Wood Science and Technology (2021) 55:919–936
https://doi.org/10.1007/s00226-021-01306-5

ORIGINAL

Determining the Young's modulus of solid wood by considering the fundamental frequency under the free-free flexural vibration mode

Hiroshi Yoshihara[1] · Makoto Maruta[2]

Received: 4 November 2020 / Accepted: 31 May 2021 / Published online: 14 June 2021

Abstract
The Young's modulus in the longitudinal direction of a Sitka spruce specimen was determined by conducting free-free flexural vibration tests based on Euler–Bernoulli's and Timoshenko's vibration theories. Comparing the Young's moduli obtained considering these theories, an equation to calculate the Young's modulus by considering only the fundamental frequency was formulated by modifying the equation derived from the Euler–Bernoulli's theory. The modification was conducted with the consideration that the modified equation was applicable to various wood species with a wide range of length/depth ratio. Additionally, the accuracy of the proposed equation was examined using a statistical method for comparing the Young's moduli obtained considering the aforementioned theories. Using the proposed equation, accurate Young's moduli could be determined considering only the fundamental frequency, with a reduced influence of the length/depth ratio. The statistical analysis results indicated that the proposed equation can effectively yield accurate Young's moduli for various wood species with a wide range of length/depth ratio.

Introduction

Elastic constants, including the Young's modulus, are the basic properties that define the deformation behaviours of a material; therefore, it is desirable to determine the elastic constants accurately and conveniently. Among the various methods to determine the elastic constants, conducting vibration tests is advantageous because the force required to generate the vibration is usually sufficiently low to enable non-destructive testing. In addition, the measurement time for vibration tests is usually smaller than for static tests, including tension, compression, and

✉ Hiroshi Yoshihara
yosihara@riko.shimane-u.ac.jp

1 Faculty of Science and Engineering, Shimane University, Matsue, Japan

2 Faculty of Science and Engineering, Shizuoka Institute Science and Technology, Fukuroi, Japan

bending tests. Among the vibration tests, flexural vibration (FV) tests are frequently conducted to determine the Young's modulus. It is desirable to conduct FV tests because the fundamental frequency of the FV mode can be easily identified owing to its large amplitude. Consequently, the FV test methods have been standardised for refractory materials (ASTM C1548-02 2020), metallic materials at elevated temperatures (JIS Z 2280-1993), fine ceramics (JIS R 1602-1995), and concrete (JIS A 1127-2010). Moreover, the FV test method for solid wood has been standardised as ASTM D6874-12 by ASTM International. In addition to this standard, there exist several examples for the conduction of FV tests (Hearmon 1958, 1966; Matsumoto 1962; Ono and Kataoka 1979a, b; Ono 1983; Sobue 1986; Chui and Smith 1990; Haines et al. 1996; Divós et al. 1998; Brancheriau and Bailleres 2002; Brancheriau and Baillères 2003; Divos et al. 2005; Brancheriau 2006; Murata and Kanazawa 2007; Roohnia et al. 2010, 2011; Sohi et al. 2011; Yoshihara 2011, 2012a, b, c; Shamaiirani and Roohnia 2016; Roohnia 2019) to obtain the Young's modulus of solid wood and wood-based materials.

In FV tests, the Young's modulus is often calculated considering only the fundamental frequency, based on the Euler–Bernoulli's theory (JIS Z 2280-1993; JIS R 1602-1995; JIS A 1127-2010; ASTM D6874-12; Hearmon 1958, 1966; Matsumoto 1962). In the Euler–Bernoulli's theory, only the flexure caused by the bending moment is taken into account, and the equation to derive the Young's modulus is represented in a simple form. However, in an actual FV test, the flexural components caused by the shearing force and rotatory inertia are inevitably induced, along with those induced by the bending moment. These components become more pronounced when the length/depth ratio of the specimen decreases; therefore, to obtain the accurate Young's modulus by the Euler–Bernoulli equation while reducing these components, the length/depth ratio of the specimen should be sufficiently large. As an alternative method to obtain the Young's modulus value, multiple resonance frequencies for the FV modes can be measured, and the Young's modulus can be calculated based on Timoshenko's vibration theory (Timoshenko 1921). This method is advantageous in that the length/depth ratio of the specimen does not necessarily need to be large. Furthermore, the shear modulus in the length-depth plane of the specimen can be obtained along with the Young's modulus in the length direction. However, measuring multiple resonance frequencies is time-consuming when the shear modulus does not need to be determined. It is more convenient and practical to determine the accurate value of the Young's modulus by conducting the FV test and identifying only the fundamental frequency alone.

In this study, FV tests were conducted using Sitka spruce specimens with various length/depth ratios, and Young's moduli were obtained by measuring the fundamental and multiple resonance frequencies of the FV modes. An equation to obtain the accurate value of the Young's modulus value considering only the fundamental frequency was proposed, and the validity of the equation was examined by conducting the analyses based on a statistical method and the experimentally obtained results. There are several attempts to provide such an equation for isotropic materials (Pickett 1945; Spinner et al. 1960; Spinner and Tefft 1961), the details of which are described below. However, it is difficult to find any examples in the literature listed above that propose the equation for solid wood, which possesses a strong orthotropy.

The novelty of the work conducted here is the proposal of the equation for determining the Young's modulus from the fundamental frequency in a certain range of the length/depth ratio of the specimen.

Materials and methods

Materials

The FV tests were conducted on Sitka spruce (*Picea sitchensis* Carr.) lumber with the initial dimensions of 900 mm (L) × 110 mm (T) × 110 mm (*R*). Prior to the test, the lumber was stored for approximately three years at a constant temperature of 20 °C and relative humidity of 65%, under air-dried condition. The equivalent moisture content (EMC) was 12%. These conditions were maintained throughout the duration of the tests. The density of the specimen was 404 ± 17 kg/m^3, and the moisture content was determined to be 10% by using a moisture tester (HM-530, Kett Electric Laboratory, Tokyo, Japan). All the specimens were cut from the aforementioned lumber such that they were side-matched. During the preparation of the specimens, the defects contained in the lumber, including knots, checks, and fibre distortions, were reduced; therefore, the specimens could be regarded as 'small and clear'. The specimen contained approximately 10–12 annual rings per 10 mm. The specimen was a straight beam with a square cross section, and the length, width, and depth directions, which coincided with the longitudinal (*L*), radial (*R*), and tangential (*T*) directions, respectively, were defined as *L*, *B*, and *H*, respectively. According to the method defined in JIS Z 2101-(2009), twelve specimens with an *L* value of 400 mm; and *B* and *H* values of 8 mm × 8 mm, 12 mm × 12 mm, 16 mm × 16 mm, and 20 mm × 20 mm were cut from the aforementioned lumber.

Free-free flexural vibration tests

In the FV test standardised in ASTM D6874-12 (2012), it is determined that the specimen should be simply supported with the span which exceeds 0.98 times the length. However, in this condition, it is difficult to excite the vibration with high resonance modes. To overcome this obstacle, the specimen was set upon a pair of extruded polystyrene (XPS) foams fixed on a medium-density fibreboard (MDF) plate. As shown in Fig. 1, the nodal positions of the free-free resonance vibration mode f_n, were supported using a pair of XPS foams, and the specimen was excited along the depth direction (*T* direction) by using a hammer. The locations of the nodal positions are listed in Table 1 (Roohnia 2019). The aerial vibration generated from the specimen was detected by a microphone (UC-53A, Rion Co. Tokyo), preamplified by NH-22 (Rion Co. Tokyo), and the resonance frequencies were analysed using a fast Fourier transform (FFT) analyser (SA-78, Rion Co. Tokyo). The frequencies from the 1st to 4th resonance modes of flexural vibrations were measured in the FV tests. The Young's moduli were determined from

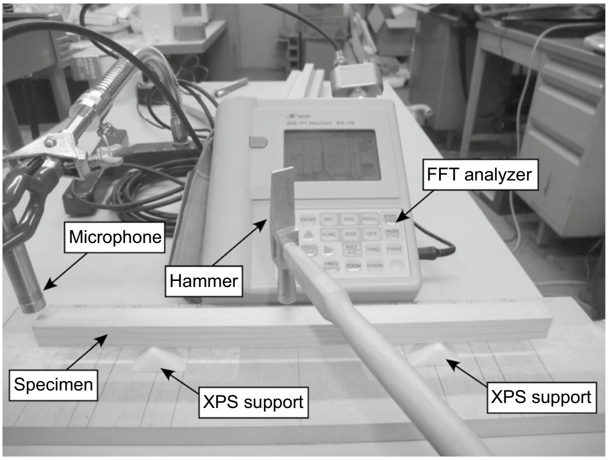

Fig. 1 Free-free flexural vibration test setup

Table 1 Location of the nodal position adopted for supporting the specimen in the FV test (Roohnia 2019)

Mode number	1	2	3	4
	0.224L	0.132L	0.073L	0.277L
	0.776L	0.868L	0.927L	0.723L

L = length of specimen

the following Euler–Bernoulli's and Timoshenko's vibration theories, and they were defined as E_L^{EB} and E_L^{TGH}, respectively.

When considering the Euler–Bernoulli's theory, the E_L^{EB} value was calculated using only the fundamental frequency for the FV mode, as follows (JIS Z 2280-1993; JIS R 1602-1995; JIS A 1127-2010; ASTM D6874-12; Hearmon 1958, 1966; Matsumoto 1962):

$$E_L^{\mathrm{EB}} = \frac{48\pi^2 \rho L^2 f_1^2}{m_1^4 H^2} \tag{1}$$

where ρ is the density of the specimen, f_1 is the fundamental frequency, and $m_1 = 4.730$. E_L^{EB} was derived using Eq. (1) considering only the bending moment induced in the specimen. However, in an actual FV test, flexural components are also induced by the shearing force and rotatory inertia, and these effects are enhanced as the length/depth ratio of the specimen L/H decreases. Consequently, the value of E_L^{EB} usually decreases as L/H value decreases. Timoshenko proposed a differential equation including the effects of the shearing force and rotatory inertia (Timoshenko 1921). Later, Goens (1931) derived an approximate solution for the Timoshenko differential equation, denoted as E_L^{TGH}, as follows:

$$E_L^{\mathrm{TGH}} = \frac{48\pi^2 \rho L^2 f_n^2}{m_n^4 H^2} T_n \tag{2}$$

where n is the mode number for the FV, f_n is the resonance frequency at the nth mode, and T_n is derived as follows:

$$\begin{aligned} T_n = {} & 1 + 2m_n F(m_n) \frac{1}{12} \left(\frac{H}{L}\right)^2 \left(3 - s\frac{E_L^{\mathrm{TGH}}}{G_{\mathrm{LT}}}\right) \\ & + m_n^2 F^2(m_n) \frac{1}{12} \left(\frac{H}{L}\right)^2 \left(1 + s\frac{E_L^{\mathrm{TGH}}}{G_{\mathrm{LT}}}\right) - s\frac{4\pi^2 \rho L^2 f_n^2}{G_{\mathrm{LT}}} \end{aligned} \tag{3}$$

where G_{LT} is the shear modulus in the LT plane, s is Timoshenko's shear factor, which is considered as 1.2 for a specimen with a rectangular cross section, and m_n and $F(m_n)$, which are the coefficients corresponding to each resonance mode, are defined as follows:

$$\begin{cases} m_1 = 4.730 \\ m_2 = 7.853 \\ m_n = \frac{(2n+1)\pi}{2} \quad (n \geq 3) \end{cases} \tag{4}$$

and

$$\begin{cases} F(m_1) = 0.9825 \\ F(m_2) = 1.0008 \\ F(m_n) = 1 (n \geq 3) \end{cases} \tag{5}$$

The $E_L{}^{\mathrm{TGH}}$ and G_{LT} values were determined using the iterative method proposed by Hearmon using Eqs. (2) and (3). In the iterative method (Hearmon 1958, 1966), which is known as the Timoshenko-Goens-Hearmon (TGH) method, the X and Y values corresponding to each mode are derived using Eqs. (2) and (3) as follows:

$$\begin{cases} X = \frac{48\pi^2 \rho L^2 f_n^2}{m_n^4 H^2} \left[-2m_n F(m_n) + m_n^2 F^2(m_n)\right] \\ Y = \frac{48\pi^2 \rho L^2 f_n^2}{m_n^4 H^2} \left[\frac{H^2}{12L^2} + 6m_n F(m_n) + m_n^2 F^2(m_n) - \frac{4\pi^2 \rho L^2 f_n^2 s}{G_{xy}}\right] \end{cases} \tag{6}$$

The X–Y relationship corresponding to each mode is regressed into the linear function $Y = q - pX$, and the Young's modulus $E_L{}^{\mathrm{TGH}}$ and shear modulus G_{LT} can be determined as

$$\begin{cases} E_L^{\mathrm{TGH}} = q \\ G_{\mathrm{LT}} = s\frac{q}{p} \end{cases} \tag{7}$$

Subsequently, the refined G_{LT} value is substituted into Eq. (6), and the iteration procedure is continued until the E_L^{TGH} and G_{LT} values converge. As described above, the resonance frequencies from the 1st to 4th flexural vibration modes were measured and analysed using a fast Fourier transform (FFT) analyser (SA-78, Rion Co., Tokyo). Recently, a method for determining the Young's modulus and shear modulus values from the multiple FV testing data was proposed as Roohnia et al. (2010), Shamaiirani and Roohnia (2016), and Roohnia (2019). However, in this study, the above-mentioned TGH method was adopted.

After the FV tests were conducted, the length of the specimen was reduced, and the succeeding FV tests were conducted using the specimen with the reduced length. In particular, the value of L was decreased from 400 to 200 mm in intervals of 50 mm. Therefore, the L/H ranged from 10 to 50.

Results and discussion

Comparison of the Young's moduli obtained using the Euler–Bernoulli's method and TGH method

Figure 2 shows the f_n values corresponding to the L and L/H values for the 1st–4th flexural vibration modes. All the E_L^{EB}, E_L^{TGH}, and G_{LT} values demonstrated below were calculated from the f_n values shown in Fig. 2.

Before discussing the results of the Young's modulus, the dependence of the G_{LT} value determined using the TGH method on the L/H value is considered, as shown in Fig. 3. It can be noted that the G_{LT} value obtained from the specimens with a similar H tends to be constant despite the variation in the L value; therefore, the G_{LT} value obtained using the TGH method in the L/H range examined in this study can be considered to be accurate.

Figure 4 shows the comparison of the E_L^{EB} and E_L^{TGH} values corresponding to the L and L/H values. Similar to the results obtained in previous studies (Hearmon 1958, 1966; Matsumoto 1962; Ono and Kataoka 1979a, b; Ono 1983; Sobue 1986; Chui and Smith 1990; Haines et al. 1996; Divós et al. 1998; Brancheriau and Bailleres 2002; Brancheriau and Baillères 2003; Divos et al. 2005; Brancheriau 2006; Murata and Kanazawa 2007; Roohnia et al. 2011; Sohi et al. 2011; Yoshihara 2011, 2012a, b, c), the E_L^{TGH} value tends to be constant in the range of L/H examined in this work. Therefore, the Young's modulus determined using the TGH method while reducing the effects of the shearing force and rotatory inertia can be considered to be accurate and recommended to be used as the reference value in discussing the accuracy of the E_L^{EB} value. In this case, E_L^{TGH}/G_{LT} is 11.6 ± 1.0. In contrast, the difference between the E_L^{EB} values decreases as the L/H value decreases. The results of the unpaired t tests, which are discussed in detail in the following subsection, indicate that the difference between E_L^{EB} and E_L^{TGH} is often significant when the L/H value is less than 20. The experimental results also suggest that the E_L^{EB} value converges to the E_L^{TGH} value when the L/H value is greater than 20. However, when the E_L^{TGH}/G_{LT}

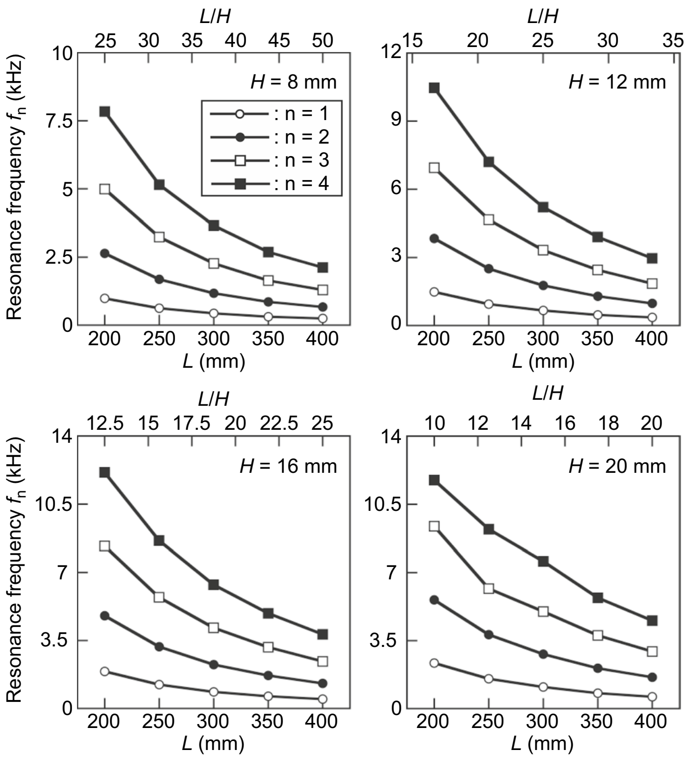

Fig. 2 f_n value corresponding to the L and L/H values. n = mode number. The SD values cannot be demonstrated because of the low variation in the f_n value

value is higher than that obtained in this study, there is a concern that the E_L^{EB} value does not agree with the E_L^{TGH} value even if L/H greater than 20.

Modification of Euler–Bernoulli's equation

As shown in Fig. 4, when Eq. (1) is used for the calculation, the dependence of E_L^{EB} on the value of L/H is significant for small L/H values. However, it is more convenient to determine the Young's modulus by measuring only the fundamental frequency alone while reducing this dependence. Based on this concept, Eq. (1) can be modified to a form that can be used to calculate the Young's modulus of the specimen in a wide range of L/H.

In the FV test methods for refractory materials and fine ceramics, standardised in ASTM C1548-02 and JIS R 1602-(1995), respectively, the Young's modulus modified using the E_L^{EB} value, defined as E_L^{MOD}, is derived as follows:

$$E_L^{\text{MOD}} = E_L^{\text{EB}} T_c \tag{8}$$

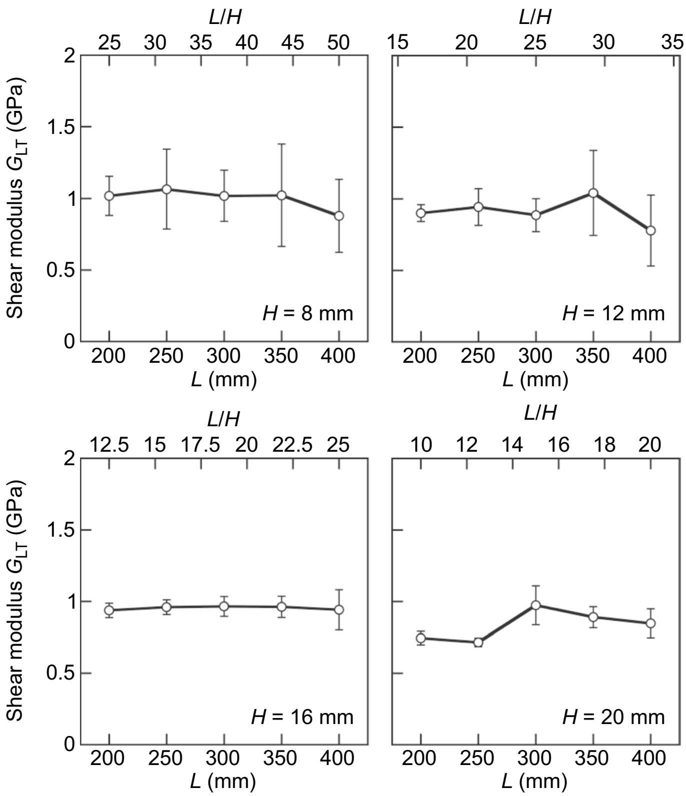

Fig. 3 G_{LT} value corresponding to the L and L/H values determined using the TGH method. The results represent the average ± SD

where T_c is the correction factor. According to the analyses by Pickett (1945), Spinner et al. (1960), and Spinner and Tefft (1961), T_c can be expressed as $T_c = 1 + 6.59(H/L)^2$ on the condition that $L/H \geq 20$. However, this relationship cannot be directly used to determine the correction factor for solid wood because the orthotropy of solid wood is usually stronger than those of refractory materials and fine ceramics. In addition, the degree of orthotropy varies among the wood species. Considering the unique characteristics of solid wood, T_c was modified as:

$$T_c = 1 + \alpha\left(\frac{H}{L}\right)^{\beta} \quad (9)$$

The α and β values appropriate for solid wood were determined by comparing the values with those pertaining to the correction factor used in the TGH method. From Eqs. (1) and (2), $E_L{}^{TGH}$ was represented using $E_L{}^{EB}$ as follows:

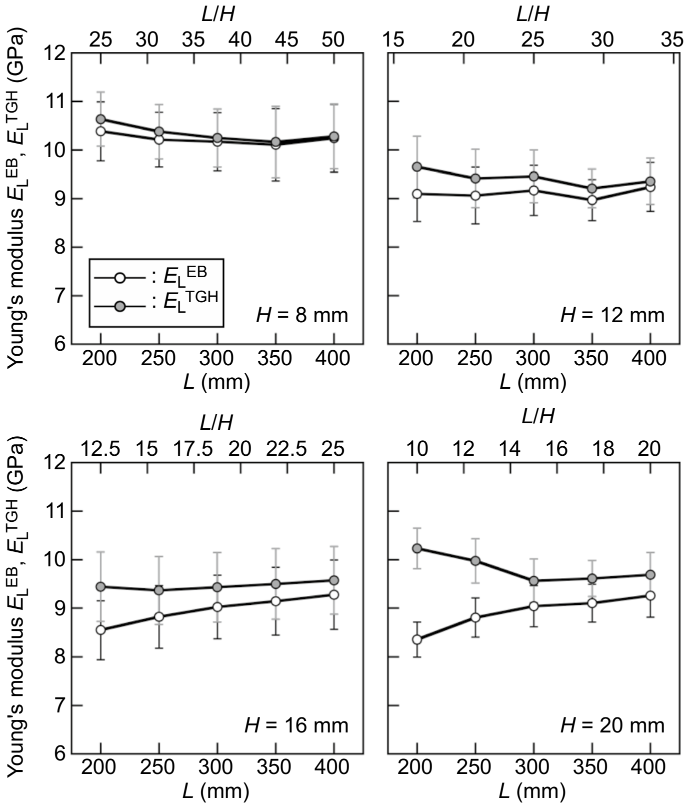

Fig. 4 Comparisons of the E_L^{EB}-L (L/H) and E_L^{TGH}-L (L/H) relationships obtained from the FV tests. The results represent the average ± SD

$$E_L^{\mathrm{TGH}} = E_L^{\mathrm{EB}} T_1 \tag{10}$$

Equations (3)–(5) can be used to determine T_1, which is the value of T_n at the fundamental frequency, as follows:

$$T_1 = 1 + \frac{1}{12}\left(\frac{H}{L}\right)^2 \left[49.48 + 12.3s\frac{E_L^{\mathrm{TGH}}}{G_{\mathrm{LT}}} - \frac{40.49}{T_1}\left(\frac{H}{L}\right)^2\right] \tag{11}$$

The modification using Eq. (9) can be effectively used when E_L^{MOD} is sufficiently close to E_L^{TGH}; therefore, according to Eqs. (8) and (11), T_c should be similar to T_1. However, as described above, the value of E_L^{TGH}/G_{LT} in Eq. (11) varies among various wood species; therefore, the α and β values in Eq. (9) are inevitably influenced by the variation in E_L^{TGH}/G_{LT}. In this study, a probable value of E_L^{TGH}/G_{LT} was determined using the results demonstrated in several existing studies (Hearmon 1948; Kollmann and Côté 1968; Handbook of Wood Industry 1982; Sawada 1983;

Table 2 Minimum, mean, and maximum E_L^{TGH}/G_{LT} for various wood species, reported in the literature

References	Minimum	Mean	Maximum
Hearmon (1948)	6.97	18.4	31.0
Kollmann and Côté (1968)	7.45	18.8	27.7
Handbook of Wood Industry (1982)	6.93	19.7	42.3
Sawada (1983)	9.09	19.3	30.3
Guitard (1987)	5.34	16.2	31.2

Guitard 1987), and the α and β values were subsequently determined by the method described below.

Table 2 lists the minimum, mean, and maximum values of E_L^{TGH}/G_{LT} for various wood species, as reported in Hearmon (1948), Kollmann and Côté, (1968), Handbook of Wood Industry (1982), Sawada (1983), and Guitard (1987). The probable value of E_L^{TGH}/G_{LT} was provisionally derived as 18.5 by averaging the mean values listed in Table 2. Subsequently, T_1 was calculated by substituting $E_L^{TGH}/G_{LT} = 18.5$ into Eq. (11) until its convergence. The converged T_1-L/H relationship was regressed into Eq. (9) with L/H ranging from 10 to 50, which is similar to that adopted in the actual FV test in this study. The α and β values were obtained as 8.72 and 1.65, respectively, and the T_1-L/H and T_c-L/H relationships are shown in Fig. 5. From Eqs. (1), (8), and (10), E_L^{MOD} was derived as follows:

$$E_L^{MOD} = \frac{48\pi^2 \rho L^2 f_1^2}{m_1^4 H^2}\left[1 + 8.72\left(\frac{H}{L}\right)^{1.65}\right] = \frac{48\pi^2 \rho L^2 f_1^2}{m_1^4 H^2} T_c \quad (12)$$

Figure 6 shows the comparisons of the E_L^{TGH}-L (E_L^{TGH}-L/H) and E_L^{MOD}-L (E_L^{MOD}-L/H) relationships derived using Eqs. (7) and (12), respectively. As previously mentioned, the E_L^{TGH}/G_{LT} value of the material used in this study was 11.6 ± 1.0, which significantly deviates from the value of 18.5 provisionally used to derive Eq. (12). However, Fig. 6 suggests the satisfactory agreement of the E_L^{MOD} and E_L^{TGH} values despite the discrepancy in the E_L^{TGH}/G_{LT} values.

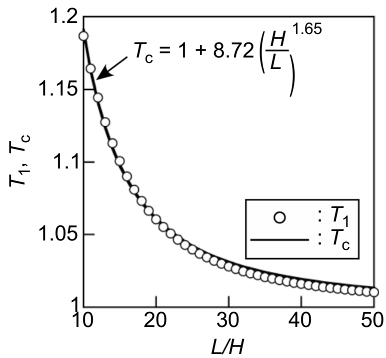

Fig. 5 T_1-L/H and T_c-L/H relationships. $E_L^{TGH}/G_{LT} = 18.5$ to determine the T_1-L/H relationship

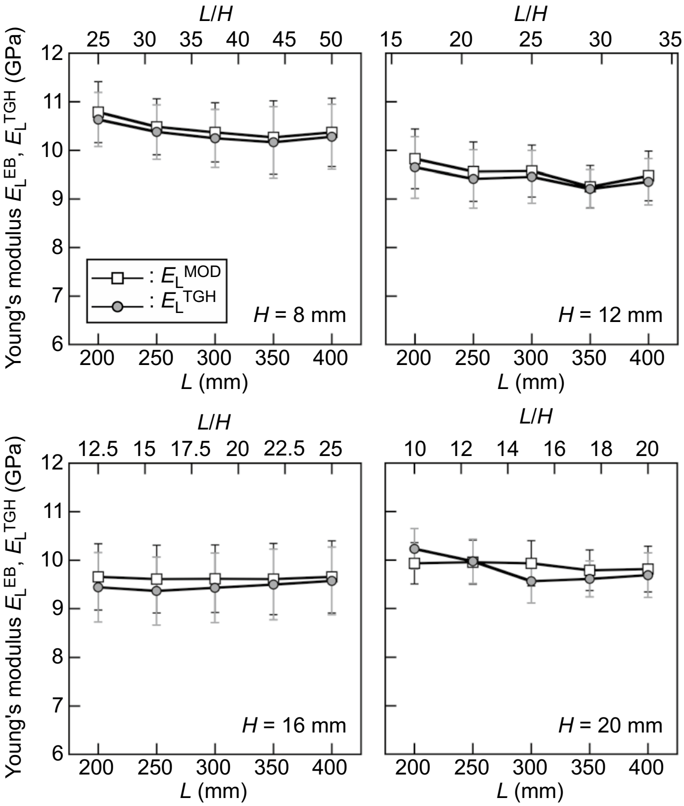

Fig. 6 Comparisons of the E_L^{MOD}-L (L/H) and E_L^{TGH}-L (L/H) relationships obtained from the FV tests. The results represent the average ± SD

Figure 7 shows the coefficient of variations (COV) for the E_L^{EB}, E_L^{MOD}, and E_L^{TGH} values. The COV value was used for the statistical analysis, which is discussed in detail in the following subsection. It should be noted that the COVs corresponding to the E_L^{EB}, E_L^{MOD}, and E_L^{TGH} values are close to each other under similar test conditions.

Figure 8 shows the p-values obtained from the unpaired t tests of the difference between the averages of the values of (a) E_L^{EB} and E_L^{TGH} and (b) E_L^{MOD} and E_L^{TGH} corresponding to L/H. The difference between E_L^{EB} and E_L^{TGH} is often significant when the L/H is less than 20 because the p-value is often less than 0.05 in this L/H range. In contrast, the difference between E_L^{MOD} and E_L^{TGH} is not significant over the entire range of L/H examined in this study. Therefore, the Young's modulus can be derived effectively as E_L^{MOD} with the L/H ranging from 10 to 50.

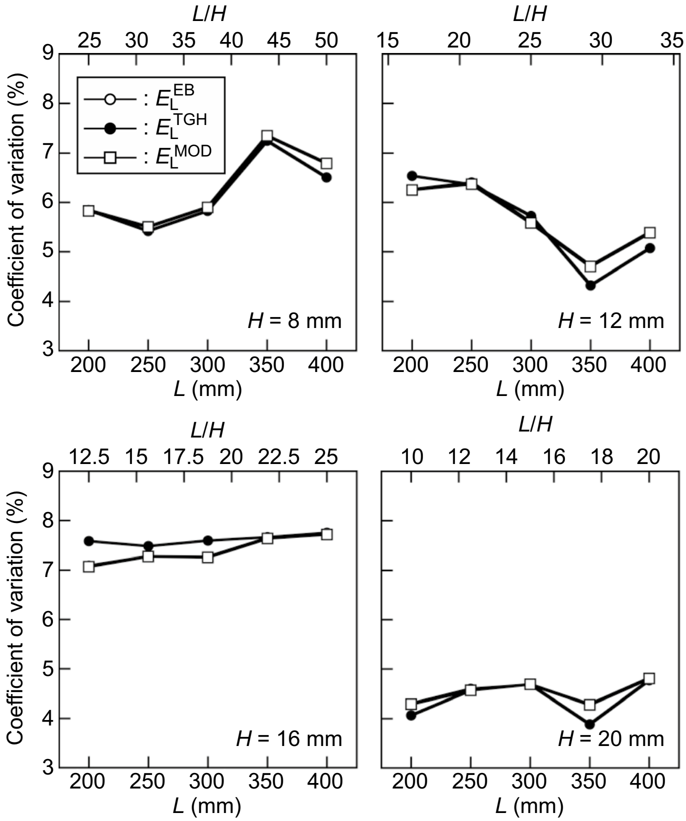

Fig. 7 Coefficient of variations (COV) for the E_L^{EB}, E_L^{MOD}, and E_L^{TGH} values

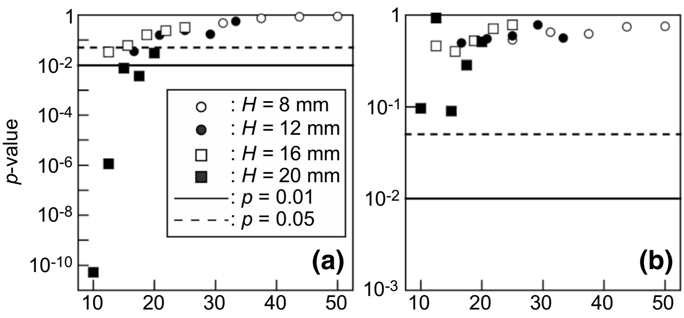

Fig. 8 Relationships between the *p*-value and *L/H* value obtained from the unpaired *t* tests of the average values of **a** E_L^{EB} and E_L^{TGH} values and **b** E_L^{MOD} and E_L^{TGH} values

Examining the applicability of the modified equation by conducting a statistical analysis

As discussed, Eq. (12) can be used to determine the Young's modulus of the material examined in this study, even though its ${E_L}^{TGH}/G_{LT}$ value is significantly different from 18.5. However, Eq. (12) should be applicable to various materials with various ${E_L}^{TGH}/G_{LT}$ values. This applicability was examined by conducting an unpaired t test on the Young's moduli determined by measuring only the fundamental frequency alone (${E_L}^{EB}$ and ${E_L}^{MOD}$) and that obtained using the TGH method, ${E_L}^{TGH}$.

The average values of the Young's modulus obtained from the fundamental frequency, ${E_L}^{EB}$ and ${E_L}^{MOD}$, are denoted as $\overline{E_L^{EB}}$ and $\overline{E_L^{MOD}}$, respectively, whereas the Young's modulus obtained using the TGH method is defined as $\overline{E_L^{TGH}}$. Hereafter, ${E_L}^{EB}$ and ${E_L}^{MOD}$ are represented as ${E_L}^{FF}$. Similarly, $\overline{E_L^{EB}}$ and $\overline{E_L^{MOD}}$ are represented as $\overline{E_L^{FF}}$. The unpaired t test of the difference between $\overline{E_L^{FF}}$ and $\overline{E_L^{TGH}}$ is conducted as follows. When the number of specimens is n, the test statistic indicating the difference in the $\overline{E_L^{FF}}$ and $\overline{E_L^{TGH}}$ values, defined as t_0, is obtained as follows (Fisher and Yates 1963):

$$t_0 = \frac{\left|\overline{E_L^{TGH}} - \overline{E_L^{FF}}\right|}{s_p\sqrt{\frac{2}{n}}} \tag{13}$$

where s_p is the pooled standard deviations. As shown in Fig. 7, the COV for the ${E_L}^{FF}$ values is close enough to that for the ${E_L}^{TGH}$ values; therefore, these COVs are considered to be similar and are collectively denoted as V. Using V, s_p is derived as follows:

$$s_p = \sqrt{\frac{\left(\overline{E_L^{TGH}}V\right)^2 + \left(\overline{E_L^{FF}}V\right)^2}{2}} = V\sqrt{\frac{\left(\overline{E_L^{TGH}}\right)^2 + \left(\overline{E_L^{FF}}\right)^2}{2}} \tag{14}$$

${E_L}^{FF}/{E_L}^{TGH}$ value is defined as c obtained using Eqs. (8) and (10), as follows:

$$c = \frac{E_L^{FF}}{E_L^{TGH}} = \begin{cases} \frac{E_L^{EB}}{E_L^{TGH}} = \frac{1}{T_1} & \left(E_L^{FF} = E_L^{EB}\right) \\ \frac{E_L^{MOD}}{E_L^{TGH}} = \frac{T_c}{T_1} & \left(E_L^{FF} = E_L^{MOD}\right) \end{cases} \tag{15}$$

Consequently, Eqs. (13) and (14) can be transformed using the c value as:

$$t_0 = \overline{E_L^{TGH}}\frac{|1 - c|}{s_p\sqrt{\frac{2}{n}}} \tag{16}$$

and

$$s_p = \overline{E_L^{\mathrm{TGH}}} V \sqrt{\frac{1 + c^2}{2}} \tag{17}$$

From Eqs. (16) and (17), t_0 can be obtained by eliminating $\overline{E_L^{\mathrm{TGH}}}$ as follows:

$$t_0 = \frac{|1 - c|}{V\sqrt{\frac{1+c^2}{n}}} \tag{18}$$

The value of c is calculated by determining the n, V, and t_0 values in Eq. (18). Subsequently, the p-value corresponding to c is obtained from the t-distribution. The E_L^{FF} and E_L^{TGH} values at the corresponding c value exhibit no significant difference when the p-value obtained from this procedure is greater than 0.05.

According to the mechanical sampling method defined in JIS Z 2101-2009 and ISO 3129:2012, the minimum number of specimens, n_{min}, can be obtained as follows:

$$n_{\mathrm{min}} \geq \left(\frac{Vt_{\mathrm{inf}}}{p_i}\right)^2 \tag{19}$$

where p_{i} is the index of the test precision, which is set as 0.05, and t_{inf} is the index of the result authenticity, which is 1.960 when $p_i = 0.05$ (Fisher and Yates 1963). In JIS Z 2101-2009, the number of specimens is considered to be 12 unless otherwise specified. Based on this concept, n and n_{min} are set as 12. By substituting $n_{\mathrm{min}} = 12$ in Eq. (19), the value of V can be obtained as 8.837×10^{-2}. Additionally, when $n = 12$, t_0 in Eq. (18) is 2.074 (Fisher and Yates 1963). The range of c values to ensure that the p-value is larger than 0.05 is 0.927–1.078. The critical length/depth ratio $(L/H)_c$ is defined as the L/H value at $c = 0.927$ or 1.078.

The c value corresponding to the L/H value was calculated by varying the $E_L^{\mathrm{TGH}}/G_{\mathrm{LT}}$ values. Table 3 lists the $E_L^{\mathrm{TGH}}/G_{\mathrm{LT}}$ values used to calculate the c value. These values were determined with reference to the $E_L^{\mathrm{TGH}}/G_{\mathrm{LT}}$ values listed in Table 3. This method was similar to those adopted for determining the specimen configuration to obtain the Young's modulus by the longitudinal vibration (LV) test (Yoshihara and Maruta 2021) and shear modulus by the torsional vibration (TV) test (Yoshihara and Maruta 2020). Figure 9 represents the c-L/H relationships corresponding to the $E_L^{\mathrm{TGH}}/G_{\mathrm{LT}}$ values listed in Table 3. The difference between E_L^{FF}

Table 3 $E_L^{\mathrm{TGH}}/G_{\mathrm{LT}}$ values used to calculate c

Model	A	B	C	D	E
$E_L^{\mathrm{TGH}}/G_{\mathrm{LT}}$ value	5.34	11.9	18.5	30.4	42.3

The $E_L^{\mathrm{TGH}}/G_{\mathrm{LT}}$ values for Models C, A, and E were determined considering the average of five values reported in the literature, minimum value presented by Guitard (1987), and maximum value indicated in the Handbook of Wood Industry (1982), respectively. The value for B was determined by averaging the values for A and C, and the value for D was determined by averaging the values of C and E

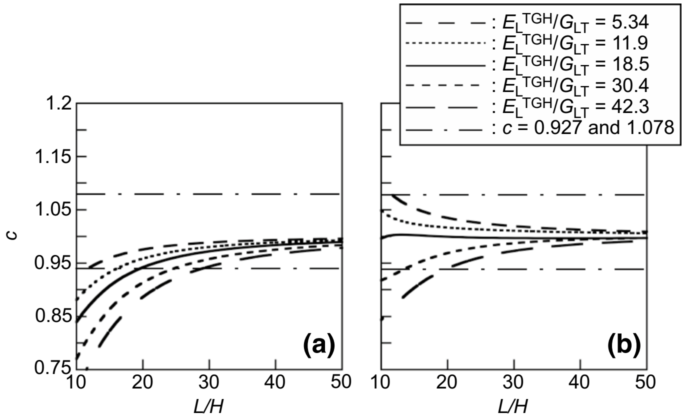

Fig. 9 The *c-L/H* relationships obtained using Eq. (18) under various E_L^{TGH}/G_{LT} values. **a** $c=E_L^{EB}/E_L^{TGH}$ and **b** $c=E_L^{MOD}/E_L^{TGH}$

Table 4 Critical length/depth ratio $(L/H)_c$

	$E_L^{FF}=E_L^{EB}$		$E_L^{FF}=E_L^{MOD}$	
Model	$c=0.927$	$c=1.078$	$c=0.931$	$c=1.074$
A ($E_L^{TGH}/G_{LT}=5.34$)	10.3	<10	<10	11.9
B ($E_L^{TGH}/G_{LT}=11.9$)	14.2	<10	<10	<10
C ($E_L^{TGH}/G_{LT}=18.5$)	17.5	<10	<10	<10
D ($E_L^{TGH}/G_{LT}=30.4$)	22.1	<10	11.3	<10
E ($E_L^{TGH}/G_{LT}=42.3$)	26.1	<10	17.7	<10

The *L/H* value used for the analysis ranges from 10 to 50

($=E_L^{EB}$ or E_L^{MOD}) and E_L^{TGH} is not significant when the *c* value is located between the semi-solid lines, and the $(L/H)_c$ value can be determined considering the intersectional point of the *c-L/H* curve and a semi-solid line. Figure 9 indicates that the range of *L/H* to determine the accurate Young's modulus is markedly extended by the modification.

Table 4 lists the $(L/H)_c$ values when $E_L^{FF}=E_L^{EB}$ and $E_L^{FF}=E_L^{MOD}$. Even when the *L/H* value is greater than 20, as indicated in ASTM D6874-12, the E_L^{EB} value often deviates from the E_L^{TGH} value when the E_L^{TGH}/G_{LT} value is high. Therefore, the Euler–Bernoulli equation (Eq. (1)) is not always suitable to obtain the accurate value of the Young's modulus. In contrast, this table indicates the reasonable agreement between E_L^{MOD} and E_L^{TGH} when *L/H* is determined according to ASTM D6874-12 (2012). Additionally, the *L/H* value standardised for the static bending test methods, as reported in ISO 3133:(1975a); ISO 3349:(1975b); JIS Z 2101-(2009); and ASTM D143-14 (2016) ranges from 16 to 18 ($L=$span length + overhang lengths). The $(L/H)_c$ values for the E_L^{MOD}, listed in Table 4, indicate that the specimens having the configurations reported in the standards for the static bending tests can be used to conduct the FV test. This aspect renders it convenient to adopt the modified equation. The aforementioned results demonstrate that Eq. (12) can be used to determine the accurate values of the Young's modulus by measuring only the fundamental frequency in a wide range of *L/H* values.

To enhance the accuracy of the modified equation, further research should be conducted considering various wood species with various configurations. If the accuracy can be obtained in all the cases, the FV test using the modified equation can be established as a standardised method to measure the Young's modulus of solid wood, in addition to the existing standards.

Conclusion

FV tests were conducted on Sitka spruce specimens to determine the Young's modulus in the longitudinal direction. The Young's moduli were calculated using the resonance frequency at only the 1st FV mode (fundamental frequency) and the frequencies pertaining to the 1st–4th FV modes. By comparing these values, the method for determining the Young's modulus using only the fundamental frequency was evaluated.

The results of the FV tests indicated that the Young's modulus can be determined using only the fundamental frequency by modifying the Euler–Bernoulli equation. The results of the statistical analyses based on the sampling methods recommended in JIS Z 2101-(2009) indicated that the modified equation could be used to determine the Young's modulus of various kinds of solid wood. Therefore, to standardise the FV test, the modified equation is a promising candidate to enable the determination of the Young's modulus of solid wood using only the fundamental frequency.

Acknowledgements This work was supported by JSPS KAKENHI Grant Number 20K06165.

Declaration

Conflict of interest The authors declare that they have no conflict of interest.

References

ASTM (2012) ASTM D6874-12, Standard test methods for nondestructive evaluation of wood-based flexural members using transverse vibration. West Conshohocken, PA, ASTM International. https://doi.org/10.1520/D6874-12

ASTM (2016) ASTM D143-14, Standard test methods for small clear specimens of timber. West Conshohocken, PA, ASTM International. https://doi.org/10.1520/D0143-14

ASTM (2020) ASTM C1548-02, Standard test method for dynamic Young's modulus, shear modulus, and Poisson's ratio of refractory materials by impulse excitation of vibration. West Conshohocken, PA, ASTM International. https://doi.org/10.1520/C1548-02R20

Brancheriau L (2006) Influence of cross section dimensions on Timoshenko's shear factor. Application to wooden beams in free-free flexural vibration. Ann For Sci 63:319–321

Brancheriau L, Bailleres H (2002) Natural vibration analysis of clear wooden beams: a theoretical review. Wood Sci Technol 36:347–365

Brancheriau L, Baillères H (2003) Use of the partial least squares method with acoustic vibration spectra as a new grading technique for structural timber. Holzforschung 57:644–652

Chui YH, Smith I (1990) Influence of rotatory inertia, shear deformation and support condition on natural frequencies of wooden beams. Wood Sci Technol 24:233–245

Divós F, Tanaka T, Nagao H, Kato H (1998) Determination of shear modulus on construction size timber. Wood Sci Technol 32:393–402

Divos F, Denes L, Iñigues G (2005) Effect of cross-sectional change of a board specimen on stress wave velocity determination. Holzforschung 59:230–231

Fisher RA, Yates F (1963) Statistical tables for biological, agricultural and medical research, 6th edn. Oliver and Boyd, Edinburgh

Goens E (1931) Über die Bestimmung des Elastizitätsmoduls von Stäben mit Hilfe von Biegungsschwingungen [Determination of Young's modulus from flexural vibrations]. Ann Phys Ser 7(11):649–678 (in German)

Guitard D (1987) Mécanique du matériau bois et composites [Mechanics of wood and composite materials]. Editions Cépaduès, Toulouse, France

Haines DW, Leban JM, Herbé C (1996) Determination of Young's modulus for spruce, fir and isotropic materials by the resonance flexure method with comparisons to static flexure and other dynamic method. Wood Sci Technol 30:253–263

Handbook of wood industry (1982) Handbook of wood industry 3rd edition, Forestry and forest products research institute Japan, Maruzen, Tokyo, Japan

Hearmon RFS (1948) Elasticity of wood and plywood. HM Stationary Office, London

Hearmon RFS (1958) The influence of shear and rotatory inertia on the free flexural vibration of wooden beams. Brit J Appl Phys 9:381–388

Hearmon RFS (1966) Vibration testing of wood. For Prod J 16(8):29–40

ISO (1975), ISO 3133:1975, Wood—Determination of ultimate strength in static bending. International Organization for Standardization, Geneva

ISO (1975), ISO 3349:1975, Wood—Determination of modulus of elasticity in static bending. International Organization for Standardization, Geneva

ISO (2012) ISO 3129:2012, Wood—Sampling methods and general requirements for physical and mechanical testing of small clear wood specimens. International Organization for Standardization, Geneva

JIS (1993) JIS Z 2280-1993, Test method for Young's modulus of metallic materials at elevated temperature. Japanese Standards Association, Tokyo

JIS (1995) JIS R 1602-1995, Testing methods for elastic modulus of fine ceramics. Japanese Standards Association, Tokyo

JIS (2009) JIS Z 2101-2009, Methods of test for woods. Japanese Standards Association, Tokyo

JIS (2010) JIS A 1127-2010, Methods of test for dynamic modulus of elasticity, rigidity and Poisson's ratio of concrete by resonance vibration. Japanese Standards Association, Tokyo

Kollmann FFP, Côté WA (1968) Principles of wood science and technology: I solid wood. Springer, Berlin

Matsumoto T (1962) Studies on the dynamic modulus E and the logarithmic decrement of wood by transverse vibration. Bull Kyushu Univ For 36:1–86

Murata K, Kanazawa T (2007) Determination of Young's modulus and shear modulus by means of deflection curves for wood beams obtained in static bending tests. Holzforschung 61:589–594

Ono T (1983) Effect of grain angle on dynamic mechanical properties of wood. J Mater Res Soc Jpn 32:108–113

Ono T, Kataoka A (1979a) The frequency dependence of the dynamic Young's modulus and internal friction of wood used for the soundboards of musical instruments I. Effect of rotatory inertia and shear on the flexural vibration of free-free beams. Mokuzai Gakkaishi 25:461–468

Ono T, Kataoka A (1979b) The frequency dependence of the dynamic Young's modulus and internal friction of wood used for the soundboards of musical instruments II. The dependence of the Young's modulus and internal friction on frequency, and the mechanical frequency dispertion. Mokuzai Gakkaishi 25:535–542

Pickett G (1945) Equations for computing elastic constants from flexural and torsional resonant frequencies of vibration of prisms and cylinders. Proc ASTM 45:846–865

Roohnia M (2019) Wood: acoustic properties. In: Hashmi S (ed) Reference module in materials science and materials engineering. Elsevier, Oxford, pp 1–13. ISBN: 978-0-12-803581-8.

Roohnia M, Yavari A, Tajdini A (2010) Elastic parameters of poplar wood with end-cracks. Ann for Sci 67:409

Roohnia M, Manouchehri N, Tajdini A, Yaghmaeipour A, Bayramzadeh V (2011) Modal frequencies to estimate the defect position in a flexural wooden beam. BioResources 6:3676–3686
Sawada M (1983) Deformation behavior of wood. J Soc Mater Sci Jpn 32:838–847
Shamaiirani S, Roohnia M (2016) Dynamic modulus of wood containing water-resistant glue finger joint after severe steaming. BioResources 11:8900–8913
Sobue N (1986) Instantaneous measurement of elastic constants by analysis of the tap tone of wood: application to flexural vibration of beams. Mokuzai Gakkaishi 32:274–279
Sohi AMA, Khademi-Eslam H, Hemmasi AH, Roohnia M, Talaiepour M (2011) Nondestructive detection on the effect of drilling on acoustic performance of wood. BioResources 6:2632–2646
Spinner S, Tefft WE (1961) A method for determining mechanical resonance frequencies and for calculating elastic moduli from these frequencies. Proc ASTM 61:1221–1238
Spinner S, Reichard TW, Tefft WE (1960) A comparison of experimental and theoretical relations between Young's modulus and the flexural and longitudinal resonance frequencies of uniform bars. J Res Natl Bur Stand 64A:147–155
Timoshenko SP (1921) On the correction for shear of the differential equation for transverse vibrations of prismatic bars. Phil Mag 41:744–746
Yoshihara H (2011) Measurement of the Young's modulus and shear modulus of in-plane quasi-isotropic medium-density fiberboard by flexural vibration. BioResources 6:4871–4885
Yoshihara H (2012a) Off-axis Young's modulus and off-axis shear modulus of wood measured by flexural vibration tests. Holzforschung 66:207–213
Yoshihara H (2012b) Examination of the specimen configuration and analysis method in the flexural vibration test of solid wood and wood-based materials. For Prod J 62(3):191–200
Yoshihara H (2012c) Influence of the specimen depth to length ratio and lamination construction on Young's modulus and in-plane shear modulus of plywood measured by flexural vibration. BioResources 7:1337–1351
Yoshihara H, Maruta M (2020) Effect of the specimen configuration on the accuracy in measuring the shear modulus of western hemlock by torsional vibration test. Wood Sci Technol 54:1479–1496
Yoshihara H, Maruta M (2021) Effect of specimen configuration and orthotropy on the Young's modulus of solid wood obtained from a longitudinal vibration test. Holzforschung 75:428–435. https://doi.org/10.1515/hf-2020-0115