_\infty/(j^{|\alpha|+|\beta|} M_{\alpha+\beta}) < \infty \text{ for some } j \in \mathbb{N}\}.$$

and

$$\mathcal{S}_{(M_\alpha)} := \{f : \mathbb{R}^n \to \mathbb{C} \mid$$
$$|f|_{\infty,j} := \sup_{\alpha,\beta\in\mathbb{N}_0^n} \|x^\alpha \partial^\beta f\|_\infty j^{|\alpha|+|\beta|}/M_{\alpha+\beta} < \infty \text{ for any } j \in \mathbb{N}\}.$$

Condition (1.2) will be needed for any $C > 0$ if $\mathcal{S}_{(M_\alpha)}$ is considered (and for some $C > 0$ if $\mathcal{S}_{\{M_\alpha\}}$ is considered).

Several modern applications of the structure theory of Frechet spaces such as splitting theory for power series spaces of finite type are based on precise (so called tame) continuity estimates for the linear operators in question. In fact, our study of right inverses for partial differential operators and convolution operators on Fourier hyperfunctions will also rely on tame splitting theory and (linearly) tame mappings (see Langenbruch [5, 6]). We will therefore aim at tame estimates in this paper.

Let us recall the precise definitions: let $(E, (|\ |_j)_{j\in\mathbb{N}})$ and $(F, (\|\ \|_j)_{j\in\mathbb{N}})$ be Frechet spaces with fixed increasing systems of semi norms defining the topology. A continuous linear mapping

$$T : (E, (|\ |_j)_{j\in\mathbb{N}}) \to (F, (\|\ \|_j)_{j\in\mathbb{N}})$$

is called tame if there are $C, j_0 \in \mathbb{N}$ such that for any $j \in \mathbb{N}$ with $j \geq j_0$ there is $C_1 > 0$ such that for any $f \in E$

$$\|T(f)\|_j \leq C_1 |f|_{Cj}.$$

Similarly, for (LB)-spaces $E = \lim\operatorname{ind}_{j\to\infty} E_j$ and $F = \lim\operatorname{ind}_{j\to\infty} F_j$ with fixed increasing systems of Banach spaces $(E_j, |\ |_j)_{j\in\mathbb{N}}$ and $(F_j, \|\ \|_j)_{j\in\mathbb{N}}$, a continuous linear mapping

$$T : E = \lim_{j\to\infty}\operatorname{ind} E_j \to F = \lim_{j\to\infty}\operatorname{ind} F_j$$

is called tame if there are $C, j_0 \in \mathbb{N}$ such that for any $j \in \mathbb{N}$ with $j \geq j_0$ there is $C_1 > 0$ such that for any $f \in E_j$

$$\|T(f)\|_{Cj} \leq C_1 |f|_j.$$

A linear mapping is called a tame isomorphism if it is bijective and if both T and T^{-1} are tame.

Two systems $\{|\ |_j \mid j \in \mathbb{N}\}$ and $\{\|\ \|_j \mid j \in \mathbb{N}\}$ of semi norms on a Frechet space E are called tamely equivalent if

$$\mathrm{id} : (E, (|\ |_j)_{j\in\mathbb{N}}) \to (E, (\|\ \|_j)_{j\in\mathbb{N}}) \text{ is a tame isomorphism.}$$

For $(LB)-$spaces this notion is defined similarly.

Since Hermite functions are an orthonormal basis in $L_2(\mathbb{R}^n)$ it is more convenient to work with L_2-norms instead of sup $-$norms. This is allowed by the following

Remark 2.1. Let (M_α) satisfy (1.2) and (2.1). Then the system $(|\ |_{\infty,j})_{j\in\mathbb{N}}$ of semi norms on $\mathcal{S}_{(M_\alpha)}$ is tamely equivalent to the system $(|\ |_{2,j})_{j\in\mathbb{N}}$ defined by

$$|f|_{2,j} := \sup_{\alpha,\beta\in\mathbb{N}_0^n} \|x^\alpha \partial^\beta f\|_2 j^{|\alpha|+|\beta|}/M_{\alpha+\beta}$$

and the system $(\|\ \|_{\infty,j})_{j\in\mathbb{N}}$ on $\mathcal{S}_{\{M_\alpha\}}$ is tamely equivalent to the system $(\|\ \|_{2,j})_{j\in\mathbb{N}}$ defined by

$$\|f\|_{2,j} := \sup_{\alpha,\beta\in\mathbb{N}_0^n} \|x^\alpha \partial^\beta f\|_2 / \big(j^{|\alpha|+|\beta|} M_{\alpha+\beta}\big).$$

Proof. We will use the following well known estimate for functions f in the Schwartz space $\mathcal{S}$:

$$\begin{aligned} \|f\|_\infty &\le C_1 \|\widehat{f}\,\|_1 \le C_2 \|(1+|x|^2)^{n+1}\widehat{f}\,\|_2 \\ &\le C_3 \|(1-\Delta)^{n+1} f\|_2 \le C_4 \sup_{|\gamma|_\infty \le 2n+2} \|\partial^\gamma f\|_2 \end{aligned} \tag{2.3}$$

where $\widehat{\ }$ is the Fourier transformation and Δ denotes the Laplacian.

The claim now easily follows from the following estimates for functions f in the respective spaces

$$\|x^\alpha \partial^\beta f\|_2 \le C_5 \|(1+|x|^2)^{(n+1)/2} x^\alpha \partial^\beta f\|_\infty$$

and

$$\|x^\alpha \partial^\beta f\|_\infty \le C_6 \sup_{|\gamma|_\infty \le 2n+2} \|\partial^\gamma \big(x^\alpha \partial^\beta f\big)\|_2$$

by (2.3). To estimate $M_{\alpha+\beta+\gamma}$ for finitely many $\gamma \in \mathbb{Z}^n$, we use (1.2) and (2.1). □

3. Hermite functions

In this section we will show that Hermite functions are a basis in $\mathcal{S}_{\{M_\alpha\}}$ and in $\mathcal{S}_{(M_\alpha)}$, respectively. Recall that for $\gamma \in \mathbb{N}_0^n$ the Hermite function H_γ is defined by

$$H_\gamma(x) := (2^{|\gamma|}\gamma!\pi^{n/2})^{-1/2} \exp\Big(-\sum_{j\le n} x_j^2/2\Big) h_\gamma(x), \tag{3.1}$$

where the Hermite polynomial h_γ is defined by

$$h_\gamma(x) := (-1)^{|\gamma|} \exp\Big(\sum_{j\le n} x_j^2\Big) \partial^\gamma \exp\Big(-\sum_{j\le n} x_j^2\Big),\ x \in \mathbb{R}^n.$$

The proofs in this section are based on the well known connection of Hermite functions to certain partial differential operators (see e.g. Meise and Vogt [7, p.362] for the case of one variable) which we study first:

For $i \leq n, \alpha \in \mathbb{N}_0^n$ and $f \in C^\infty(\mathbb{R}^n)$ let

$$A_{\pm,i}(f) := \mp\partial_i(f) + x_i f \text{ and } A_\pm^\alpha := \prod_{i\leq n} A_{\pm,i}^{\alpha_i}(f)$$

where $A_{\pm,i}^0 := \mathrm{id}$.

Notice that the operators $A_{+,i}$ and $A_{-,i}$ are commuting, e.g.

$$A_{+,i}A_{+,j} = A_{+,j}A_{+,i} \text{ for any } 1 \leq i, j \leq n. \tag{3.2}$$

Lemma 3.1. *Let $f \in C^\infty(\mathbb{R}^n)$.*

a) For any $\gamma \in \mathbb{N}_0^n$ and any $x \in \mathbb{R}^n$ we have

$$(A_+^\gamma f)(x) = \sum_{\alpha+\beta\leq\gamma} c_{\alpha,\beta}(\gamma)x^\alpha\partial^\beta f(x),$$

where

$$|c_{\alpha,\beta}(\gamma)| \leq d_{\alpha,\beta}(\gamma) := 3^{|\gamma|}(\gamma!/(\alpha+\beta)!)^{1/2}.$$

b) Assume that

$$\sup_{\alpha,\beta\in\mathbb{N}_0^n} \|x^\alpha\partial^\beta f\|_2/(C^{|\alpha|+|\beta|}M_{\alpha+\beta}) \leq C_1. \tag{3.3}$$

and that (M_α) satisfies (1.2) for this constant C. Then we have for any $\gamma \in \mathbb{N}_0^n$

$$\|A_+^\gamma f\|_2 \leq C_1 B(9HC)^{|\gamma|}M_\gamma$$

Proof. a) The estimate for $|c_{\alpha,\beta}(\gamma)|$ is trivial if $\gamma = e_j$, $j \leq n$, is a canonical unit vector. Let it hold for any $\tilde{\gamma} \in \mathbb{N}_0^n$ with $|\tilde{\gamma}| \leq m$ and fix $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = m$. For $j \leq n$ we then get by (3.2) and the induction hypothesis

$$(A_+^{\gamma+e_j} f)(x) = (A_+^{e_j}(A_+^\gamma f))(x) = \sum_{\alpha+\beta\leq\gamma} c_{\alpha,\beta}(\gamma)x^{\alpha+e_j}\partial^\beta f(x)$$
$$- \sum_{\alpha+\beta\leq\gamma} c_{\alpha,\beta}(\gamma)x^\alpha\partial^{\beta+e_j} f(x) - \sum_{\alpha+\beta\leq\gamma, 1\leq\alpha_j} c_{\alpha,\beta}(\gamma)\alpha_j x^{\alpha-e_j}\partial^\beta f(x).$$

The estimates for $|c_{\alpha,\beta}(\gamma + e_j)|$ now follow from the induction hypothesis since for $\alpha + \beta \leq \gamma$,

$$d_{\alpha,\beta}(\gamma) \leq d_{\alpha+e_j,\beta}(\gamma + e_j)/3 = d_{\alpha,\beta+e_j}(\gamma + e_j)/3$$

and

$$d_{\alpha,\beta}(\gamma)\alpha_j \leq d_{\alpha-e_j,\beta}(\gamma + e_j)/3 \quad \text{if } 1 \leq \alpha_j.$$

b) By a) and (1.2) we get

$$\|A_+^\gamma f\|_2 \le C_1 3^{|\gamma|} \sum_{\alpha+\beta\le\gamma} \binom{\gamma}{\alpha+\beta}^{1/2} (\gamma-\alpha-\beta)!^{1/2} C^{|\alpha+\beta|} M_{\alpha+\beta}$$
$$\le C_1 B (3HC)^{|\gamma|} M_\gamma \sum_{\alpha+\beta\le\gamma} \binom{\gamma}{\alpha+\beta} \le C_1 B (9HC)^{|\gamma|} M_\gamma,$$

since

$$\sum_{\alpha+\beta\le\gamma} \binom{\gamma}{\alpha+\beta} \le \sum_{\alpha\le\gamma} \Big(\sum_{\beta\le\gamma-\alpha} \binom{\gamma-\alpha}{\beta} \frac{\gamma!\beta!}{(\alpha+\beta)!(\gamma-\alpha)!} \Big)$$
$$\le \sum_{\alpha\le\gamma} 2^{|\gamma-\alpha|} \binom{\gamma}{\alpha} = 3^{|\gamma|}.$$

□

We also need an appropriate weighted L_2- estimate for the derivatives of the Hermite functions which especially shows that the Hermite functions are contained in $\mathcal{S}_{(M_\alpha)}$ and $\mathcal{S}_{\{M_\alpha\}}$.

Lemma 3.2. *a) For $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ we have*

$$\|x^\alpha \partial^\beta H_\gamma\|_2 \le 2^{|\alpha+\beta|/2} \left(\frac{(\alpha+\beta+\gamma)!}{\gamma!} \right)^{1/2} \tag{3.4}$$

b) Let (1.2) *be satisfied for some* $C > 0$. *Then*

$$\sup_{\alpha,\beta\in\mathbb{N}_0^n} \|x^\alpha \partial^\beta H_\gamma\|_2 / ((2HC)^{|\alpha|+|\beta|} M_{\alpha+\beta}) \le B e^{M(\gamma^{1/2}/C)}$$

for any $\gamma \in \mathbb{N}_0^n$ (here $\gamma^{1/2} := (\gamma_1^{1/2}, \ldots, \gamma_n^{1/2})$).

Proof. a) (3.4) is trivial for $\alpha = \beta = 0$, since $\|H_\gamma\|_2 = 1$ for any γ. For $|\alpha + \beta| = 1$ this directly follows from orthogonality and the following equations (see Meise,Vogt [7, (***) on p. 362] for the case of one variable and set $H_\beta := 0$ if $\beta_j = -1$ for some j)

$$\partial_j H_\gamma = \big(\sqrt{\gamma_j} H_{\gamma-e_j} - \sqrt{\gamma_j+1} H_{\gamma+e_j} \big)/\sqrt{2} \tag{3.5}$$

and

$$x_j H_\gamma = \big(\sqrt{\gamma_j} H_{\gamma-e_j} + \sqrt{\gamma_j+1} H_{\gamma+e_j} \big)/\sqrt{2}. \tag{3.6}$$

Let (3.4) be true for any $\tilde\alpha, \tilde\beta \in \mathbb{N}_0^n$ with $|\tilde\alpha + \tilde\beta| \le m$ for some $m \ge 1$ and let $|\alpha+\beta| = m$. We then get from (3.5) and the assumption

$$\|x^\alpha \partial^{\beta+e_j} H_\gamma\|_2 \le \frac{1}{2^{1/2}} \left(\sqrt{\gamma_j} \|x^\alpha \partial^\beta H_{\gamma-e_j}\|_2 + \sqrt{\gamma_j+1} \|x^\alpha \partial^\beta H_{\gamma+e_j}\|_2 \right)$$
$$\le 2^{(|\alpha+\beta|-1)/2} \left(\sqrt{\gamma_j} \left(\frac{(\alpha+\beta+\gamma-e_j)!}{(\gamma-e_j)!} \right)^{1/2} + \left(\frac{(\alpha+\beta+\gamma+e_j)!}{\gamma!} \right)^{1/2} \right)$$
$$\le 2^{|\alpha+\beta+e_j|/2} \left(\frac{(\alpha+\beta+e_j+\gamma)!}{\gamma!} \right)^{1/2}.$$

We finally have to consider the case where $\beta = 0$ and $|\alpha| = m$. Using (3.6) we get similarly as above

$$\begin{aligned}\|x^{\alpha+e_j}H_\gamma\|_2 &\leq \frac{1}{2^{1/2}}\Big(\sqrt{\gamma_j}\|x^\alpha H_{\gamma-e_j}\|_2 + \sqrt{\gamma_j+1}\|x^\alpha H_{\gamma+e_j}\|_2\Big)\\ &\leq 2^{(|\alpha|-1)/2}\left(\left(\frac{(\alpha+\gamma)!}{(\gamma-e_j)!}\right)^{1/2} + \left(\frac{(\alpha+\gamma+e_j)!}{\gamma!}\right)^{1/2}\right)\\ &\leq 2^{|\alpha+e_j|/2}\left(\frac{(\alpha+e_j+\gamma)!}{\gamma!}\right)^{1/2}.\end{aligned}$$

b) For fixed $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ let

$$J := \{j \leq n \mid \alpha_j + \beta_j \leq \gamma_j\} \text{ and } \widetilde{J} := \{j \leq n \mid \alpha_j + \beta_j > \gamma_j\}$$

and set

$$\delta_J := \sum_{j\in J}\delta_j e_j \text{ for } \delta \in \mathbb{N}_0^n.$$

$\delta_{\widetilde{J}}$ is defined similarly. By a) and (1.2) we get

$$\begin{aligned}\|x^\alpha\partial^\beta H_\gamma\|_2 &\leq 2^{|\alpha+\beta|/2}(\alpha+\beta+\gamma)^{(\alpha+\beta)/2}\\ &\leq 2^{|\alpha+\beta|}(\alpha_{\widetilde{J}}+\beta_{\widetilde{J}})^{(\alpha_{\widetilde{J}}+\beta_{\widetilde{J}})/2}\gamma_J^{(\alpha_J+\beta_J)/2}\\ &\leq B(2HC)^{|\alpha+\beta|}M_{\alpha+\beta}\gamma_J^{(\alpha_J+\beta_J)/2}/\big(M_{\alpha_J+\beta_J}C^{|\alpha_J+\beta_J|}\big).\end{aligned}$$

This implies that

$$\begin{aligned}&\sup_{\alpha,\beta\in\mathbb{N}_0^n}\|x^\alpha\partial^\beta H_\gamma\|_2/((2HC)^{|\alpha+\beta|}M_{\alpha+\beta})\\ &\quad\leq B\gamma^{(\alpha_J+\beta_J)/2}/\big(C^{|\alpha_J+\beta_J|}M_{\alpha_J+\beta_J}\big) \leq Be^{M(\gamma^{1/2}/C)}.\end{aligned}$$

by the definition of the associated function M(t) (see section 2). □

We will show now that our special assumption (1.2) is optimal for the results we want to prove if we additionally assume that there is $\widetilde{H} > 0$ such that

$$M_\alpha M_\beta \leq \widetilde{H}^{|\alpha+\beta|}M_{\alpha+\beta}. \tag{3.7}$$

For sequences $(M_p)_{p\in\mathbb{N}_0}$ this assumption is obviously satisfied if

$$M_j^2 \leq M_{j-1}M_{j+1} \text{ if } j \in \mathbb{N}, \tag{3.8}$$

that is, if the sequence is logarithmically convex (this is Komatsu's condition $(M1)$).

Remark 3.3. Let (M_α) satisfy (2.1) and (3.7). The following are equivalent:

a) The Hermite functions are contained in $\mathcal{S}_{\{M_\alpha\}}$ (or $\mathcal{S}_{(M_\alpha)}$, respectively)
b) There are $C > 0$ and $C_1 > 0$ (or for any $C > 0$ there is $C_1 > 0$) such that

$$\alpha^{\alpha/2} \leq C_1 C^{|\alpha|}M_\alpha \text{ if } \alpha \in \mathbb{N}_0^n. \tag{3.9}$$

c) (M_α) satisfies (1.2) for some $C > 0$ (or for any $C > 0$, respectively).

Proof. "$c) \Rightarrow a)$" This follows follows from Lemma 3.2.
"$a) \Rightarrow b)$" If $H_0(x) := \pi^{-n/4}\exp(-\sum_{j\le n} x_j^2/2)$ satisfies

$$\sup_{\alpha,\beta\in\mathbb{N}_0^n} \|x^\alpha \partial^\beta H_0\|_2/(C^{|\alpha|+|\beta|}M_{\alpha+\beta}) < \infty \tag{3.10}$$

for some $C > 0$, we get

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n} \exp\big(M(x/(AC)) - \sum_{j\le n} x_j^2/2\big) \\
&= \sup_\alpha \|x^\alpha \exp(-\sum_{j\le n} x_j^2/2)/(M_\alpha (AC)^{|\alpha|})\|_\infty \\
&\le C_4 \sup_{|\gamma|_\infty\le 2n+2,\alpha} \|\partial^\gamma (x^\alpha \exp(-\sum_{j\le n} x_j^2/2))/(M_\alpha (AC)^\alpha)\|_2 \\
&\le C_5 \sup_{\alpha,\beta} \|x^\alpha \partial^\beta H_0(x)/(M_{\alpha+\beta} C^{|\alpha+\beta|})\|_2 < \infty
\end{aligned}$$

for suitable $A > 0$ (independent of C) by the Leibniz rule, (3.7), (1.2), (3.10) and (2.1). Thus,

$$M(x/(AC)) \le \sum_{j\le n} x_j^2/2$$

and we get by the definition of the associated function $M(t)$

$$\begin{aligned}
M_\alpha/(AC)^{|\alpha|} &\ge \sup_{x\in\mathbb{R}^n} |x^\alpha e^{-M(x/(AC))}| \\
&\ge \sup_{x\in\mathbb{R}^n} |x^\alpha \exp\big(-\sum_{j\le n} x_j^2/2\big)| = \alpha^{\alpha/2} e^{-|\alpha|/2}.
\end{aligned}$$

"$b) \Rightarrow c)$" This is evident by (3.7). □

A (multi) sequence $(e_\gamma)_{\gamma\in\mathbb{N}_0^n}$ in a locally convex space E is called an absolute (Schauder) basis if for any $x \in E$ there are unique $\xi_\gamma(x) \in \mathbb{C}$ such that $x = \sum_\gamma \xi_\gamma(x) e_\gamma$ and such that for any continuous semi norm p on E there are a continuous semi norm q on E and $C > 0$ such that

$$\sum_\gamma |\xi_\gamma(x)| p(e_\gamma) \le Cq(x) \text{ for all } x \in E.$$

The continuous linear mappings $(\xi_\gamma)_{\gamma\in\mathbb{N}_0^n}$ are called the coefficient functionals of the basis. Set

$$\Lambda_{\{M_\alpha\}} := \{(c_\alpha)_{\alpha\in\mathbb{N}_0^n} \mid \exists j \in \mathbb{N}_0^n : \|(c_\alpha)\|_j := \sup_{\alpha\in\mathbb{N}_0^n} |c_\alpha| e^{M(\alpha^{1/2}/j)} < \infty\}$$

and

$$\Lambda_{(M_\alpha)} := \{(c_\alpha)_{\alpha\in\mathbb{N}_0^n} \mid \forall j \in \mathbb{N}_0^n : |(c_\alpha)|_j := \sup_{\alpha\in\mathbb{N}_0^n} |c_\alpha| e^{M(j\alpha^{1/2})} < \infty\}.$$

Let

$$\Xi : L_2(\mathbb{R}^n) \to l_2,\ \Xi(f) := (\xi_\gamma(f))_{\gamma\in\mathbb{N}_0^n} := \Big(\int f(x)H_\gamma(x)\,dx\Big)_{\gamma\in\mathbb{N}_0^n}.$$

The following is the main result of this section:

Theorem 3.4. *Let* (M_α) *satisfy* (2.1) *and* (1.2) *for some* $C > 0$ *(and for any* $C > 0$*, respectively, if* $\mathcal{S}_{(M_\alpha)}$ *is considered). Then the Hermite functions are an absolute basis in* $\mathcal{S}_{\{M_\alpha\}}$ *(and in* $\mathcal{S}_{(M_\alpha)}$*) with coefficient functionals* $(\xi_\gamma)_{\gamma\in\mathbb{N}_0^n}$ *and* Ξ *defines a tame isomorphism*

$$\Xi : \mathcal{S}_{\{M_\alpha\}} \to \Lambda_{\{M_\alpha\}} \textit{ (and } \Xi : \mathcal{S}_{(M_\alpha)} \to \Lambda_{(M_\alpha)}, \textit{ respectively).}$$

Proof. Since the Hermite functions are a real valued basis of the Hilbert space $L_2(\mathbb{R}^n)$ and since $\mathcal{S}_{\{M_\alpha\}}$ and $\mathcal{S}_{(M_\alpha)}$ are continuously embedded in $L_2(\mathbb{R}^n)$, we need to prove only the last statement.

I) Let $f \in C^\infty(\mathbb{R}^n)$ satisfy

$$\sup_{\alpha,\beta\in\mathbb{N}_0^n} \|x^\alpha \partial^\beta f\|_2/(C^{|\alpha+\beta|}M_{\alpha+\beta}) =: C_1 < \infty$$

for some $C > 0$.

Let $H_\beta := 0$ if $\beta_j = -1$ for some j. We then have

$$A_{-,i}(H_\alpha) = \sqrt{2\alpha_i}\,H_{\alpha-e_i} \text{ if } \alpha\in\mathbb{N}_0^n$$

(see Meise/Vogt [7, Example 29.5(2)]). This implies by Lemma 3.1b)

$$\begin{aligned}|\xi_\gamma(f)|^2\gamma^\alpha &\le |\langle f, H_\gamma\rangle|^2 2^{|\alpha|}\frac{(\alpha+\gamma)!}{\gamma!} = |\langle f, A_-^\alpha(H_{\gamma+\alpha})\rangle|^2\\ &= |\langle A_+^\alpha(f), H_{\gamma+\alpha}\rangle|^2 \le \|A_+^\alpha(f)\|_2^2 \le C_1^2B^2(9HC)^{2|\alpha|}M_\alpha^2\end{aligned}$$

by the Cauchy-Schwarz inequality and since $\|H_{\gamma+\alpha}\|_2 = 1$. By the definition of the associated function we thus get

$$\sup_{\gamma\in\mathbb{N}_0^n} |\xi_\gamma(f)|e^{M(\gamma^{1/2}/(9HC))} \le C_1B.$$

II) Let $(c_\gamma)_{\gamma\in\mathbb{N}_0^n}$ satisfy

$$\sup_{\gamma\in\mathbb{N}_0^n} |c_\gamma|e^{M(\gamma^{1/2}/C)} = C_1 < \infty$$

for some $C > 0$ and let (1.2) hold for B_2C where B_2 is chosen from (2.2). Then Lemma 3.2b) implies that

$$\begin{aligned}&\sum_{\gamma\in\mathbb{N}_0^n} |c_\gamma| \sup_{\alpha,\beta} \|x^\alpha\partial^\beta H_\gamma\|_2/((2B_2HC)^{|\alpha+\beta|}M_{\alpha+\beta})\\ &\le B\sum_{\gamma\in\mathbb{N}_0^n} |c_\gamma|e^{M(\gamma^{1/2}/(B_2C))}\\ &\le BC_1\sum_{\gamma\in\mathbb{N}_0^n} e^{M(\gamma^{1/2}/(B_2C))-M(\gamma^{1/2}/C)}.\end{aligned} \tag{3.11}$$

Since the sum in (3.11) is finite by (2.2), the theorem is proved. □

Since the Hermite functions are a real valued basis of the Hilbert space $L_2(\mathbb{R}^n)$, Theorem 3.4 implies by duality

Corollary 3.5. *Under the assumptions of Theorem 3.4, the Hermite functions are an absolute basis in* $(\mathcal{S}_{\{M_\alpha\}})'_b$ *(and in* $(\mathcal{S}_{(M_\alpha)})'_b$*) with coefficient functionals* $(\widetilde{\xi}_\gamma)_{\gamma\in\mathbb{N}_0^n}$, $\widetilde{\xi}_\gamma(T) := T(H_\gamma)$ *for* $T \in (\mathcal{S}_{\{M_\alpha\}})'_b$ *(and* $T \in (\mathcal{S}_{(M_\alpha)})'_b$*) and* $\widetilde{\Xi} := (\widetilde{\xi}_\gamma)_{\gamma\in\mathbb{N}_0^n}$ *defines a tame isomorphism*

$$\widetilde{\Xi} : (\mathcal{S}_{\{M_\alpha\}})'_b \to (\Lambda_{\{M_\alpha\}})'_b \text{ (and } \widetilde{\Xi} : (\mathcal{S}_{(M_\alpha)})'_b \to (\Lambda_{(M_\alpha)})'_b, \text{ respectively).}$$

Spaces of the type $\mathcal{S}_{\{M_\alpha\}}$ and $\mathcal{S}_{(M_\alpha)}$ have been introduced in the literature since in these spaces, Fourier transformation is a topological isomorphism which can be extended by duality to an isomorphism in spaces of ultradistributions, (Fourier) hyperfunctions and analytic functionals (see Roumieu [8], Gelfand and Shilov [1] and Sato [9]).

It is well known that Hermite functions are well adapted to Fourier transformation, namely

$$\widehat{H_\gamma} = (2\pi)^{-n/2}(-i)^\gamma H_\gamma \text{ for } \gamma \in \mathbb{N}_0^n \tag{3.12}$$

(see Meise and Vogt [7, Corollary 14.9]). Thus Theorem 3.4 directly implies

Corollary 3.6. *Let* (M_α) *satisfy the assumptions of Theorem 3.4. Then the Fourier transformation is a tame isomorphism in* $\mathcal{S}_{\{M_\alpha\}}$ *(and in* $\mathcal{S}_{(M_\alpha)}$*, respectively).*

Proof. The mapping

$$(c_\gamma)_\gamma \to ((-i)^\gamma c_\gamma)_\gamma$$

clearly is a tame isomorphism in $\Lambda_{\{M_\alpha\}}$ and $\Lambda_{(M_\alpha)}$. The claim thus follows from Theorem 3.4 and (3.12). □

It is often useful to separate the bounds at ∞ from the bounds on the derivatives, i.e. the following spaces are considered

$$W_{\{M_\alpha\}} := \{f : \mathbb{R}^n \to \mathbb{C} \mid \|f\|_j := \sup_{x\in\mathbb{R}^n,\beta\in\mathbb{N}_0^n} \exp(M(x/j))|\partial^\beta f(x)|/(j^{|\beta|}M_{|\beta|}) < \infty \text{ for some } j \in \mathbb{N}\}.$$

and

$$W_{(M_\alpha)} := \{f : \mathbb{R}^n \to \mathbb{C} \mid |f|_j := \sup_{x\in\mathbb{R}^n,\beta\in\mathbb{N}_0^n} \exp(M(xj))|\partial^\beta f(x)|j^{|\beta|}/M_{|\beta|} < \infty \text{ for any } j \in \mathbb{N}\}.$$

To include these spaces into the setting of this paper, Komatsu's condition $(M2)$ is needed, that is, we will assume that there is $H > 0$ such that

$$M_{\alpha+\beta} \le H^{|\alpha+\beta|+1} M_\alpha M_\beta \text{ for any } \alpha, \beta \in \mathbb{N}_0^n. \tag{3.13}$$

Theorem 3.7. *Let* (M_α) *satisfy* (3.13). *Then the Hermite expansion from Theorem 3.4 defines a tame isomorphism*

a) from $W_{\{M_\alpha\}}$ *onto* $\Lambda_{\{M_\alpha\}}$*, if* (M_α) *also satisfies* (1.2) *for some* $C > 0$*,*
b) from $W_{(M_\alpha)}$ *onto* $\Lambda_{(M_\alpha)}$*, if* (M_α) *also satisfies* (1.2) *for any* $C > 0$*.*

Proof. (3.13) clearly implies (2.1). By Theorem 3.4 it suffices to show that the identity is a tame isomorphism from $S(\{M_\alpha\})$ onto $W_{\{M_\alpha\}}$ (and from $S((M_\alpha))$ onto $W_{(M_\alpha)}$, respectively). This easily follows from (3.13) and the definitions involved. □

4. Power series spaces

For the existence of continuous linear right inverses or the solvability of vector valued problems in analysis it is important to know if the spaces which are involved are isomorphic to power series spaces (or to their dual spaces, respectively). We will consider this question here for the weighted spaces $\mathcal{S}_{\{M_\alpha\}}$ and $\mathcal{S}_{(M_\alpha)}$. Since we already have the isomorphism from Theorem 3.4 we thus mainly have to consider the sequence spaces $\Lambda_{(M_\alpha)}$ and $\Lambda_{\{M_\alpha\}}$. Similar sequence spaces occur when sequence space representations for spaces of periodic ultradifferentiable functions are considered (see Langenbruch [3]), so we will profit here a lot from the detailed study in that paper.

We therefore restrict our considerations to functions of one variable in this section, that is, $n = 1$ and $(M_p)_{p\in\mathbb{N}_0}$ is an ordinary sequence. The results can easily transferred to the following two standard cases in several variables:

a) $(M_\alpha)_{\alpha\in\mathbb{N}_0}$ is isotropic, i.e. $M_\alpha = N_{|\alpha|}$ for some sequence $(N_p)_{p\in\mathbb{N}_0}$. Notice that then

$$M(x) = N(|x|) \text{ for } x \in \mathbb{R}^n. \tag{4.1}$$

b) $(M_\alpha)_{\alpha\in\mathbb{N}_0}$ is a product of the form $M_\alpha = \prod_{j\le n} M_{j,\alpha_j}$ with sequences $(M_{j,p})_{p\in\mathbb{N}_0}$, $j \le n$. Notice that then

$$M(x) = \sum_{j\le n} M_j(x_j) \text{ for } x \in \mathbb{R}^n. \tag{4.2}$$

Recall that power series spaces and their canonical semi norm systems are defined as follows: Let $(a_k)_{k\in\mathbb{N}_0}$ be sequence of positive numbers. Then

$$\Lambda_0(a_k) := \{(c_k)_{k\in\mathbb{N}_0} \mid \forall j \in \mathbb{N} : |(c_k)|_j := \sup_{k\in\mathbb{N}_0} |c_k| e^{-a_k/j} < \infty\}$$

and

$$\Lambda_\infty(a_k) := \{(c_k)_{k\in\mathbb{N}_0} \mid \forall j \in \mathbb{N} : \|(c_k)\|_j := \sup_{k\in\mathbb{N}_0} |c_k| e^{j a_k} < \infty\}.$$

For the dual spaces we use the corresponding dual norms.

$\Lambda_0(a_k)$ (and $\Lambda_\infty(a_k)$) are called power series spaces of finite type (and of infinite type, respectively).

Since we are dealing with functions of one variable, the natural general assumptions are $(M1)$ (see (3.8)), $(M2')$ (see (2.1)) and (3.9) (see Remark 3.5).

To state our first results concerning $\mathcal{S}_{(M_\alpha)}$ we need some more standard notation: Let

$$m_p := M_p/M_{p-1} \text{ for } p \in \mathbb{N}$$

and

$$m(t) := \max\{p \mid m_p \le |t|\} \text{ for } t \in \mathbb{R}.$$

Theorem 4.1. *Let* $(M_p)_{p\in\mathbb{N}_0}$ *satisfy* (2.1), (3.8) *and also* (3.9) *for any* $C > 0$. *The following are equivalent:*

a) $\mathcal{S}_{(M_\alpha)}$ *is isomorphic to a power series space.*
b) There is $C \in \mathbb{N}$ *such that*

$$2m_p \le m_{Cp} \text{ if } p \in \mathbb{N} \text{ is large .} \tag{4.3}$$

c) The mapping

$$T : f \to (\langle f, H_k\rangle e^{M(k^{1/2})})_{k\in\mathbb{N}_0},$$

is a topological isomorphism from $\mathcal{S}_{(M_\alpha)}$ *onto* $\Lambda_\infty(m(k^{1/2}))$.

Proof. For any $j \in \mathbb{N}$ and $x \in \mathbb{R}$

$$M(x) + jm(jx) \le M(je^j x) \text{ and } M(jx) \le M(x) + \ln(j)m(jx) \tag{4.4}$$

(see Langenbruch [3, Lemma 1.2c)]). By Theorem 3.4, T is an isomorphism from $\mathcal{S}_{(M_\alpha)}$ onto

$$\Lambda := \{(c_k)_{k\in\mathbb{N}_0} \mid \forall j \in \mathbb{N} : \sup_{k\in\mathbb{N}_0} |c_k|e^{M(jk^{1/2})-M(k^{1/2})} < \infty\},$$

which by (4.4) is isomorphic to

$$\Lambda^1 := \{(c_k)_{k\in\mathbb{N}_0} \mid \forall j \in \mathbb{N} : \sup_{k\in\mathbb{N}_0} |c_k|e^{jm(jk^{1/2}))} < \infty\}.$$

"$a) \Rightarrow b)$" By a) and the above remarks, Λ^1 is isomorphic to a power series space. This implies (4.3) by the proof of Langenbruch [3, Theorem 3.1].

"$b) \Rightarrow c)$" This is evident by the isomorphism above.

"$c) \Rightarrow a)$" This is trivial. □

Especially, $\mathcal{S}_{(M_\alpha)}$ never is isomorphic to a power series space of finite type.

For $\mathcal{S}_{\{M_\alpha\}}$ we get the following:

Theorem 4.2. *Let* $(M_p)_{p\in\mathbb{N}_0}$ *satisfy* (2.1), (3.8) *and also* (3.9) *for some* $C > 0$. *The following are equivalent:*

a) $\mathcal{S}_{\{M_\alpha\}}$ *is isomorphic to the dual of a power series space of infinite type.*
b) For any $j \in \mathbb{N}$ *there is* $k \in \mathbb{N}$ *such that*

$$jm(t) \le M(t) - M(t/k) \text{ if } t \text{ is large .} \tag{4.5}$$

c) The mapping

$$T : f \to (\langle f, H_k \rangle e^{M(k^{1/2})})_{k \in \mathbb{N}_0},$$

is an isomorphism from $\mathcal{S}_{\{M_\alpha\}}$ *onto* $\Lambda_\infty(m(k^{1/2}))'_b$.

Proof. This again follows from Theorem 3.4 and the results of Langenbruch [3] (use the proof of Theorem 4.4 in loc. cit.) □

A useful sufficient condition for (4.5) is the following: There is $C \in \mathbb{N}$ such that for any $j \geq 1$.

$$j m_p \leq m_{Cp} \text{ if } p \in \mathbb{N} \text{ is large} \tag{4.6}$$

(see Langenbruch [3, (4.5')] and compare (4.3)). Thus, (M_p) is rapidly increasing and $\mathcal{S}_{\{M_\alpha\}}$ is large in this case (see 5.1 for a typical example).

In canonical cases like the Gevrey sequence, $\mathcal{S}_{\{M_\alpha\}}$ is isomorphic to the dual of a power series space of finite type:

Theorem 4.3. *Let* $(M_p)_{p \in \mathbb{N}_0}$ *satisfy* (2.1), (3.8) *and also* (3.9) *for some* $C > 0$. *The following are equivalent:*

a) $\mathcal{S}_{\{M_\alpha\}}$ *is isomorphic to the dual of a power series space of finite type.*
b) $(M_p)_{p \in \mathbb{N}_0}$ *satisfies condition* $(M2)$ *(see* (3.13)*) and* (4.3).
c) The Hermite expansion from Theorem 3.4 is an isomorphism from $\mathcal{S}_{\{M_\alpha\}}$ *onto* $\Lambda_0(m(k^{1/2}))'_b$.

Proof. This follows from Theorem 3.4 and Langenbruch [3, 4.3]. □

5. Examples

In this section, we will present some instructive and typical examples and we will also calculate explicitly the corresponding sequence spaces and counting of semi norms. More examples can be found in the literature (e.g. in Langenbruch [3, 4]). For non quasi analytic classes of functions some of the sequence space representations of this paper can also be proved using Pelczynski's trick. However, no explicit bases nor coefficient functionals can be obtained in that way.

Theorem 4.2 usually applies to rapidly increasing sequences (M_p) and large spaces $\mathcal{S}_{\{M_\alpha\}}$. (4.4) indicates that it might be difficult to obtain a tame isomorphism in Theorem 4.2. However, for specific examples this is possible:

Example 5.1. Let $M_p := e^{|p|^r}$ for $p \in \mathbb{N}_0^n$. Then (M_p) satisfies (1.2) and (2.1) iff $1 < r \leq 2$. For these r, the mapping

$$\widetilde{T} : f \to (\langle f, H_k \rangle e^{-M(|k|^{1/2})})_{k \in \mathbb{N}_0^n},$$

defines tame isomorphisms

$$\widetilde{T} : \mathcal{S}_{\{M_\alpha\}} \to \Lambda_\infty(\ln(|k|))^{1/(r-1)})'_b$$

and

$$\widetilde{T} : \mathcal{S}_{(M_\alpha)} \to \Lambda_\infty(\ln(|k|))^{1/(r-1)}),$$

if we use the systems

$$\{\| \ \|_{e^j} \mid j \in \mathbb{N}\} \text{ in } \mathcal{S}_{\{M_\alpha\}} \text{ and } \{| \ |_{e^j} \mid j \in \mathbb{N}\} \text{ in } \mathcal{S}_{(M_\alpha)}.$$

Proof. The first claim is trivial. A direct calculation shows that there are $C_r, C > 0$ such that

$$C_r(\ln(|t|))^{r/(r-1)} - \ln(|t|) - C \le M(t) \le C_r(\ln(|t|))^{r/(r-1)} \tag{5.1}$$

if $t \in \mathbb{R}^n$ and $|t|$ is large. The claim now follows from Theorem 3.4, (4.1) and the mean value theorem since $r \le 2$. Finally, the standard enumeration of $\mathbb{N}_0^n$ is used. □

More examples of the type considered in Theorem 4.2 are given in Langenbruch [4, Examples 1.3].

$M_p := e^{p^2}$ is an extreme case for the spaces considered here. $\mathcal{S}_{(e^{\alpha^2})}$ is isomorphic to the space (s) of rapidly decreasing sequences, hence also to the Schwartz space $\mathcal{S}$ and to the space of 2π-periodic C^∞-functions. $\mathcal{S}_{\{e^{\alpha^2}\}}$ is isomorphic to the space $(s)'_b$.

Since the Hermite functions are also a basis in $\mathcal{S}$ (and in $\mathcal{S}'$), the mapping

$$L : f \to \sum e^{-(\ln(k))^2/16} \langle f, H_k \rangle H_k$$

provides the isomorphisms of $\mathcal{S}_{(e^{\alpha^2})}$ and $\mathcal{S}$ (and of $\mathcal{S}_{\{e^{\alpha^2}\}}$ and $\mathcal{S}'$, respectively) since $C_2 = 1/4$ in (5.1).

Theorem 4.1 and Theorem 4.3 can be applied e.g. for the classical Gevrey type spaces:

Remark 5.2. Let $M_p := (p!)^r (ln(2+p))^{ps}$ for $p \in \mathbb{N}_0$.

a) M_p satisfies (3.9) for any $C > 0$ iff $r > 1/2$ and $s \in \mathbb{R}$ (and for some $C > 0$ iff $r > 1/2$ and $s \in \mathbb{R}$ or $r = 1/2$ and $s > 0$). For these r and s, also the other assumptions of Theorems 4.1 and 4.3 (including (3.13)) are satisfied.

b) There are constants $0 < C_1 < C_2$ such that

$$C_1 t^{1/r}\big(\ln(t)\big)^{-s/r} \le m(t) \le C_2 t^{1/r}\big(\ln(t)\big)^{-s/r} \text{ if t is large.} \tag{5.2}$$

This estimate also holds for $M(t)$ (with different constants C_j).

Proof. a) This is easy.

b) (3.13) implies that there is $k \in \mathbb{N}$ such that

$$2M(t) \le M(kt) \text{ if t is large.}$$

By (4.4) this implies that

$$m(t/e) \le \int_{\ln(t)-1}^{\ln(t)} m(e^\tau)\,d\tau \le M(t) - M(t/e) \le M(t)$$
$$\le M(kt) - M(t) \le \ln(k) m(kt)$$

if t is large. It is therefore sufficient to prove (5.2) for $m(t)$.

We first consider m_p. Clearly,

$$\frac{1}{2} p^r (\ln(p))^s \le m_p = M_p/M_{p-1} \le 2p^r (\ln(p))^s \text{ if } p \text{ is large.}$$

This implies that

$$\max\{p \mid 2p^r(\ln(p))^s \le t\} \le \max\{p \mid m_p \le t\} = m(t)$$
$$\le \max\{p \mid \frac{1}{2} p^r (\ln(p))^s \le t\} \text{ if } t \text{ is large.}$$

We first assume that $s \ge 0$ and set $p := [C_2 t^{1/r} (\ln(t))^{-s/r}] + 1$ where [ ] is the Gauss bracket. Then

$$2p^r(\ln(p))^s \ge 2\big(C_2 t^{1/r}(\ln(t))^{-s/r}\big)^r \left(ln\big(C_2 t^{1/r}(\ln(t))^{-s/r}\big)\right)^s$$
$$= 2C_2^r t(\ln(t))^{-s}\big(\ln(C_2) + \ln(t)/r - s\ln(\ln(t))/r)\big)^s$$
$$\ge 2C_2^r t/(2r)^s > t \text{ if } t \text{ is large}$$

for suitable $C_2 > 0$. This shows that

$$m(t) \le C_2 t^{1/r} (\ln(t))^{-s/r}$$

i.e. the right hand side of (5.2) is proved. The left hand side and the case $s < 0$ are treated similarly. □

The following example includes the spaces of type S_r^r of Gelfand and Shilov [1].

Example 5.3. For $0 < r$ and $s \in \mathbb{R}$ let $M_p := (p!)^r (\ln(2+p))^{ps}$, $p \in \mathbb{N}_0$, and let

$$S_{\{r,s\}} := \{f : \mathbb{R}^n \to \mathbb{C} \mid \exists j \in \mathbb{N} :$$
$$|f\|_j := \sup_{x\in\mathbb{R}^n, \beta\in\mathbb{N}_0^n} \exp\big(\frac{1}{j}|x|^{1/r}/\ln(|x|)^{s/r}\big)|\partial^\beta f(x)|/(j^{r|\beta|} M_{|\beta|}) < \infty\}$$

and

$$S_{(r,s)} := \{f : \mathbb{R}^n \to \mathbb{C} \mid \forall j \in \mathbb{N} :$$
$$|f|_j := \sup_{x\in\mathbb{R}^n, \beta\in\mathbb{N}_0^n} \exp\big(j|x|^{1/r}/\ln(|x|)^{s/r}\big)|\partial^\beta f(x)| j^{r|\beta|}/M_{|\beta|} < \infty.\}$$

Then the Hermite expansion from Theorem 3.4 is a tame isomorphism

a) from $S_{(r,s)}$ onto $\Lambda_\infty(\frac{k^{1/(2rn)}}{\ln(k)^{s/r}})$ if $r > 1/2$

b) from $S_{\{r,s\}}$ onto $\Lambda_0(\frac{k^{1/(2rn)}}{\ln(k)^{s/r}})'_b$ if $r > 1/2$ or $r = 1/2$ and $s > 0$.

Proof. a) By Theorem 3.7, (4.1) and Remark 5.2 we know that $W_{(M_p)}$ is tamely isomorphic to $\Lambda_{(M_p)}$ which is tamely isomorphic to

$$\widetilde{\Lambda}_{(M_p)} := \{(c_k)_{k\in\mathbb{N}_0} \mid \forall j \in \mathbb{N}_0 : |(c_k)|_j := \sup_{k\in\mathbb{N}_0^n} |c_k| e^{j^{1/r}|k|^{1/(2r)}/\ln(|k|)^{s/r}} < \infty\}.$$

We thus have to use the grading $\{| \ |_{j^r} \mid j \in \mathbb{N}\}$ in $W_{(M_p)}$ to obtain the grading of a power series space for $\widetilde{\Lambda}_{(M_p)}$. Finally, we use the standard enumeration of $\mathbb{N}_0^n$ to obtain the tame isomorphism to the above power series spaces. This shows a).

b) This is follows as in a). □

Notice the different skaling of j in $S_{\{r,s\}}$ and $S_{(r,s)}$ for the growth of the variable and the derivatives, respectively.

We could also treat non isotropic versions of these spaces (use (4.2)).

The spaces $S_{\{r,0\}}$ are the well known spaces S_r^r of Gelfand and Shilov [1]. The Hermite functions cannot be a basis in the spaces S_r^ρ (see Gelfand and Shilov [1, chapter IV, section 2.3]) for $r \neq \rho$ since this would imply that the Fourier transformation is an isomorphism in S_r^ρ by (3.12) which is false for $r \neq \rho$.

For $1/2 \leq r < 1$ the functions in S_r^r can be extended to entire functions with non radial growth conditions and we get a tame sequence space representation for these from Example 5.3. For $\rho > 1$ let

$$\begin{aligned} &H_{\{\rho\}} := \{f \in H(\mathbb{C}^n) \mid \exists j \in \mathbb{N} : \\ &\|f\|_j := \sup_{z\in\mathbb{C}^n} |f(z)| \exp\big(|\Re(z)|^\rho/j - j^{1/(\rho-1)}|\Im z)|^{\rho/(\rho-1)}\big) < \infty\}. \end{aligned}$$

and

$$\begin{aligned} &H_{(\rho)} := \{f \in H(\mathbb{C}^n) \mid \forall j \in \mathbb{N} : \\ &|f|_j := \sup_{z\in\mathbb{C}^n} |f(z)| exp\big(j|\Re(z)|^\rho - j^{-1/(\rho-1)}|\Im(z)|^{\rho/(\rho-1)}\big) < \infty\}. \end{aligned}$$

Notice again the different skaling of j for the growth in real and imaginary direction.

Example 5.4. The Hermite expansion from Theorem 3.4 is a tame isomorphism

a) from $H_{(\rho)}$ onto $\Lambda_\infty(k^{\rho/(2n)})$ if $1 < \rho < 2$
b) from $H_{\{\rho\}}$ onto $\Lambda_0(k^{\rho/(2n)})'_b$ if $1 < \rho \leq 2$

Proof. By Gelfand and Shilov [1, (2) on p. 208 and chapter IV,section 7.5,Theorem 3], the identity is a tame isomorphism from $H_{\{\rho\}}$ onto $S_{\{1/\rho,0\}}$ and from $H_{(\rho)}$ onto $S_{(1/\rho,0)}$ if $1 < \rho \leq 2$. The claim thus follows from Example 5.3. □

Again, a non isotropic version of this example can be treated.

Finally, the interesting example of Fourier hyperfunctions is considered: the space of test functions for the Fourier hyperfunctions is usually denoted by P_* (see

Kaneko [2]) and corresponds to the holomorphic extension of the space $S_{\{1,0\}}$ (= S_1^1 in Gelfand and Shilov's notation). It is defined as follows:

$$P_* := \liminf_{j\to\infty} P_{*,j},$$

where

$$P_{*,j} := \{f \in H(U_j) \mid \|f\|_j := \sup_{z\in U_j} |f(z)| \exp(|\Re(z)|/j) < \infty\},$$

where U_j is the strip

$$U_j := \{z \in \mathbb{C}^n \mid |\Im(z)| < 1/j\}.$$

Similarly, let

$$P_{**} := \{f \in H(\mathbb{C}^n) \mid |f|_j := \sup_{|\Im(z)|<j} |f(z)| \exp(j|\Re(z)|) < \infty\}.$$

Example 5.5. The Hermite expansion from Theorem 3.4 is a tame isomorphism from P_* onto $\Lambda_0(k^{1/(2n)})'_b$ and from P_{**} onto $\Lambda_\infty(k^{1/(2n)})$, respectively.

Proof. It is easily seen that P_* (and P_{**}) are tamely isomorphic to $W_{\{\alpha!\}}$ (and $W_{(\alpha!)}$, respectively). The claim thus follows from Theorem 3.7 and Remark 5.2. □

Using the results of the present paper, convolution operators and their continuous linear right inverses are studied on Fourier hyperfunctions in the forthcoming papers Langenbruch [5, 6].

References

[1] Gelfand, I.M., Shilov, G.E.: Generalized functions. Vol. **2**, Academic Press, New York/London, 1968

[2] Kaneko, A.: Introduction to hyperfunctions.Kluwer, Dordrecht/Boston, 1988

[3] Langenbruch, M.: Ultradifferentiable functions on compact intervals. Math. Nachr. **140**, 109–126 (1989)

[4] Langenbruch, M.: Surjectivity of partial differential operators. Note di Mat. **Vol. X** (Suppl. n. 2), 369–388 (1990)

[5] Langenbruch, M.: Convolution operators on Fourier hyperfunctions. Preprint

[6] Langenbruch, M.: Right inverses for partial differential operators on Fourier hyperfunctions. Manuscript

[7] Meise, R., Vogt, D.: Introduction to functional analysis. Clarendon Press, Oxford, 1997

[8] Roumieu, C.: Ultra-distributions definies sur $\mathbb{R}^n$ et sur certains classes des varietes differentiables. J. Analyse Math. **10**, 153–192 (1962-63)

[9] Sato, M.: Theory of hyperfunctions (in Japanese). Sugaku **10**, 1–27 (1958)