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On Fourier coefficients of automorphic forms of symplectic groups

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Abstract. In this paper we study certain properties of Fourier coefficients of cuspidal representations on symplectic groups. We prove that every cuspidal representation has a nontrivial Fourier coefficient with respect to a certain type of unipotent class.

0. Introduction

An important problem in the theory of automorphic representations is to determine the Fourier coefficients that a representation has. This knowledge is a basic tool in many applications such as constructions of Rankin-Selberg integrals or studying liftings by using automorphic representations as kernel functions.

The most well known Fourier coefficient is the so-called Whittaker Fourier coefficient and it is known that every cuspidal representation on $\mathrm{GL}_n(\mathbb{A})$ has such a Fourier coefficient. However for other classical groups this is not the case. That is, there are cuspidal representations which have no nontrivial Whittaker Fourier coefficients.

Maybe the most convenient way to parameterize Fourier coefficients is by using the parameterization of unipotent orbits. For basic properties of unipotent orbits we refer the reader to [C] or [C-M]. In Section 2 we show how to associate to each unipotent orbit a set of Fourier coefficients on a given automorphic representation of the group Sp_{2n} or its double cover. In [M-W] this kind of association is done for representations of classical groups over p-adic fields.

As explained in [C] and [C-M] the set of unipotent orbits admits a partial ordering. Given an automorphic representation π of $G = \mathrm{Sp}_{2n}(\mathbb{A})$ or on its double cover we define $\mathcal{O}_G(\pi)$ to be the set of all unipotent orbits of G with the following property. $\mathcal{O} \in \mathcal{O}_G(\pi)$ if π has a nontrivial Fourier coefficient corresponding to $\mathcal{O}$ and for all $\widetilde{\mathcal{O}} > \mathcal{O}$ (relative to the above partial ordering) π has no nontrivial Fourier coefficients corresponding to $\widetilde{\mathcal{O}}$.

For example, if π has a Whittaker Fourier coefficient then $\mathcal{O}_G(\pi) = ((2n))$. If θ denotes the minimal representation on $\widetilde{\mathrm{Sp}}_{2n}$ then $\mathcal{O}_G(\theta) = (21^{2n-2})$.

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In [M] it is proven (for any classical group) that if π is a representation over a local p-adic field and if $\mathcal{O} \in \mathcal{O}_G(\pi)$ then $\mathcal{O}$ must be special (See [C-M] for definition.). It follows from [M-W] that if π is supercuspidal and u is a unipotent element associated with $\mathcal{O} \in \mathcal{O}_G(\pi)$ then u cannot be conjugated into a Levi part of the group G. (As before this is true for any classical group). In this paper we prove part of these two results for automorphic representations π of G.

In Section 2 (Theorem 2.1) we show that if π is an automorphic representation of $\mathrm{Sp}_{2n}(\mathbb{A})$ and if $\mathcal{O} \in \mathcal{O}_G(\pi)$ then $\mathcal{O}$ is special. The proof is somewhat similar to the local case as done in [M].

In case π is cuspidal, the result we prove is weaker than the one in the local p-adic case. Before stating it let us emphasize that since we are working over the rational points, then for a given unipotent class one can possibly associate infinitely many nonconjugate Fourier coefficients. Hence it can happen that over the rational points some unipotent elements u in $\mathcal{O}$ could be conjugated into a Levi part and some may not. We did not prove it but we believe that if u belongs to $\mathcal{O}$ which contains an odd number (see [C] or [C-M]) then u can be conjugated into a Levi part of G. Motivated by the local results of [M-W] we conjecture that if $\mathcal{O} \in \mathcal{O}_G(\pi)$ and π is cuspidal then $\mathcal{O}$ is even i.e. a partition which consists of even numbers.

In this paper we prove (Theorem 2.7) that there is an even unipotent orbit $\mathcal{O} \in \mathcal{O}_G(\pi)$. We do however expect that every $\mathcal{O} \in \mathcal{O}_G(\pi)$ will be even.

Finally, we mention the conjecture stated in [M1] p. 259 (and the references cited there) that $\mathcal{O}_G(\pi)$ consists of a unique unipotent class. We do expect this to happen also over global fields but we have no evidence of that.

1. Fourier coefficients and unipotent classes

In this section we shall show how to associate to a given unipotent class a set of Fourier coefficients of a given automorphic form. In [M-W] a similar construction is defined over p-adic fields.

The basic references for the classification of unipotent classes can be found in [C] or [C-M]. We recall that unipotent classes of classical groups are parameterized by partitions. For $G = \mathrm{Sp}_{2n}$ unipotent classes are parameterizes by all partitions of $2n$ where odd numbers occur with even multiplicity. To each partition $\mathcal{O}$ we associate a one-dimensional torus $h_{\mathcal{O}}(t)$ as explained in [C] or [C-M] page 80. We shall always choose $h_{\mathcal{O}}(t)$ to be written in decreasing order of the exponents of t. For example if $\mathcal{O} = (3^2 2)$ in Sp_8 then $h_{\mathcal{O}}(t) = \mathrm{diag}(t^2, t^2, t, 1, 1, t^{-1}, t^{-2}, t^{-2})$.

In terms of matrices, Sp_{2n} is defined with respect to the matrix $\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$ where $J_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. The torus acts on each one parameter subgroup $x_\alpha(r)$ associated with any root α as

$$h_{\mathcal{O}}(t) x_\alpha(r) h_{\mathcal{O}}(t)^{-1} = x_\alpha(t^i r).$$

By our choice if $x_\alpha(r)$ is upper unipotent then $i \geq 0$. Let $V_1(\mathcal{O})$ be the group generated by all $x_\alpha(r)$ such that $i \geq 1$. Then $V_1(\mathcal{O})$ is a unipotent radical of a parabolic subgroup $P(\mathcal{O}) = M(\mathcal{O})V_1(\mathcal{O})$ of G. We denote by $V_2(\mathcal{O})$ the group generated by all $x_\alpha(r)$ such that $i \geq 2$. Clearly $V_2(\mathcal{O}) \subset V_1(\mathcal{O})$. The group $V_2(\mathcal{O})/[V_2(\mathcal{O}), V_2(\mathcal{O})]$ can be identified with an abelian matrix group and it is well known that over the closure $M(\mathcal{O})$ acts on this quotient with an open orbit. If M^0 is the stabilizer of this open orbit then M^0 is reductive.

Let ψ be a nontrivial additive character of $F\backslash\mathbb{A}$, where F is a number field and $\mathbf{A}$ is its ring of adeles. The group $M(\mathcal{O})(F)$ acts on the group of all characters of $V_2(\mathcal{O})/[V_2(\mathcal{O}), V_2(\mathcal{O})]$ with points in $F\backslash\mathbb{A}$. Let $\psi_{V_2(\mathcal{O})}$ be such a character whose stabilizer in $M(\mathcal{O})(F)$ is of type M^0. The choice of $\psi_{V_2(\mathcal{O})}$ is not unique and in fact there could be infinitely many such characters which are not conjugate under $M(\mathcal{O})(F)$.

Let $\varphi(g)$ be an automorphic function on G. To the unipotent class $\mathcal{O}$ we associate the set of Fourier coefficients given by

$$f_1(g) = \int\limits_{V_2(\mathcal{O})(F)\backslash V_2(\mathcal{O})(\mathbb{A})} \varphi(vg)\psi_{V_2(\mathcal{O})}(v)dv \qquad (1.1)$$

where $g \in G(\mathbb{A})$.

Example. Let $G = \mathrm{Sp}_4$ and $\mathcal{O} = (2^2)$. Then $h_{\mathcal{O}}(t) = \mathrm{diag}(t, t, t^{-1}, t^{-1})$, and $M(\mathcal{O}) = \mathrm{GL}_2$. $V_1(\mathcal{O}) = V_2(\mathcal{O}) = \left\{\begin{pmatrix} 1 & & x & y \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{pmatrix}\right\}$. In this case the set of Fourier coefficients associated with $\mathcal{O}$ is given by

$$\int\limits_{(F\backslash\mathbb{A})^3} \varphi\left[\begin{pmatrix} 1 & & x & y \\ & 1 & z & x \\ & & 1 & \\ & & & 1 \end{pmatrix} g\right]\psi(\alpha y + \beta z)dxdydz$$

where $\alpha, \beta \in (F^*)^2\backslash F^*$. Indeed the stabilizer in $M(\mathcal{O})(F)$ of each such character is a one dimensional torus. □

If $V_1(\mathcal{O})/[V_2(\mathcal{O}), V_2(\mathcal{O})]$ is a generalized Heisenberg group then we can write this quotient as $X \oplus Y \oplus Z$ where $Z = V_2(\mathcal{O})/[V_2(\mathcal{O}), V_2(\mathcal{O})]$ and X and Y are maximal abelian subgroups of $V_1(\mathcal{O})/V_2(\mathcal{O})$. There is always a choice of such a polarization so that X and Y are subgroups of $V_1(\mathcal{O})$ that preserve $\psi_{V_2(\mathcal{O})}$ and hence we can define

$$f_2(g) = \int\limits_{Y(F)\backslash Y(\mathbb{A})} \int\limits_{V_2(\mathcal{O})(F)\backslash V_2(\mathcal{O})(\mathbb{A})} \varphi(vyg)\psi_{V_2(\mathcal{O})}(v)dvdy. \qquad (1.2)$$

Notice that $f_1(g)$ defines an automorphic function on $M^0(F)\backslash M^0(\mathbb{A})$ whereas $f_2(g)$ does not. To fix it we define a third integral as follows. Since $V_1(\mathcal{O})/[V_2(\mathcal{O}), V_2(\mathcal{O})]$ is a generalized Heisenberg group we can find a homomorphism $\sigma : V_1(\mathcal{O})(\mathbb{A}) \to H_{2m+1}(\mathbb{A})$ where H_{2m+1} is the Heisenberg group with $2m+1$ variables. Thus every element in $H_{2m+1}(\mathbb{A})$ can be written as $(x|y|z)$ where $x, y \in \mathbb{A}^n$

and $z \in \mathbb{A}$ is the center of $H_{2m+1}(\mathbb{A})$. We can choose σ so that it maps $V_1(\mathcal{O})/V_2(\mathcal{O})$ onto the set $(x|y|0)$. Also if $x_\alpha(r)$ is in $V_2(\mathcal{O})$ such that $\psi_{V_2(\mathcal{O})}(x_\alpha(r))$ is nontrivial and $\alpha = \beta_1 + \beta_2$ where $x_{\beta_1}(r_1)$ and $x_{\beta_2}(r_2)$ are in $V_1(\mathcal{O})/V_2(\mathcal{O})$ then σ maps $x_\alpha(r)$ onto the center of H_{2m+1}. All this means that M^0 can be embedded in Sp_{2m} in such a way that its action on H_{2m+1} is compatible with its action on $V_1(\mathcal{O})/V_2(\mathcal{O})$.

Let $\widetilde{\theta}_\phi^{\psi_\mathcal{O}}$ denote a theta function on $\widetilde{\mathrm{Sp}}_{2m}(\mathbf{A})$. Here $\psi_\mathcal{O}$ is chosen to be compatible with $\psi_{V_2(\mathcal{O})}$. That is $\widetilde{\theta}_\phi^{\psi_\mathcal{O}}(\sigma(z)g) = \psi_{V_2(\mathcal{O})}(z)\widetilde{\theta}_\phi^{\psi_\mathcal{O}}(g)$ where $z \in V_2(\mathcal{O})$ is mapped to $(0|0|*)$ under σ. Recall that the theta function is a function on $H_{2m+1}(\mathbf{A})\widetilde{\mathrm{Sp}}_{2m}(\mathbf{A})$. Thus we can define the integral

$$\widetilde{f}_3(h) = \int\limits_{V_1(\mathcal{O})(F)\backslash V_1(\mathcal{O})(\mathbb{A})} \widetilde{\theta}_\phi^{\psi_\mathcal{O}}(\sigma(v)h)\varphi(vh)\psi_{V_2(\mathcal{O})}(\ker\sigma(v))dv \tag{1.3}$$

where $h \in M^0(F)\backslash\widetilde{M}^0(\mathbb{A})$ i.e. it is automorphic function on the inverse image of M^0 in $\widetilde{\mathrm{Sp}}_{2m}(\mathbf{A})$. Here $ker(v)$ denotes the projection of v on $V_2(\mathcal{O})/[V_2(\mathcal{O}), V_2(\mathcal{O})]$. Note that $\widetilde{f}_3$ need *not* be a genuine function. If $\varphi \in \pi$ we denote the space of functions generated by $\widetilde{f}_3$ by $D_\mathcal{O}(\pi)$. Thus $D_\mathcal{O}(\pi)$ is an automorphic representation of $M^0(F)\backslash\widetilde{M}^0(\mathbb{A})$. We have,

Lemma 1.1. *If one of the integrals (1.1), (1.2) or (1.3), is zero for all choice of data then the other two are also zero for all choice of data.*

Proof. We may assume $g = h = e$. Starting with integral (1.1) and given a polarization $X \oplus Y$ of $V_1(\mathcal{O})/V_2(\mathcal{O})$ we write a Fourier expansion along $Y(F)\backslash Y(\mathbb{A})$. Thus (1.1) equals

$$\sum_{\xi\in F^m} \int\limits_{Y(F)\backslash Y(\mathbb{A})} \int\limits_{V_2(\mathcal{O})(F)\backslash V_2(\mathcal{O}(\mathbb{A})} \varphi(vy)\psi_{V_2(\mathcal{O})}(v)\psi(y\cdot\xi)dvdy$$

where $y \cdot \xi$ is the standard bilinear product of y with ξ when we identify $Y(\mathbb{A})$ with $\mathbb{A}^m$. Let $\xi_1 \in X(F)$. Since φ is automorphic it is left invariant under ξ_1. Conjugating ξ_1 across v and y, the above integral equals

$$\sum_{\xi\in F^m} \int \varphi(v(\xi_1, y)y\xi_1)\psi_{V_2(\mathcal{O})}(v)\psi(y\cdot\xi)dvdy$$

where (ξ_1, y) is the matrix obtained from the conjugation of ξ_1 across y. Here v and y are integrated as before. Clearly (ξ_1, y) lies in $V_2(\mathcal{O})$ modulo $[V_2(\mathcal{O}), V_2(\mathcal{O})]$ and we can choose ξ_1 such that the change of variables $v \to v(\xi_1, y)^{-1}$ will give $\psi_{V_2(\mathcal{O})}(v(\xi_1, y)^{-1}) = \psi_{V_2(\mathcal{O})}(v)\overline{\psi}(y\cdot\xi)$. Thus we may conclude that (1.1) equals

$$\sum_{\xi\in F^m} \int\limits_{Y(F)\backslash Y(\mathbb{A})} \int\limits_{V_2(\mathcal{O})(F)\backslash V_2(\mathcal{O})(\mathbb{A})} \varphi(vy\xi_1)\psi_{V_2(\mathcal{O})}(v)dvdy$$

where ξ_1 depends on ξ. From this it follows that (1.1) is zero for all choice of data if and only if (1.2) is zero for all choice of data. Next we start with (1.3). It equals

$$\int_{V_2(\mathcal{O})(\mathbb{A})V_1(\mathcal{O})(F)\backslash V_1(\mathcal{O})(\mathbb{A})} \int_{V_2(\mathcal{O})(F)\backslash V_2(\mathcal{O})(\mathbb{A})} \widetilde{\theta}_{\phi}^{\psi_{\mathcal{O}}}(\sigma(v_2)\sigma(v_1))\varphi(v_2v_1) \times \psi_{V_2(\mathcal{O})}(\ker\sigma(v_2))dv_2dv_1$$

Identifying $V_2(\mathcal{O})(\mathbb{A})V_1(\mathcal{O})(F)\backslash V_1(\mathcal{O})(\mathbb{A})$ with $X \oplus Y$ where X, Y are identified with $(F\backslash\mathbb{A})^m$ we can write the above integral as

$$\int \widetilde{\theta}_{\phi}^{\psi_{\mathcal{O}}}(\sigma(v_2)\sigma(y)\sigma(x))\varphi(v_2yx)\psi_{V_2(\mathcal{O})}(\ker\sigma(v_2))dv_2dydx$$

where x is integrated over $X(F)\backslash X(\mathbb{A})$, y over $Y(F)\backslash Y(\mathbb{A})$ and v_2 as before. Unfolding the theta function we have

$$\begin{aligned}\widetilde{\theta}_{\phi}^{\psi_{\mathcal{O}}}(\sigma(v_2)\sigma(y)\sigma(x)) &= \sum_{\xi\in F^m} \omega_{\psi_{\mathcal{O}}}(\sigma(v_2)\sigma(y)\sigma(x))\phi(\xi)\\ &= \sum_{\xi\in F^m} \omega_{\psi_{\mathcal{O}}}(\sigma(\xi)\sigma(v_2)\sigma(y)\sigma(x))\phi(0)\end{aligned}$$

Here we identify $X(F)$ with F^m, and view ξ accordingly. Since φ is automorphic it is left invariant under ξ. Conjugating in the above integral ξ and $\sigma(\xi)$ across, changing variables and collapsing summation with integration we obtain

$$\int_{X(\mathbb{A})} \int \omega_{\psi_{\mathcal{O}}}(\sigma(v_2)\sigma(y)\sigma(x))\phi(0)\varphi(v_2yx)\psi_{V_2(\mathcal{O})}(\ker\sigma(v_2))dv_2dydx$$

where y and v_2 are integrated as before. Using the Weil representation action we conclude that (1.3) equals

$$\int_{X(\mathbb{A})} \phi(x) \int_{Y(F)\backslash Y(\mathbb{A})} \int_{V_2(\mathcal{O})(F)\backslash V_2(\mathcal{O})(\mathbb{A})} \varphi(v_2yx)\psi_{V_2(\mathcal{O})}(v_2)dv_2dydx$$

Thus if (1.2) is zero for all choice of data so is (1.3). Since $\phi \in \mathcal{S}(\mathbb{A}^m)$ is an arbitrary Schwartz function, the vanishing of (1.3) for all choice of data implies that (1.2) is zero for all choice of data. □

Let G be a reductive group and let π be an automorphic representation of $G(\mathbb{A})$. It is clear that π has a nontrivial Fourier coefficient which corresponds to the minimal unipotent class. Since there is a partial order on the set of unipotent classes corresponding to G, it is natural to ask which are the largest unipotent classes such that π has a nontrivial Fourier coefficient corresponding to these classes. More precisely,

Definition. *Let G be a reductive group and π an automorphic representation of π. We shall denote by $\mathcal{O}_G(\pi)$ the set of all unipotent classes such that π has a non-trivial Fourier coefficient corresponding to these classes, and if $\mathcal{O}$ is any unipotent class which is larger than some member in $\mathcal{O}_G(\pi)$ then integral* (1.1) *vanishes for all choice of data for this unipotent class $\mathcal{O}$.* □

Example. Let $G = \widetilde{\mathrm{Sp}}_4$ (double cover of Sp_4). Let π be the theta function on G. Then $\mathcal{O}_G(\pi) = (21^2)$. □

It will be convenient to use the following terminology. Given a unipotent class $\mathcal{O}$ of a group G and an automorphic representation π of $G(\mathbb{A})$ we will say that π supports $\mathcal{O}$ if there is a choice of data such that the Fourier coefficient of π corresponding to $\mathcal{O}$ as defined in (1.1), is not zero. Similarly, we say that π vanishes on $\mathcal{O}$ if integral (1.1) which corresponds to $\mathcal{O}$ is zero for all choice of data. It is important to realize that this nonvanishing or vanishing might depend on the additive character in question. In other words, it is possible that π will vanish on $\mathcal{O}$ for a certain character and will not vanish on $\mathcal{O}$ for another character.

Let $\mathbf{f}$ be a unipotent class corresponding to G. If $V_1(\mathbf{f}) \neq V_2(\mathbf{f})$ then integral (1.3), which corresponds to $\mathbf{f}$ defines an automorphic function (genuine or not) on the group $M^0(F)\backslash\widetilde{M}^0(\mathbb{A})$. If $\mathbf{g}$ is a unipotent class of the group M^0 we let $\mathbf{f} \circ \mathbf{g}$ denote the Fourier coefficient corresponding to the integration of (1.3) composed with the Fourier coefficient corresponding to the unipotent class $\mathbf{g}$. Let U_{M^0} be the maximal unipotent subgroup of M^0. With a certain choice of Y integral (1.2) is a function defined over $U_{M^0}(F)\backslash U_{M^0}(\mathbb{A})$. One can check following the proof of Lemma 1.1 that

$$\int\limits_{V(F)\backslash V(\mathbb{A})} f_2(v)\psi_V(v)dv$$

is zero for all choice of data if and only if

$$\int\limits_{V(F)\backslash V(\mathbb{A})} f_3(v)\psi_V(v)dv \tag{1.4}$$

is zero for all choice of data. Here V is a unipotent subgroup of U_{M^0} and ψ_V any additive character on $V(F)\backslash V(\mathbb{A})$. Hence, as far as vanishing or nonvanishing properties, $\mathbf{f} \circ \mathbf{g}$ can be studied by either (1.2) or (1.3). However this is *not* the case with integral (1.1) which is also a function on $U_{M^0}(F)\backslash U_{M^0}(\mathbb{A})$. All we can say is that if

$$\int\limits_{V(F)\backslash V(\mathbb{A})} f_1(v)\psi_V(v)dv$$

is zero for all choice of data then (1.4) is zero for all choice of data. But the converse need not be true.

2. Fourier coefficients on symplectic groups

Let G denote the group Sp_{2k} or the double cover of the symplectic group. We shall now describe and make a certain choice of subgroups of G in order to describe integrals (1.1)–(1.3). Let $\mathcal{O}$ be a unipotent class of G given by

$$\mathcal{O} = \left(1^{2(n_1-n_2)} 2^{m_1-m_2} 3^{2(n_2-n_3)} \dots (2k-1)^{2n_k} (2k)^{m_k}\right) \tag{2.1}$$

where $n_i \geq n_{i+1}$ and $m_j \geq m_{j+1}$. With these notations we have $M(\mathcal{O}) = \mathrm{Sp}_{2n_1} \times \prod_{i=1}^{k} \mathrm{GL}_{m_i} \times \prod_{j=2}^{k} \mathrm{GL}_{2n_j}$ where GL_0 means the identity group. We also have

$$V_1(\mathcal{O})/V_2(\mathcal{O}) = \Big(\bigoplus_{i=1}^{k} M_{m_i \times 2n_i}\Big) \bigoplus \Big(\bigoplus_{j=2}^{k} M_{2n_j \times m_{j-1}}\Big)$$

and

$$V_2(\mathcal{O})/[V_2(\mathcal{O}), V_2(\mathcal{O})] = M'_{m_1 \times m_1} \bigoplus \Big(\bigoplus_{i=2}^{k} M_{2n_i \times 2n_{i-1}}\Big) \bigoplus \Big(\bigoplus_{j=2}^{k} M_{m_j \times m_{j-1}}\Big).$$

Here $M_{\ell_1 \times \ell_2}$ is the group of all $\ell_1 \times \ell_2$ matrices and $M'_{m_1 \times m_1} = \{\ell \in M_{m_1 \times m_1} : J_{m_1}\ell - \ell^t J_{m_1} = 0\}$. The action of $M(\mathcal{O})$ on $V_2(\mathcal{O})/[V_2(\mathcal{O}), V_2(\mathcal{O})]$ is given as follows. The group Sp_{2n_1} act on $M_{2n_2 \times 2n_1}$ as $x \to xg^{-1}$. The group GL_{m_1} acts on $M'_{m_1 \times m_1} \oplus M_{m_2 \times m_1}$ as $g : (x, y) \to (gxg^*, yg^{-1})$ where $g^* = Jg^t J$. For $i > 1$, GL_{m_i} and GL_{2n_i} acts on $M_{m_i \times m_{i-1}} \oplus M_{m_{i+1} \times m_i}$ and $M_{2n_i \times 2n_{i-1}} \oplus M_{2n_{i+1} \times 2n_i}$ respectively as $g : (x, y) \to (gx, yg^{-1})$.

Next we describe $\psi_{V_2(\mathcal{O})}$. We start with the groups $M_{2n_i \times 2n_{i-1}}$. Define for a given matrix $\ell \in M_{2n_i \times 2n_{i-1}}$

$$\begin{aligned}\psi_{V_2(\mathcal{O})}(\ell) = \psi(&\ell_{1,1} + \ell_{2,2} + \dots + \ell_{n_i,n_i} + \ell_{n_i+1,2n_{i-1}-n_i+1} \\ &+ \ell_{n_i+2,2n_{i-1}-n_i+2} + \dots + \ell_{2n_i,2n_{i-1}})\end{aligned}$$

With this definition the stabilizer in $\mathrm{GL}_{2n_2} \times \mathrm{Sp}_{2n_1}$ is isomorphic to $\mathrm{Sp}^{\triangle}_{2n_2} \times \mathrm{Sp}_{2(n_2-n_1)}$ and the stabilizer in $\mathrm{GL}_{2n_i} \times \mathrm{GL}_{2n_{i-1}}$ for $i \geq 2$ is $\mathrm{Sp}^{\triangle}_{2n_i} \times \mathrm{Sp}_{2(n_{i-1}-n_i)}$ where $\mathrm{Sp}^{\triangle}_{2n_i}$ is the diagonal embedding. Continuing this process for all i we obtain that the stabilizer in $M(\mathcal{O})$ of $\psi_{V_2(\mathcal{O})}$ restricted to $\bigoplus_{i=2}^{k} M_{2n_i \times 2n_{i-1}}$ is $\prod_{i=2}^{k} \mathrm{Sp}^{\triangle}_{2(n_{i-1}-n_i)}$ as given in [C-M]. Next, for $\ell \in M'_{m_1 \times m_1}$ we define

$$\psi_{V_2(\mathcal{O})}(\ell) = \psi(\epsilon_1 \ell_{1,m_1} + \dots + \epsilon_{m_1}\ell_{m_1,1})$$

where $\epsilon_i \in F^*$. For $\ell \in M_{m_j \times m_{j-1}}$ we define $\psi_{V_2(\mathcal{O})}(\ell) = \psi(\ell_{1,1} + \dots + \ell_{m_j,m_j})$. The stabilizer of $\psi_{V_2(\mathcal{O})}$ in $M(\mathcal{O})$ for this part is $\prod_{j=2}^{k} O^{\triangle}_{2(m_{j-1}-m_j)}$, again as given in [C-M]. Finally, we describe the group Y which is contained in $V_1(\mathcal{O})/V_2(\mathcal{O})$.

In $M_{m_i\times 2n_i}$ we choose all matrices of the form (0ℓ) where $\ell\in M_{m_i\times n_i}$ and in $M_{2n_j\times m_{j-1}}$ we choose all matrices of the form $\begin{pmatrix}\ell\\0\end{pmatrix}$ where $\ell\in M_{n_j\times m_{j-1}}$. This choice of group Y preserves $\psi_{V_2(\mathcal{O})}$ and if we let U_{M^0} be the maximal unipotent subgroup of M^0 which consists of upper unipotent matrices then U_{M^0} normalizes Y and hence integrals (1.1)–(1.3) are well defined for any subgroup of $U_{M^0}(F)\backslash U_{M^0}(\mathbb{A})$. (Of course for some $\mathcal{O}$, U_{M^0} may be trivial). The following Theorem is the global version of Theorem 1.4 in [M].

Theorem 2.1. *Let π be an irreducible automorphic representation of $G=\mathrm{Sp}_{2k}(\mathbf{A})$. Then $\mathcal{O}_G(\pi)$ consists of unipotent classes which are special.*

Proof. We sketch the idea. For definition of a special unipotent class see [C-M]. Suppose that $\mathcal{O}$ in $\mathcal{O}_G(\pi)$ which is not special is given by (2.1). Let $(2r-1)$ be the largest integer such that $2(n_r-n_{r+1})$ is nonzero and the number of even numbers, in the partition, which are larger than $2r-1$ is odd. Such a $2r-1$ exists by definition of special partitions. As explained in [M] (see also [N]) it follows that the representation given by $D_{\mathcal{O}}(\pi)$ as defined by (1.3) is a genuine representation of $\widetilde{\mathrm{Sp}}_{2(n_r-n_{r+1})}(\mathbf{A})$ which is a subgroup of M^0. Let $x_\alpha(\ell)$ denote the one parameter unipotent subgroup of $\mathrm{Sp}_{2(n_r-n_{r+1})}$ corresponding to the highest weight root vector in this group. Since $D_{\mathcal{O}}(\pi)$ is genuine there is $a\in F^*$ such that

$$\int\limits_{F\backslash\mathbb{A}} \widetilde{f}_3(x_\alpha(\ell))\psi(a\ell)d\ell \tag{2.2}$$

is nonzero for some choice of data. Let $\widetilde{\mathcal{O}}$ be the unipotent class obtained from $\mathcal{O}$ by replacing $(2r-1)^2$ by $(2r-2)(2r)$. Thus $\widetilde{\mathcal{O}}>\mathcal{O}$ and applying Fourier expansions one can check that the nonvanishing of (2.2) is equivalent to the nonvanishing of (1.3), for some choice of data, where the Fourier coefficient corresponds to the unipotent class $\widetilde{\mathcal{O}}$. This contradicts the maximality of $\mathcal{O}$. □

Remark. With the appropriate definition of special representations, a similar Theorem is valid also for automorphic representations on the group $\widetilde{\mathrm{Sp}}_{2k}(\mathbf{A})$.

We now assume that π is a cusp form on the group $G=\mathrm{Sp}_{2k}(\mathbf{A})$ or on its double cover. To prove our main theorem we start with a few lemmas. Let U_r, for $r\le k$ be the unipotent radical of the parabolic subgroup of G whose Levi part is $\mathrm{GL}_1^r\times\mathrm{Sp}_{2(k-r)}$. We define a character ψ_{U_r} of U_r as follows. If $u\in U_r$ then $\psi_{U_r}(u)=\psi(u_{1,2}+\cdots+u_{r-1,r})$. We start with:

Lemma 2.2. *Let $\varphi\in\pi$. If the integral*

$$\int\limits_{U_r(F)\backslash U_r(\mathbb{A})} \varphi(ug)\psi_{U_r}(u)du \tag{2.3}$$

is nonzero for some choice of data and for $r<k$, then there exists a number $k\ge m>r$ such that π has a nontrivial Fourier coefficient corresponding to the unipotent class $((2m)1^{2(k-m)})$.

Proof. Define the matrix $x_\alpha(\ell) = I_{2k} + \ell e_{r+1,2k-r}$ where $e_{i,j}$ is the $2k \times 2k$ matrix with one at the (i, j) position and zero otherwise. Thus $x_\alpha(r)$ is the one-dimensional unipotent subgroup which corresponds to the highest weight root vector in $\mathrm{Sp}_{2(k-r)}$ (embedded in Sp_{2k}). Thus we can expand (2.3) with respect to $x_\alpha(\ell)$ where ℓ is in $F\backslash\mathbb{A}$. Since (2.3) is nonzero for some choice of data it follows that there is an $a \in F$ such that the integral

$$\int_{F\backslash\mathbb{A}} \int_{U_r(F)\backslash U_r(\mathbb{A})} \varphi(ux_\alpha(\ell)g)\psi_{U_r}(u)\psi(a\ell)du d\ell$$

is nonzero for some choice of data. If $a \neq 0$ then the lemma follows with $m = r+1$. If the above integral is nonzero only when $a = 0$ define $z(\ell_1, \dots, \ell_{2(k-r-1)}) = I_{2k} + \ell_1 e'_{r+1,r+1} + \cdots + \ell_{2(k-r-1)} e'_{r+1,2k-r-1}$. Here $e'_{ij} = e_{ij} - e_{2k-j+1,2k-i+1}$. We now expand the above integral, with $a = 0$, along $z(\ell_1, \dots, \ell_{2(k-r-1)})$ with points in $F\backslash\mathbb{A}$. The group $\mathrm{Sp}_{2(k-r-1)}$ acts on the characters of $z(\ell_1, \dots, \ell_{2(k-r-1)})$ with two orbits. The trivial orbit will contribute zero by cuspidality whereas the nontrivial orbit will give the nonvanishing of the integral

$$\int_{U_{r+1}(F)\backslash U_{r+1}(\mathbb{A})} \varphi(ug)\psi_{U_{r+1}}(u)du.$$

Continuing by induction the result follows. □

Next we prove:

Lemma 2.3. *Let $\mathcal{O} = ((2r)1^{2(k-r)})$ for $r < k$. Let V be any unipotent radical subgroup of a maximal parabolic subgroup of* $\mathrm{Sp}_{2(k-r)}$. *If the integral* $\int_{V(F)\backslash V(\mathbb{A})} f_2(vg)dv$ *(where f_2 is defined in* (1.2) *corresponding to $\mathcal{O}$) is nonzero for some choice of data then there exists $k \geq m > r$ such that* $((2m)1^{2(k-m)})$ *supports π.*

Proof. This follows as in the proof of Theorem 8 in [G-R-S1]. □

Lemma 2.4. *Let $\mathcal{O} = ((2r+1)^2 d_2 \dots d_s)$ with $2r+1 \geq d_i$ for all i, and suppose that $\mathcal{O}$ supports π. Then there exists a number m such that $2m > 2r+1$ and that $((2m)1^{2(k-m)})$ supports π.*

Proof. From the structure of $\mathcal{O}$ we deduce that

$$h_{\mathcal{O}}(t) = \mathrm{diag}(t^{2r}, t^{2r}, \dots, t^{d_2-1}, \dots, t^{-(d_2-1)}, \dots, t^{-2r}, t^{-2r}).$$

There is a Weyl group element w which conjugates $h_{\mathcal{O}}(t)$ to the torus

$$h(t) = \mathrm{diag}(t^{2r}, t^{2r-2}, \dots, t^{-(2r-2)}, t^{-2r}, t^{d_2-1}, \dots, t^{-(d_2-1)}, t^{2r}, \dots, t^{-2r})$$

Consider the integral (1.2) corresponding to the unipotent class $\mathcal{O}$. Conjugating it by the above Weyl element we deduce that the integral

$$\int \varphi\left[\begin{pmatrix} u_1 & \ell_1 & \ell_2 \\ 0 & u_2 & \ell_1^* \\ 0 & 0 & u_1^* \end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ q_1 & I & 0 \\ q_2 & q_1^* & I \end{pmatrix}\right]\psi_{U_1}(u_1)\psi_{U_2}(u_2)du_i d\ell_j dq_n \qquad (2.4)$$

is nonzero for some choice of data. We now describe the domain of integration. To determine it we conjugate the above variables by $h(t)$. Since, in (1.2), we integrate over upper unipotent matrices generated by all positive roots, for which $h_{\mathcal{O}}(t)$ acts with powers t^i for $i \geq 2$ and by half the numbers of all roots for which $h_{\mathcal{O}}(t)$ acts with a power of t, the same will be true when we conjugate the variables in (2.4) by $h(t)$. We can choose the Weyl element so that the following happens. First notice that $h(t)$ acts on u_1 with powers of t^i with $i \geq 2$. This means that u_1 is integrated over all upper unipotent matrices in GL_{2r+1}. Moreover, if $u_1 = (u_{ij})$ then $h(t)$ acts on $u_{i,i+1}$ with power of t^2. Thus, we may choose w so that $\psi_{U_1}(u_1) = \psi(\Sigma u_{i,i+1})$. Let $\widetilde{\mathcal{O}} = (d_2 \cdots d_s)$. We have $h_{\widetilde{\mathcal{O}}}(t) = \mathrm{diag}(t^{d_2-1}, \ldots, t^{-(d_2-1)})$ and hence u_2 is integrated over the subgroup of $V_1(\widetilde{\mathcal{O}})$ corresponding to $\widetilde{\mathcal{O}}$ and $\psi_{U_2}(u_2) = \psi_{V_2(\widetilde{\mathcal{O}})}(u_2)$. From the action of $h(t)$ we also deduce that $\ell_2, q_2 \in M^0_{(2r+1)\times(2r+1)}$ where $M^0_{(2r+1)\times(2r+1)} = \{\ell \in M'_{(2r+1)\times(2r+1)} : \ell_{ij} = 0 \text{ if } i \geq j\}$. Indeed $h(t)$ acts on matrices in $M^0_{(2r+1)\times(2r+1)}$ with powers of t^i with $i \geq 2$. Next, we consider the variables ℓ_1 and q_1. Let $h_1(t) = \mathrm{diag}(t^{2r}, t^{2r-2}, \ldots, t^{-2r})$. For $e_{ij} \in M_{(2r+1)\times(d_2+\cdots+d_s)}$ we let p be the integer defined by $h_1(t)e_{ij}h_{\widetilde{\mathcal{O}}}(t)^{-1} = t^p e_{ij}$. Set $M^1_{(2r+1)\times(d_2+\cdots+d_s)} = \{\ell = (\ell_{ij}) \in M_{(2r+1)\times(d_2+\cdots+d_s)} : \ell_{ij} = 0$ if $p \leq 0\}$. In these notations the above matrix $\ell_1 \in M^1_{(2r_1+1)\times(d_2+\cdots+d_s)}$. Similarly, if $e_{ij} \in M_{(d_2+\cdots+d_s)\times(2r+1)}$ let p be the integer defined by $h_{\widetilde{\mathcal{O}}}(t)e_{ij}h_1(t)^{-1} = t^p e_{ij}$. Set $M^2_{(d_2+\cdots d_s)\times(2r+1)} = \{\ell = (\ell_{ij}) \in M_{(d_2+\cdots+d_s)\times(2r+1)} : \ell_{ij} = 0 \text{ if } p \leq 1\}$. With the notations the above matrix $q_1 \in M^1_{(d_2+\cdots+d_s)\times(2r+1)}$. Let us mention that the difference between these two groups (i.e. the fact that $p \leq 0$ in the first and $p \leq 1$ in the second) is due to fact that we integrate only "half" of the roots in $V_1(\mathcal{O})/V_2(\mathcal{O})$, that is over Y and not over X. We choose the Weyl element w to conjugate Y to upper unipotent matrices. In integral (2.4) all variables are integrated in their groups with points in $F\backslash\mathbb{A}$. We will show, using Fourier expansions, that (2.3) is an inner integration of (2.4) and hence it is nonzero for some choice of data. The lemma will follow using Lemma 2.2. For all $1 \leq j \leq i \leq r$ let $z(\ell'_{ij}) = \ell'_{ij}(e_{ij} + e_{2r-j+2,2r-i+2})$. Thus $z(\ell'_{ij})$ is a matrix in $M'_{(2r+1)\times(2r+1)}$ (embedded in the "l_2" corner). We let $\ell'_2 = \oplus z(\ell'_{ij})$ where the sum runs over $1 \leq j \leq i \leq r$. We expand (2.4) along ℓ'_2 where ℓ'_{ij} are in $F\backslash\mathbb{A}$. We obtain

$$\sum_{\beta_{ij}} \int \varphi \left[\begin{pmatrix} u_1 & \ell_1 & \ell'_2 + \ell_2 \\ 0 & u_2 & \ell_1^* \\ 0 & 0 & u_1^* \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ q_1 & I & 0 \\ q_2 & q_1^* & I \end{pmatrix} \right] \psi_{U_1}(u_1)\psi_{U_2}(u_2)\psi(\Sigma\beta_{ij}\ell'_{ij})d(\ldots)\,.$$

Here $\beta_{i,j} \in F$ for all $1 \leq j \leq i \leq r$. Next, for $1 \leq j \leq i \leq r$, define the matrix $\overline{z}(q'_{ij}) = q'_{ij}(e_{j,i+1} + e_{2r-i+1,2r-j+2})$. Denote $q'_2 = \oplus\overline{z}(\beta_{i,j})$ where the sum is over $1 \leq j \leq i \leq r$. Thus $q'_2 \in M^0_{(2r+1)\times(2r+1)}$. The matrix $\begin{pmatrix} I & & \\ & I & \\ q'_2 & & I \end{pmatrix}$ is a rational matrix and hence φ is left invariant by this matrix. Conjugating this matrix from left to right in the above integral, changing variables and collapsing summation with integration we obtain

$$\int \varphi \left[\begin{pmatrix} u_1 & \ell_1 & \overline{\ell}_2 \\ 0 & u_2 & \ell_1^* \\ 0 & 0 & u_1^* \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ q_1 & I & 0 \\ \overline{q}_2 & q_1^* & I \end{pmatrix} \right] \psi_{U_1}(u_1)\psi_{U_2}(u_2)d(\ldots)$$

Here $\overline{\ell}_2$ is in $M''_{(2r+1)\times(2r+1)} = \{\ell \in M'_{(2r+1)\times(2r+1)} : \ell_{ij} = 0 \text{ if } r+1 \le i \text{ and } j \le r+1\}$ with points in $F\backslash\mathbb{A}$. Also $\overline{q}_2 \in M^0_{(2r+1)\times(2r+1)}$ where some variables are integrated over $F\backslash\mathbb{A}$ and some over $\mathbb{A}$. Since it will not matter to us we shall not be more explicit. Next we proceed as above with the matrix ℓ_1. More precisely, suppose that for $\ell_1 = (\ell_{ij})$ we have $\ell_{i,j} = 0$ where $i \le r$. This means that $h_1(t)e_{ij}h_{\widetilde{\mathcal{O}}}(t)^{-1} = t^{-p}e_{ij}$ with $p \ge 0$. If we expand the above integral along $\widetilde{\ell}_1 = \ell_{ij}e_{ij}$ where $\ell_{ij} \in F\backslash\mathbb{A}$ we obtain

$$\sum_{\beta_{ij}} \varphi \left[\begin{pmatrix} u_1 & \widetilde{\ell}_1 + \ell_1 & \overline{\ell}_2 \\ 0 & u_2 & * \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ q_1 & I & 0 \\ \overline{q}_2 & q_1^* & I \end{pmatrix} \right] \psi_{U_1}(u_1)\psi_{U_2}(u_2)\psi(\beta_{ij}\ell_{ij})d(\ldots)$$

where $\beta_{ij} \in F$. From the condition on (i, j) and p we obtain that $h_{\widetilde{\mathcal{O}}}(t)e_{j,i+1}$ $h_1(t)^{-1} = t^{p+2}e_{ij}$. This means that $\widetilde{q}_1 = \epsilon e_{j,i+1}$ is a matrix in $M^2_{(d_2+\cdots+d_s)\times(2r+1)}$ where ϵ is any variable. Setting $\epsilon = \beta_{i,j}$, conjugating this matrix from left to right and repeating this process for all i, j with $i \le r$ we obtain

$$\int \varphi \left[\begin{pmatrix} u_1 & \overline{\ell}_1 & \overline{\ell}_2 \\ 0 & u_2 & * \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ \overline{q}_1 & I & 0 \\ \overline{q}_2 & q_1^* & I \end{pmatrix} \right] \psi_{U_1}(u_1)\psi_{U_2}(u_2)d(\ldots).$$

where now $\overline{\ell}_1$ is integrated over all first r rows of $M_{(2r+1)\times(d_2+\cdots+d_s)}$ with points in $F\backslash\mathbb{A}$. Combining the relevant integrations of u_1 $\overline{\ell}_1$ and $\overline{\ell}_2$ we obtain the integral over $U_r(F)\backslash U_r(\mathbb{A})$ as inner integration. The lemma now follows from Lemma 2.2. □

Lemma 2.5. *Let $n_2 > n_1$ and suppose that the representation π has a nonzero Fourier coefficient with respect to the unipotent class $((2n_1)1^{2(k-n_1)}) \circ ((2n_2)1^{2(k-n_1-n_2)})$. (The notation $\circ$ was defined at the end of section one). Then there exists a unipotent class $(d_1d_2\ldots d_s)$ with $d_i \ge d_{i+1}$ which supports π and where $d_1 > 2n_1$.*

Proof. Let

$$h(t) = \text{diag}(t^{2n_1-1}, \ldots, t, t^{2n_2-1}, \ldots, t, 1, \ldots, 1, t^{-1}, \ldots, t^{-(2n_1-1)})$$

where we have $2(k-n_1-n_2)$ ones. We start with integral (1.2) corresponding to (see the end of section one) $((2n_1)1^{2(k-n_1)}) \circ ((2n_2)1^{2(k-n_1-n_2)})$. Let w_1 be the minimal Weyl element of Sp_{2k} which conjugates $h(t)$ to the torus

$$h_1(t) = \text{diag}(t^{2n_1-1}, \ldots, t^3, t^{2n_2-1}, \ldots, t^3, t, t, 1, \ldots, 1, t^{-1}, t^{-1}, \ldots, t^{-(2n_1-1)}) .$$

Applying w_1 to integral (1.2) we obtain

$$\int \varphi\left[\begin{pmatrix} u & \ell & q \\ & v & \ell^* \\ & & u^* \end{pmatrix} w_1\right] \psi_1(z(u, v, \ell))d(\ldots) \tag{2.5}$$

where the variables are integrated over points in $F\backslash\mathbb{A}$ of the following groups. First $q \in M'_{(n_1+n_2-2)\times(n_1+n_2-2)}$. The variable u is integrated over all upper unipotent matrices in $\mathrm{GL}_{n_1+n_2-2}$. The variable v is integrated over the unipotent group corresponding to the partition $\mathcal{O}_1 = (2^2 1^{2(k-n_1-n_2)})$ of the group $\mathrm{Sp}_{2(k-n_1-n_2+2)}$. Finally, ℓ is integrated over $\{\ell \in M_{(n_1+n_2-2)\times 2(k-n_1-n_2+2)} : \ell_{i,1} = 0 \text{ for } i \geq n_1\}$. The character ψ_1 is defined as follows. On $u = (u_{ij})$ we let $\psi_1(u) = \psi(u_{1,2} + \cdots + u_{n_1-3,n_1-1} + u_{n_1,n_1+1} + \cdots + u_{n_1+n_2-3,n_1+n_2-2})$. On v the character $\psi_1(v)$ agrees with $\psi_{V_2(\mathcal{O}_1)}$ for the unipotent class $\mathcal{O}_1$. Finally, for $\ell = (\ell_{ij})$ we have $\psi_1(\ell) = \psi(\ell_{n_1-1,1} + \ell_{n_1+n_2-2,2})$.

In (2.5) we expand along $F\backslash\mathbb{A}$ over the missing variables of ℓ i.e. over $\ell_{i,1}$ with $i \geq n_1$. Using the $n_1 - 1$ row in u as was done in Lemma 2.3, the nonvanishing of (2.5) implies the nonvanishing of

$$\int \varphi\left[\begin{pmatrix} u & \ell & q \\ 0 & v & \ell^* \\ 0 & 0 & u^* \end{pmatrix}\right] \psi_1(z(u, v, \ell))d(\ldots) \tag{2.6}$$

where now u is integrated over all upper unipotent matrices of $\mathrm{GL}_{n_1+n_2-2}$ such $u_{n_1-1,j} = 0$ for $j \geq n_1$. Also ℓ is integrated over all $M_{(n_1+n_2-2)\times 2(k-n_1-n_2+2)}$. The character ψ_1 is not changed. Next let w_2 be the minimal Weyl element which conjugates $h_1(t)$ to

$$h_2(t) = \mathrm{diag}(t^{2n_1-1}, \ldots, t^5, t^{2n_2-1}, \ldots, t^3, t^3, t, t, 1, \ldots, 1, \ldots)$$

Applying this Weyl element to (2.6) we deduce that the integral

$$\int \varphi\left[\begin{pmatrix} u & \ell & q \\ 0 & v & \ell^* \\ 0 & 0 & u^* \end{pmatrix}\right] \psi_2(z(u, \ell, v))d(\ldots)$$

is nonzero for some choice of data. Here u is integrated over all upper unipotent matrices in $\mathrm{GL}_{n_1+n_2-4}$ and v is integrated over the unipotent group corresponding to the partition $\mathcal{O}_2 = (4^2 1^{2(k-n_1-n_2)})$ of the group $\mathrm{Sp}_{2(k-n_1-n_2+4)}$. The variable ℓ is integrated over $\{\ell \in M_{(n_1+n_2-4)\times 2(k-n_1-n_2+4)} : \ell_{i,1} = 0 \text{ for } i \geq n_1 - 1\}$. The character ψ_2 is defined as follows. On $u = (u_{ij})$, $\psi_2(u) = \psi(u_{1,2} + \cdots + u_{n_1-3,n_1-2} + u_{n_1-1,n_1} + \cdots + u_{n_1+n_2-5,n_1+n_2-4})$ and v is the character $\psi_2(v)$ which agrees with $\psi_{V_2(\mathcal{O}_2)}$. On $\ell = (\ell_{ij})$ we have $\psi_2(\ell) = \psi(\ell_{n_1-2,1} + \ell_{n_1+n_2-4,2})$. All variables are integrated over $F\backslash\mathbb{A}$. Continuing this process $n_1 - 1$ times we obtain the nonvanishing of the integral

$$\int \varphi\left[\begin{pmatrix} u & \ell & q \\ 0 & v & \ell^* \\ 0 & 0 & u^* \end{pmatrix}\right] \psi_{n_1-1}(z(u, \ell, v))d(\ldots) \tag{2.7}$$

where u is integrated over the upper unipotent subgroup of $GL_{n_2-n_1}$ and v is integrated over the unipotent group corresponding to the partition $\mathcal{O}_{n_1-1} = ((2n_1)^2 1^{2(k-n_1-n_2)})$. The variable ℓ is integrated over $\{\ell \in M_{(n_2-n_1)\times 2(k+n_1-n_2)} : \ell_{i,1} = 0$ for all $i\}$. The character ψ_{n_1-1} is defined on $u = (u_{ij})$ as $\psi_{n_1-1}(u) = \psi(u_{1,2} + \cdots + u_{n_2-n_1-1,n_2-n_1})$ and on v, ψ_{n_1-1} is defined as $\psi_{V_2(\mathcal{O}_{n_1-1})}$. On $\ell = (\ell_{ij})$ we have $\psi_{n_1-1}(\ell) = \psi(\ell_{n_2-n_1,2})$. Next we expand (2.7) along the variables $\ell_{i,1}$ for $1 \le i \le n_2 - n_1$. However, because of the u integration we need to do it one variable at the time starting from $i = 1$. Thus the nonvanishing of (2.7) implies that at least one of the integrals

$$\int \varphi\left[\begin{pmatrix} u & \ell_m & q \\ 0 & v & \ell_m^* \\ 0 & 0 & u^* \end{pmatrix}\right] \psi_{n_1-1}(z(u,\ell_m,v))\psi(\alpha_m \ell_{m,1})d(\ldots) \tag{2.8}$$

is nonzero where $1 \le m \le n_2 - n_1$. Here $\ell_m = (\ell_{i,j})$ such that $\ell_{i,1} = 0$ if $i > m$ and if $i \ne n_2 - n_1$ then $\alpha_m \in F^*$. If $m = n_2 - n_1$ then $\alpha \in F$. ψ_{n_1-1} is defined on ℓ_m by restricting it to the variable ℓ as defined in integral (2.7). Let us first treat the case when $m = n_2 - n_1$. For some α_m (2.8) will represent integral (1.2) for the partition $((2n_2)(2n_1)1^{2(k-n_1-n_2)})$. However, depending also on the character ψ_{n_1-1} restricted to v, this integral might be different. To explain this write v as $v = \begin{pmatrix} v_1 & p_1 & q_1 \\ 0 & I & p_1^* \\ 0 & 0 & v_1^* \end{pmatrix}$ where $v_1 = \begin{pmatrix} I_2 & x_1 & & & \\ & I_2 & x_2 & & \\ & & \ddots & & \\ & & & I_2 & x_{n_1-1} \\ & & & & I_2 \end{pmatrix}$ is a matrix of size $2n_1 \times 2n_1$ and $x_i \in M_{2\times 2}$. The matrix $p_1 = (p_{ij}) \in M_{2n_1 \times 2(k-n_1-n_2)}$ such that $p_{ij} = 0$ if $i = 2n_1 - 1, 2n_1$ and $j \le k - n_1 - n_2$. Also, I is the identity matrix of size $2(k - n_1 - n_2)$. The character $\psi_{V_2(\mathcal{O}_{n_1-1})}$ restricted to v_1 is given by $\psi(tr(x_1 + \cdots + x_{n_1-1}))$ and when restricted to q_1 we have $\psi_{V_2(\mathcal{O}_{n_1-1})}(q_1) = \psi(\epsilon_1 q_1(2n_1-1,2) + \epsilon_2 q_1(2n_1,1))$. Here $q_1(i,j)$ is the $(i,j)-th$ entry of the matrix q_1. If $\epsilon_1\epsilon_2 = -\beta^2$ for some $\beta \in F^*$ and α_m is chosen suitably then there is a rational matrix which when conjugating (2.8) by it we obtain

$$\int \varphi\left[\begin{pmatrix} u & \ell & q \\ 0 & v & \ell^* \\ 0 & 0 & u^* \end{pmatrix}\right] \widetilde{\psi}_{n_1-1}(z(u,\ell,v))d(\ldots) \tag{2.9}$$

where now $\ell \in M_{(n_2-n_1)\times 2(k-n_1-n_2)}$ and $\widetilde{\psi}_{n_1-1}$ is defined on $\ell = (\ell_{ij})$ as $\psi_{n_1-1}(\ell) = \psi(\ell_{n_2-n_1,1})$ and $\widetilde{\psi}_{n_1-1}$ restricted to $q_1 = (q_{ij})$ (as defined in the above matrix for v) is given by $\widetilde{\psi}_{n_1-1}(q) = \psi(q_{2n_1-1,1})$. In this case one can find a Weyl element to conjugate (2.9) and that after a suitable Fourier expansion (similar to the one done in lemma 2.3) we shall obtain $\int_{U_{n_2}(F)\backslash U_{n_2}(\mathbb{A})} \varphi(u)\psi_{U_{n_2}}(u)du$ as inner integration. Applying Lemma 2.2 our result follows.

If in (2.8) $m < n_2 - n_1$ we consider the rational matrix $I_{2k} + \alpha_m e'_{m+1,n_2-n_1+1}$ in Sp_{2k}. Conjugating by this matrix and changing variables we obtain the integral

$$\int \varphi \left[\begin{pmatrix} u & \ell_m & q \\ 0 & v & \ell_m^* \\ 0 & 0 & u^* \end{pmatrix} \right] \psi_m(z(u, \ell_m, v)d(\dots)$$

where all groups are as in (2.8) and ψ_m is defined as follows. On v we integrate as before, on $u = (u_{ij})$ we have $\psi_m(u) = \psi(u_{1,1} + \cdots + u_{m-1,m} + u_{m+1,m} + \cdots + u_{n_2-n_1-1,n_2-n_1})$ and on $\ell_m = (\ell_{ij})$ we have $\psi_m(\ell_m) = \psi(\alpha_m \ell_{m,1} + \ell_{n_2-n_1,2})$. Thus we get exactly an integral of the type (2.6). Continuing by induction we will either get that $(2(n_2 - i)2(n_1 + i)1^{2(k-n_1-n_2)})$ supports π for suitable values of i, or we will get as inner integration on integral over U_{2i} with the character ψ_{2i} for some $i > n_1$. In either case our result follows. □

Lemma 2.6. *Let $\mathcal{O} = ((2n)d_2 \cdots d_s)$ where $2n \geq d_i$, for all i and $d_2 + \cdots + d_s + 2n = 2k$. Then for compatible characters, the unipotent class $\mathcal{O}$ supports π if and only if $((2n)1^{2k-(d_2+\cdots+d_s)}) \circ (d_2 \cdots d_s)$ supports π.*

Proof. The idea is similar to the proof of Lemma 2.4. Let

$$h_{\mathcal{O}}(t) = \mathrm{diag}(t^{2n-1}, \dots, t^{d_2-1}, \dots, t^{-(d_2-1)}, \dots, t^{-(2n-1)}).$$

Since the number $2n$ appears in $\mathcal{O}$ at least once then in $h_{\mathcal{O}}(t)$ we have the following powers $t^{2n-1}, t^{2n-3}, \dots, t$ at least once. Let w be the Weyl element of minimal length which conjugates $h_{\mathcal{O}}(t)$ to the torus

$$h(t) = \mathrm{diag}(t^{2n-1}, t^{2n-3}, \dots, t, \dots, t^{d_2-1}, \dots, t^{-(d_2-1)}, t^{-1}, \dots, t^{-(2n-1)}).$$

Consider integral (1.2) corresponding to the unipotent class $\mathcal{O}$. Conjugating the integral by w we obtain the integral

$$\int \varphi \left[\begin{pmatrix} u & \ell & x \\ 0 & v & \ell^* \\ 0 & 0 & u^* \end{pmatrix} \begin{pmatrix} I & & \\ q & I & \\ 0 & q^* & I \end{pmatrix} \right] \psi_1(z(u, v, x))d(\dots) \tag{2.10}$$

Here u is integrated over the upper unipotent matrices in GL_n, v is integrated as in integral (1.2) corresponding to the partition $\widetilde{\mathcal{O}} = (d_2 \dots d_s)$ inside the group Sp_{2k_1} where $k_1 = d_2 + \cdots + d_s$ and x is integrated over $M'_{n \times n}$. Finally ℓ is integrated over $M^1_{n \times 2k_1}$ as defined in the proof of lemma 2.4 and q is integrated over $M^2_{2k_1 \times n}$ as also defined in the proof of that Lemma. All variables are integrated over $F\backslash\mathbb{A}$. The character ψ_1 is defined as follows. For $u = (u_{ij})$ we let $\psi_1(u) = \psi(u_{1,2} + \cdots + u_{n-1,n})$. For $x = (x_{ij})$ we let $\psi_1(x) = \psi(\epsilon x_{n,1})$ for some $\epsilon \in F^*$ and for v we let ψ_1 be the character $\psi_{V_2(\widetilde{\mathcal{O}})}$ as defined by (1.2) for $\widetilde{\mathcal{O}}$. Using similar Fourier expansion as in Lemma 2.4, (2.10) equals

$$\int \varphi \left[\begin{pmatrix} u & \ell & x \\ 0 & v & \ell^* \\ 0 & 0 & u^* \end{pmatrix} \begin{pmatrix} I & & \\ q & I & \\ 0 & q^* & I \end{pmatrix} \right] \psi_1(z(u, v, x))d(\dots) \tag{2.11}$$

where now $\ell \in M_{n\times 2k}$ such that $\ell_{n,j} = 0$ for $1 \le j \le k_1$ and the integration over q is over points in $\mathbb{A}$. Using the ideas as in Lemma 1 in [G-R-S2] p. 895 we can "get rid" of the adelic integration and show that (2.11) is nonzero for some choice of data if and only if

$$\int \varphi \left[\begin{pmatrix} u & \ell & x \\ 0 & v & \ell^* \\ 0 & 0 & u^* \end{pmatrix} \right] \psi_1(z(u,v,x))d(\ldots)$$

is nonzero for some choice of data. But this last integral is exactly $((2n)1^{2(k-k_1)}) \circ (d_2 \ldots d_s)$ where the ψ which is used to define $((2n)1^{2(k-k_1)})$ is compatible with $\psi_{V_2(\mathcal{O})}$. □

We now prove our main result,

Theorem 2.7. *Let π be a cusp form on $G = \mathrm{Sp}_{2k}(\mathbf{A})$ or on its double cover. Then there exists a unipotent class $\mathcal{O}$ in $\mathcal{O}_G(\pi)$ such that $\mathcal{O} = ((2n_1)(2n_2)\ldots(2n_r))$ and $n_i \ge n_{i+1}$.*

Proof. It is easy to see that every automorphic representation has a nonzero Fourier coefficient corresponding to the unipotent class $(21^{2(k-1)})$. Since we may assume that π is not generic then there exists a unipotent class $((2n_1)1^{2(k-n_1)})$ which supports π such that $n_1 < k$ and that π vanishes on *any* unipotent class of the form $((2m)1^{2(k-m)})$ with $m > n_1$. Every nonzero Fourier coefficient corresponding to the unipotent class above gives, using (1.3), an automorphic representation which by Lemma 2.3 using the maximality of n_1, is a cuspidal representation on the group $\mathrm{Sp}_{2(k-n_1)}(\mathbf{A})$ or on its double cover. The point is that to the unipotent class $((2n_1)1^{2(k-n_1)})$ there may correspond many nonzero Fourier coefficients depending on the additive character. Let n_2 be the number such that at least one of these automorphic cuspidal representations is supported by $((2n_2)1^{2(k-n_1-n_2)})$ and such that *all* the automorphic cuspidal representations above vanish on $((2m)1^{2(k-n_1-m)})$ if $m > n_2$. In other words n_2 is defined so that $((2n_1)1^{2(k-n_1)}) \circ ((2n_2)1^{2(k-n_1-n_2)})$ supports π and if $m > n_2$ then $((2n_1)1^{2(k-n_1)}) \circ ((2m)1^{2(k-n_1-m)})$ vanishes on π. We claim that $n_1 \ge n_2$. Indeed, if $n_2 > n_1$, then by Lemma 2.5 there is a unipotent class $(d_1 \ldots d_s)$ which supports π such that $d_1 > 2n_1$. If $d_1 = 2\ell + 1$ is odd, then by Lemma 2.4 there is a number m such that $2m > 2\ell + 1$ and such that $((2m)1^{2(k-m)})$ supports π. But this contradicts the maximality of n_1. If $d_1 = 2m$ with $m > n_1$, then applying Lemma 2.6 we deduce that $((2m)1^{2(k-m)})$ supports π and once again we derive a contradiction to the maximality of n_1. Thus $n_1 \ge n_2$. Continuing this process we obtain a set of numbers $n_1 \ge n_2 \cdots \ge n_r$ such that the unipotent class

$$((2n_1)1^{2(k-n_1)}) \circ ((2n_2)1^{2(k-n_1-n_2)}) \circ \cdots \circ (2n_r)$$

supports π and it is maximal at every stage. (Since, by Lemma 2.3, we get a cusp form at every stage we must eventually obtain a generic cusp form.) The maximality at every stage means that if $m > n_i$ then π vanishes on the unipotent class

$$((2n_1)1^{2(k-n_1)}) \circ \cdots \circ ((2n_{i-1})1^{2(k-(n_1+\cdots+n_{i-1}))}) \circ ((2m)1^{2(k-(n_1+\cdots n_{i-1}+m))}) .$$

Using Lemma 2.6 inductively we deduce that $\mathcal{O} = ((2n_1)(2n_2)\cdots(2n_r))$ supports π. To show that $\mathcal{O}$ is in $\mathcal{O}_G(\pi)$, let $\widetilde{\mathcal{O}} = (d_1 \ldots d_s)$ be a unipotent class which supports π and that $\widetilde{\mathcal{O}} \geq \mathcal{O}$. We will show that $\widetilde{\mathcal{O}} = \mathcal{O}$. Suppose that $d_1 > 2n_1$. If $d_1 = 2\ell + 1$ then $\widetilde{\mathcal{O}} = ((2\ell+1)^2 d_3 \ldots d_s)$. Using Lemma 2.4 there is a number m such that $2m > 2\ell + 1$ such that $((2m)1^{2(k-n)})$ supports π. Since $m > n_1$ this contradicts the maximality of n_1. If $d_1 = 2m$ with $m > n_1$ then using Lemma 2.6 we derive a contradiction once again. Thus $d_1 = 2n_1$. Using Lemma 2.6 we deduce that $((2n_1)1^{2(k-n_1)}) \circ (d_2 \ldots d_s)$ supports π. Arguing by induction we obtain that $s = r$ and $d_i = 2n_i$ for all i. Thus $\widetilde{\mathcal{O}} = \mathcal{O}$. □

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