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A Tight Upper Bound on Kolmogorov Complexity and Uniformly Optimal Prediction

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Abstract. This paper links the concepts of Kolmogorov complexity (in complexity theory) and Hausdorff dimension (in fractal geometry) for a class of recursive (computable) ω-languages.

It is shown that the complexity of an infinite string contained in a Σ_2-definable set of strings is upper bounded by the Hausdorff dimension of this set and that this upper bound is tight. Moreover, we show that there are computable gambling strategies guaranteeing a uniform prediction quality arbitrarily close to the optimal one estimated by Hausdorff dimension and Kolmogorov complexity provided the gambler's adversary plays according to a sequence chosen from a Σ_2-definable set of strings.

We provide also examples which give evidence that our results do not extend further in the arithmetical hierarchy.

Introduction

The *Kolmogorov complexity* of a finite word w, $K(w)$, is, roughly speaking, the length of a shortest input (program) π for which a universal algorithm $\mathfrak{U}$ prints w (see, e.g., [Ch] and [LV]).[1] For infinite words (ω-words) its (normalized) Kolmogorov complexity might be thought of as

$$\lim_{n\to\infty} \frac{K(\xi/n)}{n}, \tag{1}$$

where ξ/n denotes the prefix of length n of ξ.

[1] As in [S4] we require that w and π be words over the same finite (not necessarily binary) alphabet X of cardinality ≥ 2.

Although this limit need not exist, Cai and Hartmanis [CH] proved that the graph of the function $t\colon \xi \to [0, 1]$ is a fractal regardless with which intermediate value

$$\underline{\kappa}(\xi) := \liminf_{n\to\infty} \frac{K(\xi/n)}{n} \le t(\xi) \le \kappa(\xi) := \limsup_{n\to\infty} \frac{K(\xi/n)}{n}$$

the possibly not existing limit in (1) is replaced.

Other evidence of the intimate relationship between fractals and Kolmogorov complexity was given in [B], [R1]–[R3], [S1], and [S4] where a different approach was pursued: Given a set of infinite words (a so-called ω-language) F, bound the maximum possible Kolmogorov complexity $\underline{\kappa}(\xi)$ and $\kappa(\xi)$ for $\xi \in F$.

Here Ryabko [R1] showed that for arbitrary ω-languages F the Hausdorff dimension, $\dim F$, is a lower bound to $\underline{\kappa}(F) := \sup\{\underline{\kappa}(\xi) : \xi \in F\}$, but as Example 3.18 of [S4] already shows for simple computable ω-languages the Hausdorff dimension is not an upper bound to $\kappa(F) := \sup\{\kappa(\xi) : \xi \in F\}$. Instead the results of [B], [S1], and [S4] show that the upper Kolmogorov complexity of ω-words in F is closely related to (metric) entropy, an information size measure different from Hausdorff dimension (see [F]).

In [S4], however, we showed that for restricted classes of computable ω-languages F (that is, ω-languages satisfying additionally certain combinatorial properties) its Hausdorff dimension is also an upper bound to $\underline{\kappa}(F)$, thus giving a partial completion to Ryabko's lower bound.

After having introduced some notation and preliminaries in Section 1, we show in the the second part that the Hausdorff dimension is an upper bound to lower Kolmogorov complexity for arbitrary Σ_2-definable ω-languages. Moreover, we give evidence that, unless other structural properties of ω-languages are involved, our upper bound does not extend further in the arithmetical hierarchy of ω-languages. This result tightens Ryabko's lower bound in the range of computable (recursive) ω-languages.

Then the third section considers the following prediction problem for infinite sequences (see [R2] and [R3]): A gambler plays a fair game against an adversary who draws his outcomes according to the symbols of an arbitrarily chosen infinite sequence $\xi \in X^\omega$. The prediction quality of the gambler's strategy is measured by the exponent of the increase of the gambler's capital. Utilizing Levin's universal semicomputable semimeasure it was shown in [R2] and [R3] that this exponent $\lambda(\xi)$ is bounded from above by $1 - \underline{\kappa}(\xi)$ provided the gambler plays according to a computable strategy.

In this paper we investigate the case when the adversary is bound to draw his outcomes from a sequence $\eta \in F$ where $F \subseteq X^\omega$ is Σ_2-definable. We show that then there are computable strategies guaranteeing a prediction quality $\lambda(\eta)$ arbitrarily close to the value $(1 - \dim F)$ uniformly on F, that is, regardless from which $\eta \in F$ the adversary draws his outcomes. Moreover, if the Hausdorff dimension of F, $\dim F$, is a semicomputable from the above real number, then there is a strategy guaranteeing $\lambda(\eta) \ge 1 - \dim F$ for $\eta \in F$.

Finally, Section 4 gives some connections to previous work, in particular, upper bounds on Kolmogorov complexity via Hausdorff dimension for ω-languages satisfying additional combinatorial properties are considered.

1. Notation and Preliminary Results

By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers. We consider the space X^ω of infinite strings (sequences, ω-words) on a finite alphabet of cardinality $r := card\ X \geq 2$. By X^* we denote the set (monoid) of finite strings (words) on X, including the *empty* word e. For $w \in X^*$ and $b \in X^* \cup X^\omega$ let $w \cdot b$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. As usual we denote subsets of X^* as languages and subsets of X^ω as ω-languages.

Furthermore, $|w|$ is the *length* of the word $w \in X^*$, hence $X^n = \{w : w \in X^* \wedge |w| = n\}$. As introduced above, by b/n we denote the *length n prefix* of a string $b \in X^*,\ |b| \geq n$, or $b \in X^\omega$, and $\mathbf{A}(b) := \{b/n : n \in \mathbb{N} \wedge n \leq |b|\}$ and $\mathbf{A}(B) := \bigcup_{b \in B} \mathbf{A}(b)$ are the sets of all finite prefixes of $b \in X^* \cup X^\omega$ and $B \subseteq X^* \cup X^\omega$, respectively.

As usual we define Π_1-*definable ω-languages* $E \subseteq X^\omega$ as

$$E = \{\xi : \forall n (n \in \mathbb{N} \rightarrow \xi/n \notin W_E)\}, \tag{2}$$

where $W_E \subseteq X^*$ is a recursive language, and we define Σ_2-*definable ω-languages* $F \subseteq X^\omega$ as

$$F = \{\xi : \exists i (i \in \mathbb{N} \wedge \forall n (n \in \mathbb{N} \rightarrow (i, \xi/n) \notin M_F))\}, \tag{3}$$

where M_F is a recursive subset of $\mathbb{N} \times X^*$.

Define $L_i := \{w : (i, w) \in M_F\}$. Then (2) and (3) may be alternatively written as

$$E = X^\omega \backslash W_E \cdot X^\omega \tag{4}$$

and

$$F = \bigcup_{i \in \mathbb{N}} (X^\omega \backslash L_i \cdot X^\omega) = \bigcup_{k \in \mathbb{N}} \left(X^\omega \backslash \left(\bigcap_{i=0}^{k} L_i \cdot X^* \right) \cdot X^\omega \right). \tag{5}$$

In particular, every Σ_2-definable ω-language is a countable union of Π_1-definable ω-languages.

From the definitions and (4) and (5) it is clear that we may think of W_E and M_F as upward closed, that is, $W_E = W_E \cdot X^*$ and M_F satisfies the condition

$$(k, w) \in M_F \Rightarrow \forall v (v \in X^* \rightarrow (k, w \cdot v) \in M_F)$$

and

$$(k, w) \in M_F \Rightarrow \forall i (i \leq k \rightarrow (i, w) \in M_F),$$

that is, $L_i = L_i \cdot X^*$ and $L_i \supseteq L_{i+1}$.

In order to give a more detailed analysis we consider X^ω as a topological space (Cantor space), where the open subsets are defined as $W \cdot X^\omega$ ($W \subseteq X^*$). It should be mentioned that this space is compact, hence for every closed subset $F \subseteq X^\omega$ the inclusion $F \subseteq W \cdot X^\omega$ implies that there is already a finite subset $W' \subseteq W$ for which $F \subseteq W' \cdot X^\omega$.

It is known (e.g., [S2, Theorem 6.2] and [S5, Lemma 3.12]) that a subset $F \subseteq X^\omega$ is Π_1-definable iff F is closed and $X^*\setminus\mathbf{A}(F)$ is recursively enumerable, consequently, a Σ_2-definable subset $E \subseteq X^\omega$ is a countable union of closed sets (a so-called $\mathbf{F}_\sigma$*-set*).

The Hausdorff dimension of an ω-language $F \subseteq X^\omega$ may be defined using the entropy of languages. To this end we introduce the *structure function* of a language $L \subseteq X^*$, $\boldsymbol{s}_L \colon \mathbb{N} \to \mathbb{N}$, as

$$\boldsymbol{s}_L(n) := \mathit{card}\, L \cap X^n,$$

and we define its *entropy*

$$H_L := \limsup_{n\to\infty} \frac{\log_r(1+\boldsymbol{s}_L(n))}{n}.$$

In other words, we have the following equivalences for $\alpha \neq H_L$:

$$\sum_{n\in\mathbb{N}} \boldsymbol{s}_L(n)\cdot r^{-\alpha\cdot n} = \sum_{v\in L} r^{-\alpha\cdot|v|} < \infty \leftrightarrow \alpha > H_L, \tag{6}$$

$$\sum_{n\in\mathbb{N}} \boldsymbol{s}_L(n)\cdot r^{-\alpha\cdot n} = \sum_{v\in L} r^{-\alpha\cdot|v|} = \infty \leftrightarrow \alpha < H_L. \tag{7}$$

In case $\alpha = H_L$ we may have $\sum_{v\in L} r^{-\alpha\cdot|v|} < \infty$ as well as $\sum_{v\in L} r^{-\alpha\cdot|v|} = \infty$.

Moreover, let $V^\delta := \{\xi : \xi \in X^\omega \wedge \mathbf{A}(\xi)\cap V \text{ is infinite}\}$ be the δ*-limit* of the language $V \subseteq X^*$. Then according to (3.11) of [S4]

$$\dim F := \inf\{H_W : W \subseteq X^* \wedge F \subseteq W^\delta\} \tag{8}$$

is the *Hausdorff dimension* of $F \subseteq X^\omega$. In particular, we have

$$\dim V^\delta \leq H_V. \tag{9}$$

Moreover, following Lemma 3.10 of [S4], we have, for $\gamma > \dim F$,

$$\forall\varepsilon \left(\varepsilon > 0 \to \exists W \left(W \subseteq X^* \wedge F \subseteq W\cdot X^\omega \wedge \sum_{w\in W} (r^{-\gamma})^{|w|} < \varepsilon\right)\right). \tag{10}$$

We mention some relations between lower Kolmogorov complexity $\underline{\kappa}$ and Hausdorff dimension or the entropy of languages. First we have a general lower bound to $\underline{\kappa}$ by Hausdorff dimension, and second we give an upper bound on $\underline{\kappa}$ by the entropy of languages.

Theorem 1 [R1, Theorem 2]. *If* $F \subseteq X^\omega$, *then* $\dim F \leq \underline{\kappa}(F)$.

Lemma 2 [S4,Proposition 2.14]. *If* $L \subseteq X^*$ *and* L *or its complement* $X^*\setminus L$ *is a recursively enumerable language, then*

$$\underline{\kappa}(\xi) \leq H_L \qquad \textit{for all} \quad \xi \in L^\delta.$$

2. An Upper Bound for Kolmogorov Complexity

In this section we derive the upper bound for $\underline{\kappa}(\xi)$ by the Hausdorff dimension of $F \subseteq X^\omega$, $\dim F$, provided $\xi \in F$ and F is Σ_2-definable. Our result will be an immediate consequence of the following recursive analogue to the definition of the Hausdorff dimension (see (8)).

To this end we need the definition of semicomputable real numbers. As in Section 3.8 of [W] we call a real number $\alpha \in \mathbb{R}$ left- (right-)computable iff the set of all rational numbers smaller (larger) than α is recursively enumerable. An equivalent definition is the following one:

Definition 1. A real number $\alpha \in \mathbb{R}$ is called *left-computable* iff there is a recursive function $f_\alpha \colon \mathbb{N} \to \mathbb{Q}$ such that

1. $\forall i (i \in \mathbb{N} \to f_\alpha(i) < f_\alpha(i+1) < \alpha)$, and
2. $\lim_{i\to\infty} f_\alpha(i) = \alpha$.

We also say that f_α *approximates* α *from below*.

Right-computable real numbers are defined accordingly.

It should be noted that $-\alpha$ is left-computable iff α is right-computable, and that the commonly used monotone functions in analysis like $x + y$, $x \cdot y$, r^x, $\log_r x$ preserve left-computable as well as right-computable real numbers.

Theorem 3. *If $F \subseteq X^\omega$ is a Σ_2-definable ω-language, then*

$$\dim F = \inf\{H_V : F \subseteq V^\delta \wedge V \subseteq X^* \text{ is recursive}\}.$$

If, moreover, $\dim F$ *is a right-computable real number, then there is a recursive language $V \subseteq X^*$ such that $F \subseteq V^\delta$ and* $\dim F = H_V$.

Remark. It should be mentioned that an ω-language $F \subseteq X^\omega$ is Σ_2-definable iff there is a recursive language $W \subseteq X^*$ such that $F = X^\omega \setminus W^\delta$. Accordingly, an ω-language $E \subseteq X^\omega$ is Π_2-*definable* iff there is a recursive language $V \subseteq X^*$ such that $E = V^\delta$ (see [S2, Theorem 7.4] or [S5, Theorem 3.12]).

In view of (9) this leads to the following consequence of Theorem 3.

Corollary 4. *If $F \subseteq X^\omega$ is a Σ_2-definable ω-language, then*

$$\dim F = \inf\{\dim E : F \subseteq E \wedge E \text{ is } \Pi_2\text{-definable}\}.$$

As Corollary 4 is trivally true for Π_2-definable ω-languages $F \subseteq X^\omega$, it would be interesting to know whether Theorem 3 holds also for Π_2-definable ω-languages. We shall see below in Lemma 6 that our recursive analogue to the definition of the Hausdorff dimension (Theorem 3) is not valid for Π_2-definable ω-languages.

Proof of Theorem 3. For the first part it suffices to assign to every rational number $\alpha > \dim F$ a recursive language $V_\alpha \subseteq X^*$ such that $F \subseteq V_\alpha^\delta$ and $\dim F \leq H_{V_\alpha}$. In order to cover both cases of the theorem we may assume $\alpha \geq \dim F$ to be a right-computable real number and describe an algorithm which constructs a recursive language V_α such that $F \subseteq V_\alpha^\delta$ and $H_{V_\alpha} \leq \alpha$.

Since then $r^{-\alpha}$ is a left-computable real number, we may assume $g_\alpha\colon \mathbb{N} \to \mathbb{Q}$ to be a recursive function approximating $r^{-\alpha}$ from below, that is, $g_\alpha(i-1) < g_\alpha(i) \underset{i\to\infty}{\longrightarrow} r^{-\alpha}$, and since $0 \leq \alpha \leq 1$, we may assume $g_\alpha(i) > r^{-2}$.

Furthermore, let $(U_j)_{j\in\mathbb{N}}$ be an effective enumeration of all finite prefix codes in X^* such that $\sup\{|v|\colon v \in U_j\} \leq \sup\{|v|\colon v \in U_{j+1}\}$, and let the languages $L_k \subseteq X^*$ defining F be given according to (5) as $L_k := (\bigcap_{i=0}^{k} L_i \cdot X^*)$. Define

$$\mathbf{test}(k, j, n)\colon \Leftrightarrow \left((U_j \cup (L_k \cap X^n)) \cdot X^\omega = X^\omega \ \wedge \sum_{v\in U_j} g_\alpha(k)^{|v|} < r^{-k} \right).$$

Observe that $\mathbf{test}(k, j, n)$ is effectively computable and that if $\mathbf{test}(k, j, n)$ is true, then the following conditions for U_j are satisfied:

$$\begin{array}{lll} F \subseteq U_j \cdot X^\omega, & \text{because} \quad L_k \cdot X^\omega \cap F = \emptyset, & \text{and} \\ \forall v(v \in U_j \to k < 2\cdot|v|), & \text{because} \quad g_\alpha(i) > r^{-2}. & \end{array} \tag{11}$$

Now the following algorithm, when given M_F, computes a finite prefix code C_k satisfying $F \subseteq C_k \cdot X^\omega$ and $\sum_{w\in C_k} g_\alpha(k)^{|w|} < r^{-k}$:

Algorithm C_k

```
input   k
        n = 0
        repeat j = −1
               repeat j = j + 1
               until  test(k, j, n) ∨ (sup{|v|: v ∈ U_j} > n)
               n = n + 1
        until  test(k, j, n)
output  C_k := U_j
```

Informally, for every $n \geq 0$ our algorithm successively searches for a U_j satisfying the condition $\mathbf{test}(k, j, n)$, more precisely, it searches until such a U_j is found or else all U_j having $\sup\{|v|\colon v \in U_j\} \leq n$ fail to satisfy $\mathbf{test}(k, j, n)$. In the latter case the value of n is increased (thus allowing for a larger maximum codeword length and a larger complementary ω-language $(L_k \cap X^n) \cdot X^\omega$) and the search starts anew. Consequently, the algorithm terminates iff there is a finite prefix code U such that $\sum_{v\in U} g_\alpha(k)^{|v|} < r^{-k}$ and $U \cdot X^\omega \cup (L_k \cap X^n) \cdot X^\omega = X^\omega$ for some $n \in \mathbb{N}$.

We still have to verify that our algorithm always terminates provided $\alpha \geq \dim F$. To this end we observe that in view of $g_\alpha(k) < r^{-\alpha}$ according to (10) for every $\varepsilon > 0$ there is a $W \subseteq X^*$ such that $F \subseteq W \cdot X^\omega$ and $\sum_{w\in W} g_\alpha(k)^{|w|} < \varepsilon$.

Since $X^\omega \backslash L_k \cdot X^\omega$ is a closed subset of F, for $\varepsilon \leq r^{-k}$ we find a finite subset

$W_k \subseteq W$ such that $X^\omega \setminus L_k \cdot X^\omega \subseteq W_k \cdot X^\omega$. Consequently, there is also a finite prefix code U_j satisfying $(U_j \cup L_k) \cdot X^\omega = X^\omega$ and thus $(U_j \cup (L_k \cap X^n)) \cdot X^\omega = X^\omega$ for n large enough.

Now let $V_\alpha := \bigcup_{k\in\mathbb{N}} C_k$. Obviously, V_α is recursively enumerable. Note that V_α is even recursive: Since $w \in C_k$ implies $2 \cdot |w| > k$ (see (11)), we have $w \in V_\alpha$ iff $\exists k(k < 2\cdot|w| \wedge w \in C_k)$. The predicate $k < 2\cdot|w|$ is recursive and bounds the quantifier $\exists k$ from above.

Next we show that $F \subseteq V_\alpha^\delta$. If $\xi \in F$ there is a k such that $\xi \in X^\omega \setminus L_i \cdot X^\omega$ for all $i \geq k$. Hence, for every $i \geq k$ the ω-word ξ has a prefix $w_i \in C_i$. As was observed above, $2 \cdot |w_i| > i$. Consequently, ξ has infinitely many prefixes in $V_\alpha = \bigcup_{i\in\mathbb{N}} C_i$.

Finally it remains to show that $H_{V_\alpha} \leq \alpha$. By virtue of (6) it suffices to verify the inequality $\sum_{w\in V_\alpha} r^{-\gamma\cdot|w|} < \infty$ for every $\gamma > \alpha$.

If $\gamma > \alpha$, then there are only finitely many $i \in \mathbb{N}$ such that $g_\alpha(i) < r^{-\gamma}$. Let i_0 be the largest such i. Consequently,

$$\begin{aligned}\sum_{w\in V_\alpha} r^{-\gamma\cdot|w|} &\leq \sum_{k\in\mathbb{N}} \sum_{w\in C_k} r^{-\gamma\cdot|w|} \\ &\leq \sum_{k\leq i_0} \sum_{w\in C_k} r^{-\gamma\cdot|w|} + \sum_{k>i_0} \sum_{w\in C_k} g_\alpha(i)^{|w|} \\ &\leq \sum_{k\leq i_0} \sum_{w\in C_k} r^{-\gamma\cdot|w|} + \sum_{k>i_0} r^{-k} \quad < \infty. \end{aligned}$$ □

Utilizing Theorem 1 and Lemma 2 we obtain the announced upper bound on $\underline{\kappa}(F)$ by $\dim F$ as a direct consequence of Theorem 3.

Theorem 5. *If $F \subseteq X^\omega$ is a Σ_2-definable ω-language, then $\underline{\kappa}(F) = \dim F$.*

Observe that, since the Hausdorff dimension is countably stable, that is,

$$\dim \bigcup_{i\in\mathbb{N}} F_i = \sup_{i\in\mathbb{N}} \dim F_i,$$

this result immediately extends to arbitrary countable unions of Σ_2-definable (or, equivalently, to countable unions of Π_1-definable) ω-languages $\bigcup_{j\in\mathbb{N}} F_j$.

The bound of Theorem 5 cannot be tightened to ensure that there is indeed a $\xi \in F$ such that $\underline{\kappa}(\xi) = \dim F$, as the following example shows.

Example 1. Let $E := \{0\}^\omega \cup \bigcup_{n\in\mathbb{N}} 0^n \cdot (1 \cdot X^n)^\omega$. Then E and also the ω-languages $\{0\}^\omega$ and $0^n \cdot (1 \cdot X^n)^\omega$ are Π_1-definable. Now, similar to Example 4.11 of [S4] one calculates

$$\dim\ 0^n \cdot (1 \cdot X^n)^\omega = \frac{n}{n+1}.$$

Consequently, $\dim E = 1 > n/(n+1) \geq \underline{\kappa}(\xi)$ for $\xi \in 0^n \cdot (1 \cdot X^n)^\omega$.

Next, we show that the upper bound of Theorem 5 cannot be extended further to higher classes of the arithmetical hierarchy of ω-languages. To this end we consider the class

$$\mathfrak{S} := \{E\colon\ E \subseteq X^\omega \wedge E \text{ is closed } \wedge \mathbf{A}(E) \text{ is recursively enumerable}\},$$

which in some sense seems to be dual to the class of Π_1-definable ω-languages. Since $E \in \mathfrak{S}$ implies $E = \mathbf{A}(E)^\delta$, every $E \in \mathfrak{S}$ is Π_2-definable, but as the remark following Corollary 7.2 in [S2] shows, not every Π_2-definable and closed ω-language belongs to $\mathfrak{S}$.

In [S2] a "nice" characterization of the class $\mathfrak{S}$ was asked for. In Example 1.15 of [S4] it was already explained that there are ω-languages in $\mathfrak{S}$ which are not Σ_2-definable.

Here we give a proof that an analogue of Theorem 5 cannot hold for the class $\mathfrak{S}$. More precisely, we present a countable ω-language $E \in \mathfrak{S}$ which except for a single limit point $\xi \in E$ is Σ_2-definable but does not satisfy $\underline{\kappa}(E) \leq \dim E$. This does not only prove that Theorem 5 and hence also Theorem 3 do not hold for Π_2-definable ω-languages but also gives further evidence of the fact that there seems to be no "nice" characterization of the class $\mathfrak{S}$ in terms of the arithmetical hierarchy.

Lemma 6. *There is a countable ω-language $E \in \mathfrak{S}\setminus\Sigma_2$ such that E is the closure* (*in Cantor space*) *of a Σ_2-definable ω-language E' with $\underline{\kappa}(\xi) = 1$ for $\xi \in E\setminus E'$.*

Proof. P. Martin-Löf (see [LV] or [Ca]) proved that a random sequence $\zeta \in X^\omega$ has $\underline{\kappa}(\zeta) = 1$ and that the set of all random sequences $\mathfrak{N}_{\mathrm{rand}}$ is Σ_2-definable. Hence, $\mathfrak{N}_{\mathrm{rand}}$ contains a nonempty Π_1-definable subset $F = X^\omega\setminus W_F \cdot X^\omega$, where $W_F \subseteq X^*$ is a recursive language.

Fix the natural order on $X = \{0, 1, \ldots, r-1\}$ and extend it to a quasi-lexicographical order on X^*. Since F is closed in the Cantor topology of X^ω, it contains a leftmost sequence ζ_{left}.[2]

We show that there is a countable ω-language $E \in \mathfrak{S}$ such that E contains the leftmost sequence $\zeta_{\mathrm{left}} \in F$.

Define for every $n \in \mathbb{N}$ the word v_n as the lexicographically first word in $X^n\setminus W_F \cdot X^*$. Obviously, the language $V := \{v_n\colon\ n \in \mathbb{N}\}$ is recursive, and $\zeta_n := v_n \cdot 0^\omega$ is the leftmost ω-word in $(X^n\setminus W_F \cdot X^*) \cdot X^\omega \supseteq F$.

Since $(X^n\setminus W_F \cdot X^*) \cdot X \supseteq X^{n+1}\setminus W_F \cdot X^*$, the family $((X^n\setminus W_F \cdot X^*) \cdot X^\omega)_{n\in\mathbb{N}}$ of closed subsets converges to $F = \bigcap_{n\in\mathbb{N}}(X^n\setminus W_F \cdot X^*) \cdot X^\omega$, and since ζ_n is the leftmost point in $(X^n\setminus W_F \cdot X^*) \cdot X^\omega \supseteq F$, we also have $\lim_{n\to\infty} \zeta_n = \zeta_{\mathrm{left}}$ whence $\bigcup_{n\in\mathbb{N}} \mathbf{A}(\zeta_n) \supseteq \mathbf{A}(\zeta_{\mathrm{left}})$.

Defining $E' := \{\zeta_n\colon\ n \in \mathbb{N}\} = V \cdot 0^\omega$ and $E := E' \cup \{\zeta_{\mathrm{left}}\}$ yields that E is closed, $\mathbf{A}(E) = \mathbf{A}(V) \cup V \cdot 0^*$, and E' is Σ_2-definable. □

Since E is countable, $\dim E = 0$, and, consequently, neither the analogue of Theorem 5 nor the recursive version of the definition of the Hausdorff dimension (Theorem 3)

[2] This corresponds to the obvious interpretation of ω-words $\zeta \in X^\omega$ as r-ary expansions of real numbers $\iota(\zeta) := 0{\cdot}\zeta \in [0, 1]$. The mapping $\iota\colon\ X^\omega \to [0, 1]$ is continuous and is one-to-one on $X^\omega\setminus X^* {\cdot} \{0^\omega, (r-1)^\omega\} \supseteq \mathfrak{N}_{\mathrm{rand}}$.

can hold for the class $\mathfrak{S}$. One can only show the following upper bound on $\underline{\kappa}(E)$ related to the structure function $\mathbf{s}_{\mathbf{A}(E)}$.

Theorem 7 [S4, Proposition 2.11]. *If $E \subseteq X^\omega$ is an ω-language such that $\mathbf{A}(E)$ or the complement $X^* \setminus \mathbf{A}(E)$ is recursively enumerable, then*

$$\underline{\kappa}(\xi) \leq \liminf_{n\to\infty} \frac{\log_r \mathbf{s}_{\mathbf{A}(E)}(n)}{n} \qquad \textit{for all} \quad \xi \in E.$$

As a further consequence of the proof of Lemma 6 one obtains the following result on ω-languages in the arithmetical hierarchy.

Corollary 8. *There is an ω-language $E \in \mathfrak{S}$ which is not representable as a countable union of Σ_2-definable ω-languages.*

3. Computable Martingales and Prediction

Quite recently Ryabko [R2], [R3] proved another kind of relationship between Hausdorff dimension and Kolmogorov complexity. He considered a problem which is related to the prediction problem of infinite sequences. This problem, addressed in the Introduction, can be easily described using martingales, Ville's formalism of the concept of gambling strategy.

Here we consider martingales as functions $\mathcal{V}\colon X^* \to [0, \infty)$ which satisfy

$$\mathcal{V}(e) > 0 \quad \text{and} \quad \mathcal{V}(w) = \frac{\sum_{x\in X} \mathcal{V}(wx)}{\mathit{card}\ X}, \tag{12}$$

and we denote by $\Lambda_{\mathcal{V}}(w) := \log_r \mathcal{V}(w)$ the *exponent of the increase* (*pay-off*) of $\mathcal{V}$.

More general is the notion of *submartingale* where in (12) we have "$\geq$" instead of equality. Using Levin's universal semicomputable (left-computable) measure $\mathcal{R}$ one can show (see Theorem 4.10 of [LV]) that there is a universal left-computable submartingale $\mathcal{U}$ which majorizes every left-computable submartingale $\mathcal{V}$, that is,

$$\exists c_{\mathcal{V}}(c_{\mathcal{V}} > 0 \wedge \forall w(w \in X^* \to \mathcal{U}(w) \geq c_{\mathcal{V}} \cdot \mathcal{V}(w))). \tag{13}$$

Utilizing Theorem 3.4 of [ZL] one has

$$|\log_r \mathcal{U}(w) + |w| - K(w)| = o(|w|). \tag{14}$$

Let $\mathcal{V}$ be a computable martingale (or as in [R2] and [R3]: a Turing prediction strategy). Then (13) and (14) imply

$$\forall \xi \left(\xi \in X^\omega \to \lambda_{\mathcal{V}}(\xi) := \limsup_{w\to\xi} \frac{\Lambda_{\mathcal{V}}(w)}{|w|} \leq 1 - \underline{\kappa}(\xi) \right). \tag{15}$$

It is mentioned in [R3] that there is no optimal prediction method, that is, there is no computable martingale $\mathcal{V}_{\text{opt}}$ such that $\lambda_{\mathcal{V}_{\text{opt}}}(\xi) \geq \lambda_{\mathcal{V}}(\xi)$ for all $\xi \in X^\omega$ and all computable martingales $\mathcal{V}$, and it is proved there that for

$$\lambda(\xi) := \sup\{\lambda_{\mathcal{V}}(\xi)\colon \mathcal{V} \text{ is a computable martingale}\}$$

the set of all ω-words with prediction quality $\geq \gamma$ has Hausdorff dimension

$$\dim\{\xi\colon \xi \in X^{\omega} \wedge \lambda(\xi) \geq \gamma\} = 1 - \gamma.$$

Our aim is to prove a certain constructive converse of this general proposition: We show that for every Σ_2-definable ω-language F and every $\alpha > \dim F$ there is a computable martingale $\mathcal{V}$ such that $\lambda_{\mathcal{V}}(\xi) \geq 1 - \alpha$ for all $\xi \in F$, thus being nearly uniformly optimal for the ω-language F. Similar to Theorem 3 we also achieve the inequality $\lambda_{\mathcal{V}}(\xi) \geq 1 - \dim F$ if $\dim F$ is a right-computable real number.

In order to achieve our goal we introduce families of covering codes as follows.

For a prefix code $C \subseteq X^*$ we define its *minimal complementary code* as

$$\widehat{C} := (X \cup \mathbf{A}(C) \cdot X) \setminus \mathbf{A}(C).$$

If $C = \emptyset$ we have $\widehat{C} = X$, and if $C \neq \emptyset$ the set $\widehat{C}$ consists of all words $w \cdot x \notin \mathbf{A}(C)$ where $w \in \mathbf{A}(C)$ and $x \in X$. It is readily seen that $C \cup \widehat{C}$ is a maximal prefix code, $C \cap \widehat{C} = \emptyset$, and $\mathbf{A}(C \cup \widehat{C}) = \{e\} \cup \mathbf{A}(C) \cup \widehat{C}$.

We call $\mathfrak{C} := (C_w)_{w \in X^*}$ a *family of covering codes* provided each C_w is a finite prefix code. Since then the set $C_w \cup \widehat{C}_w$ is a finite maximal prefix code, every word $u \in X^*$ has a uniquely specified $\mathfrak{C}$-factorization $u = u_1 \cdots u_n \cdot u'$ where $u_{i+1} \in C_{u_1 \cdots u_i} \cup \widehat{C}_{u_1 \cdots u_i}$ for $i = 0, \ldots, n-1$ ($u_1 \cdots u_i = e$, if $i = 0$) and $u' \in \mathbf{A}(C_{u_1 \cdots u_n} \cup \widehat{C}_{u_1 \cdots u_n})$. Analogously, every $\xi \in X^{\omega}$ has a uniquely specified $\mathfrak{C}$-factorization $\xi = u_1 \cdots u_i \cdots$ where $u_{i+1} \in C_{u_1 \cdots u_i} \cup \widehat{C}_{u_1 \cdots u_i}$ for $i = 1, \ldots$.

In what follows we use martingales derived from prefix codes in the following manner.

Lemma 9. *Let* $0 \leq \alpha \leq 1$ *and* $\emptyset \neq C \subseteq X^*$ *be a prefix code satisfying* $\sum_{v \in C} r^{-\alpha|v|} < \infty$. *Then there is a martingale* $\mathcal{V}_C^{(\alpha)}\colon X^* \to \mathbb{R}_+$ *such that*

$$\mathcal{V}_C^{(\alpha)}(w) = \begin{cases} \dfrac{r^{(1-\alpha)|w|}}{\sum_{v \in C} r^{-\alpha|v|} + \sum_{u \in \widehat{C}} r^{-|u|}} & \text{for} \quad w \in C, \\ \dfrac{1}{\sum_{v \in C} r^{-\alpha|v|} + \sum_{u \in \widehat{C}} r^{-|u|}} & \text{for} \quad w \in \widehat{C}. \end{cases} \tag{16}$$

Proof. Set $\Gamma := \sum_{v \in C} r^{-\alpha|v|} + \sum_{u \in \widehat{C}} r^{-|u|}$, and define for $u \in \mathbf{A}(C \cup \widehat{C})$ and $w \in C \cup \widehat{C}$, $v \in X^*$,

$$\mathcal{V}_C^{(\alpha)}(u) := \frac{r^{|u|}}{\Gamma} \cdot \left(\sum_{u \cdot w \in C} r^{-\alpha|u \cdot w|} + \sum_{u \cdot w \in \widehat{C}} r^{-|u \cdot w|} \right),$$

$$\mathcal{V}_C^{(\alpha)}(w \cdot v) := \mathcal{V}_C^{(\alpha)}(w).$$

Then $\mathcal{V}_C^{(\alpha)}$ fulfills (16). We still have to show the property $\mathcal{V}_C^{(\alpha)}(u) = (1/r) \sum_{x \in X} \mathcal{V}_C^{(\alpha)}(ux)$. This identity is obvious if $u \notin \mathbf{A}(C \cup \widehat{C})$.

Now, let $u \in \mathbf{A}(C \cup \widehat{C})$. Then

$$\sum_{x \in X} \frac{\mathcal{V}_C^{(\alpha)}(ux)}{r} = \sum_{x \in X} \frac{r^{|ux|}}{r \cdot \Gamma} \cdot \left(\sum_{uxw \in C} r^{-\alpha|uxw|} + \sum_{uxw \in \widehat{C}} r^{-|uxw|} \right)$$
$$= \frac{r^{|u|}}{\Gamma} \cdot \sum_{x \in X} \left(\sum_{uxw \in C} r^{-\alpha|uxw|} + \sum_{uxw \in \widehat{C}} r^{-|uxw|} \right),$$

because for $u \in \mathbf{A}(C \cup \widehat{C}) \setminus (C \cup \widehat{C})$ the set $\{w\colon\ w \in C \cup \widehat{C} \wedge u \sqsubseteq w\}$ partitions into the sets $\{w\colon\ w \in C \cup \widehat{C} \wedge ux \sqsubseteq w\}$ $(x \in X)$, and the required equation follows. □

Remark. If C is a finite prefix code and $r^{-\alpha}$ is a rational number, then $\mathcal{V}_C^{(\alpha)}$ is a computable (recursive) martingale.

For a recursive function $g\colon \mathbb{N} \to \mathbb{Q}$ let a family of covering codes $\mathfrak{C} := (C_w)_{w \in X^*}$ such that $\sum_{v \in C_w} g(|w|)^{|v|} < \infty$ be given, and let $\mathcal{V}_{C_w}^{(g)} := \mathcal{V}_{C_w}^{(\alpha_w)}$ be as defined above for $\alpha_w := -\log_r g(|w|)$. Then we define our martingale $\mathcal{V}_{\mathfrak{C}}$ as follows: For $u \in X^*$ consider the $\mathfrak{C}$-factorization $u_1 \cdots u_n \cdot u'$, and put

$$\mathcal{V}_{\mathfrak{C}}^{(g)}(u) := \left(\prod_{i=0}^{n-1} \mathcal{V}_{C_{u_1 \cdots u_i}}^{(g)}(u_{i+1}) \right) \cdot \mathcal{V}_{C_{u_1 \cdots u_n}}^{(g)}(u'),$$

that is, $\mathcal{V}_{\mathfrak{C}}^{(g)}$ is in some sense the concatenation of the martingales $\mathcal{V}_{C_w}^{(\alpha_w)}$. Observe that $\mathcal{V}_{\mathfrak{C}}^{(g)}$ is computable if only g is a recursive function and the function which assigns to every w the corresponding code C_w is computable.

We have the following.

Lemma 10. *Let $g\colon \mathbb{N} \to \mathbb{Q}$ be a recursive function and let $\mathfrak{C} = (C_w)_{w \in X^*}$ be a family of covering codes such that $\sum_{v \in C_w} g(|w|)^{|v|} < r^{-|w|}$. If for some $\alpha \in [0, 1]$ the ω-word $\xi \in X^\omega$ has a $\mathfrak{C}$-factorization $\xi = u_1 \cdots u_i \cdots$ such that for some $n_\xi \in \mathbb{N}$ and all $i \geq n_\xi$ it holds that*

1. *$r^{-\alpha} < g(|u_1 \cdots u_i|)$ and*
2. *all factors u_{i+1} belong to $C_{u_1 \cdots u_i}$,*

then there is a constant $c_\xi > 0$ not depending on i for which

$$\mathcal{V}_{\mathfrak{C}}(u_1 \cdots u_i) \geq c_\xi \cdot r^{(1-\alpha) \cdot |u_1 \cdots u_i|}.$$

Proof. Since $\widehat{C}_w$ is a code, we have $\sum_{v \in \widehat{C}_w} r^{-|v|} \leq 1$, and from the assumption we obtain

$$\sum_{v \in C_w} g(|w|)^{|v|} + \sum_{v \in \widehat{C}_w} r^{-|v|} \leq r^{-|w|} + 1.$$

Now $|u_i| \geq 1$ implies $|u_1 \cdots u_i| \geq i$, and (16) yields

$$\mathcal{V}^{(\alpha)}_{C_{u_1 \cdots u_i}}(u_{i+1}) \geq \begin{cases} \dfrac{1}{r^{-i}+1} & \text{if} \quad i \leq n_\xi, \\ \dfrac{r^{(1-\alpha)\cdot|u_{i+1}|}}{r^{-i}+1} & \text{if} \quad i > n_\xi. \end{cases} \tag{17}$$

Put

$$c_\xi := \prod_{i=0}^{\infty} \frac{1}{r^{-i}+1} \cdot \prod_{i=0}^{n_\xi} r^{-(1-\alpha)\cdot|u_{i+1}|}.$$

Clearly, $c_\xi > 0$, and using (17) by induction on i the assertion is easily verified. □

Now we prove the announced result.

Theorem 11. *For every Σ_2-definable ω-language $F \subseteq X^\omega$ and every computable real number $\alpha \geq \dim F$ there is a recursive martingale $\mathcal{V}$ such that*

$$\lambda_{\mathcal{V}}(\xi) \geq 1 - \alpha \qquad \textit{for all} \quad \xi \in F.$$

Proof. We use the same recursive approximation from below of $r^{-\alpha}$, $g_\alpha \colon \mathbb{N} \to \mathbb{Q}$, as in the proof of Theorem 3.

By virtue of Lemma 10 it suffices to construct a computable family of covering codes $\mathfrak{C} = (C_w)_{w \in X^*}$ such that the function which assigns to every w the corresponding finite prefix code C_w is computable. To this end we modify the predicate **test** introduced in the proof of Theorem 5 as follows:

$$\mathbf{test}'(w, j, n) \colon \Leftrightarrow \Bigl((w \cdot U_j \cup (L_{|w|} \cap X^{|w|+n})) \cdot X^\omega \supseteq w \cdot X^\omega \\ \wedge \sum_{v \in U_j} g_\alpha(|w|)^{|v|} < r^{-|w|} \Bigr).$$

In the same way we modify the algorithm presented there.

Algorithm C_w

input w
$n = 0$
repeat $j = -1$
  repeat $j = j + 1$
  until $\mathbf{test}'(w, j, n) \vee (\sup\{|v| \colon v \in U_j\} > n)$
  $n = n + 1$
until $\mathbf{test}'(w, j, n)$
output $C_w := U_j$

Similarly to the proof of Theorem 3 this algorithm computes a prefix code C_w with $\sum_{v\in C_w} g_\alpha(|w|)^{|v|} < r^{-|w|}$ and $w\cdot C_w\cdot X^\omega \supseteq w\cdot X^\omega\setminus L_{|w|}\cdot X^\omega$, and it terminates, because $\dim(F\cap w\cdot X^\omega) \le \dim F < -\log_r g_\alpha(|w|)$.

It remains to show that every $\xi\in F$ has a $\mathfrak{C}$-factorization $\xi = u_1\cdots u_i\cdots$ such that almost all factors u_{i+1} belong to the corresponding $C_{u_1\cdots u_i}$.

Let $\xi\in F$. Then there is a $k\in\mathbb{N}$ such that $\xi\in X^\omega\setminus L_i\cdot X^\omega$ for all $i\ge k$. Consequently, $w\in\mathbf{A}(\xi)$ implies $w\notin L_k = L_k\cdot X^*$, and according to the definition of $\mathfrak{C}$ there is a $u\in C_w$ such that $w\cdot u\in\mathbf{A}(\xi)$ whenever $|w|\ge k$. □

We cannot improve Theorem 11 much further, because on the one hand Theorem 5 and (15) show that it is not possible to construct a computable martingale $\mathcal{V}$ such that $\forall\xi(\xi\in F\to\lambda_{\mathcal{V}}(\xi)\ge 1-\alpha)$ when $\alpha<\dim F$, and on the other hand again the ω-language E defined in the proof of Lemma 6 shows that the result of Theorem 11 cannot be extended to further classes of the arithmetical hierarchy.

4. Relation to Previous Results

In the Introduction we mentioned that under certain additional structural constraints the result of Theorem 5 extends beyond the range of Σ_2-definable ω-languages. In this section we recall some of the results of [S4] giving evidence of this fact, but showing also that the constraints involved are not recursion-theoretic ones.

The first two classes of ω-languages satisfying properties analogous to Theorems 3 and 5 are the classes of recursive ω-power languages and of δ-limits of recursive submonoids.

An *ω-power language* is an ω-language of the form

$$W^\omega := \{\xi\colon \xi\in X^\omega\wedge\exists(w_i)_{i\in\mathbb{N}}(w_i\in W\wedge w_i\ne e\wedge\xi = w_0\cdots w_i\cdots)\},$$

where $W\subseteq X^*$, and the *submonoid* of X^* generated by W, W^*, is the language $W^* := \{w_1\cdots w_n\colon n\in\mathbb{N}\wedge w_i\in W\}$.

Now (6.2) of [S4] is an analogue to Theorem 3 for submonoids $W^*\subseteq X^*$:

$$\dim W^\omega = \dim(W^*)^\delta = H_{W^*}.$$

Then Theorem 1 and Lemma 2 show the following.

Proposition 12. *If $W^*\subseteq X^*$ or its complement $X^*\setminus W^*$ are recursively enumerable languages, then $\underline{\kappa}(W^\omega) = \underline{\kappa}((W^*)^\delta) = \dim W^\omega$.*[3]

According to the discussion in Section 6 of [S4], the class $\{W^\omega\colon W\subseteq X^*\wedge W \text{ is recursive}\}$ is incomparable with the class of all Σ_2-definable subsets of X^ω.

[3] It is interesting to note that one can easily derive from Theorem 6.1 of [S4] that $\dim W^\omega$ is always a left-computable real number provided W^* is recursively enumerable.

A second class which exhibits a similar property is the class of so-called balanced ω-languages introduced in [S3]. We call a subset F of X^ω *balanced* if and only if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(n) = o(2^{c\cdot n})$ for arbitrary $c > 0$ and for which the inequality

$$\boldsymbol{s}_F(w, |w| + n) \leq f(|w|) \cdot \frac{\boldsymbol{s}_{\mathbf{A}(F)}(|w| + n)}{\boldsymbol{s}_{\mathbf{A}(F)}(|w|)} \tag{18}$$

holds for $w \in X^*$ and $n \in \mathbb{N}$. Here $\boldsymbol{s}_F(w, \cdot)$ is shorthand for the structure function of the language $U(F, w) := \mathbf{A}(F) \cap w \cdot X^*$, $\boldsymbol{s}_{U(F,w)}$. Roughly speaking, our (18) means that for arbitrary $w \in X^*$ the function $\boldsymbol{s}_F(w, \cdot)$ does not grow much faster than the average taken over all functions $\boldsymbol{s}_F(v, \cdot)$ such that $|v| = |w|$ and $v \in \mathbf{A}(F)$.

Then Theorem 4 of [S3] proves that for closed and balanced $F \subseteq X^\omega$ the following identity holds true:

$$\dim F = \liminf_{n\to\infty} \frac{\log_r \boldsymbol{s}_{\mathbf{A}(F)}(n)}{n}.$$

Consequently, Theorem 7 implies an upper bound on $\underline{\kappa}(F)$ for balanced ω-languages in $\mathfrak{S}$.

Lemma 13. *If $F \in \mathfrak{S}$ is a balanced ω-language, then $\underline{\kappa}(F) \leq \dim F$.*

We conclude this section by mentioning Corollary 3.17 of [S4] which proves that for closed and balanced ω-languages F there is always a $\xi \in F$ such that $\underline{\kappa}(\xi) \geq \dim F$, thus yielding, in contrast to Example 1 above, that for balanced closed[4] Σ_2-definable ω-languages F as well as for balanced ω-languages $F \in \mathfrak{S}$ there is indeed a $\xi \in F$ such that $\underline{\kappa}(\xi) = \underline{\kappa}(F) = \dim F$.

5. Concluding Remark

Our Theorems 3, 5, and 11 in connection with previous results of Ryabko [R1]–[R3] and this author [S4] give evidence that there is a strong coincidence between the concepts of Kolmogorov complexity (in complexity theory) and Hausdorff dimension (in fractal geometry) for a class of recursive (computable) ω-languages. The results of this paper show a borderline in the arithmetical hierarchy up to which this coincidence holds true, and our examples give evidence that it does not extend much further.

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[4] This property, however, does not hold in general for nonclosed balanced Σ_2-definable ω-languages, e.g., it does not hold for $E' := \bigcup_{n\in\mathbb{N}}(1 \cdot X^n)^\omega$.

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