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Linear Codes Correcting Repeated Bursts Equipped with Homogeneous Distance

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Abstract

The homogeneous weight (metric) is useful in the construction of codes over a ring of integers $\mathbb{Z}_{p^l}$ (p prime and $l \geq 1$ an integer). It becomes Hamming weight when the ring is taken to be a finite field and becomes Lee weight when the ring is taken to be $\mathbb{Z}_4$. This paper presents homogeneous weight distribution and total homogeneous weight of burst and repeated burst errors in the code space of n-tuples over $\mathbb{Z}_{p^l}$. Necessary and sufficient conditions for existence of an (n, k) linear code over $\mathbb{Z}_{p^l}$ correcting the error patterns with respect to the homogeneous weight are derived.

Keywords Burst · Repeated burst · Bound · Homogeneous weight · Codes over $\mathbb{Z}_{p^l}$

1 Introduction

Fire [5] in 1959 observed that errors in some communication channels were not random in nature, they occurred in clustered way, known as burst error. In [5, 13] necessary and sufficient conditions for existence of a linear code over Galois field $GF(q)$ which corrects bursts were studied. As the days passed, more and more advanced communication channels keep on appearing in practice which keep on producing different types of errors like CT-burst, cyclic burst, periodic error, etc. In 2009, Berardi [3] et al. observed that the burst error repeats itself in a busy channel which they called as "2-repeated bursts". Further, Dass and Verma [4] observed that if the channel becomes more busy, the frequency of repetition of burst increased and they are referred to as

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"m-repeated bursts". Such errors occur in the channels like lutamate-injured networks and glutamate-injured networks [16]. In [4], existence of repeated burst correcting linear codes over $GF(q)$ was studied.

The homogeneous weight (metric), introduced for integer residue rings in [2] and extended for finite rings in [6, 7], has been gaining attention in the context of ring-linear coding. It is an extension of Hamming and Lee weight [9, 10]. It becomes Hamming weight when the ring taken to be a finite field and becomes Lee weight when the ring taken to be $\mathbb{Z}_4$. Ring of integers $\mathbb{Z}_{p^l}$ (p prime and $l \geq 1$ an integer) is a generalization of $GF(q)$. The rich algebraic structure of rings increases the popularity of linear codes over rings. The homogeneous weight is useful in construction of codes over $\mathbb{Z}_{p^l}$ (see [18–20]). Many classical results/bounds which were true for linear codes with Hamming weight are investigated for this homogeneous weight [7, 8, 11, 12, 17].

In [14, 15], Hamming weight distribution of bursts and m-repeated bursts are studied. In this paper, we present homogeneous weight distribution and total homogeneous weight of these error patterns. In [17], Temiz and Siap presented necessary condition only for linear codes over $\mathbb{Z}_{p^l}$ correcting repeated CT-bursts with homogeneous weight. In this paper, we present necessary as well as sufficient conditions for linear codes over $\mathbb{Z}_{p^l}$ correcting bursts and m-repeated bursts along with homogeneous weight constraint.

2 Definitions and Notations

Let $\mathbb{Z}_{p^l}^n$ denote the code space of all n-tuples over $\mathbb{Z}_{p^l}$. Then $\mathbb{Z}_{p^l}^n$ becomes a module over $\mathbb{Z}_{p^l}$.

Definition 1 [17] A subset C of $\mathbb{Z}_{p^l}^n$ is called an (n, M) linear block code if C is a submodule of $\mathbb{Z}_{p^l}^n$ of size M. A submodule of $\mathbb{Z}_{p^l}^n$ is called an (n, k) linear code if it is spanned by k elements. Its dual code ($C^{\perp}$) can be defined in the similar way as linear code over $GF(q)$.

Note 1 In a linear code over $\mathbb{Z}_{p^l}$, the rows of the generator matrix would span the code, but the rows need not be linearly independent. If the rows are linearly independent, then the code is said to be a *free code*.

Definition 2 [6] The homogeneous weight w_{hom} of an element $x \in \mathbb{Z}_{p^l}$ is defined as

$$w_{hom}(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ p^{l-1} & \text{if} \quad x \in (p^{l-1})\backslash\{0\} \\ (p-1)p^{l-2} & \text{otherwise} \end{cases},$$

where (p^{l-1}) denotes the ideal of $\mathbb{Z}_{p^l}$ generated by p^{l-1}.

Definition 3 [6] The homogeneous weight of a vector $v = (v_1, v_2, ..., v_n) \in \mathbb{Z}_{p^l}^n$ is defined as

$$w_{hom}(v) = \sum_{i=1}^{n} w_{hom}(v_i).$$

Definition 4 [6] The homogeneous distance d_{hom} between any two vectors $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ in $\mathbb{Z}_{p^l}^n$ is defined by

$$d_{hom}(u, v) = w_{hom}(u - v) = \sum_{i=1}^{n} w_{hom}(u_i - v_i).$$

Remark 1 Since $(p^{l-1}) = \left\{0, p^{l-1}, 2p^{l-1}, \ldots, (p-1)p^{l-1}\right\}$, there are $p-1$ elements of homogeneous weight p^{l-1} and $p^l - p$ elements of homogeneous weight $(p-1)p^{l-2}$ in $\mathbb{Z}_{p^l}$.

Definition 5 [5] A burst of length b is an n-tuple whose all nonzero components are confined to some b consecutive components, the first and the last of which are nonzero.

Definition 6 [4] An m-repeated burst of length b is an n-tuple whose only nonzero components are confined to m distinct sets of some b successive positions, the first and the last component of each set being nonzero.

(00102002410030103110) is an example of 4-repeated burst of length 3 over $GF(5)$.

Note 2 Since for $b = 1$, m-repeated bursts of length b turn out to be simply m random errors, we confine our study to the case $b > 1$. Further, for $l = 1$, the ring of integers $\mathbb{Z}_{p^l}$ becomes always a field $\mathbb{Z}_p$ of prime order, so we consider our study to the case $l > 1$.

Observation 1 For a burst error of length b (> 1) in $\mathbb{Z}_{p^l}^n$ $(l > 1)$ having homogeneous weight w_{hom}, the minimum value of w_{hom} is $2(p-1)p^{l-2}$, and the maximum value of w_{hom} is bp^{l-1} for any b. Further, for an m-repeated burst of length b (> 1) in $\mathbb{Z}_{p^l}^n$ $(l > 1)$ having homogeneous weight w_{hom}, the minimum value of w_{hom} is $2m(p-1)p^{l-2}$ and the maximum value of w_{hom} is mbp^{l-1}.

Now, we quote two results from [1] which give us the order of generator and parity check matrices of a linear code over $\mathbb{Z}_{p^l}$. We shall use the notation and format of the matrices in our examples.

Theorem 1 [1] *A nonzero (n, k) linear code C over $\mathbb{Z}_{p^l}$ has a generator matrix which after a suitable permutation of the coordinates can be written in the form*

$$G = \begin{pmatrix} I_{k_0} & A_{0,1} & A_{0,2} & A_{0,3} & \ldots & A_{0,l-1} & A_{0,l} \\ 0 & pI_{k_0} & pA_{1,2} & pA_{1,3} & \ldots & pA_{1,l-1} & pA_{1,l} \\ 0 & 0 & p^2 I_{k_0} & p^2 A_{2,3} & \ldots & p^2 A_{2,l-1} & p^2 A_{2,l} \\ . & . & . & . & \ldots & . & . \\ 0 & 0 & 0 & 0 & \ldots & p^{l-1} I_{k_0} & p^{l-1} A_{l-1,l} \end{pmatrix},$$

where the columns are grouped into blocks of sizes $k_0, k_1, ..., k_{l-1}, k_l$ *and* k_i *are non-negative integers adding to n. This means that C consists of all codewords*

$$[v_0 v_1 v_2 ... v_{l-1}]G,$$

where each v_i *is a vector of length* k_i *with components from* $Z_{p^{l-i}}$ *so that C contains* p^k *codewords, where* $k = \sum_{i=0}^{l-1}(l-i)k_i$.

Theorem 2 [1] *The parity check matrix of the code C with generator matrix G given in Theorem 1 has the form*

$$H = \begin{pmatrix} B_{0,l} & B_{0,l-1} & B_{0,l-2} & \dots & B_{0,3} & B_{0,2} & B_{0,1} & I \\ pB_{1,l} & pB_{1,l-1} & pB_{1,l-2} & \dots & pB_{1,3} & pB_{1,2} & pI & 0 \\ p^2B_{2,l} & p^2B_{2,l-1} & p^2B_{2,l-2} & \dots & p^2B_{2,3} & p^2I & 0 & 0 \\ . & . & . & \dots & . & . & . & . \\ p^{l-1}B_{l-1,l} & p^{l-1}I & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the column blocks have the same sizes as in G. Then, the dual code contains $p^{k_\perp}$ *codewords, where* $k_\perp = \sum_{i=1}^{l} ik_i$. *Also* $|C||C^\perp| = p^{k+k_\perp} = p^{ln}$, *i.e.,* $|C^\perp| = p^{ln-k}$ *and* $(C^\perp)^\perp = C$.

The two results lead to the following observations:

Observation 2 The orders of G and H are respectively $(k_0 + k_1 + \cdots + k_{l-1}) \times n$ and $(k_l + k_{l-1} + \cdots + k_1) \times n$, where $n = k_0 + k_1 + \cdots + k_{l-1} + k_l$.

Observation 3 As there are p^k codewords in an (n, k) linear code C over $\mathbb{Z}_{p^l}$, the number of available cosets is $\frac{(p^l)^n}{p^k} = p^{nl-k}$.

3 Results on Bursts

This section provides the total homogeneous weight of all vectors of $\mathbb{Z}_{p^l}^n$ having burst with or without weight constraint. It also gives the necessary and sufficient conditions for existence of an (n, k) linear code C over $\mathbb{Z}_{p^l}$ correcting these errors with respect to homogeneous weight.

Lemma 1 *The number of all bursts of length* b (> 1) *in the code space of n-tuples over* $\mathbb{Z}_{p^l}$ $(l > 1)$ *is given by*

$$\mathbb{A}_n^b = (n - b + 1)(p^l - 1)^2(p^l)^{b-2}.$$

Proof The burst of length b can start from the 1^{st} to $(n-b+1)^{th}$ positions, and the first and the last component in each burst can be any $p^l - 1$ nonzero elements of the ring $\mathbb{Z}_{p^l}$ and the other $b-2$ components can be any p^l elements of the ring. This follows the lemma. □

Now, we enumerate the number of bursts with homogeneous weight constraint.

Lemma 2 *The number of all burst errors of length b (>1) having homogeneous weight w_{hom} in the code space of n-tuples over $\mathbb{Z}_{p^l}$ $(l>1)$ is equal to*

$$\begin{aligned}\mathbb{A}_n^b(w_{hom}) =&(n-b+1)\Bigg[\sum_{u_0,v_0}\binom{b-2}{u_0}\binom{b-2-u_0}{v_0}(p-1)^{u_0}(p^l-p)^{v_0+2}\\ &+2\sum_{u_1,v_1}\binom{b-2}{u_1}\binom{b-2-u_1}{v_1}(p-1)^{u_1+1}(p^l-p)^{v_1+1}\\ &+\sum_{u_2,v_2}\binom{b-2}{u_2}\binom{b-2-u_2}{v_2}(p-1)^{u_2+2}(p^l-p)^{v_2}\Bigg], \qquad (1)\end{aligned}$$

where $u_0, u_1, u_2, v_0, v_1, v_2$ are non-negative integers such that

$$\begin{aligned}&u_0+v_0\le b-2,\\ &u_1+v_1\le b-2,\\ &u_2+v_2\le b-2,\\ &w_{hom}=u_0p^{l-1}+(v_0+2)(p-1)p^{l-2},\\ &w_{hom}=(u_1+1)p^{l-1}+(v_1+1)(p-1)p^{l-2},\\ &w_{hom}=(u_2+2)p^{l-1}+v_2(p-1)p^{l-2}.\end{aligned}$$

$\Big[$ Note that the terms, for which homogeneous weight is not equal to w_{hom}, are absent in the formula $\Big]$

Proof Since a burst of length b having homogeneous weight w_{hom} has nonzero components in its first and last positions of the b consecutive positions and there are two different nonzero homogeneous weights $(p-1)p^{l-2}$ and p^{l-1} in $\mathbb{Z}_{p^l}$, we consider the following three cases.

Case 1: Suppose that both the first and last nonzero components of the burst of length b are of weight $(p-1)p^{l-2}$. Among the in-between $b-2$ components, let the number of components of weight p^{l-1} is u_0 and that of weight $(p-1)p^{l-2}$ is v_0 such that $u_0+v_0\le b-2$ and $w_{hom}=u_0p^{l-1}+(v_0+2)(p-1)p^{l-2}$.

Since there are $p-1$ elements of weight p^{l-1} and p^l-p elements of weight $(p-1)p^{l-2}$ in $\mathbb{Z}_{p^l}$, the number of bursts of length b having homogeneous weight w_{hom} is

$$(n-b+1)\binom{b-2}{u_0}\binom{b-2-u_0}{v_0}(p-1)^{u_0}(p^l-p)^{v_0+2}.$$

Case 2: Suppose that one of the first and the last nonzero components of the burst of length b is of weight p^{l-1} and the other is of weight $(p-1)p^{l-2}$. Let the number of components of weight p^{l-1} is u_1 and that of weight $(p-1)p^{l-2}$ is v_1 such that $u_1 + v_1 \leq b - 2$ and $w_{hom} = (u_1 + 1)p^{l-1} + (v_1 + 1)(p-1)p^{l-2}$.

Then, the number of components of weight p^{l-1} is $u_1 + 1$ and the number of components of weight $(p-1)p^{l-2}$ is $v_1 + 1$. So, the number of bursts of length b having homogeneous weight w_{hom} is

$$2(n-b+1)\binom{b-2}{u_1}\binom{b-2-u_1}{v_1}(p-1)^{u_1+1}(p^l-p)^{v_1+1}.$$

Case 3: Suppose that both the first and last nonzero components of the burst of length b are of weight p^{l-1}. Then, we can assume the number of components of weight p^{l-1} is u_2 and that of weight $(p-1)p^{l-2}$ is v_2 such that $u_2 + v_2 \leq b - 2$ and $w_{hom} = (u_2 + 2)p^{l-1} + v_2(p-1)p^{l-2}$.

By the same logic as earlier two cases, the number of bursts of length b having homogeneous weight w_{hom} is

$$(n-b+1)\binom{b-2}{u_2}\binom{b-2-u_2}{v_2}(p-1)^{u_2+2}(p^l-p)^{v_2}.$$

□

In the following two results, we present the total homogeneous weight of all vectors having burst of length b with and without homogeneous weight constraints. The first result (Theorem 3) immediately follows from the notations introduced earlier, and the second one (Theorem 4) is an immediate consequence of Lemma 2.

Theorem 3 *The total homogeneous weight of all vectors having burst of length b (> 1) with homogeneous weight w_{hom} or less in the code space of n-tuples over $\mathbb{Z}_{p^l}$ $(l > 1)$ is given by*

$$\sum_{i=(p-1)p^{l-2}}^{w_{hom}} i\mathbb{A}_n^b(i).$$

Theorem 4 *The total homogeneous weight of all vectors having burst of length b (> 1) in the code space of n-tuples over $\mathbb{Z}_{p^l}$ $(l > 1)$ is*

$$W_n^b = (n-b+1)\left[\sum_{i=0}^{2}\sum_{u_i,v_i}\binom{2}{i}\binom{b-2}{u_i}\binom{b-2-u_i}{v_i}(p-1)^{u_i+i}(p^l-p)^{v_i-i+2}\times \left[(u_i+i)p^{l-1}+(v_i-i+2)(p-1)p^{l-2}\right]\right],$$

where $u_0, u_1, u_2, v_0, v_1, v_2$ are non-negative integers such that

$$u_0 + v_0 \leq b-2,$$
$$u_1 + v_1 \leq b-2,$$
$$u_2 + v_2 \leq b-2.$$

Note: If $w_{hom} = bp^{l-1}$, Theorems 3 and 4 coincide.

Example 1 For $p = l = 2$, $b = 3$, $w_{hom} = 4$ and $n = 5$ in Lemma 2, we enumerate the number of all bursts of length b having homogeneous weight w_{hom} in $\mathbb{Z}^n_{p^l}$ by finding the non-negative integers $u_0, u_1, u_2, v_0, v_1, v_2$:

$$\begin{aligned} 4 &= u_0 2 + (v_0 + 2), \\ 4 &= (u_1 + 1)2 + (v_1 + 1), \\ 4 &= (u_2 + 2)2 + v_2, \\ 1 &\geq u_0 + v_0, \\ 1 &\geq u_1 + v_1, \\ 1 &\geq u_2 + v_2. \end{aligned}$$

Then, by solving, we get $u_0 = 1, v_0 = 0$; $u_1 = 0, v_1 = 1$ and $u_2 = 0, v_2 = 0$. Substituting these values into (1) of Lemma 2, we get the following result:

$$\mathbb{A}^3_5(4) = (3)\left[\binom{1}{1}\binom{0}{0}(2)^2 + 2\binom{1}{0}\binom{1}{1}(2)^2 + \binom{1}{0}\binom{1}{0}(2)^0\right] = 39.$$

These 39 bursts of length 3 with homogeneous weight 4 in $\mathbb{Z}^5_{2^2}$ are listed below.

11200 01120 00112 13200 01320 00132 12100 01210 00121
21300 02130 00213 21100 02110 00211 32300 03230 00323
20200 02020 00202 33200 03320 00332 12300 01230 00123
32100 03210 00321 23300 02330 00233 31200 03120 00312
23100 02310 00231.

Now, we count (in the similar way) the number of bursts of length 3 with other possible homogeneous weights.

The number of bursts of length 3 with homogeneous weight 3 as

$$\mathbb{A}^3_5(3) = 36,$$

which are listed as

11100 01110 00111 33300 03330 00333 13300 01330 00133
10200 01020 00102 20100 02010 00201 30200 03020 00302
20300 02030 00203 33100 03310 00331 11300 01130 00113
13100 01310 00131 31100 03110 00311 31300 03130 00313.

The number of burst errors of length 3 with homogeneous weight 2 is $\mathbb{A}^3_5(2) = 12$ and they are

10100 01010 00101 30300 03030 00303 10300 01030 00103

30100 03010 00301.

The number of bursts of length 3 with homogeneous weight 6 is $\mathbb{A}_5^3(6) = 3$ and they are

22200 02220 00222.

The number of bursts of length 3 with homogeneous weight 5 is $\mathbb{A}_5^3(5) = 18$ and they are

12200 01220 00122 32200 03220 00322 21200 02120 00212
23200 02320 00232 22100 02210 00221 22300 02230 00223.

Thus, the total homogeneous weight of all bursts of length 3 is

$$2 \times \mathbb{A}_5^3(2) + 3 \times \mathbb{A}_5^3(3) + 4 \times \mathbb{A}_5^3(4) + 5 \times \mathbb{A}_5^3(5) + 6 \times \mathbb{A}_5^3(6) = 396.$$

This can be verified by Theorem 4. According to Theorem 4, the total homogeneous weight of all bursts of length 3 in $\mathbb{Z}_{p^l}^n$ is given by

$$\begin{aligned} W_5^3 = 3\Bigg[&\left\{\binom{1}{0}\binom{1}{0}(2)^2 2 + \binom{1}{1}\binom{0}{0}(2)^2(4) + \binom{1}{0}\binom{1}{1}(2)^3(3)\right\} \\ &+ 2\left\{\binom{1}{0}\binom{1}{0}(2)^1 3 + \binom{1}{1}\binom{0}{0}(2)^1(5) + \binom{1}{0}\binom{1}{1}(2)^2(4)\right\} \\ &+ \left\{\binom{1}{0}\binom{1}{0}(2)^0 4 + \binom{1}{1}\binom{0}{0}(2)^0(6) + \binom{1}{0}\binom{1}{1}(2)^1(5)\right\}\Bigg] \\ = 3\times 132 = 396. \end{aligned}$$

Now, we give necessary and sufficient conditions for an (n, k) linear code C over $\mathbb{Z}_{p^l}$ that corrects burst of length b with homogeneous weight constraints.

Theorem 5 *A necessary condition for an (n, k) linear code C over $\mathbb{Z}_{p^l}$ $(l > 1)$ correcting any burst of length b (> 1) with homogeneous weight w_{hom} or less $(w_{hom} \leq bp^{l-1})$ is*

$$p^{ln-k} \geq 1 + \sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_n^b(\rho).$$

Proof From Lemma 2, the total number of burst errors of length b (> 1) having homogeneous weight w_{hom} or less, including the zero vector, is

$$1 + \sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_n^b(\rho).$$

The code corrects all bursts of length b with homogeneous weight w_{hom} or less, so all such bursts must be in different cosets. As the total cardinality of cosets of the code is p^{ln-k}, we have

$$p^{ln-k} \geq 1 + \sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_n^b(\rho).$$

□

Theorem 6 *The sufficient condition for the existence of an* (n, k) *linear code* C *over* $\mathbb{Z}_{p^l}$ $(l > 1)$ *correcting any burst of length* b (> 1) *with homogeneous weight* w_{hom} *or less* $(w_{hom} \leq bp^{l-1})$ *is given by*

$$p^{ln-k} > 1 + \sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_b^b(\rho) \times \sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_{n-b}^b(\rho).$$

Proof For the existence of the code, we follow the technique of Varshamov-Gilbert-Sacks Bound [13] where we can keep on adding the columns to the parity check matrix of the code one after another untill it produces distinct syndromes by the error patterns. Let us assume that first $j-1$ columns of parity check matrix H are added suitably and the j^{th} column h_j to H can be added such that αh_j $(\alpha \in \mathbb{Z}_{p^l}\backslash\{0\})$ should not be a linear combination of immediate previous $b-1$ consecutive columns having homogeneous weight w_{hom} or less, together with a linear combination of b consecutive columns having homogeneous weight w_{hom} or less among the first $j-b$ columns. That is

$$\begin{aligned} \alpha h_j \neq (&\alpha_1 h_{j-1} + \alpha_2 h_{j-2} + \cdots + \alpha_{b-1} h_{j-b+1}) \\ &+ (\beta_i h_i + \beta_{i+1} h_{i+1} + \cdots + \beta_{i+b-1} h_{i+b-1}), \end{aligned} \tag{2}$$

where $i+b-1 < j-b+1$; $\alpha, \alpha_{b-1} \in \mathbb{Z}_{p^l}\backslash\{0\}$; $\alpha_i, \beta_{i+r} \in \mathbb{Z}_{p^l}$ such that α_i's along with α, and β_{i+r}'s respectively form a burst of length b in a vector of length b and $j-b$ with homogeneous weight w_{hom} or less.

This condition ensures that any two bursts of length b with homogeneous weight w_{hom} or less can not have same syndrome. In other words, they will have distinct syndromes. We now calculate the number of ways the coefficients $\alpha, \alpha_i, \beta_{i+r}$ can be chosen.

The number of coefficients α and α_i's that form a burst of length b with homogeneous weight w_{hom} or less in a vector of length b (by Lemma 2) is

$$\sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_b^b(\rho).$$

Again, the number of different ways in which the coefficient β_{i+r}'s can be selected is (by Lemma 2)

$$\sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_{j-b}^{b}(\rho).$$

Thus, the total number of linear combinations in Expression (2), including the zero combination, is

$$1 + \sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_{b}^{b}(\rho) \times \sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_{j-b}^{b}(\rho).$$

As the number of available cosets is p^{nl-k}, the number of available distinct syndromes is p^{nl-k}. So, we can add the j^{th} column h_j to H provided it produces distinct syndromes. Therefore, adding of h_j is possible provided

$$p^{ln-k} > 1 + \sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_{b}^{b}(\rho) \times \sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_{j-b}^{b}(\rho).$$

Replacing j by n completes the proof. □

Example 2 For $p = l = 2, b = 3, w_{hom} = 4$ and $n = k_0+k_1+k_2 = 1+2+5 = 8$ in Theorem 6, we get the inequalities for the non-negative integers $u_0, u_1, u_2, v_0, v_1, v_2$ as

$$\begin{aligned}
&u_0 2 + (v_0 + 2) \leq 4,\\
&(u_1 + 1)2 + (v_1 + 1) \leq 4,\\
&(u_2 + 2)2 + v_2 \leq 4,\\
&u_0 + v_0 \leq 1,\\
&u_1 + v_1 \leq 1,\\
&u_2 + v_2 \leq 1.
\end{aligned}$$

Then, by solving, we get $(u_0, v_0) = \Big\{(0, 0), (0, 1), (1, 0)\Big\}$, $(u_1, v_1) = \Big\{(0, 0), (0, 1)\Big\}$; and $(u_2, v_2) = (0, 0)$.

By Lemma 2, we have

$$\sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_{b}^{b}(\rho) = \sum_{\rho=2}^{4} \mathbb{A}_{3}^{3}(\rho) = 29$$

and

$$\sum_{\rho=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_{n-b}^{b}(\rho) = \sum_{\rho=2}^{4} \mathbb{A}_{5}^{3}(\rho) = 3 \times 29.$$

Therefore, $p^{nl-k} > 1 + 29 \times 3 \times 29 = 2524$ implies $nl - k \geq 12$ implies $k \leq 4$. This gives rise to a (8, 4) linear code over $\mathbb{Z}_{2^2}$, where $k = 2k_0 + k_1 = 4$ and $k_\perp = k_1 + 2k_2 = 12$ (see Theorems 1-2). Here $k_1 + k_2 = 7$, so the order of the parity check matrix of the (8, 4) linear code is 7 by 8.

As every parity check matrix of linear code over $\mathbb{Z}_{p^l}$ can be reduced to the form H given in Theorem 2, we construct a parity check matrix $H_{7\times 8}$ of the (8, 4) linear block code by adding the columns one after another satisfying Condition (2) and maintaining the form given in Theorem 2. The form in Theorem 2 helps us to find the parity check matrix with less difficulty as we have to look for less number of components in the matrix. We derive one such parity check matrix $H_{7\times 8}$ of the (8, 4) linear block code as below:

$$H_{7\times 8} = \begin{bmatrix} (B_{0,2})_{5\times 1} & (B_{0,1})_{5\times 2} & I_{5\times 5} \\ 2(B_{1,2})_{2\times 1} & 2I_{2\times 2} & 0_{2\times 5} \end{bmatrix} = \begin{bmatrix} 1&2&1&1&0&0&0&0 \\ 0&3&0&0&1&0&0&0 \\ 1&0&3&0&0&1&0&0 \\ 0&3&1&0&0&0&1&0 \\ 1&2&3&0&0&0&0&1 \\ 2&2&0&0&0&0&0&0 \\ 2&0&2&0&0&0&0&0 \end{bmatrix}.$$

We can verify that syndromes of bursts of length 3 with homogeneous weight 4 or less are all nonzero and distinct (checked by MS-Excel). So, the (8, 4) linear code over $\mathbb{Z}_{2^2}$ which is the null space of the matrix $H_{7\times 8}$ can correct all such errors.

4 Results on *m*-Repeated Bursts

This section extends the results of Section 3 for m-repeated bursts. We take $m \geq 2$ throughout the section. For $m = 1$, repeated burst simply becomes a burst. We first count the number of m-repeated bursts without homogeneous weight constraint.

Lemma 3 *The number of all m-repeated bursts of length $b(> 1)$, $\mathbb{A}_n^{m,b}$, in the code space of n-tuples over $\mathbb{Z}_{p^l}$ $(l > 1)$ is given by*

$$\mathbb{A}_n^{m,b} = \binom{n - m(b-1)}{m} (p^l - 1)^{2m} (p^l)^{mb-2m}. \tag{3}$$

Proof First, we count the number of m distinct sets of b consecutive positions in a vector of length n. Let
y_1 be the number of positions before 1^{st} set of b consecutive positions,
y_2 be the number of positions before 2^{nd} set of b consecutive positions,
$\vdots$
y_m be the number of positions before the m^{th} set of b consecutive positions, and
y_{m+1} be the number of positions after the m^{th} set of b consecutive positions.

Then

$$y_1 + y_2 + \cdots + y_{m+1} = n - mb.$$

The number of non-negative integer solutions of the above equation is equal to the number of m distinct sets of b consecutive positions. This number is given by

$$\binom{n-mb+m+1-1}{m+1-1} = \binom{n-m(b-1)}{m}.$$

As the first and last components of each set of m sets of b consecutive positions are nonzero and the in-between $b-2$ components in each set can be any element from $\mathbb{Z}_{p^l}$, the required number of m-repeated bursts in $\mathbb{Z}_{p^l}^n$ is given by (3). □

Lemma 4 *The number of all m-repeated bursts of length $b(>1)$ having homogeneous weight w_{hom} in the code space of n-tuples over $\mathbb{Z}_{p^l}$ $(l>1)$ is given by*

$$\mathbb{A}_n^{m,b}(w_{hom}) = \binom{n-m(b-1)}{m} \times \sum_{i=0}^{2m}\sum_{u_i,v_i}\binom{2m}{i}\binom{mb-2m}{u_i}\binom{mb-2m-u_i}{v_i} \times (p-1)^{u_i+i}(p^l-p)^{v_i+2m-i},$$

where $u_i+v_i \le m(b-2)$ and $w_{hom} = (u_i+i)p^{l-1} + (v_i+2m-i)(p-1)p^{l-2}$ for $0 \le i \le 2m$.

Proof For the proof, we need to consider $2m+1$ cases depending upon weights of the first components of each b consecutive positions. We already know by Lemma 3 that the number of m distinct sets of b consecutive positions is $\binom{n-m(b-1)}{m}$.

Case 1: Assuming all the nonzero components among the first and the last components in each m distinct sets of b consecutive positions are of weight $(p-1)p^{l-2}$, then the numbers of m-repeated bursts of length b is counted as

$$\binom{mb-2m}{u_0}\binom{mb-2m-u_0}{v_0}(p-1)^{u_0}(p^l-p)^{v_0+2m},$$

where u_0 and v_0 represent the number of positions having components of weight p^{l-1} and $(p-1)p^{l-2}$, respectively within $b-2$ components of each m distinct sets of b consecutive positions such that $u_0+v_0 \le m(b-2)$ and $w_{hom} = u_0p^{l-1} + (v_0 + 2m)(p-1)p^{l-2}$.

Case 2: Assuming one of nonzero components among the first and the last components in each m distinct sets of b consecutive positions are of weight p^{l-1} and all others are of weight $(p-1)p^{l-2}$, we have the numbers of m-repeated bursts of length b for suitable pairwise (u_1, v_1) (like Case 1) as

$$\binom{2m}{1}\binom{mb-2m}{u_1}\binom{mb-2m-u_1}{v_1} \times (p-1)^{u_1+1}(p^l-p)^{v_1+2m-1},$$

where $u_1+v_1 \le m(b-2)$ and $w_{hom} = (u_1+1)p^{l-1} + (v_1+2m-1)(p-1)^{l-2}$.

Case 3: Assuming any two of nonzero components among the first and the last components in each m distinct sets of b consecutive positions are of weight p^{l-1} and remaining are of weight $(p-1)p^{l-2}$, we have the numbers of m-repeated bursts of length b for pairwise (u_2, v_2) as

$$\binom{2m}{2}\binom{mb-2m}{u_2}\binom{mb-2m-u_2}{v_2} \times (p-1)^{u_2+2}(p^l-p)^{v_2+2m-2},$$

where $u_2 + v_2 \leq m(b-2)$ and $w_{hom} = (u_2+2)p^{l-1} + (v_2+2m-2)(p-1)^{l-2}$.

Continuing in this way, we get the last $(2m+1)^{th}$ case.

Case 2m+1: Assuming all nonzero components among the first and the last components in each m distinct sets are of weight p^{l-1}, then the numbers of m-repeated bursts of length b for convenient pair (u_{2m}, v_{2m}) is given by

$$\binom{mb-2m}{u_{2m}}\binom{mb-2m-u_{2m}}{v_{2m}} \times (p-1)^{u_{2m}+2m}(p^l-p)^{v_{2m}},$$

where $u_{2m} + v_{2m} \leq m(b-2)$ and $w_{hom} = (u_{2m}+2m)p^{l-1} + v_{2m}(p-1)p^{l-2}$.

This completes the proof. □

Next two results give the total homogeneous weight of m-repeated bursts without and with weight constraint.

Theorem 7 *The total homogeneous weight, $\mathbb{W}_n^{m,b}$, of all m-repeated bursts of length $b\ (>1)$ in the code space of n-tuples over $\mathbb{Z}_{p^l}$ $(l>1)$ is given by*

$$\mathbb{W}_n^{m,b} = \binom{n-m(b-1)}{m} \times \sum_{i=0}^{2m}\sum_{u_i,v_i}\binom{2m}{i}\binom{mb-2m}{u_i}\binom{mb-2m-u_i}{v_i} \times (p-1)^{u_i+i}(p^l-p)^{v_i+2m-i}\Big[(u_i+i)p^{l-1} + (v_i+2m-i)(p-1)p^{l-2}\Big],$$

where u_i and v_i for $0 \leq i \leq 2m$ are non-negative integers such that $u_i+v_i \leq m(b-2)$.

Proof We consider following $2m+1$ cases as Lemma 4.

For $0 \leq i \leq 2m$, if i numbers of nonzero components among the first and the last components in each m distinct sets of b consecutive positions are of weight p^{l-1} and all others are of weight $(p-1)p^{l-2}$ and further if u_i and v_i are being the number of components of weight p^{l-1} and $(p-1)p^{l-2}$ respectively within the $b-2$ components in each set such that $u_i+v_i \leq m(b-2)$, then the homogeneous weight of all m-repeated bursts of length b for each i is

$$\binom{2m}{i}\binom{mb-2m}{u_i}\binom{mb-2m-u_i}{v_i}(p-1)^{u_i+i}(p^l-p)^{v_i+2m-i} \times \Big[(u_i+i)p^{l-1} + (v_i+2m-i)(p-1)p^{l-2}\Big].$$

Therefore, the total homogeneous weight of all m-repeated bursts of length b in $\mathbb{Z}_{p^l}^n$ is

$$\mathbb{W}_n^{m,b} = \binom{n-m(b-1)}{m} \times \sum_{i=0}^{2m} \sum_{u_i, v_i} \binom{2m}{i} \binom{mb-2m}{u_i} \binom{mb-2m-u_i}{v_i} \times (p-1)^{u_i+i} (p^l - p)^{v_i+2m-i} \Big[(u_i+i)p^{l-1} + (v_i+2m-i)(p-1)p^{l-2}\Big].$$

□

Theorem 8 *The total homogeneous weight of the set of all m-repeated bursts of length b with homogeneous weight w_{hom} or less in the code space of n-tuples over $\mathbb{Z}_{p^l}$ $(l > 1)$ is*

$$\sum_{i=m(p-1)p^{l-2}}^{w_{hom}} i\mathbb{A}_n^{m,b}(i).$$

Proof It follows from Lemma 4. □

Note: If $w_{hom} = mbp^{l-1}$, Theorems 7 and 8 coincide.

Lastly, we give necessary and sufficient conditions for codes over $\mathbb{Z}_{p^l}$ correcting m-repeated burst with homogeneous weight at most w_{hom}. The necessary condition follows immediately from Lemma 4.

Theorem 9 *An (n, k) linear code C over $\mathbb{Z}_{p^l}$ $(l > 1)$ that corrects all m-repeated bursts of length $b(> 1)$ with homogeneous weight at most w_{hom} always satisfies*

$$p^{ln-k} \geq 1 + \sum_{i=2(p-1)p^{l-2}}^{w_{hom}} \mathbb{A}_n^{m,b}(\rho),$$

where $\mathbb{A}_n^{m,b}(\rho)$ is given by Lemma 3.

Theorem 10 *A sufficient condition to exist an (n, k) linear code C over $\mathbb{Z}_{p^l}$ $(l > 1)$ that corrects all m-repeated bursts of length b (> 1) with homogeneous weight at most w_{hom} $(n > 2mb;\ w_{hom} \leq bp^{l-1})$ is*

$$p^{ln-k} > 1 + \sum_{r+s \leq 2w_{hom};\, r \leq bp^{l-1}} \left\{ \sum_{i=2(p-1)p^{l-2}}^{r} \mathbb{A}_b^b(i) \times \sum_{\rho=2(2m-1)(p-1)p^{l-2}}^{s} \mathbb{A}_{n-b}^{2m-1,b}(\rho) \right\},$$

where $\mathbb{A}_b^b(i)$ and $\mathbb{A}_{n-b}^{2m-1,b}(\rho)$ are from Lemmas 2 and 4 respectively.

Proof The proof of this theorem follows the same process as Theorem 6. After choosing the first $j-1$ columns of H and the j^{th} column h_j can be added provided αh_j $(\alpha \in \mathbb{Z}_{p^l} \backslash \{0\})$ should not be a linear combination of immediate previous $b-1$ consecutive columns having homogeneous weight r $(\leq bp^{l-1})$ or less, together with a linear combination of $(2m-1)$ sets of previous b consecutive columns among the first $j-b$ columns, each set having homogeneous weight s or less such that $r+s \leq 2w_{hom}$. That is

$$\begin{aligned}\alpha h_j \neq\ & (\alpha_1 h_{j-1} + \alpha_2 h_{j-2} + \cdots + \alpha_{b-1} h_{j-b+1}) \\ & + (\beta_{11} h_{i_1} + \beta_{12} h_{i_1+1} + \cdots + \beta_{1b} h_{i_1+b-1}) \\ & + (\beta_{21} h_{i_2} + \beta_{22} h_{i_2+1} + \cdots + \beta_{2b} h_{i_2+b-1}) \\ & + \\ & \vdots \\ & + (\beta_{(2m-1)1} h_{i_{2m-1}} + \beta_{(2m-1)2} h_{i_{2m-1}+1} + \cdots + \beta_{(2m-1)b} h_{i_{2m-1}+b-1}),\end{aligned} \tag{4}$$

where $j-b+1 > i_1+b-1$; $i_r > i_{r+1}+b-1$ (where $1 \leq r \leq 2m-2$); α, α_i, $\beta_{i,j} \in \mathbb{Z}_{p^l}$ such that α and α_i together form a burst of length b with homogeneous weight at most r and $\beta_{i,j}$'s are such that they form a $(2m-1)$- repeated burst of length b with homogeneous weight at most s in a vector of length $j-b$ such that $r+s \leq 2w_{hom}$ and $r \leq bp^{l-1}$.

This condition ensures that any two m-repeated bursts of length b with homogeneous weight w_{hom} or less will have distinct syndromes.

The number of coefficients α_i's by Lemma 2 is

$$\sum_{i=2(p-1)p^{l-2}}^{r} \mathbb{A}_b^b(i)$$

and the number of $\beta_{i,j}$'s by Lemma 4 is

$$\sum_{\rho=2(2m-1)(p-1)p^{l-2}}^{s} \mathbb{A}_{j-b}^{2m-1,b}(\rho).$$

So, the total number of linear combinations in Expression (4), including the zero combination, is

$$1 + \sum_{r+s \leq 2w_{hom};\, r \leq bp^{l-1}} \left\{ \sum_{i=2(p-1)p^{l-2}}^{r} \mathbb{A}_b^b(i) \times \sum_{\rho=2(2m-1)(p-1)p^{l-2}}^{s} \mathbb{A}_{j-b}^{2m-1,b}(\rho) \right\}.$$

As the available number of distinct syndromes is p^{nl-k}, the j^{th} column h_j can be added to H provided

$$p^{ln-k} > 1 + \sum_{r+s \leq 2w_{hom};\, r \leq bp^{l-1}} \left\{ \sum_{i=2(p-1)p^{l-2}}^{r} \mathbb{A}_b^b(i) \times \sum_{\rho=2(2m-1)(p-1)p^{l-2}}^{s} \mathbb{A}_{j-b}^{2m-1,b}(\rho) \right\}.$$

Replacing j by n proves the theorem. □

Example 3 Let us take $p = l = m = 2$, $b = 3$, $w_{hom} = 4$ and $n = k_0 + k_1 + k_2 = 4 + 3 + 6 = 13$ in the Theorem 10. Then

$$
\begin{aligned}
& p^{ln-k} > 1 + \sum_{i=2}^{2} \mathbb{A}_b^b(i) \times \sum_{\rho=6}^{6} \mathbb{A}_{n-b}^{2m-1,b}(\rho) \\
\implies & 2^{ln-k} > 1 + \mathbb{A}_3^3(2) \times \mathbb{A}_{10}^{3,3}(6) \\
\implies & 2^{ln-k} > 1 + 4 \times 4 \times 2^6 = 1 + 2^{10} \\
\implies & ln - k \geq 11 \\
\implies & k \leq 15.
\end{aligned}
$$

Considering $k = 11$, we get a (13, 11) linear code over $\mathbb{Z}_{2^2}$ where $k = 2k_0 + k_1 = 11$ (refer Theorems 1-2). The order of the parity check matrix of the (13, 11) linear code is 9 by 13 as $k_1 + k_2 = 9$. Like Example 2, we construct a parity check matrix $H_{9\times 13}$ satisfying Condition (4) and maintaining the format given in Theorem 2 as follows:

$$
\begin{aligned}
H_{9\times 13} &= \begin{bmatrix} (B_{0,2})_{6\times 4} & (B_{0,1})_{6\times 3} & I_{6\times 6} \\ 2(B_{1,2})_{3\times 4} & 2I_{3\times 3} & 0_{3\times 6} \end{bmatrix} \\
&= \begin{bmatrix}
1&1&1&3&2&1&1&1&0&0&0&0&0 \\
2&0&3&2&1&0&2&0&1&0&0&0&0 \\
3&1&0&0&1&2&3&0&0&1&0&0&0 \\
1&0&2&1&3&0&3&0&0&0&1&0&0 \\
2&1&1&2&3&3&1&0&0&0&0&1&0 \\
3&3&1&1&1&1&3&0&0&0&0&0&1 \\
2&0&0&2&2&0&0&0&0&0&0&0&0 \\
2&2&0&0&0&2&0&0&0&0&0&0&0 \\
2&0&2&0&0&0&2&0&0&0&0&0&0
\end{bmatrix}.
\end{aligned}
$$

All the 576 syndromes of the discussed errors are found to be non-zero and distinct (checked by MS-Excel). This shows that the (13, 11) linear code over $\mathbb{Z}_{2^2}$ with parity check matrix $H_{9\times 13}$ corrects all 2-repeated bursts of length 3 having homogeneous weight at most 4.

5 Conclusion

In this paper, we studied linear codes correcting burst and repeated burst in homogeneous metric sense along with its necessary and sufficient conditions. Total homogeneous weight of these errors with respect to homogeneous metric are also obtained. The study can further be extended if the errors are confined to a sub-block of code length. Location of such errors in homogeneous metric sense can also be investigated.

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