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RESEARCH

# Expansivity and Periodicity in Algebraic Subshifts

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## Abstract

A $d$-dimensional configuration $c : \mathbb{Z}^d \longrightarrow A$ is a coloring of the $d$-dimensional infinite grid by elements of a finite alphabet $A \subseteq \mathbb{Z}$. The configuration $c$ has an annihilator if a non-trivial linear combination of finitely many translations of $c$ is the zero configuration. Writing $c$ as a $d$-variate formal power series, the annihilator is conveniently expressed as a $d$-variate Laurent polynomial $f$ whose formal product with $c$ is the zero power series. More generally, if the formal product is a strongly periodic configuration, we call the polynomial $f$ a periodizer of $c$. A common annihilator (periodizer) of a set of configurations is called an annihilator (periodizer, respectively) of the set. In particular, we consider annihilators and periodizers of $d$-dimensional subshifts, that is, sets of configurations defined by disallowing some local patterns. We show that a $(d-1)$-dimensional linear subspace $S \subseteq \mathbb{R}^d$ is expansive for a subshift if the subshift has a periodizer whose support contains exactly one element of $S$. As a subshift is known to be finite if all $(d-1)$-dimensional subspaces are expansive, we obtain a simple necessary condition on the periodizers that guarantees finiteness of a subshift or, equivalently, strong periodicity of a configuration. We provide examples in terms of tilings of $\mathbb{Z}^d$ by translations of a single tile.

**Keywords** Symbolic dynamics · Annihilator · Periodicity · Expansivity · Golomb-Welch conjecture · Periodic tiling problem

## 1 Introduction

A configuration in this paper is a coloring of the $d$-dimensional grid $\mathbb{Z}^d$ using finitely many colors. Our colors are integers. A configuration $c$ has an annihilator if the zero configuration can be obtained as a non-trivial linear combination of suitable translations of $c$. In other terms, annihilation means that a linear cellular automaton maps the configuration $c$ to the zero configuration. This mapping, in the terminology of digital

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signal processing, is filtering by a $d$-dimensional discrete-time finite-extend impulse response (FIR) filter. Writing $c$ as a $d$-variate formal power series, the annihilator is conveniently expressed as a $d$-variate Laurent polynomial $f$ whose formal product with $c$ is the zero power series.

Configurations that have annihilators come up in several contexts. Every low-complexity configuration has an annihilator, where low-complexity means that the number of patterns in the configuration of some finite fixed shape $D \subseteq \mathbb{Z}^d$ is at most the size $|D|$ of the shape [1]. Low-complexity configurations are the object of interest in the unsolved Nivat's conjecture [2], and also in the recently solved periodic tiling problem [3] where tilings of $\mathbb{Z}^d$ by translates of a single tile are low-complexity configurations [1]. Also so-called perfect colorings of grid graphs have annihilators [4].

Configurations with annihilators have global rigidity, although they are not necessarily periodic. In the two-dimensional case, periodicity in all directions is known to be enforced if the annihilator has no line polynomial factors, that is, an annihilating polynomial does not have a non-monomial factor whose monomials are on a single line [4–6]. In this paper we present a similar condition that works in all dimensions $d$. More generally, we provide a condition on the annihilator that enforces expansivity: this is a directional determinism property studied in multidimensional symbolic dynamics. Expansivity in all directions is known to imply strong periodicity [7].

The article is organized as follows. In Section 2 we present necessary terminology, our notations and some results we need from literature. In Section 3 we discuss a particular application: tilings of $\mathbb{Z}^d$ by translated copies of a single tile. Throughout the article, we demonstrate our methods with examples that come from this setup. Section 4 contains the new contributions. We prove a condition on annihilators that guarantees expansivity, and consequently obtain a condition that implies strong periodicity of configurations. We provide several examples, including a discussion on the relation to the Golomb-Welch conjecture. We finish with some concluding remarks in Section 5.

## 2 Preliminaries

We start by defining the necessary terminology and concepts. This part is included for the convenience of the reader although it greatly repeats what is written, for example, in [6].

### 2.1 Configurations and Periodicity

A $d$-dimensional *configuration* over a finite alphabet $A$ is an assignment

$$c : \mathbb{Z}^d \longrightarrow A$$

of symbols of $A$ on the infinite grid $\mathbb{Z}^d$. For any configuration $c \in A^{\mathbb{Z}^d}$ and any cell $\mathbf{u} \in \mathbb{Z}^d$, we denote by $c_{\mathbf{u}}$ the letter $c(\mathbf{u})$ that $c$ has in the cell $\mathbf{u}$.

For a vector $\mathbf{t} \in \mathbb{Z}^d$, the *translation* $\tau^{\mathbf{t}}$ shifts a configuration $c$ so that the cell $\mathbf{t}$ is moved to the cell $\mathbf{0}$, that is, $\tau^{\mathbf{t}}(c)_{\mathbf{u}} = c_{\mathbf{u}+\mathbf{t}}$ for all $\mathbf{u} \in \mathbb{Z}^d$. We say that $c$ is *periodic* if

$\tau^{\mathbf{t}}(c) = c$ for some non-zero $\mathbf{t} \in \mathbb{Z}^d$. In this case $\mathbf{t}$ is a *vector of periodicity* and $c$ is also called $\mathbf{t}$-*periodic*. If there are $d$ linearly independent vectors of periodicity (viewed as elements of the vector space $\mathbb{R}^d$) then $c$ is called *strongly periodic*. We denote by $\mathbf{e}_i = (0, \ldots, 0, 1, 0 \ldots, 0)$ the basic $i$'th unit coordinate vector, for $i = 1, \ldots, d$. A strongly periodic $c \in A^{\mathbb{Z}^d}$ has automatically, for some $k > 0$, vectors of periodicity $k\mathbf{e}_1, k\mathbf{e}_2, \ldots, k\mathbf{e}_d$ in the $d$ coordinate directions.

### 2.2 Patterns and Pattern Complexity

Let $D \subseteq \mathbb{Z}^d$ be a finite set of cells, a *shape*. A $D$-*pattern* is an assignment $p \in A^D$ of symbols in the shape $D$. A *(finite) pattern* is a $D$-pattern for some shape $D$. We call $D$ the *domain* of the pattern. Notation $A^*$ is used for the set of all finite patterns over the alphabet $A$ (where the dimension $d$ is assumed to be known).

We say that a finite pattern $p$ of shape $D$ *appears* in a configuration $c$ if for some $\mathbf{t} \in \mathbb{Z}^d$ we have $\tau^{\mathbf{t}}(c) \restriction_D = p$. We also say that $c$ *contains* the pattern $p$ in the position $\mathbf{t}$. For a fixed $D$, the set of $D$-patterns that appear in a configuration $c$ is denoted by $\mathcal{L}_D(c)$. We denote by $\mathcal{L}(c)$ the set of all finite patterns that appear in $c$, i.e., the union of $\mathcal{L}_D(c)$ over all finite $D \subseteq \mathbb{Z}^d$.

The *pattern complexity* of a configuration $c$ with respect to a shape $D$ is the number of different $D$-patterns that $c$ contains. A sufficiently low pattern complexity forces global regularities in a configuration. A relevant threshold happens when the pattern complexity is at most $|D|$, the number of cells in shape $D$. Hence we say that $c$ has *low complexity* with respect to shape $D$ if

$$|\mathcal{L}_D(c)| \leq |D|.$$

We call $c$ a *low complexity configuration* if it has low complexity with respect to some finite shape $D$.

### 2.3 Subshifts

Let $p \in A^D$ be a finite pattern of a shape $D$. The set $[p] = \{c \in A^{\mathbb{Z}^d} \mid c \restriction_D = p\}$ of configurations that have $p$ in the domain $D$ is called the *cylinder* determined by $p$. The collection of cylinders $[p]$ is a base of a compact topology on $A^{\mathbb{Z}^d}$, the *prodiscrete* topology. See, for example, the first few pages of [8] for details. The topology is equivalently defined by a metric on $A^{\mathbb{Z}^d}$ where two configurations are close to each other if they agree with each other on a large region around the cell $\mathbf{0}$. Cylinders are clopen in the topology: they are both open and closed.

A subset $X$ of $A^{\mathbb{Z}^d}$ is called a *subshift* if it is closed in the topology and closed under translations. Note that – somewhat nonstandardly – we allow $X$ to be the empty set. By a compactness argument one has that every configuration $c$ that is not in $X$ contains a finite pattern $p$ that prevents it from being in $X$: no configuration that contains $p$ is in $X$. We can then as well define subshifts using forbidden patterns: given a set $P$ of finite patterns we define

$$\mathcal{X}_P = \{c \in A^{\mathbb{Z}^d} \mid \mathcal{L}(c) \cap P = \emptyset\},$$

the set of configurations that do not contain any of the patterns in $P$. The set $\mathcal{X}_P$ is a subshift, and every subshift is $\mathcal{X}_P$ for some $P$. If $X = \mathcal{X}_P$ for some finite $P$ then $X$ is a *subshift of finite type* (SFT).

For a subshift $X \subseteq A^{\mathbb{Z}^d}$ (or actually for any set $X$ of configurations) we define its language $\mathcal{L}(X) \subseteq A^*$ to be the set of all finite patterns that appear in some element of $X$, that is, the union of sets $\mathcal{L}(c)$ over all $c \in X$. For a fixed shape $D$, we analogously define $\mathcal{L}_D(X) = \mathcal{L}(X) \cap A^D$, the union of all $\mathcal{L}_D(c)$ over $c \in X$. We say that $X$ has low complexity with respect to shape $D$ if $|\mathcal{L}_D(X)| \leq |D|$. For example, if we fix shape $D$ and a small set $P \subseteq A^D$ of at most $|D|$ allowed patterns of shape $D$, then $X = X_{A^D \setminus P} = \{c \in A^{\mathbb{Z}^d} \mid \mathcal{L}_D(c) \subseteq P\}$ is a low complexity SFT since $\mathcal{L}_D(X) \subseteq P$ and $|P| \leq |D|$.

The *orbit* of a configuration $c$ is the set $\mathcal{O}(c) = \{\tau^{\mathbf{t}}(c) \mid \mathbf{t} \in \mathbb{Z}^2\}$ of all its translates, and the *orbit closure* $\overline{\mathcal{O}(c)}$ of $c$ is the topological closure of its orbit. The orbit closure is a subshift, and in fact it is the intersection of all subshifts that contain $c$. In terms of finite patterns, $c' \in \overline{\mathcal{O}(c)}$ if and only if $\mathcal{L}(c') \subseteq \mathcal{L}(c)$. Of course, the orbit closure of a low complexity configuration is a low complexity subshift.

### 2.4 Annihilators and Periodizers

To use commutative algebra we assume that $A \subseteq \mathbb{Z}$, i.e., the symbols in the configurations are integers. We also maintain the assumption that $A$ is finite. We express a $d$-dimensional configuration $c \in A^{\mathbb{Z}^d}$ as a formal power series over $d$ variables $x_1, \ldots x_d$ where the monomials address cells in a natural manner $x_1^{u_1} \cdots x_d^{u_d} \longleftrightarrow (u_1, \ldots, u_d) \in \mathbb{Z}^d$, and the coefficients of the monomials in the power series are the symbols at the corresponding cells. Using the convenient vector notation $\mathbf{x} = (x_1, \ldots x_d)$ we write $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \cdots x_d^{u_d}$ for the monomial that represents cell $\mathbf{u} = (u_1, \ldots u_d) \in \mathbb{Z}^d$. Note that all our power series and polynomials are *Laurent* as we allow negative as well as positive powers of variables. Now the configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ can be coded as the formal power series

$$c(\mathbf{x}) = \sum_{\mathbf{u} \in \mathbb{Z}^d} c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}.$$

The power series $c(\mathbf{x})$ is *integral* (the coefficients are integers) and because $A \subseteq \mathbb{Z}$ is finite, it is *finitary* (there are only finitely many different coefficients). Henceforth we treat configurations as integral, finitary power series. By default, for any Laurent power series or polynomial $f$ we denote by $f_{\mathbf{u}}$ the coefficient of $\mathbf{x}^{\mathbf{u}}$.

Note that the power series are indeed formal: the role of the variables is only to provide the position information on the grid. We may sum up two power series, or multiply a power series with a polynomial, but we never plug in any values in the variables. Multiplying a power series $c(\mathbf{x})$ by a monomial $\mathbf{x}^{\mathbf{t}}$ simply adds $\mathbf{t}$ to the exponents of all monomials, thus producing the power series of the translated configuration $\tau^{\mathbf{t}}(c)$. Hence the configuration $c(\mathbf{x})$ is $\mathbf{t}$-periodic if and only if $\mathbf{x}^{\mathbf{t}}c(\mathbf{x}) = c(\mathbf{x})$, that is, if and only if $(\mathbf{x}^{\mathbf{t}}-1)c(\mathbf{x}) = 0$, the zero power series. Thus we can express the periodicity of a configuration in terms of its *annihilation* under the multiplication with a *difference binomial* $\mathbf{x}^{\mathbf{t}} - 1$. Very naturally then we introduce the *annihilator ideal*

$$\mathrm{Ann(c)} = \{\mathrm{f} \in \mathbb{C}[\mathbf{x}^{\pm 1}] \mid \mathrm{fc} = 0\}$$

containing all the polynomials that annihilate $c$. Here we use the notation $\mathbb{C}[\mathbf{x}^{\pm 1}]$ for the set of Laurent polynomials with complex coefficients. Note that Ann(c) is indeed an ideal of the Laurent polynomial ring $\mathbb{C}[\mathbf{x}^{\pm 1}]$.

Let us denote the *support* of a Laurent polynomial $f \in \mathbb{C}[\mathbf{x}^{\pm 1}]$ by

$$\mathrm{Supp(f)} = \{\mathbf{u} \in \mathbb{Z}^{\mathrm{d}} \mid \mathrm{f}_{\mathbf{u}} \neq 0\}.$$

***Remark 1*** If a configuration $c$ has an annihilator $f$ with complex coefficients then it also has an annihilator $f'$ with integer coefficients that satisfies $\mathrm{Supp(f')} = \mathrm{Supp(f)}$.

To see why the remark is true, note that the annihilation condition $fc = 0$ can be viewed as a homogeneous system of linear equations for the coefficients of the annihilating polynomial $f$. The coefficients of the variables in the equations come from the configuration $c$ and are hence integers. It is easy to see that for any (complex valued) solution of a homogeneous linear system with integer coefficients there is also an integer valued solution with the property that each variable that had a non-zero value in the original complex solution also has a non-zero value in the new integral solution. The integral solution provides the coefficients of an integral annihilator $f'$ that satisfies $\mathrm{Supp(f')} = \mathrm{Supp(f)}$.

We find it sometimes convenient to work with the *periodizer ideal*

$$\mathrm{Per(c)} = \{\mathrm{f} \in \mathbb{C}[\mathbf{x}^{\pm 1}] \mid \mathrm{fc\ is\ strongly\ periodic}\}$$

that contains those Laurent polynomials whose product with configuration $c$ is strongly periodic. Clearly also Per(c) is an ideal of the Laurent polynomial ring $\mathbb{C}[\mathbf{x}^{\pm 1}]$, and we have $\mathrm{Ann(c)} \subseteq \mathrm{Per(c)}$. Moreover, if Per(c) contains non-zero polynomials, so does Ann(c). Indeed, if $f \in \mathrm{Per(c)}$ then $fc$ is annihilated by $\mathbf{x}^{\mathbf{t}} - 1$ for any period $\mathbf{t}$ of the strongly periodic $fc$, and thus $f(\mathbf{x})(\mathbf{x}^{\mathbf{t}} - 1)$ is an annihilator of $c$.

Our first observation relates the low complexity assumption to annihilators. Namely, it is easy to see using elementary linear algebra that any low complexity configuration has at least some non-trivial annihilators:

**Lemma 1** ([1]) *Let c be a low complexity configuration. Then* Ann(c) *contains a non-zero polynomial. More precisely, if c has low complexity with respect to a shape $D \subseteq \mathbb{Z}^d$ then there is a non-zero $f \in$* Per(c) *with* $-\mathrm{Supp(f)} \subseteq \mathrm{D}$.

The minus sign in front of the support of $f$ in the statement of the lemma comes from the manner the convolutions in the product $fc$ are computed: For all $\mathbf{u} \in \mathbb{Z}^d$

$$(fc)_{\mathbf{u}} = \sum_{\mathbf{v} \in \mathrm{Supp(f)}} f_{\mathbf{v}} c_{\mathbf{u}-\mathbf{v}},$$

so that the pattern of shape $-\mathrm{Supp(f)}$ in $c$ at position $\mathbf{u}$ determines the new value $(fc)_{\mathbf{u}}$ at position $\mathbf{u}$. We see the analogous minus sign also in other statements in the rest of the article.

One of the main results of [1] states that if a configuration $c$ is annihilated by a non-zero polynomial (e.g., due to low complexity) then it is automatically annihilated by a product of difference binomials. This result is fundamental to our approach.

**Theorem 1** ([1, 5]) *Let $c$ be a configuration and $f \in \mathrm{Ann(c)}$. For every $\mathbf{u} \in \mathrm{Supp(f)}$ there exist pairwise linearly independent $\mathbf{t}_1, \ldots, \mathbf{t}_m \in \mathbb{Z}^d$ such that each $\mathbf{t}_i$ is parallel to $\mathbf{u}_i - \mathbf{u}$ for some $\mathbf{u}_i \in \mathrm{Supp(f)} \setminus \{\mathbf{u}\}$, and*

$$(\mathbf{x}^{\mathbf{t}_1} - 1) \cdots (\mathbf{x}^{\mathbf{t}_m} - 1) \in \mathrm{Ann(c)}.$$

In [1] the statement of Theorem 1 is given without reference to elements of Supp(f) but the given proof provides for an arbitrary $\mathbf{u}$ in Supp(f) the vectors $\mathbf{u}_i \in \mathrm{Supp(f)}$ as in the statement above. In Theorem 12 of [5] the result is stated in this stronger form. In the present paper the directions $\mathbf{u}_i - \mathbf{u}$ of $\mathbf{t}_i$ between positions in the support of the annihilating polynomial $f$ play a central role. Note also that by Remark 1 the annihilating polynomial $f$ does not need to be integral: there always exists one with the same support and with integer coefficients.

For a subshift $X \subseteq A^{\mathbb{Z}^d}$, we denote by Ann(X) the set of Laurent polynomials that annihilate all elements of $X$, and we call Ann(X) the annihilator ideal of $X$. Similarly, Per(X) is the intersection of sets Per(c) over $c \in X$. All results stated above for Ann(c) and Per(c) for a single configuration $c$ work just as well for Ann(X) and Per(X) for a subshift $X$, with similar proofs. In particular, we have the following subshift variant of Theorem 1.

**Theorem 1′** *Let $X$ be a subshift and $f \in \mathrm{Ann(X)}$. For every $\mathbf{u} \in \mathrm{Supp(f)}$ there exist pairwise linearly independent $\mathbf{t}_1, \ldots, \mathbf{t}_m \in \mathbb{Z}^d$ such that each $\mathbf{t}_i$ is parallel to $\mathbf{u}_i - \mathbf{u}$ for some $\mathbf{u}_i \in \mathrm{Supp(f)} \setminus \{\mathbf{u}\}$, and*

$$(\mathbf{x}^{\mathbf{t}_1} - 1) \cdots (\mathbf{x}^{\mathbf{t}_m} - 1) \in \mathrm{Ann(X)}.$$

## 3 Tilings by Translations of a Single Tile

As a specific setup and a convenient source of examples throughout the article we consider tilings of $\mathbb{Z}^d$ using translated copies of a single finite shape $D \subseteq \mathbb{Z}^d$. In this context we call $D$ a *tile*. A tiling by $D$ is expressed as a binary configuration where symbols 1 identify the positions where copies of $D$ are placed to fully cover $\mathbb{Z}^d$ without overlaps. More precisely, $c \in \{0,1\}^{\mathbb{Z}^d}$ is a *tiling* by $D$ if and only if $c(\mathbf{x}) f_D(\mathbf{x}) = \mathbb{1}(\mathbf{x})$ where

$$f_D(\mathbf{x}) = \sum_{\mathbf{u} \in D} \mathbf{x}^{\mathbf{u}}$$

is the characteristic polynomial of $D$, and

$$\mathbb{1}(\mathbf{x}) = \sum_{\mathbf{u} \in \mathbb{Z}^d} \mathbf{x}^{\mathbf{u}}$$

is the uniform configuration of 1's. The polynomial $f_D$ is thus a periodizer of every tiling by $D$.

Let $T_D \subseteq \{0,1\}^{\mathbb{Z}^d}$ be the set of tilings by $D$. Clearly $T_D$ is a low-complexity subshift of finite type: the elements of $T_D$ are exactly the binary configurations whose $(-D)$-patterns have precisely one occurrence of symbol 1, and there exist $|-D|$ such patterns in total.

***Example 1*** In illustrations we draw tiles in two and three dimensions as unions of unit squares and cubes. For example, Fig. 1(a) shows the tile $D = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$. This tile admits tilings of $\mathbb{Z}^3$ that are not strongly periodic. One may, for example, start with any tiling $a \in \{0,1\}^{\mathbb{Z}^2}$ of $\mathbb{Z}^2$ by the $2 \times 2$ square tile $S = \{(0,0), (0,1), (1,0), (1,1)\}$. Then $c \in \{0,1\}^{\mathbb{Z}^3}$ defined by $c(x_1, x_2, x_3) = a(x_1 + x_3, x_2 + x_3)$ is a $(1,1,-1)$-periodic tiling of $\mathbb{Z}^3$ by $D$, whose slice $(x_1, x_2) \mapsto c(x_1, x_2, 0)$ on $\mathbb{Z} \times \mathbb{Z} \times \{0\}$ is equal to $a$. If $a$ is not strongly periodic then $c$ is not strongly periodic either. See Fig. 1(b) and (c).

***Example 2*** For a dimension $d$ and radius $r \in \mathbb{Z}_+$, let us denote

$$B_r^d = \{(n_1, \ldots, n_d) \in \mathbb{Z}^d \mid \sum_{i=1}^{d} |n_i| \le r\}$$

for the $d$-dimensional radius-$r$ sphere under the Lee metric (also known as the Manhattan metric). See Fig. 2 for illustrations of $B_2^3$ and $B_3^2$.

If $d \le 2$ or if $r = 1$ then there are strongly periodic tilings by tile $B_r^d$ [9]: these are perfect codes under the Lee metric. In [9] it was conjectured that for other values of $d$ and $r$ the tile $B_r^d$ does not tile $\mathbb{Z}^d$. There are two natural variants of the conjecture: the strong Golomb-Welch conjecture states that no tiling exists, while the weak Golomb-Welch conjecture postulates that no strongly periodic tiling exists. The conjectures are still open for dimensions $d \ge 6$. It is known that the conjecture is true in every dimension for sufficiently large radiuses, and so the case of radius $r = 2$ seems most challenging. See [10] for more details.

It was recently proved in [3] that for some dimension $d$, there exists a tile $D \subseteq \mathbb{Z}^d$ such that $T_D$ is an aperiodic SFT, *i.e.*, such that there exists a tiling but no strongly

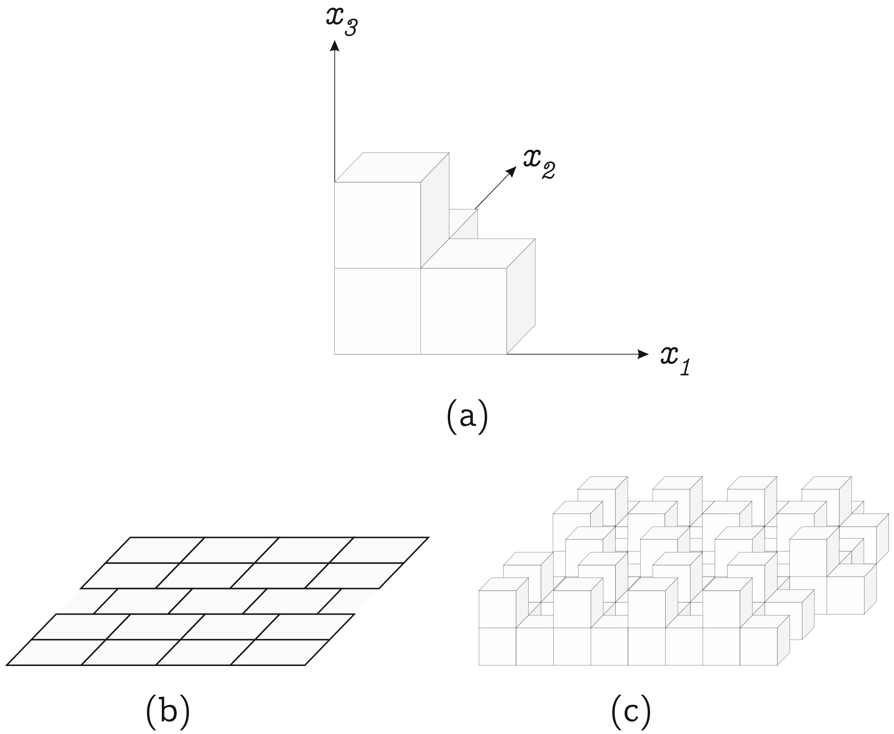


**Fig. 1** (a) The tile $D = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of Example 1, (b) a tiling of $\mathbb{Z}^2$ by $2 \times 2$ squares that is not strongly periodic since one row of tiles is shifted by one, (c) the corresponding layer of a tiling of $\mathbb{Z}^3$ by $D$. A tiling of $\mathbb{Z}^3$ is obtained by repeating the layer $(1, 1, -1)$-periodically

periodic tiling exists. This provided a negative answer to the Periodic tiling problem [11]. In contrast, any two-dimensional tile $D \subseteq \mathbb{Z}^2$ that tiles $\mathbb{Z}^2$ also tiles $\mathbb{Z}^2$ periodically [12, 13].

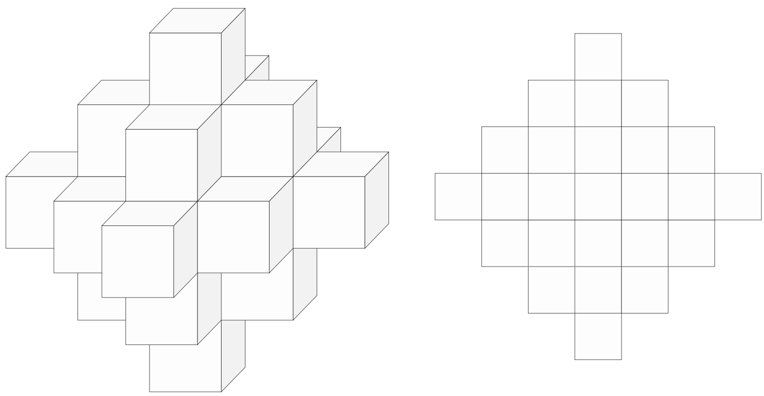

**Fig. 2** The radius-2 Lee sphere $B_2^3$ in dimension $d = 3$ (on the left), and the radius-3 Lee sphere $B_3^2$ in dimension $d = 2$ (on the right)

Interestingly, if $|D|$ is a prime number then every tiling by $D$ is strongly periodic [14]. This fact has also a simple proof using our algebraic approach, see Example 2 in [1]. In [14] it was also shown that $T_D = T_{-D}$ for all tiles $D$, *i.e.*, rotating each tile in place turns a tiling by $D$ into a tiling by $-D$. Thus both $f_D(\mathbf{x})$ and $f_{-D}(\mathbf{x})$ are periodizers of valid tilings by $D$.

## 4 Expansivity and Determinism

We need some basic concepts of discrete geometry of $\mathbb{Z}^d \subseteq \mathbb{R}^d$. We use the notation $\langle \mathbf{u}, \mathbf{v} \rangle$ for the inner product of vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. For a non-zero vector $\mathbf{u} \in \mathbb{R}^d$ we denote

$$H_{\mathbf{u}} = \{\mathbf{x} \in \mathbb{Z}^d \mid \langle \mathbf{x}, \mathbf{u} \rangle < 0\}$$

for the open discrete half space in the direction $\mathbf{u}$. See Fig. 3 for a two-dimensional illustration.

A subshift $X$ is *deterministic* in the direction of $\mathbf{u}$ if for all $c, c' \in X$

$$c \restriction_{H_{\mathbf{u}}} = c' \restriction_{H_{\mathbf{u}}} \quad \Longrightarrow \quad c = c',$$

that is, if the contents of a configuration in the discrete half space $H_{\mathbf{u}}$ uniquely determines the contents in the rest of the cells. Note that it is enough to verify that the value $c_{\mathbf{0}}$ on the boundary of the half space is uniquely determined by $c \restriction_{H_{\mathbf{u}}}$ — the rest follows by the fact that $X$ is topologically closed and translation invariant.

The following observation is immediate and well known. It states that if a subshift has as an annihilator (or even as a periodizer) a polynomial $f$ whose negative support $-\text{Supp(f)}$ contains a unique position $\mathbf{v}$ maximally in the direction of a vector $\mathbf{u}$ (meaning that the inner product $\langle \mathbf{v}, \mathbf{u} \rangle$ has maximal value) then $X$ is deterministic in the direction of $\mathbf{u}$. In the terminology of [15], the set $-\text{Supp(f)}$ is generating for the subshift, as knowing all but one symbol of a pattern of shape $-\text{Supp(f)}$ in $\mathcal{L}(X)$

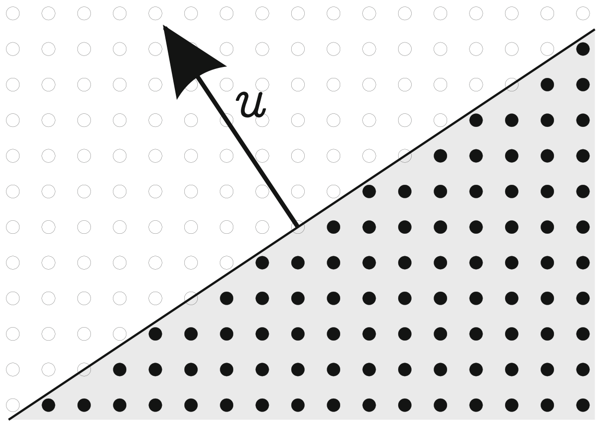


**Fig. 3** The open discrete half space $H_{\mathbf{u}}$ in dimension $d = 2$

uniquely identifies also the unknown symbol of the pattern. In Theorem 3 we generalize this lemma to the case where $-\mathrm{Supp(f)}$ contains a position with a unique (but not necessarily maximal) inner product with $\mathbf{u}$.

**Lemma 2** *Let $X$ be a $d$-dimensional subshift and let $f \in \mathrm{Per(X)}$ be such that $\mathbf{0} \in \mathrm{Supp(f)}$. Let $\mathbf{u} \in \mathbb{R}^d$ be a non-zero vector such that $-\mathrm{Supp(f)} \setminus \{\mathbf{0}\} \subseteq \mathrm{H}_{\mathbf{u}}$. Then $X$ is deterministic in the direction of $\mathbf{u}$.*

***Proof*** Let $c, c' \in X$ be such that $c \restriction_{H_{\mathbf{u}}} = c' \restriction_{H_{\mathbf{u}}}$. By replacing the polynomial $f(\mathbf{x})$ by $f(\mathbf{x})(\mathbf{x}^{\mathbf{t}} - 1)$ where $\mathbf{t} \in -H_{\mathbf{u}}$ is a common period of $f(\mathbf{x})c(\mathbf{x})$ and $f(\mathbf{x})c'(\mathbf{x})$, we may assume that $f(\mathbf{x}) \in \mathrm{Ann(c)}$ and $f(\mathbf{x}) \in \mathrm{Ann(c')}$. From

$$0 = (fc)_{\mathbf{0}} - (fc')_{\mathbf{0}} = \sum_{\mathbf{x} \in \mathrm{Supp(f)}} f_{\mathbf{x}} c_{-\mathbf{x}} - \sum_{\mathbf{x} \in \mathrm{Supp(f)}} f_{\mathbf{x}} c'_{-\mathbf{x}} = f_{\mathbf{0}} c_{\mathbf{0}} - f_{\mathbf{0}} c'_{\mathbf{0}}$$

we obtain by dividing with $f_{\mathbf{0}} \neq 0$ that $c_{\mathbf{0}} = c'_{\mathbf{0}}$. □

If a subshift $X$ is deterministic in directions $\mathbf{u}$ and $-\mathbf{u}$ then the $(d-1)$-dimensional subspace $S = \langle \mathbf{u} \rangle^{\perp} = \{\mathbf{v} \in \mathbb{R}^d \mid \langle \mathbf{u}, \mathbf{v} \rangle = 0\}$ is called an *expansive* space for $X$. Otherwise it is *non-expansive*. Using the compactness of $X$ one easily sees that the content of a configuration $c \in X$ within bounded distance from the expansive space $S$ uniquely identifies $c$: There exists $\delta > 0$ such that for all $c, c' \in X$,

$$c \restriction_B = c' \restriction_B \quad \Longrightarrow \quad c = c',$$

where $B = \cup_{\mathbf{s} \in S} B^{\delta}(\mathbf{s})$ and $B^{\delta}(\mathbf{s}) = \{\mathbf{v} \in \mathbb{Z}^d \mid \langle \mathbf{v} - \mathbf{s}, \mathbf{v} - \mathbf{s} \rangle < \delta^2\}$ is the ball of radius $\delta$ around $\mathbf{s}$ under the usual Euclidean metric. See [7] for results concerning expansive spaces of multidimensional subshifts. In particular, the following classical result from [7] is central to us, stating that if all $(d-1)$-dimensional subspaces are expansive for a $d$-dimensional subshift $X$, then $X$ contains only strongly periodic configurations. This result is our link from deterministic directions to periodicity.

**Theorem 2** ([7]) *A subshift that is deterministic in every direction is finite, and hence only contains strongly periodic configurations.*

***Example 3*** Consider a tiling $c \in \{0, 1\}^{\mathbb{Z}^3}$ of $\mathbb{Z}^3$ by translations of the tile $D = (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ from Example 1, illustrated in Fig. 1(a). Suppose that $c$ is $\mathbf{t}$-periodic for $\mathbf{t} = k(1, 1, 1)$ for some $k \in \mathbb{Z}_+$. Let us prove that $c$ is strongly periodic. Polynomials $f_D(\mathbf{x}) = 1 + x_1 + x_2 + x_3$ and $f_{-D}(\mathbf{x}) = 1 + x_1^{-1} + x_2^{-1} + x_3^{-1}$, as well as $\mathbf{x}^{\mathbf{t}} - 1$ and $\mathbf{x}^{-\mathbf{t}} - 1$ are periodizers of $c$, and hence they are also in $\mathrm{Per(X)}$ for the orbit closure $X = \overline{\mathcal{O}(c)}$ of $c$. Lemma 2 with the periodizers (in fact, annihilators) $\mathbf{x}^{\mathbf{t}} - 1$ and $\mathbf{x}^{-\mathbf{t}} - 1$ shows that $X$ is deterministic in every direction $\mathbf{u}$ that is not perpendicular to $\mathbf{t}$. Consider then any non-zero $\mathbf{u} \perp \mathbf{t}$, meaning that $\mathbf{u} = (a, b, c)$ with $a + b + c = 0$. If $a = 0$ then $\mathbf{u} = (0, b, -b)$ for $b \neq 0$. Either $(0, 1, 0)$ (if $b > 0$) or $(0, 0, 1)$ (if $b < 0$) is the unique $\mathbf{v} \in D$ with the largest inner product with $\mathbf{u}$. Thus the periodizer $\mathbf{x}^{-\mathbf{v}} f_D(\mathbf{x})$ of $X$ shows, by Lemma 2, that $X$ is deterministic in the direction $\mathbf{u}$. Cases $b = 0$ and $c = 0$ are similar. Finally, if $a$, $b$ and $c$ are all non-zero then one of them, say $a$, has different sign than the other two. Thus $\mathbf{v} = (1, 0, 0)$ is the unique element of $D$ with the maximal or the minimal inner product with $\mathbf{u}$. Hence $\mathbf{x}^{-\mathbf{v}} f_D(\mathbf{x})$ or $\mathbf{x}^{-\mathbf{v}} f_{-D}(\mathbf{x})$ confirms, by Lemma 2, that $X$ is deterministic in the direction $\mathbf{u}$. We have shown that $X$ is deterministic in every direction. By Theorem 2 all elements of $X$, including $c$, are strongly periodic.

### 4.1 A Sufficient Condition for Expansivity

Now we are ready to develop our main tool for establishing expansive spaces of a subshift with annihilators, and consequently strong periodicity of configurations. We start by noting how the special annihilator $(\mathbf{x}^{\mathbf{t}_1} - 1) \cdots (\mathbf{x}^{\mathbf{t}_m} - 1)$ provided by Theorem 1' gives that $\langle \mathbf{u} \rangle^{\perp}$ is expansive for $X$ if $\mathbf{u}$ is such that $\langle \mathbf{u}, \mathbf{t}_i \rangle \neq 0$ for all $i \in \{1, \ldots, m\}$.

**Lemma 3** *Let $X$ be a $d$-dimensional subshift and $(\mathbf{x}^{\mathbf{t}_1} - 1) \cdots (\mathbf{x}^{\mathbf{t}_m} - 1) \in \mathrm{Ann}(X)$. For every $(d-1)$-dimensional linear subspace $S \subseteq \mathbb{R}^d$, if $\mathbf{t}_i \notin S$ for all $i \in \{1, \ldots, m\}$ then $S$ is an expansive space for $X$.*

***Proof*** This is an immediate corollary of Lemma 2. Let $\mathbf{u} \in \mathbb{R}^d$ be such that $S = \langle \mathbf{u} \rangle^{\perp}$. Noting that $\mathbf{x}^{\mathbf{t}} - 1 = -\mathbf{x}^{\mathbf{t}}(\mathbf{x}^{-\mathbf{t}} - 1)$, we may replace any $\mathbf{t}_i$ by $-\mathbf{t}_i$ in the annihilator $f(\mathbf{x}) = (\mathbf{x}^{\mathbf{t}_1} - 1) \cdots (\mathbf{x}^{\mathbf{t}_m} - 1)$.

By the assumption, for all $i$ we have that $\langle \mathbf{u}, \mathbf{t}_i \rangle \neq 0$. If $\langle \mathbf{u}, \mathbf{t}_i \rangle < 0$ we replace $\mathbf{t}_i$ by $-\mathbf{t}_i$ in the annihilator $f$. So we may assume that $\langle \mathbf{u}, \mathbf{t}_i \rangle > 0$ for all $i \in \{1, \ldots, m\}$. But now the annihilator $f$ satisfies $\mathbf{0} \in \mathrm{Supp(f)}$ and $-\mathrm{Supp(f)} \setminus \{\mathbf{0}\} \subseteq \mathrm{H}_{\mathbf{u}}$, so that by Lemma 2 the subshift $X$ is deterministic in the direction of $\mathbf{u}$. Since $S = \langle -\mathbf{u} \rangle^{\perp}$ we also have determinism in the direction of $-\mathbf{u}$. □

The following theorem states a sufficient condition for expansivity in terms of annihilating (or peridizing) polynomials. It generalizes Lemma 2.

**Theorem 3** *Let $X$ be a $d$-dimensional subshift and let $S$ be a proper linear subspace of $\mathbb{R}^d$. If $f \in \mathrm{Per(X)}$ is such that*

$$\mathrm{Supp(f)} \cap \mathrm{S} = \{\mathbf{0}\} \tag{1}$$

*then there exist pairwise linearly independent $\mathbf{t}_1, \ldots, \mathbf{t}_m \in \mathbb{Z}^d$ such that $\mathbf{t}_i \notin S$ for all $i \in \{1, \ldots, m\}$ and $(\mathbf{x}^{\mathbf{t}_1} - 1) \cdots (\mathbf{x}^{\mathbf{t}_m} - 1) \in \mathrm{Ann(X)}$. In particular, if $S$ is $(d-1)$-dimensional then $S$ is expansive for $X$.*

***Proof*** Let us first prove that there exists $g \in \mathrm{Ann(X)}$ that satisfies $\mathrm{Supp(g)} \cap \mathrm{S} = \{\mathbf{0}\}$, *i.e.*, the same (1) that the periodizer $f$ satisfies. Set $Y = \{fc \mid c \in X\}$ is a subshift that only contains strongly periodic configurations. Such a subshift is finite. (This is proved in [16] for two-dimensional subshifts of finite type, but the proof directly generalizes to subshifts in any dimension $d$.) As the dimension of $S$ is at most $d - 1$, some unit coordinate vector $\mathbf{e}$ is not in $S$. Because $Y$ is a finite set of strongly periodic configurations, its elements have a common period in the direction of $\mathbf{e}$. Multiples of the period are also periods, so that there are arbitrarily large integers $k$ such that

$(\mathbf{x}^{k\mathbf{e}}-1)f(\mathbf{x}) \in \mathrm{Ann}(X)$. Because $\mathbf{e} \notin S$, for all large enough $k$ the support of $\mathbf{x}^{k\mathbf{e}} f(\mathbf{x})$ has an empty intersection with $S$. Consequently, some $g(\mathbf{x}) = (\mathbf{x}^{k\mathbf{e}}-1)f(\mathbf{x})$ satisfies $\mathrm{Supp(g)} \cap \mathrm{S} = \{\mathbf{0}\}$ and $g \in \mathrm{Ann}(X)$.

Applying Theorem 1' with the annihilator $g$ and $\mathbf{u} = \mathbf{0}$ gives the desired special annihilator $(\mathbf{x}^{\mathbf{t}_1}-1)\cdots(\mathbf{x}^{\mathbf{t}_m}-1)$, as $\mathbf{u}_i - \mathbf{u} \notin S$ for $\mathbf{u}_i \in \mathrm{Supp(g)} \setminus \{\mathbf{u}\}$. The last claim now directly follows from Lemma 3. □

Theorems 2 and 3 directly give the following tool for forced strong periodicity.

**Corollary 1** *Let $X$ be a $d$-dimensional subshift such that for every non-zero $\mathbf{u} \in \mathbb{R}^d$ there exists $f \in \mathrm{Per(X)}$ and $\mathbf{v} \in \mathrm{Supp(f)}$ such that $\langle \mathbf{v}, \mathbf{u}\rangle \neq \langle \mathbf{v}', \mathbf{u}\rangle$ for all $\mathbf{v}' \in \mathrm{Supp(f)} \setminus \{\mathbf{v}\}$. Then $X$ is finite and thus only contains strongly periodic configurations.*

***Proof*** For every $(d-1)$-dimensional subspace $S$ we take $\mathbf{u} \in \mathbb{R}^d$ such that $S = \langle \mathbf{u}\rangle^{\perp}$. Letting $f$ and $\mathbf{v}$ be as in the statement of the corollary, we have that $\mathbf{x}^{-\mathbf{v}} f(\mathbf{x})$ is a periodizer of $X$ that satisfies (1). By Theorem 3 the subspace $S$ is expansive for $X$. Since $S$ was arbitrary, the claim now follows from Theorem 2. □

We can also obtain the following corollary for lower dimensional subspaces.

**Corollary 2** *Let $X$ be a $d$-dimensional subshift, and let $k \leq d-2$. Suppose that for every $k$-dimensional linear subspace $S \subseteq \mathbb{R}^d$ there exists $f \in \mathrm{Per(X)}$ such that $\mathrm{Supp(f)} \cap \mathrm{S} = \{\mathbf{0}\}$. Then there exist $(k+1)$-dimensional linear subspaces $S_1, \ldots, S_n$, finitely many, such that every $(d-1)$-dimensional non-expansive space contains some $S_i$ as its subspace.*

***Proof*** We use mathematical induction on $k$. The base case $k = 0$ is easy: The assumption that for $S = \{\mathbf{0}\}$ there exists $f \in \mathrm{Per(X)}$ such that $\mathrm{Supp(f)} \cap \mathrm{S} = \{\mathbf{0}\}$ means that $X$ has a non-zero annihilator. By Theorem 1' there is a special annihilator $(\mathbf{x}^{\mathbf{t}_1}-1)\cdots(\mathbf{x}^{\mathbf{t}_m}-1)$. By Lemma 3, a $(d-1)$-dimensional space that does not contain any of the vectors $\mathbf{t}_i$ is expansive for $X$, so the spaces $S_i = \langle \mathbf{t}_i\rangle$ for $i \in \{1, \ldots, m\}$ satisfy the claim.

Consider then $k \geq 1$ and suppose the claim is true with $k-1$ in place of $k$. The assumption is that for every $k$-dimensional linear subspace $S \subseteq \mathbb{R}^d$ there exists $f \in \mathrm{Per(X)}$ such that $\mathrm{Supp(f)} \cap \mathrm{S} = \{\mathbf{0}\}$. Then the analogous assumption with $k-1$ in place of $k$ holds, so that by the inductive hypothesis there exist $k$-dimensional linear subspaces $S_1, \ldots, S_n$ such that every non-expansive space contains some $S_i$. By the assumption, for every $S_i$ there exists $f_i \in \mathrm{Per(X)}$ such that $\mathrm{Supp(f_i)} \cap \mathrm{S_i} = \{\mathbf{0}\}$. This means, by Theorem 3, that for every $i \in \{1, \ldots, m\}$ the subshift $X$ has a special annihilator $(\mathbf{x}^{\mathbf{t}_1^{(i)}}-1)\cdots(\mathbf{x}^{\mathbf{t}_{m_i}^{(i)}}-1)$ such that $\mathbf{t}_j^{(i)} \notin S_i$ for all $j \in \{1, \ldots, m_i\}$. Again, by Lemma 3, a $(d-1)$-dimensional space that for some $i$ does not contain any of the vectors $\mathbf{t}_j^{(i)}$ for $j \in \{1, \ldots, m_i\}$ is expansive for $X$. We conclude that every non-expansive $(d-1)$-dimensional subspace $S$ contains for some $i \in \{1, \ldots n\}$ the $k$-dimensional subspace $S_i$, and for some $j \in \{1, \ldots, m_i\}$ the vector $\mathbf{t}_j^{(i)}$. Consequently, $S$ contains the $(k+1)$-dimensional subspace generated by $S_i$ and $\mathbf{t}_j^{(i)} \notin S_i$. There are finitely many choices of $i$ and $j$. □

In particular, if a $d$-dimensional subshift $X$ has the property that for every $(d-2)$-dimensional subspace $S$ of $\mathbb{R}^d$ there exists $f \in \mathrm{Per(X)}$ that satisfies (1), then all but finitely many $(d-1)$-dimensional spaces are expansive for $X$.

### 4.2 Fibers

The existence of $f \in \mathrm{Per(X)}$ that satisfies the condition (1) can often be conveniently deduced in terms of linear combinations of "slices" of periodizers parallel to $S$. Let $S \subseteq \mathbb{R}^d$ be a fixed linear subspace. We call a Laurent polynomial $f$ an $S$-*fiber* if $\mathrm{Supp(f)} \subseteq \mathrm{S}$. Since products and sums of $S$-fibers are $S$-fibers, all $S$-fibers form a subring of the Laurent polynomial ring.

By the *restriction* of a Laurent polynomial $f$ in a subspace $S$ we mean the $S$-fiber

$$\sum_{\mathbf{u} \in \mathrm{Supp(f)} \cap \mathrm{S}} f_{\mathbf{u}} \mathbf{x}^{\mathbf{u}},$$

and we denote it by $f \restriction S$. Thus the restriction is the sum of those monomials of $f$ that lie in $S$. We are actually interested in $S$-fibers of $f$ in all "slices" parallel to $S$: for any $\mathbf{u} \in \mathbb{Z}^d$, restrict $f$ in the translated subspace $\mathbf{u} + S$ and translate the restriction by $-\mathbf{u}$ to make it an $S$-fiber. The same fiber is obtained as well by restricting the translated polynomial $\mathbf{x}^{-\mathbf{u}} f(\mathbf{x})$ in the space $S$. We call these the $S$-fibers of $f$ and denote their collection

$$\mathcal{F}_S(f) = \{(\mathbf{x}^{-\mathbf{u}} f(\mathbf{x})) \restriction S \mid \mathbf{u} \in \mathbb{Z}^d\}.$$

In the examples below we informally may call $f \restriction (\mathbf{u} + S)$ an $S$-fiber of $f$ although, more precisely, it is a monomial multiple of an $S$-fiber of $f$.

The $S$-fibers of a Laurent polynomial ideal $I$ is the set

$$I \restriction S = \{f \restriction S \mid f \in I\}$$

of the restrictions of all $f \in I$ in $S$. As for all $f \in I$ also $\mathbf{x}^{-\mathbf{u}} f(\mathbf{x})$ is in $I$, we have that the $S$-fibers of an ideal $I$ are precisely the $S$-fibers of its elements:

$$I \restriction S = \bigcup_{f \in I} \mathcal{F}_S(f).$$

Note that the $S$-fibers of a product $fg$ of two polynomials are linear combinations of $S$-fibers of $f$ (and also linear combinations of $S$-fibers of $g$). This easily implies that $I \restriction S$ is an ideal of the ring of $S$-fibers.

The condition (1) that $\mathrm{Supp(f)} \cap \mathrm{S} = \{\mathbf{0}\}$ for some element $f$ of an ideal $I$ is simply stating that $I \restriction S$ contains the monomial 1, *i.e.*, it is the complete $S$-fiber ring. In practice then, verifying this condition for the periodizer ideal $\mathrm{Per(X)}$ of a subshift amounts to expressing the monomial 1 – or any other non-zero monomial – as a linear combination of $S$-fibers of various $f \in \mathrm{Per(X)}$.

***Example 4*** Let $d = 3$ and $D = \{1, \dots, n_1\} \times \{1, \dots, n_2\} \times \{1, \dots, n_3\} \setminus \{(n_1, n_2, n_3)\}$ be a tile for some $n_1, n_2, n_3 \geq 2$. The tile is a rectangular parallelepiped of size $n_1 \times n_2 \times n_3$ with the missing corner $(n_1, n_2, n_3)$. See Fig. 4(a) for an illustration in the case $n_1 = n_2 = n_3 = 2$. Let us prove that every tiling $c \in \{0, 1\}^{\mathbb{Z}^3}$ of $\mathbb{Z}^3$ by translations of $D$ is strongly periodic. We prove this by showing that for every two-dimensional linear subspace $S = \langle \mathbf{u} \rangle^{\perp}$ the $S$-fibers of the periodizer $f_D$ generate a non-zero monomial. Then there is also a periodizer $f$ that satisfies $\mathrm{Supp(f)} \cap \mathrm{S} = \{\mathbf{0}\}$, and we can conclude strong periodicity using Corollary 1 for $X = \overline{\mathcal{O}(c)}$.

Let $\mathbf{u} = (a, b, c)$, and consider the following case analysis based on $a$, $b$ and $c$:

- If $a \neq 0$, $b \neq 0$ and $c \neq 0$, then one of the corners $(1, 1, 1)$, $(n_1, 1, 1)$, $(1, n_2, 1)$ or $(1, 1, n_3)$ of $D$ has a unique inner product with $\mathbf{u}$, and thus provides a monomial $S$-fiber of $f_D$.
- If $a \neq 0$ and $b \neq 0$, but $c = 0$, then $f(\mathbf{x}) = 1 + x_3 + x_3^2 + \cdots + x_3^{n_3}$ is one of the fibers of $f_D$. But there is also a fiber $g(\mathbf{x}) = 1 + x_3 + x_3^2 + \cdots + x_3^{n_3 - 1} + p(\mathbf{x}) f(\mathbf{x})$ given by the slice through the missing corner of $D$, where $p(\mathbf{x})$ is some polynomial capturing the positions of full columns on the same plane as the missing corner. See Fig. 4(b) for an illustration of this case. Fibers $f$ and $g$ generate a non-zero monomial $(1 + p(\mathbf{x})) f(\mathbf{x}) - g(\mathbf{x}) = x_3^{n_3}$. The cases when $a = 0$ or $b = 0$ instead of $c = 0$ are symmetric.
- Finally, consider the case $a \neq 0$ but $b = 0$ and $c = 0$. In this case $f_D$ has fibers $f(\mathbf{x}) = \sum_{i=1}^{n_2} \sum_{j=1}^{n_3} x_2^i x_3^j$ and $f(\mathbf{x}) - x_2^{n_2} x_3^{n_3}$ whose difference is a non-zero monomial. The fibers are obtained from slices not containing the missing corner, and containing the missing corner of $D$, respectively. See Fig. 4(c) for an illustration of this case. Cases where $b \neq 0$ or $c \neq 0$ instead of $a \neq 0$ are symmetric.

***Example 5*** Let $D = B_2^d$ be the radius-2 Lee sphere in dimension $d \geq 2$, defined in Example 2. Let us prove that the subspace $S = \langle (1, 1, \dots, 1) \rangle^{\perp}$ is expansive for the subshift $X = T_D$ of valid tilings of $\mathbb{Z}^d$ by $D$. Note that the direction $\mathbf{u} = (1, 1, \dots, 1)$ of determinism is perpendicular to a $(d-1)$-dimensional discrete facet of $D$, and thus it is intuitively "maximally non-deterministic" among all directions. We show that a monomial is generated by two $S$-fibers of $f_D$ corresponding to positions of $D$ having inner products 0 and 1 with $\mathbf{u} = (1, 1, \dots, 1)$. The first fiber, capturing the monomials $\mathbf{x}^{\mathbf{v}}$ for $\mathbf{v} \in D$ with $\langle \mathbf{v}, \mathbf{u} \rangle = 0$ is

$$f(\mathbf{x}) = 1 + \sum_{\substack{1 \leq i, j \leq d \\ i \neq j}} x_i x_j^{-1}.$$

The second fiber, corresponding to positions $\mathbf{v} \in D$ with $\langle \mathbf{v}, \mathbf{u} \rangle = 1$ is (a monomial multiple of)

$$g(\mathbf{x}) = \sum_{1 \leq i \leq d} x_i.$$

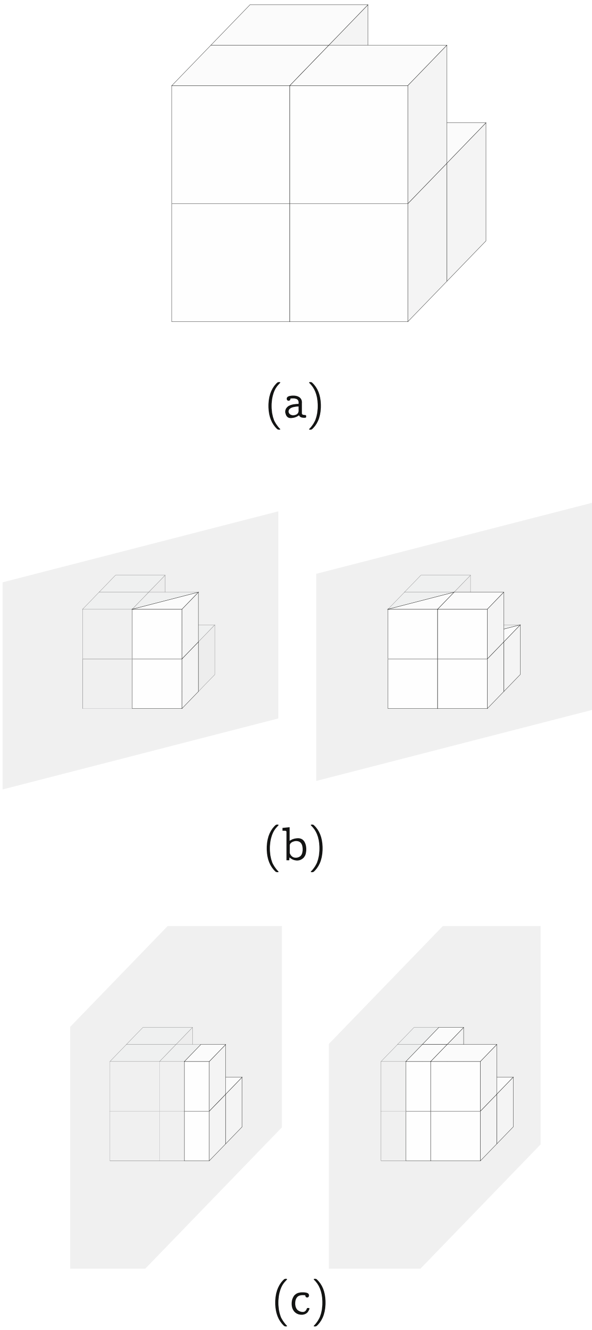


**Fig. 4** (a) The $2 \times 2 \times 2$ cube missing a corner, studied in Example 4, (b) two fibers parallel to an edge that together generate a monomial, (c) two fibers parallel to a face that generate a monomial

Because

$$(x_1^{-1} + x_2^{-1} + \dots x_d^{-1})g(\mathbf{x}) = f(\mathbf{x}) + (d-1),$$

we have that the non-zero monomial $d-1$ is an $S$-fiber of Per(X).

***Remark 2*** It remains for future research to determine whether in the case of Lee spheres $D = B_2^d$ the $S$-fibers of $f_D$ generate a non-zero monomial for all $(d-1)$-dimensional subspaces $S$. If this is the case then $D = B_2^d$ can only admit strongly periodic tilings, thus proving that the weak and the strong Golomb-Welch conjectures are equivalent for radius-2 Lee spheres.

Note that our methods show that certain subshifts can only contain strongly periodic configurations. This does not imply that there necessarily are any elements in the subshifts – the subshift may just as well be empty. For example, it is known that the Lee sphere $D = B_2^d$ considered in Example 5 does not tile $\mathbb{Z}^d$ in the cases $d \le 5$, so that in these cases Example 5 concerns the empty subshift!

***Example 6*** Let us continue with the radius-2 Lee sphere $D = B_2^3$ in dimension $d = 3$, illustrated in Fig. 2. Let us prove that for every plane $S = \langle \mathbf{u} \rangle^{\perp}$ the $S$-fibers of $f_D$ generate a non-zero monomial. This implies that valid tilings of $\mathbb{Z}^3$ by $D$ are strongly periodic. However, as pointed out above, there are no valid tilings by $D$ so this implication is uninteresting. But the result more broadly implies that all configurations $c$ that are periodized by $f_D$, not only the tilings by $D$, are strongly periodic.

Let $\mathbf{u} = (a, b, c)$. By the symmetries of $D$ we may assume that $a \ge b \ge c \ge 0$. Consider the following case analysis based on $a$, $b$ and $c$:

- If $a > b \ge c \ge 0$ then $\mathbf{v} = (2, 0, 0)$ is the unique element of $D$ such that $\langle \mathbf{v}, \mathbf{u} \rangle = 2a$, so that $\mathbf{x}^{\mathbf{v}} = x_1^2$ provides a monomial $S$-fiber.
- If $a = b > c > 0$ we take the two $S$-fibers of $f_D$ corresponding to positions of $D$ having inner products $2a$ and $a + c$ with $\mathbf{u}$. The first fiber, capturing the monomials $\mathbf{x}^{\mathbf{v}}$ for $\mathbf{v} \in D$ with $\langle \mathbf{v}, \mathbf{u} \rangle = 2a$ is (a monomial multiple) of

$$f(\mathbf{x}) = x_1^2 + x_1 x_2 + x_2^2,$$

while the second fiber, corresponding to positions $\mathbf{v} \in D$ with $\langle \mathbf{v}, \mathbf{u} \rangle = a + c$ is (a monomial multiple of)

$$g(\mathbf{x}) = x_1 x_3 + x_2 x_3.$$

Their linear combination $f(\mathbf{x}) - x_1 x_3^{-1} g(\mathbf{x}) = x_2^2$ is a monomial.

- If $a = b > c = 0$ then we need three fibers, corresponding to inner product values $2a$, $a$ and 0. The fibers are (monomial multiples of)

$$\begin{aligned} f(\mathbf{x}) &= x_1^2 + x_1 x_2 + x_2^2, \\ g(\mathbf{x}) &= x_1 + x_2 + x_1 x_3 + x_2 x_3 + x_1 x_3^{-1} + x_2 x_3^{-1}, \\ h(\mathbf{x}) &= x_3^2 + x_3 + 1 + x_3^{-1} + x_3^{-2} + x_1 x_2^{-1} + x_1^{-1} x_2. \end{aligned}$$

See Fig. 5. As a linear combination of these we obtain the fiber

$$p(\mathbf{x}) = x_1^{-2}(1 + x_3 + x_3^2) f(\mathbf{x}) - x_1^{-2} x_2 x_3 g(\mathbf{x}) = 1 + x_3 + x_3^2,$$

and then further

$$h(\mathbf{x}) - (1 + x_3^{-2}) p(\mathbf{x}) - x_1^{-1} x_2^{-1} f(\mathbf{x}) = -2,$$

a non-zero monomial.

- The case $a = b = c > 0$ was demonstrated in Example 5.

Let us finish with some remarks concerning the two-dimensional case $d = 2$. In this case our tool to infer strong periodicity of a configuration is essentially proved in [5, 6] using the structure of the annihilator and periodizer ideals. Non-monomial $S$-fibers for one-dimensional linear subspaces $S \subseteq \mathbb{R}^2$ are called *line polynomials* as they have at least two monomials and all monomials are along the same line. For any two-dimensional configuration $c$ the periodizer ideal Per(c) is known to be a principal ideal $\langle \phi_1 \phi_2 \cdots \phi_m \rangle$ generated by a product of line polynomials $\phi_i$ [5, 6]. If $c$ has a periodizer $f$ that has no line polynomial factors in any direction then from $f \in \langle \phi_1 \phi_2 \cdots \phi_m \rangle$ we conclude that $m = 0$ so that $\mathrm{Per(c)} = \langle 1 \rangle$, implying that $c$ is strongly periodic. In [4] it was noted that this fact can also be proved without referring to the structure of Per(c) simply by noting that $f$ and the special annihilator $g(\mathbf{x}) = (\mathbf{x}^{\mathbf{t}_1} - 1) \cdots (\mathbf{x}^{\mathbf{t}_m} - 1)$ guaranteed by Theorem 1 do not have any common factors as $f$ has no line polynomial factors while all irreducible factors of $g$ are line polynomials. It follows that there are non-zero linear combinations of $f$ and $g$ where either one of the two variables has been eliminated. (These are given by the resultants of $f$ and $g$ with respect to variables $x_1$ and $x_2$, respectively.) Thus there are non-zero annihilators without variables $x_1$ or $x_2$, which implies periodicity of $c$ in horizontal and vertical directions, *i.e.*, its strong periodicity..

The present paper provides a third proof of this fact that a periodizer without line polynomial factors implies strong periodicity of a two-dimensional configuration. The present proof has the advantage that it scales to higher dimensions. One should note, however, that in higher dimensions the statement cannot be given in terms of $(d-1)$-dimensional $S$-fibers not having common factors, but rather in terms of $S$-fibers generating monomial 1, *i.e.*, generating the full ring. As line polynomials are

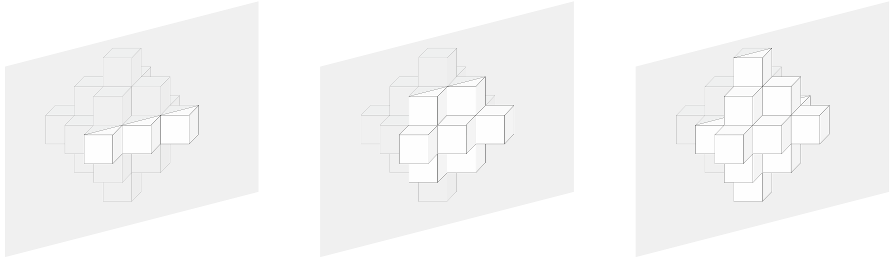

**Fig. 5** Three planes that slice fibers $f$, $g$ and $h$ in the case $a = b > c = 0$ of Example 6

essentially one-variate Laurent polynomials, the two conditions are equivalent in the two-dimensional case: a collection of one-variate Laurent polynomials generate 1 if and only if the polynomials have no non-trivial common factors. But this is no longer true for polynomials with two or more variables. (Think of $x - 1$ and $y - 1$: they have no common factors but as they have a common zero $x = 1$, $y = 1$, there is no way to express 1 as their linear combination.)

## 5 Conclusion

We have discussed a method to infer strong periodicity of a multidimensional configuration from its annihilators or periodizers. The method generalizes the two-dimensional technique used in [4–6] to arbitrary dimensions $d > 2$. The new method is in fact based on a more general condition on the annihilators or periodizers that implies expansivity of a multidimensional subshift in a given direction. We then use the well known fact that expansivity in all directions implies strong periodicity of the elements of the subshift.

We demonstrated our technique with several examples in the setup of tilings of $\mathbb{Z}^d$ by translated copies of a single tile. The famous Golomb-Welch -conjecture can be stated in this context, and we provided examples related to this conjecture. It remains an interesting topic for future research to see if our method could provide the equivalence of the weak and strong variants of the conjecture, by showing that all tilings by Lee spheres of radius $d \geq 2$ must be strongly periodic. In all the cases that we looked at, it was the case that the fibers extracted from the Lee sphere generated the monomial 1.

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**Consent to participate** Not applicable

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