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Analyzing Power in Weighted Voting Games with Super-Increasing Weights

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Abstract Weighted voting games (WVGs) are a class of cooperative games that capture settings of group decision making in various domains, such as parliaments or committees. Earlier work has revealed that the effective decision making power, or influence of agents in WVGs is not necessarily proportional to their weight. This gave rise to measures of influence for WVGs. However, recent work in the algorithmic game theory community have shown that computing agent voting power is computationally intractable. In an effort to characterize WVG instances for which polynomial-time computation of voting power is possible, several classes of WVGs have been proposed and analyzed in the literature. One of the most prominent of these are *super increasing weight sequences*. Recent papers show that when agent weights

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are super-increasing, it is possible to compute the agents' voting power (as measured by the Shapley value) in polynomial-time. We provide the first set of explicit closed-form formulas for the Shapley value for super-increasing sequences. We bound the effects of changes to the quota, and relate the behavior of voting power to a novel function. This set of results constitutes a complete characterization of the Shapley value in weighted voting games, and answers a number of open questions presented in previous work.

Keywords Weighted voting games · Shapley values

1 Introduction

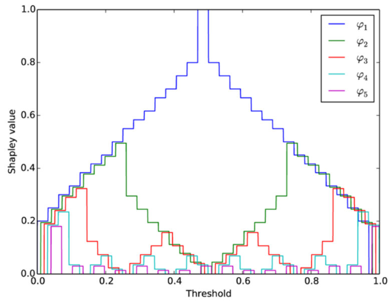
Weighted voting games (WVGs) are a class of cooperative games, commonly used to model large group decision making systems, such as parliaments. In this setting, each player i is identified with a political party; the weight of i is then the number of electoral seats its party controls. The value of a group of parties is 1 if the total number of seats they collectively control exceeds a certain threshold, and is 0 otherwise. Alternatively, one can think of each player as controlling some resource, with winning coalitions being ones that have sufficient resources in order to complete a task. One of the main challenges in the WVG setting is the measurement of *player influence*, or *power*. It is a well known fact that one's ability to affect decisions may not necessarily be proportional to one's weight. As an intuitive example, consider a parliament with three parties, A, B and C: A and B both have 50 seats, while C has 20 (a government must control a majority of the house, i.e., have at least 60 votes). If one equates voting power with weight, then A and B are significantly more powerful than C. However, a government can be formed by any two coalitions, and no single party can form a government on its own. Based on this observation, it can be reasonably argued that all parties have equal electoral power. Formal measures of voting influence, such as the *Shapley value*, aim to capture exactly these effects, providing a formal measure of player influence in WVGs. The Shapley value is considered by many to be a particularly appealing method of measuring voting power, as it satisfies several desired properties. However, it is well-known that computing the Shapley value in WVGs is computationally intractable [43]. This has naturally led to works identifying classes of WVGs for which computing voting influence is computationally tractable. In particular, an interesting sufficient condition on weights has been identified, which, if satisfied, guarantees the polynomial-time computability of the Shapley value. More formally, polynomial-time computability of the Shapley value is guaranteed if player weights are known to be *super-increasing*: a sequence of weights $w_1, \dots, w_n$ is said to be super-increasing if $w_i > \sum_{j=i+1}^{n} w_j$ for all $i \in \{1, \dots, n-1\}$ [2].

1.1 Our Contributions

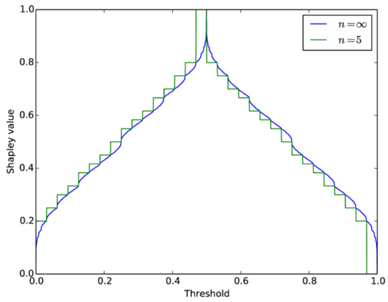
We provide a complete characterization of the Shapley values in a game in which the weights form a super-increasing sequence (Section 3). We provide a closed-form formula for the Shapley value when weights are super-increasing (extending techniques and observations on such games discussed in earlier work [2, 16, 63]). This formula is derived by exploiting an interesting relation between general super-increasing sequences, and the WVG obtained when weights are exponents of 2. We show several implications of our analysis to the results by [2] and [63], as well as a relation to a curious fractal function (Fig. 1). We significantly improve our understanding of this function, showing its various analytical properties, and its relation to Shapley values in WVGs with super-increasing weights. On a technical level, we employ several non-trivial combinatorial techniques, as well as surprising insights on the bit representation of fractions.

1.2 Related Work

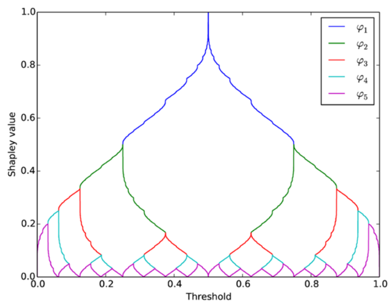
We use the Shapley value [53] to measure voting power; this follows the extensive literature in mathematical economics and, more recently, the AI community (see [17, Chapter 4] and [18] for a literature review), on measuring influence in cooperative games. The Shapley value (or as it is known in the context of voting, the Shapley–Shubik power index [54]) is the only way of measuring influence in cooperative games that satisfies certain desirable properties (see Section 2 for more details); [60]



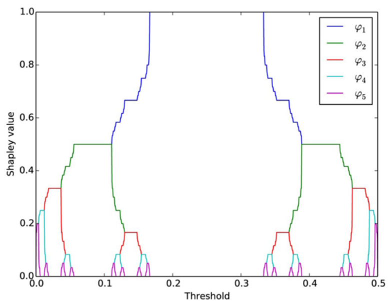
(a) Shapley values for $n = 5$, $w_i = 2^{-i}$. Values $\varphi_i(q)$ for different i are slightly nudged to show the effects of Lemma 6.



(b) Shapley values $\varphi_1(q)$ for $n = 5$, $w_i = 2^{-i}$ compared to the limiting case when n goes to ∞.



(c) Limits of the Shapley values for players $1, \ldots, 5$ in the case $w_i = 2^{-i}$, as n goes to ∞.



(d) Limits of the Shapley values for players $1, \ldots, 5$ in the case $w_i = 3^{-i}$, as n goes to ∞.

Fig. 1 Examples Shapley values corresponding to super-increasing sequences presents an alternative characterization of the Shapley value, based on a monotonicity property (see [59] for an overview).

1.2.1 Computing the Shapley Value in Specific Weight Classes

Our work generalizes several results appearing in [2, 16, 62, 63]. Aziz and Paterson [2] present algorithms for computing the Shapley value (or establish the computational intractability) for various weight types. Like this work, [2] study *super-increasing weight sequences* (though they refer to them as unbalanced weight sequences): these are weights where for every i, $w_i > \sum_{j<i} w_j$; they also study the case where each weight is an integer multiple of the weight before it. We study general super-increasing sequences of weights, showing more general algorithms and relations. Chakravarty et al. [16] study similar classes of weights, but focus on the problem of deciding whether a given player is pivotal for some set: is there a set S of players such that S is losing, but $S \cup \{i\}$ is winning? A player that does not satisfy this property for any set is called a *dummy*. Our analysis of the Shapley value in WVGs can be thought of as a *quota manipulation problem*: a parliament has a fixed set of weights (seats for each party), but the quota (the number of votes required in order to pass a bill) may be more easily changed. Is it possible for a malicious entity to choose a quota that maximizes or minimizes the influence of some specific party? Zuckerman et al. [63] consider computational aspects of quota manipulation, and establish some results for super-increasing weights, which we generalize in this paper (see Corollary 1); [62] establish some properties of the quota manipulation problem for general weight sequences, as well as some analysis of the case where weights are powers of 2 (they establish the existence of a poly-time algorithm for this case); in particular, they show that the Shapley value of a player is maximized when the quota is set to their weight, but finding a quota minimizing a player's value is computationally intractable. Zick [61] study the variance of the Shapley value as a function of the quota in WVGs, establishing both theoretical and empirical bounds on the variance. [4] analyze the expected Shapley–Shubik power index in WVGs, when players' weights are sampled from various discrete weight distributions.

1.2.2 Computing the Shapley Value

In this work, we provide closed-form formulas for the Shapley value when weights are super-increasing. The computational complexity of computing the Shapley value is a well-studied problem, with several works on either establishing its intractability. The first such attempt is probably by [42], who provide an (inefficient) algorithm for computing the Shapley–Shubik power index for large WVGs using generating functions. Another inefficient algorithm is proposed by [35], which uses multilinear extensions of WVGs. Matsui and Matsui [43] establish that computing the Shapley value for WVGs is #P complete; more precisely, they show that deciding whether a given player has no voting power is NP-complete, and that deciding whether a given player has more voting power than another given player is NP-complete. In addition, they present a dynamic-programming pseudopolynomial time algorithm for computing the Shapley value exactly: its running time is polynomial in the number of players (n) and the maximal weight of any player ($\max_i w_i$). A similar independent analysis appears in [51]. Despite their seemingly nice structure, computing solution concepts for WVGs is surprisingly hard. Elkind et al. [23] establishes the computational intractability of computing several solutions concepts for WVGs, such as the least-core, ε-core and the nucleolus of WVGs; [22] shows that computing an optimal coalition structure — i.e. a partition of the players into the maximum possible number of disjoint winning coalitions — is computationally intractable as well. In a related line of work, [24] show that given two weighted voting games, deciding whether a player has more voting power in one than in the other is computationally intractable. Given the slew of negative results on computing solution concepts for WVGs, several authors turn to approximation. The first such attempt is by [41], who present a Monte-Carlo method for computing the Shapley value; their analysis does not provide any optimality guarantees — indeed, their analysis predates results on concentration inequalities! [7] show that random sampling can be used to compute an (ε, δ) approximation of the Shapley–Shubik power index for WVGs using concentration inequalities; other sampling approaches include [25], as well as [40] who generalize the approach in [7] beyond WVGs. A similar strand of literature analyzes the behavior of the Shapley value in the face of various types of uncertainty [8, 10, 11]. Rather than trying to approximate the Shapley value, other works propose algorithms for computing it exactly for some specific class of cooperative games [3, 6, 9, 14, 20, 38, 57].

1.2.3 Measuring Influence and Power Using Power Indices

The Shapley value (or the Shapley–Shubik power index in the context of weighted voting) has been used to measure individual influence in a broad gamut of domains. Its first use case is most certainly voting; going back to canonical work of [54], many works study voting power in various political contexts (see e.g. [29, 55] and [18, 26] for an overview). We mention that the Shapley value is not the only method for measuring voting power; [13] proposed an alternative measure (later named the Banzhaf power index) (see also [21, 44]), which was also independently proposed by [49] (see [27] for an overview). Voting power is extensively studied in the context of the EU electoral system [33, 39, 55], the IMF [34], and the US Electoral College [42, 44].

Beyond voting, the Shapley value is extensively used as the standard method of dividing reveue fairly in cooperative games (see [17] for an overview). However, more recent works use the Shapley value (and other cooperative solution concepts) to measure influence in a variety of domains. These include graph centrality measures in crowdsourcing [5], proof systems [12], social network analysis [28, 37, 45, 46, 50, 57, 58], explaining the behavior of black-box decision making algorithms [19], and in biological applications [15, 30].

2 Preliminaries

We generally refer to vectors as lowercase, boldface letters and sets as uppercase letters. Given a positive integer m we denote $[m] = \{1, \ldots, m\}$.

2.1 Weighted Voting Games

A *weighted voting game* (WVG) is given by a set of agents $N = \{1, \dots, n\}$, a non-negative weight vector $\mathbf{w} = (w_1, \dots, w_n)$, where w_i is the weight of player $i \in N$ (and we let $\mathbf{w}$ denote the length-n weight vector), and a *quota* (or *threshold*) q. Thus, we refer to a WVG over N as the tuple $\langle \mathbf{w}; q \rangle$. Unless otherwise specified, we assume that $w_1 \geq \cdots \geq w_n$. For a subset of agents $S \subseteq N$ (also referred to as a *coalition*), we define $w(S) = \sum_{i \in S} w_i$.

A coalition $S \subseteq N$ is called *winning* (has a value $v(S) = 1$) if $w(S) \geq q$ and is called *losing* (has a value $v(S) = 0$) otherwise. To define the Shapley value, we require the following notation. Given a coalition $S \subseteq N$ and some $i \in N \setminus S$, we let the *marginal contribution* of i to S be

$$m_i(S) = v(S \cup \{i\}) - v(S); \tag{1}$$

for WVGs, $m_i(S) \in \{0, 1\}$, and $m_i(S) = 1$ iff $w(S) < q$ but $w(S) + w_i \geq q$. If $m_i(S) = 1$ we say that i is *pivotal* for S. Given a permutation $\sigma : N \to N$, we let $P_i(\sigma) = \{j \in N \mid \sigma(i) > \sigma(j)\}$ be the set of i's *predecessors* in σ. By letting $m_i(\sigma) = m_i(P_i(\sigma))$, we have that $m_i(\sigma) = 1$ iff i is pivotal for its predecessors in σ, in which case we simply say that i is pivotal for σ. Let Sym_n be the set of all permutations of N. The *Shapley value* of player i is the probability that i is pivotal for a randomly selected permutation $\sigma \in \mathrm{Sym}_n$:

$$\varphi_i(\mathbf{w}; q) = \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} m_i(\sigma). \tag{2}$$

For $i \in N$, we write $\varphi_i(q)$ whenever $\mathbf{w}$ is clear from the context, and assume that $q \in (0, w(N)]$ (as otherwise $\varphi_i(\mathbf{w}; q) = 0$).

2.2 The Shapley Value: a Unique Method of Value Distribution

In what follows, we provide a brief overview of the axiomatic approach that led to the formulation (and subsequent popularity) of the Shapley value. We present the axiomatic analysis here for purpose of exposition; in particular, the disinclined reader may safely skip this subsection and proceed to Section 3.

As mentioned in Section 1.2, the Shapley value is the only measure satisfying certain desirable properties. We first point out that weighted voting games are a subclass of a more general model of cooperative interaction called *cooperative games*. Much like WVGs, cooperative games are defined over a set of players $N = \{1, \dots, n\}$; in addition, there is some *characteristic function* $v : 2^N \to \mathbb{R}$ assigning a value $v(S)$ to every coalition $S \subseteq N$. Note that WVGs are cooperative games whose characteristic function takes on a very specific form (defined via $\mathbf{w}$ and q). In cooperative games, one often assumes that the coalition N (known as the *grand coalition*) forms, and the revenue $v(N)$ is to be divided amongst the players. Within this framework, the Shapley value is just one of many possible ways of dividing value amongst the players in a cooperative game. It is certainly possible to come up with many other methods (also referred to as *solution concepts*); these are mappings that take as input a cooperative game v, and output a (possibly empty) set of possible payment vectors. In addition to the Shapley value, other notable solution concepts include the *core*, the *nucleolus*, the *bargaining set* and the Banzhaf value (see [48] for an overview). A *value* is simply a mapping that takes as input a cooperative game $v: 2^N \to \mathbb{R}$, and outputs a single vector $\alpha(v) \in \mathbb{R}^n$, where the value $\alpha_i(v)$ is the payment (or value) assigned to player i. In his seminal work, Shapley [53] establishes that the value defined in (2) is the only value satisfying the following properties:

Efficiency: a value α is efficient if the total payment to all players equals $v(N)$: $\sum_{i=1}^{n} \alpha_i(v) = v(N)$.

Symmetry: two players $i, j \in N$ are symmetric if $m_i(S) = m_j(S)$ for all $S \subseteq N \setminus \{i, j\}$ (note that the definition of $m_i(S)$ in (1) still holds for general cooperative games). A value α is symmetric if $\alpha_i(v) = \alpha_j(v)$ for any two players $i, j \in N$ who are symmetric under v.

Dummy: a player $i \in N$ is a *dummy* if $m_i(S) = 0$ for all $S \subseteq N \setminus \{i\}$. A value α satisfies the *dummy* property if $\alpha_i(v) = 0$ whenever i is a dummy under v.

Additivity: A value α satisfies the *additivity* property if for any two games v, v' over N, and for any player $i \in N$, $\alpha_i(v) + \alpha_i(v') = \alpha_i(v + v')$, where $v + v'$ is the cooperative game defined with every coalition $S \subseteq N$ having a value of $v(S) + v'(S)$.

If one accepts that the above axioms make sense, then the *only* way of dividing revenue in cooperative games is using the Shapley value. Naturally, using different sets of axioms will lead one to characterizing other cooperative solution concepts. While [60] uses a different set of axioms to characterize the Shapley value, still other sets of axioms can be used to uniquely characterize the Banzhaf index [36], the nucleolus [56] and the core [47]. Axiomatic characterization of cooperative solution concepts is often useful as a means of justifying one's choice of method for revenue division (or voting power). Cooperative solution concepts can be rather obscure: the formula for the Shapley value given in (2) is not particularly meaningful in itself, and other cooperative solution concepts can have even more complex formulations. However, the axioms characterizing them are often quite easy to understand. For example, efficiency simply means that a value should divide $v(N)$, nothing more, nothing less; symmetry states that equally contributive players should be paid the same. Axiomatic analysis also provides one with a provably sound solution: any objections to the use of, say, the Shapley value, require a well-reasoned objection to one of the axioms that characterize it. Since the Shapley value axioms are all rather intuitive (the axioms proposed in [60] even more so in the authors' opinion), it has become one of the most popular methods of distributing revenue in cooperative games.

3 A Formula for the Shapley Value Under Super-Increasing Sequences

Given a vector of weights $\mathbf{w} = (w_1, \dots, w_n)$, we say that $\mathbf{w}$ is *super-increasing* (SI) if $w_i > \sum_{j=i+1}^{n} w_j$ for all $i \in \{1, \dots, n-1\}$. We henceforth assume that $\mathbf{w}$ is a super-increasing sequence.[1]

In Lemma 2, we show that computing the Shapley value for SI weight sequences is essentially equivalent to doing so for the sequence $\beta = (2^{n-1}, 2^{n-2}, \dots, 1)$ (for a subset $S \subseteq N$, recall that $\beta(S) = \sum_{i \in S} 2^{n-i}$). Given an integer value $q \in (0, 2^n - 1 = \beta(N)]$, we note that there exists a unique subset $S_q \subseteq N$ such that $\beta(S_q) = q$. Given an SI vector $\mathbf{w}$, not every number q in the range $(0, w(N)]$ can be written as a sum of members of $\{w_1, \dots, w_n\}$; however, there are certain naturally defined intervals that partition $(0, w(N)]$, as we show later in this section.

We begin by proving the following two simple lemmas.

Lemma 1 *Let $\mathbf{w}$ be an SI weight vector. For every $S, T \subseteq N$, $\beta(S) < \beta(T)$ if and only if $w(S) < w(T)$.*

Proof We first prove that $\beta(S) < \beta(T)$ implies that $w(S) < w(T)$. In order to prove this claim, it suffices to consider sets $S, T \subseteq N$ satisfying $\beta(T) = \beta(S) + 1$. Let ℓ be the agent with the smallest weight that does not belong to S; that is, $\ell = \max\{i \in N \setminus S\}$, and define $C = S \cap \{1, \dots, \ell - 1\}$; that is, C is the set of all agents in S that have weight greater than w_ℓ.

We claim that if $\beta(T) = \beta(S) + 1$, then $S = C \cup \{\ell + 1, \dots, n\}$ and $T = C \cup \{\ell\}$. Indeed, suppose that S does not contain some agent $j \in \{\ell + 1, \dots, n\}$; then ℓ is not the agent with the smallest weight that does not belong to S. The fact that $T = C \cup \{\ell\}$ is an immediate consequence of the fact that $\beta(T) = \beta(S) + 1$.

Now, $w(T) - w(S) = w_\ell - w(\{\ell + 1, \dots, n\})$; since $\mathbf{w}$ is super-increasing, it must be the case that $w_\ell > \sum_{j=\ell+1}^{n} w_j$; in particular $w(T) - w(S) > 0$, and $\beta(S) < \beta(T)$ implies $w(S) < w(T)$.

For the other direction, we arrange the 2^n subsets of $[N]$ according to β: $\emptyset = \beta(S_0) < \dots < \beta(S_{2^n-1}) = N$. The preceding argument shows that $w(S_0) < \dots < w(S_{2^n-1})$. It follows that the orders induced by β and by $\mathbf{w}$ are isomorphic, and so $\beta(S) < \beta(T)$ if and only if $w(S) < w(T)$. □

For a non-empty set of agents $S \subseteq N$, we let $S^- \subseteq N$ be the unique subset of agents satisfying $\beta(S^-) = \beta(S) - 1$. For example, assuming $n = 4$, if $S = \{1, 3, 4\}$, then $\beta(S) = 2^{4-1} + 2^{4-3} + 2^{4-4} = 2^3 + 2^1 + 2^0 = 11$; thus $S^- = \{1, 3\}$ since $\beta(\{1, 3\}) = 2^{4-1} + 2^{4-3} = 2^3 + 2^1 = 10$. Lemma 1 shows that for every quota $q \in (0, w(N)]$ there exists a *unique* set $A(q) \subseteq N$ such that q is in $(w(A(q)^-), w(A(q))]$. Whenever we write $A(q) = \{a_0, \dots, a_r\}$, we will always assume that $a_0 < \dots < a_r$.

[1]Our definition actually results in *super-decreasing* weight sequences; for consistent notation with [2, 63] and others, we refer to our sequences as super-increasing.

Lemma 2 *Given an SI vector* $\mathbf{w}$*, then for every* $i \in N$ *and* $q \in (0, w(N)]$, $\varphi_i(\mathbf{w}; q) = \varphi_i(\beta; \beta(A(q)))$.

Proof Recall that $P_i(\sigma)$ is the set of agents appearing before agent i in a given permutation $\sigma \in \mathrm{Sym}_n$. The Shapley value $\varphi_i(\mathbf{w}; q)$ is the probability that $w(P_i(\sigma)) \in [q - w_i, q)$, or equivalently, that $q \in (w(P_i(\sigma)), w(P_i(\sigma)) + w_i]$. The intervals $(w(C^-), w(C)]$ partition $(0, w(N)]$; thus q is in $(w(P_i(\sigma)), w(P_i(\sigma)) + w_i]$ if and only if $w(P_i(\sigma)) \leq w(A(q)^-)$ and $w(A(q)) \leq w(P_i(\sigma) \cup \{i\})$. Lemma 1 shows that this is equivalent to checking whether $\beta(P_i(\sigma)) \leq \beta(A(q)^-)$ and $\beta(A(q)) \leq \beta(P_i(\sigma) \cup \{i\})$. Now, note that $\beta(A(q)^-) = \beta(A(q)) - 1$, so the above condition simply states that i is pivotal for σ under β when the quota is $\beta(A(q))$. □

Lemma 2 implies that for any SI $\mathbf{w}$, computing $\varphi_i(\mathbf{w}; q)$ only requires finding $A(q)$; this can be done using Algorithm 1.

Lemma 3 *It is possible to find* $A(q)$ *in polynomial-time.*

Proof We claim that Algorithm 1 finds $A(q)$. Let $A(q) = \{a_0, \ldots, a_r\}$, so that we have $A(q)^- = \{a_0, \ldots, a_{r-1}, a_r + 1, \ldots, n\}$. Denote by A_i the value of A in the algorithm after i iterations of the loop. We prove by induction on i that $A_i = A(q) \cap \{1, \ldots, i\}$, which shows that the algorithm returns $A(q)$.

The inductive claim trivially holds for $i = 0$. Assuming that $A_{i-1} = A(q) \cap \{1, \ldots, i-1\}$, we now prove that $A_i = A(q) \cap \{1, \ldots, i\}$. We consider two cases: $i \notin A(q)$ and $i \in A(q)$. If $i \notin A(q)$ then $q \leq w(A(q)) = w(A_{i-1}) + w(A(q) \cap \{i, \ldots, n\}) \leq w(A_{i-1}) + w(\{i+1, \ldots, n\})$, and so i is not added to A_i. Suppose now that $i \in A(q)$. If $a_r = i$ then $q > w(A(q)^-) = w(A_{i-1}) + w(\{i+1, \ldots, n\})$, and so i is added to A_i. If $a_r > i$ then $q > w(A(q)^-) \geq w(A_{i-1}) + w_i > w(A_{i-1}) + w(\{i+1, \ldots, n\})$, since $\mathbf{w}$ is super-increasing, and so i is added to A_i in this case as well. □

Algorithm 1 Algorithm `Find-Set` for finding $A(q)$

Input: $\mathbf{w}$, q
$A \leftarrow \emptyset$
for $i \leftarrow 1$ ***to*** n **do**
  if $q > w(A \cup \{i+1, \ldots, n\})$ **then**
    $A \leftarrow A \cup \{i\}$
  end
end
return A

We now present our main result, a closed form formula for the Shapley values in the super-increasing case. The resulting Shapley values are illustrated in Fig. 1a.

Theorem 1 *Given an SI vector* $\mathbf{w}$ *and a threshold* q*, let* $A(q) = \{a_0, \ldots, a_r\}$.

If $i \notin A(q)$ then:

$$\varphi_i(\mathbf{w}; q) = \sum_{\substack{t \in \{0,\dots,r\}:\\ a_t > i}} \frac{1}{a_t \binom{a_t - 1}{t}}.$$

If $i \in A(q)$, say $i = a_s$, then:

$$\varphi_i(\mathbf{w}; q) = \frac{1}{a_s \binom{a_s - 1}{s}} - \sum_{t > s} \frac{1}{a_t \binom{a_t - 1}{t - 1}}.$$

Proof Lemma 2 shows that $\varphi_i(\mathbf{w}; q) = \varphi_i(\beta; \beta(A(q)))$, where $\beta = 2^{n-1}, \dots, 1$. Therefore we can assume without loss of generality that $\mathbf{w} = \beta$ and that the threshold is $q^* = \sum_{j \in A(q)} 2^{n-j}$.

Recall that $\varphi_i(\mathbf{w}; q)$ is the probability that $w(P_i(\sigma)) \in [q - w_i, q)$, where σ is chosen randomly from Sym_n, and $P_i(\sigma)$ is the set of predecessors of i in σ. The idea of the proof is to consider the maximal $\tau \in \{1, \dots, r+1\}$ such that $a_t \in P_i(\sigma)$ for all $t < \tau$. We will show that when $i \notin A(q)$, each possible value of τ corresponds to one summand in the expression for $\varphi_i(\mathbf{w}; q)$. When $i \in A(q)$, say $i = a_s$, we will show that the events that i is pivotal (w.r.t. σ) when the threshold is q and that i is pivotal when the threshold is $q - w_i$ are disjoint, and their union is an event occurring w.p. $\frac{1}{a_s \binom{a_s - 1}{s}}$.

Suppose that i is pivotal for σ. We start by showing that $\tau \le r$, ruling out the case $\tau = r + 1$. If $\tau = r + 1$ then by definition $\beta(P_i(\sigma)) \ge \sum_{j \in A(q)} 2^{n-j} = q^*$, contradicting the assumption $\beta(P_i(\sigma)) < q^*$. Thus $\tau \le r$, and so a_τ is well-defined. We claim that if $k \in P_i(\sigma)$ for some agent $k < a_\tau$ then $k \in A(q)$. Indeed, otherwise:

$$\begin{aligned}\beta(P_i(\sigma)) &\ge \sum_{t=0}^{\tau-1} 2^{n-a_t} + 2^{n-k} \ge \sum_{t=0}^{\tau-1} 2^{n-a_t} + 2^{n-a_\tau+1}\\ &\ge \sum_{t=0}^{\tau-1} 2^{n-a_t} + \sum_{j=a_\tau}^{n} 2^{n-j} \ge \beta(A(q)) = q^*,\end{aligned}$$

again contradicting $\beta(P_i(\sigma)) < q^*$; thus, if $k \in P_i(\sigma) \setminus A(q)$, then $k > a_\tau$.

Furthermore, we claim that $a_\tau \ge i$. Otherwise:

$$\begin{aligned}\beta(P_i(\sigma)) &\le \sum_{t=0}^{\tau-1} 2^{n-a_t} + \sum_{j=a_\tau+1}^{n} 2^{n-j} - 2^{n-i}\\ &< \sum_{t=0}^{\tau} 2^{n-a_t} - 2^{n-i} \le q^* - w_i,\end{aligned}$$

contradicting the assumption $w(P_i(\sigma)) \ge q^* - w_i$.

Summarizing, we have that if i is pivotal for σ, then $\tau \le r$, $a_\tau \ge i$ and

$$P_i(\sigma) \cap \{1, \dots, a_\tau\} = \{a_0, \dots, a_{\tau-1}\}. \tag{3}$$

Denote this event E_τ, and call a $\tau \le r$ satisfying $a_\tau \ge i$ *legal*.

Recall that $i \notin A(q)$; we have shown above that if i is pivotal for σ then E_τ occurs for some legal τ. We claim that the converse is also true; that is, if there exists some legal τ such that (3) holds with respect to σ, then i is pivotal for σ. Indeed, given E_τ defined with respect to a permutation σ, and for some legal τ, the weight of $P_i(\sigma)$ can be bounded as follows.

$$\sum_{t=0}^{\tau-1} 2^{n-a_t} \le \beta(P_i(\sigma)) \le \sum_{t=0}^{\tau-1} 2^{n-a_t} + \sum_{j=a_\tau+1}^{n} 2^{n-j} < \sum_{t=0}^{\tau} 2^{n-a_t},$$

where the last expression is at most q^*. The second inequality follows from the definition of τ. As $i < a_\tau$, the lower bound satisfies:

$$\sum_{t=0}^{\tau-1} 2^{n-a_t} \ge q^* - \sum_{j=a_\tau}^{n} 2^{n-j} > q^* - 2^{n-a_\tau+1} \ge q^* - 2^{n-i},$$

It remains to calculate $\Pr[E_\tau]$. The event E_τ states that the restriction of σ to $\{1, \ldots, a_\tau\}$ consists of the elements $\{a_0, \ldots, a_{\tau-1}\}$ in some order, followed by i (recall that $i \le a_\tau$). For each of the $\tau!$ possible orders, the probability of this is $1/a_\tau \cdots (a_\tau - \tau) = (a_\tau - \tau - 1)!/a_\tau!$, and so

$$\Pr[E_\tau] = \frac{\tau!(a_\tau - \tau - 1)!}{a_\tau!} = \frac{1}{a_\tau \binom{a_\tau - 1}{\tau}}. \tag{4}$$

Summing over all legal τ, we obtain the formula in the statement of the theorem. This completes the proof in the case $i \notin A(q)$.

Suppose next that $i \in A(q)$, say $i = a_s$. Since $a_\tau \ge a_s = i$ and $i \notin P_i(\sigma)$, we deduce that $\tau = s$. Therefore the event E_s happens. Conversely, when E_s happens,

$$\beta(P_i(\sigma)) \le \sum_{t=0}^{s-1} 2^{n-a_t} + \sum_{j=a_s+1}^{n} 2^{n-j} < \sum_{t=0}^{s} 2^{n-a_t} \le q^*.$$

Therefore i is pivotal (with respect to σ) if and only if E_s happens and $\beta(P_i(\sigma)) \ge q^* - 2^{n-i}$.

It is easy to check that $A(q - w_i) = A(q) \setminus \{i\} = \{a_0, \ldots, a_{s-1}, a_{s+1}, \ldots, a_r\}$. The argument above shows that if i is pivotal with respect to $q^* - 2^{n-i}$ then for some $\tau' \ge s + 1$,

$$P_i(\sigma) \cap \{1, \ldots, a_{\tau'}\} = \{a_0, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{\tau'-1}\}.$$

In particular, the event E_s happens. Conversely, when E_s happens,

$$\begin{aligned}\beta(P_i(\sigma)) &\ge \sum_{t=0}^{s-1} 2^{n-a_t} \ge q^* - 2^{n-a_s} - \sum_{j=a_s+1}^{n} 2^{n-j} \\ &> (q^* - 2^{n-a_s}) - 2^{n-a_s}.\end{aligned}$$

Therefore i is pivotal with respect to $q^* - 2^{n-i}$ if and only if E_s happens and $\beta(P_i(\sigma)) < q^* - 2^{n-i}$. We conclude that

$$\Pr[i \text{ is pivotal with respect to } q^*] = \Pr[E_s] - \Pr[i \text{ is pivotal with respect to } q^* - 2^{n-i}].$$

Above we have calculated $\Pr[E_s] = 1/a_s\binom{a_s-1}{s}$, from which the theorem follows. □

Example 1 Consider a 10 agent game where $w_i = 2^{n-i}$. Let us compute the Shapley value of agent 7 when the quota is $q = 27$. We can write $q = 16 + 8 + 2 + 1 = w_6 + w_7 + w_9 + w_{10}$, hence $A(q) = \{a_0 = 6, a_1 = 7, a_2 = 9, a_3 = 10\}$. Since agent 7 is in $A(q)$, it must be the case that: $\varphi_7(27) = \frac{1}{7\binom{6}{1}} - \frac{1}{9\binom{8}{1}} - \frac{1}{10\binom{9}{2}} \approx 0.007143$.

4 The Properties of Shapley Values Under Super-Increasing Weights

Zuckerman et al. [63] prove a nice property of super-increasing sets (Lemma 19):

Theorem 2 *(given in [63]) Suppose that $n \geq 3$; if the weights $\mathbf{w}$ are SI, then for every quota $q \in (0, w(N)]$, either $\varphi_n(q) = \varphi_{n-1}(q)$ or $\varphi_{n-1}(q) = \varphi_{n-2}(q)$.*

We generalize this result by leveraging Theorem 1. Our main technical result for this section is Lemma 6, which describes how to determine in which cases $\varphi_i(q) = \varphi_{i+1}(q)$ under super-increasing weight sequences; Corollary 1 then shows how Lemma 6 implies Theorem 2. We prove Lemma 6 using two combinatorial identities.

Lemma 4 *Let p, t be integers satisfying $p > t \geq 1$. Then*

$$\frac{1}{p\binom{p-1}{t}} + \frac{1}{p\binom{p-1}{t-1}} = \frac{1}{(p-1)\binom{p-2}{t-1}}.$$

Proof The proof is a simple calculation:

$$\begin{aligned}
\frac{1}{p\binom{p-1}{t}} + \frac{1}{p\binom{p-1}{t-1}} &= \frac{t!(p-t-1)! + (t-1)!(p-t)!}{p!} \\
&= \frac{(t-1)!(p-t-1)![t + (p-t)]}{p!} \\
&= \frac{(t-1)!(p-t-1)!}{(p-1)!} \\
&= \frac{1}{(p-1)\binom{p-2}{t-1}}.
\end{aligned}$$

□

Lemma 5 *Let p, t, k be integers satisfying $p > t \geq 0$ and $k \geq 0$. Then*

$$\frac{1}{p\binom{p-1}{t}} - \sum_{\ell=1}^{k} \frac{1}{(p+\ell)\binom{p+\ell-1}{t+\ell-1}} = \frac{1}{(p+k)\binom{p+k-1}{t+k}}.$$

In particular,

$$\frac{1}{p\binom{p-1}{t}} = \sum_{\ell=1}^{\infty} \frac{1}{(p+\ell)\binom{p+\ell-1}{t+\ell-1}}.$$

Proof The proof is by induction on k. If $k = 0$ then there is nothing to prove. For $k > 0$ we have

$$\begin{aligned}\frac{1}{p\binom{p-1}{t}} - \sum_{\ell=1}^{k} \frac{1}{(p+\ell)\binom{p+\ell-1}{t+\ell-1}} &= \frac{1}{(p+k-1)\binom{p+k-2}{t+k-1}} \\ - \frac{1}{(p+k)\binom{p+k-1}{t+k-1}} &= \frac{1}{(p+k)\binom{p+k-1}{t+k}},\end{aligned}$$

using Lemma 4. The second expression of the lemma follows from rearranging the first formula and taking the limit $k \to \infty$. □

Using Lemma 4 and Lemma 5, we are ready to prove Lemma 6.

Lemma 6 *Given a quota $q \in (0, w(N)]$, let $A(q) = \{a_0, \ldots, a_r\}$. Given some $i \in N \setminus \{n\}$,*

(a) *if $i, i+1 \in A(q)$ or $i, i+1 \notin A(q)$ then $\varphi_i(q) = \varphi_{i+1}(q)$;*
(b) *if $i \notin A(q)$ and $i+1 \in A(q)$ then $\varphi_i(q) \geq \varphi_{i+1}(q)$, with equality if and only if $i+1 = a_r$;*
(c) *if $i \in A(q)$ and $i+1 \notin A(q)$ then $\varphi_i(q) > \varphi_{i+1}(q)$.*

Proof We write $A(q) = \{a_1, \ldots, a_r\}$. Let us first assume that neither i nor $i+1$ are in $A(q)$. For every $t \in \{0, \ldots, r\}$, $a_t > i$ if and only $a_t > i+1$. Employing the formula used in Theorem 1, we have that

$$\begin{aligned}\varphi_i(q) &= \sum_{\substack{t \in \{0,\ldots,r\}: \\ a_t > i}} \frac{1}{a_t\binom{a_t-1}{t}} \\ &= \sum_{\substack{t \in \{0,\ldots,r\}: \\ a_t > i+1}} \frac{1}{a_t\binom{a_t-1}{t}} = \varphi_{i+1}(q).\end{aligned}$$

Next, if $i, i+1 \in A(q)$ then there is some s such that $i = a_s$ and $i+1 = a_{s+1}$, so:

$$\begin{aligned}
\varphi_i(q) &= \frac{1}{a_s\binom{a_s-1}{s}} - \sum_{\substack{t \in \{0,\ldots,r\}:\\ a_t > i}} \frac{1}{a_t\binom{a_t-1}{t-1}} \\
&= \frac{1}{a_s\binom{a_s-1}{s}} - \frac{1}{a_{s+1}\binom{a_{s+1}-1}{s}} - \sum_{\substack{t \in \{0,\ldots,r\}:\\ a_t > i+1}} \frac{1}{a_t\binom{a_t-1}{t-1}} \\
&= \frac{1}{a_s\binom{a_s-1}{s}} - \frac{1}{(a_s+1)\binom{a_s}{s}} - \sum_{\substack{t \in \{0,\ldots,r\}:\\ a_t > i+1}} \frac{1}{a_t\binom{a_t-1}{t-1}}
\end{aligned}$$

According to Lemma 4 this equals:

$$\begin{aligned}
\frac{1}{(a_s+1)\binom{a_s}{s+1}} - \sum_{\substack{t \in \{0,\ldots,r\}:\\ a_t > i+1}} \frac{1}{a_t\binom{a_t-1}{t-1}} &= \frac{1}{(a_{s+1})\binom{a_{s+1}-1}{s+1}} \\
&\quad - \sum_{\substack{t \in \{0,\ldots,r\}:\\ a_t > i+1}} \frac{1}{a_t\binom{a_t-1}{t-1}} = \varphi_{i+1}(q),
\end{aligned}$$

where the last equality uses Theorem 1.

Now, suppose that $i \notin A(q)$ but $i+1 \in A(q)$; writing $i+1 = a_s$, we have that $\varphi_i(q) - \varphi_{i+1}(q)$ equals

$$\begin{aligned}
&\sum_{t=s}^{r} \frac{1}{a_t\binom{a_t-1}{t}} - \left[\frac{1}{a_s\binom{a_s-1}{s}} - \sum_{t=s+1}^{r} \frac{1}{a_t\binom{a_t-1}{t-1}}\right] \\
&= \sum_{t=s+1}^{r} \left[\frac{1}{a_t\binom{a_t-1}{t}} + \frac{1}{a_t\binom{a_t-1}{t-1}}\right] = \sum_{t=s+1}^{r} \frac{1}{(a_t-1)\binom{a_t-2}{t-1}},
\end{aligned}$$

using Lemma 4 again. Therefore, $\varphi_i(q) \geq \varphi_{i+1}(q)$, with equality if and only if $s = r$.

Finally, suppose that $i \in A(q)$ and $i+1 \notin A(q)$. According to Theorem 1, $\varphi_i(q) = \frac{1}{a_s\binom{a_s-1}{s}} - \sum_{t=s+1}^{r} \frac{1}{a_t\binom{a_t-1}{t-1}}$, and $\varphi_{i+1}(q) = \sum_{a_t > i+1} \frac{1}{a_t\binom{a_t-1}{t}}$. If $s = r$, then the claim trivially holds. Since $i + 1 \notin A(q)$, it must be that $a_{s+1} \geq a_s + 2$, and in general $a_{s+\ell} \geq a_s + \ell + 1$. First, we note that $\varphi_{i+1}(q) = \sum_{a_t > i+1} \frac{1}{a_t\binom{a_t-1}{t}} \leq \sum_{t=s+1}^{r} \frac{1}{a_t\binom{a_t-1}{t}}$; therefore:

$$\begin{aligned}\varphi_i(q) - \varphi_{i+1}(q) &\geq \frac{1}{a_s\binom{a_s-1}{s}} - \sum_{t=s+1}^{r}\left(\frac{1}{a_t\binom{a_t-1}{t-1}} + \frac{1}{a_t\binom{a_t-1}{t}}\right)\\ &= \frac{1}{a_s\binom{a_s-1}{s}} - \sum_{t=s+1}^{r}\frac{1}{(a_t-1)\binom{a_t-2}{t-1}}\\ &= \frac{1}{a_s\binom{a_s-1}{s}} - \sum_{\ell=1}^{r-s}\frac{1}{(a_{s+\ell}-1)\binom{a_{s+\ell}-2}{s+\ell-1}}\end{aligned}$$

The last transition uses Lemma 4. Now, since $\frac{1}{(m-1)\binom{m-2}{k}}$ is monotone decreasing in m, and since $a_{s+\ell} \geq a_s + \ell + 1$, the last expression is at least $\frac{1}{a_s\binom{a_s-1}{s}} - \sum_{\ell=1}^{r-s} \frac{1}{(a_s+\ell-1)\binom{a_s+\ell-2}{s+\ell-1}}$ which equals $\frac{1}{(a_s+r-s)\binom{a_s+r-s-1}{r}} > 0$ by Lemma 5. □

Next, we show that Lemma 6 generalizes Theorem 2. We can, in fact, show the following stronger corollary.

Corollary 1 *Let* $\mathbf{w}$ *be a vector of SI weights. Let* $A(q) = \{a_0, \ldots, a_r\}$. *Then for all* $i \geq a_r$, *either* $\varphi_i(q) = \varphi_{i-1}(q)$, *or* $\varphi_{i-1}(q) = \varphi_{i-2}(q)$.

Proof Let $\mathbb{I}(i \in S)$ be the indicator function for the property $i \in S$. Lemma 6 states that if $\mathbb{I}(i \in A(q)) = \mathbb{I}(i - 1 \in A(q))$ we have that $\varphi_i(q) = \varphi_{i-1}(q)$, and that if $\mathbb{I}(i - 1 \in A(q)) = \mathbb{I}(i - 2 \in A(q))$ we have that $\varphi_{i-1}(q) = \varphi_{i-2}(q)$. Thus, if either holds, we are done.

Suppose that neither case holds. First, we consider the case that $i - 1 \notin A(q)$ but $i \in A(q)$. Since $i \geq a_r$, it must be the case that $a_r = i$. Invoking case (b) of Lemma 6 gives us that $\varphi_{i-1}(q) = \varphi_i(q)$.

Finally, suppose that $i - 1 \in A(q)$. This means that $i - 2, i \notin A(q)$. Since $i - 1 \in A(q)$ but $i \notin A(q)$, it must be the case that $a_r = i - 1$. We can again invoke case (b) of Lemma 6 for $i - 2$ and $i - 1$. Therefore, it must be the case that $\varphi_{i-2}(q) = \varphi_{i-1}(q)$, which concludes the proof. □

Invoking Corollary 1 with $i = n$ gives Theorem 2. We mention that it may be the case that $\varphi_{i-1}(q) < \varphi_i(q) < \varphi_{i+1}(q)$ when weights are super-increasing.

Example 2 Let us observe the 10 agent game where for all $i \in N = \{1, \ldots, 10\}$, $w_i = 2^{n-i}$. As shown in Example 1, $\varphi_7(27) \approx 0.007143$. However, $\varphi_8(27) \approx 0.005159$, and $\varphi_9(27) \approx 0.00119$. The reason that $\varphi_7(27) < \varphi_8(27) < \varphi_9(27)$ is the structure of $A(27)$. Recall that $A(27) = \{6, 7, 9, 10\}$; that is, $8 \notin A(27)$, but 7 and

9 are in $A(27)$. However, there exists an element whose index is greater than 9 in $A(27)$ (namely 10), so Corollary 1 does not hold.

Another interesting implication of Corollary 1 is the following. Suppose that $A(q) = \{a_0, \ldots, a_r\}$, then for all $i, j > a_r$, $\varphi_i(q) = \varphi_j(q)$.

It is often desirable that WVGs exhibit *separability*: if two players have different weights, then they should have different voting power. Zuckerman et al. [63] show that separability is not attainable under SI weights; Corollary 1 implies that some quotas offer more separability than others: if $A(q)$ does not consist of low-weight agents, then low-weight agents are not separable under q. For example, if weights are exponents of 2 and $q = \ell 2^{n-m}$, where ℓ is an odd number, then $\varphi_{n-m+1} = \cdots = \varphi_n(q)$. Our results allow us to bound the difference in voting power that one may achieve by changing the quota under SI weights. Recall that given a set $S \subseteq N$, S^- is the set for which $\beta(S) = \beta(S^-) + 1$. As the Shapley values are constant in the interval $(w(S^-), w(S)]$, in order to analyze the behavior of $\varphi_i(q)$, one needs only determine the rate of increase or decrease at quotas of the form $w(S)$ for $S \subseteq N$. These are given by the following lemma.

Lemma 7 *For every $S \subseteq N$, and any $i \in N$, if $i \notin S^-$ then $\varphi_i(w(S^-)) < \varphi_i(w(S))$. If $i \in S^-$ then $\varphi_i(w(S^-)) > \varphi_i(w(S))$.*

Moreover, $|\varphi_i(w(S)) - \varphi_i(w(S^-))| = \frac{1}{n}$ if one of the following holds: (a) $S = \{n\}$; (b) $i < n$ and $S = \{1, \ldots, i\}$ or $S = \{i, n\}$; or (c) $i = n$ and $S = \{n-1\}$. Otherwise, $|\varphi_i(w(S)) - \varphi_i(w(S^-))| \le \frac{1}{n(n-1)}$.

Proof Given a non-empty set $S \subseteq N$, we define $\varphi_+ = \varphi_i(w(S))$ and $\varphi_- = \varphi_i(w(S^-))$. Let $S = \{a_0, \ldots, a_r\}$. We have $S^- = \{a_0, \ldots, a_{r-1}, a_r + 1, \ldots, n\}$.

Suppose first that $i > a_r$, and let s be the index of i in the sequence S^-. According to Theorem 1, $\varphi_+ = 0$ and

$$\varphi_- = \frac{1}{i\binom{i-1}{s}} - \sum_{\ell=1}^{n-i} \frac{1}{(i+\ell)\binom{i+\ell-1}{s+\ell-1}} = \frac{1}{n\binom{n-1}{s+n-i}};$$

thus $\varphi_- > \varphi_+$. Furthermore, $|\varphi_+ - \varphi_-| \le \frac{1}{n(n-1)}$, unless $s + n - i \in \{0, n-1\}$. If $s + n - i = 0$ then $s = 0$ and $i = n$, implying $S^- = \{n\}$ and so $S = \{n-1\}$. If $s + n - i = n - 1$ then $s = i - 1$ and so $S^- = \{1, \ldots, n\}$, which is impossible.

Suppose next that $i = a_r$. Since $i = a_r$, it must be the case that $i \notin S^-$. Moreover, according to Theorem 1,

$$\varphi_+ - \varphi_- = \frac{1}{i\binom{i-1}{r}} - \sum_{\ell=1}^{n-i} \frac{1}{(i+\ell)\binom{i+\ell-1}{r+\ell-1}} = \frac{1}{n\binom{n-1}{r+n-i}}.$$

thus $\varphi_+ > \varphi_-$. Furthermore, $|\varphi_+ - \varphi_-| \le \frac{1}{n(n-1)}$ unless $r + n - i \in \{0, n-1\}$. If $r + n - i = 0$ then $r = 0$ and $i = n$, and so $S = \{n\}$. If $r + n - i = n$ then $r = i - 1$ and so $S = 1, \ldots, i$.

Finally, suppose that $i < a_r$. If $i \notin S$ then it cannot be the case that $i \in S^-$. Moreover,

$$\varphi_+ - \varphi_- = \frac{1}{a_r\binom{a_r-1}{r}} - \sum_{\ell=1}^{n-a_r} \frac{1}{(a_r+\ell)\binom{a_r+\ell-1}{r+\ell-1}} = \frac{1}{n\binom{n-1}{r+n-a_r}}.$$

so $\varphi_+ > \varphi_-$. Furthermore, $|\varphi_+ - \varphi_-| \leq \frac{1}{n(n-1)}$ unless $r + n - a_r \in \{0, n-1\}$. If $r + n - a_r = 0$ then $r = 0$ and $a_r = n$, and so $S = \{n\}$. If $r + n - a_r = n - 1$ then $a_r = r + 1$, which implies $S = \{1, \dots, r + 1\}$. However, this contradicts the assumption that $i \notin S$.

If $i < a_r$ and $i \in S$ then i must be in S^- as well. Therefore,

$$\varphi_- - \varphi_+ = \frac{1}{a_r\binom{a_r-1}{r-1}} - \sum_{\ell=1}^{n-a_r} \frac{1}{(a_r+\ell)\binom{a_r+\ell-1}{r+\ell-2}} = \frac{1}{n\binom{n-1}{r+n-a_r-1}}.$$

and $\varphi_- > \varphi_+$. Furthermore, $|\varphi_+ - \varphi_-| \leq \frac{1}{n(n-1)}$ unless $r+n-a_r-1 \in \{0, n-1\}$. If $r+n-a_r-1 = 0$ then $r = 1$ and $a_r = n$, and so $S = \{i, n\}$. If $r+n-a_r-1 = n-1$ then $a_r = r$, which is impossible. □

5 The Limiting Behavior of the Shapley Value under Super-Increasing Weights

Given a super-increasing sequence $w_1, \dots, w_n$ (where again, $w_1 > w_2 > \cdots > w_n$) and some $m \in N$, let us write $\mathbf{w}|_m$ for $(w_1, \dots, w_m)$ and $[m]$ for $\{1, \dots, m\}$. We write $\varphi_i(\mathbf{w}|_m; q)$ for the Shapley value of agent $i \in [m]$ in the weighted voting game in which the set of agents is $[m]$, the weights are $\mathbf{w}|_m$, and the quota is q. We also write $A|_m(q)$ for the set $S \subseteq [m]$ such that $q \in (w|_m(S^-), w|_m(S)]$.

The following lemma relates $\varphi_i(\mathbf{w}; q)$ and $\varphi_i(\mathbf{w}|_m; q)$.

Lemma 8 *Let $m \in N$ and $i \in [m]$, and let $q \in (0, w([m])]$. Then*

$$\varphi_i(\mathbf{w}|_m; q) = \varphi_i(\mathbf{w}; w(A|_m(q))).$$

Proof The proof makes use of Lemma 2. According to Lemma 2, $\varphi_i(\mathbf{w}; q)$ is only a function of $A(q)$. Namely, $\varphi_i(\mathbf{w}; q) = \varphi_i(\beta; \beta(A(q)))$. Now, on the one hand, $\varphi_i(\mathbf{w}|_m; q) = \varphi_i(\mathbf{b}eta; \beta(A|_m(q)))$. On the other hand, when $q = w(A|_m(q))$, then $A(q)$ under the weight vector $\mathbf{w}$ equals $A|_m(q)$. In particular, $\varphi_i(\mathbf{w}; w(A|_m(q))) = \varphi_i(\beta; \beta(A|_m(q)))$, which concludes the proof. □

Therefore the plot of $\varphi_i(\mathbf{w}|_m; q)$ (as a function of q) can be readily obtained from that of $\varphi_i(\mathbf{w}; q)$. This suggests looking at the limiting case of an *infinite* super-increasing sequence $(w_i)_{i=1}^{\infty}$, which is a sequence satisfying $w_i > 0$ and $w_i \geq \sum_{j=i+1}^{\infty} w_j$ for all $i \geq 1$. In this section we make some normalizing assumptions that will be useful. Just like in the preceding subsections, we assume that weights are arranged in decreasing order; furthermore, we assume that $w_1 = \frac{1}{2}$. This is no loss of generality: it is an easy exercise to see that given a weight vector $\mathbf{w}$

and some positive constant α, $\varphi_i(\mathbf{w}; q) = \varphi_i(\alpha\mathbf{w}; \alpha q)$. Thus, instead of the weight vector $(2^{n-1}, 2^{n-2}, \dots, 1)$, we now have $(\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^{n-1}})$. The super-increasing condition implies that the infinite sequence sums to some value $w(\infty) \leq 1$. Lemma 8 suggests how to define $\varphi_i(q)$ in this case. For $q \in (0, w(\infty))$ and $i \geq 1$, define: $\varphi_i^{(\infty)}(q) = \lim_{n\to\infty} \varphi_i(\mathbf{w}|_n; q)$.

We show that the limit exists by providing an explicit formula for it, as given in the main result of this section, Theorem 3. Under this definition, Lemma 8 easily extends to the case $n = \infty$.

Lemma 9 *Let $m \geq 1$ be an integer, let $i \in [m]$, and let $q \in (0, w([m])]$. Then $\varphi_i(\mathbf{w}|_m; q) = \varphi_i^{(\infty)}(w(A|_m(q)))$.*

Proof Lemma 8 shows that for $n \geq m$, $\varphi_i(\mathbf{w}|_m; q) = \varphi_i(\mathbf{w}|_n; w(A|_m(q)))$, and therefore $\varphi_i(\mathbf{w}|_m; q) = \lim_{n\to\infty} \varphi_i(\mathbf{w}|_n; w(A|_m(q))) = \varphi_i^{(\infty)}(w(A|_m(q)))$. □

Below, we consider possibly infinite subsets $S = \{a_0, \dots, a_r\}$ of the positive integers, ordered in increasing order; when $r = \infty$, the subset is infinite. Also, the notation $\{a, \dots, \infty\}$ (or $\{a, \dots, r\}$ when $r = \infty$) means all integers larger than or equal to a.

Given a finite sequence of integers $S = \{a_0, \dots, a_r\}$, such that $a_0 < a_1 < \dots < a_r$, we define S^- to be $\{a_0, \dots, a_{r-1}\} \cup \{a_{r+1}, \dots, \infty\}$; note the analogy to the finite case: when we had a finite sequence of agents N, S^- was the maximal weight set such that $w(S^-) < w(S)$. This is also the case for S^- as defined above. For a (possibly infinite) subset S of the positive integers, define $\beta_\infty(S) = \sum_{i\in S} 2^{-i}$. First, we show an analog of Lemma 1.

Lemma 10 *Suppose that $S, T \subseteq \mathbb{N}$ are two subsets of the positive integers. Then $\beta_\infty(S) \leq \beta_\infty(T)$ if and only if $w(S) \leq w(T)$. Further, if $\beta_\infty(S) < \beta_\infty(T)$ then $w(S) < w(T)$.*

Proof Suppose that $\beta_\infty(S) \leq \beta_\infty(T)$ and $S \neq T$. Let $i = \min_{j\in T\setminus S} j$; then

$$w(T) - w(S) \geq w_i - \sum_{j=i+1}^{\infty} w_j \geq 0.$$

Equality is only possible if $\max_{j\in T} j = i$ and $S = T \setminus \{i\} \cup \{i+1, \dots, \infty\}$. However, in that case $\beta_\infty(S) = \beta_\infty(T)$. □

There is a subtlety involved here: unlike the finite case explored in Lemma 1, we can have $\beta_\infty(S) = \beta_\infty(T)$ for $S \neq T$. This is because dyadic rationals (numbers of the form $\frac{a}{2^b}$ for some positive integer a) have two different binary expansions. For example, $\frac{1}{2} = (0.1000\dots)_2 = (0.0111\dots)_2$. The lemma states (in this case) that $w(\{1\}) \geq w(\{2, 3, 4, \dots\})$, but there need not be equality.

Next, we use the fact that any real $r \in (0, 1)$ has a binary expansion with infinitely many 0s (alternatively, a set S_r such that $\beta_\infty(S_r) = \sum_{n\in S_r} 2^{-n} = r$ and there are infinitely many $n \notin S_r$), and a binary expansion with infinitely many 1s (alternatively, a set T_r such that $\beta_\infty(T_r) = \sum_{n\in T} 2^{-n} = r$ and there are infinitely many $n \in T_r$). If r is not dyadic, then it has a unique binary expansion which has infinitely many 0s and 1s. If r is dyadic, say $r = \frac{1}{2}$, then it has one expansion $(0.1000\ldots)_2$ with infinitely many 0s and another expansion $(0.0111\ldots)_2$ with infinitely many 1s. The following lemma describes the analog of the intervals $(w(S^-), w(S)]$ in the infinite case.

Lemma 11 *Let $q \in (0, w(\infty))$. There exists a non-empty subset S of the positive integers such that either $q = w(S)$ or $S = \{a_0, \ldots, a_r\}$ is finite and $q \in (w(S^-), w(S)]$.*

Proof Since $q < w(\infty)$, there exists some finite m such that $q \le w([m])$. For any $n \ge m$, let $A|_n = A|_n(q)$. Let $Q|_n$ be the subset of $[n]$ preceding $A|_n$, and let $R|_n$ be the subset of $[n+1]$ preceding $A|_n$; here "preceding" is in the sense of $X \mapsto X^-$. The interval $(w(Q|_n), w(A|_n)]$ splits into $(w(Q|_n), w(R|_n)] \cup (w(R|_n), w(A|_n)]$, and so $A|_{n+1} \in \{R|_n, A|_n\}$. Also $\beta_\infty(A|_{n+1}) \le \beta_\infty(A|_n)$, with equality only if $A|_{n+1} = A|_n$. We consider two cases. The first case is when for some integer M, for all $n \ge M$ we have $A|_n = A = \{a_0, \ldots, a_r\}$. In that case for all $n \ge M$ $\sum_{t=0}^{r-1} w_{a_t} + \sum_{t=a_r+1}^{n} w_t < q \le \sum_{t=0}^{r} w_{a_t}$, and taking the limit $n \to \infty$ we obtain $q \in (w(A^-), w(A)]$. The other case is when $A|_n$ never stabilizes. The sequence $\beta_\infty(A|_n)$ is monotonically decreasing, and reaches a limit b satisfying $b < \beta_\infty(A|_n)$ for all n. Since $w(A|_m) \in (w(Q|_n), w(A|_n)]$ for all integers $m \ge n \ge 1$, Lemma 10 implies that $b \in [\beta_\infty(Q|_n), \beta_\infty(A|_n))$.

Let L be a subset such that $b = \beta_\infty(L)$ and there are infinitely many $i \notin L$, and define $L|_n = L \cap [n]$. We have $b \in [\beta_\infty(L|_n), \beta_\infty(L|_n) + 2^{-n})$. Thus $Q|_n = L|_n$, and so $q > w(Q|_n) = w(L|_n)$. Taking the limit $n \to \infty$, we deduce that $q \ge w(L)$. If $n \notin L$ then $A|_n = Q|_n \cup \{n\}$, and so $q \le w(A|_n) = w(L|_n) + w_n$. There are infinitely many such n, so taking the limit $n \to \infty$ we conclude that $q \le w(L)$ and so $q = w(L)$. □

We can now give an explicit formula for $\varphi_i^{(\infty)}$. We extend our notation to accommodate the notions given in Lemma 11. The proof of Theorem 3 is similar in spirit to the proof of Theorem 1, with one important subtlety: given some $q \in (0, w(\infty))$, we write $A(q) \subseteq \mathbb{N}$ to be an infinite set S such that $q = w(S)$, or the finite set S for which $q \in (w(S^-), w(S)]$. In the first case there may be *more than one set S* such that $q = w(S)$; Theorem 3 holds for any of the possible representations of q using $\mathbf{w}$.

Theorem 3 *Let $q \in (0, w(\infty))$ and let i be a positive integer. Let $A(q) = \{a_0, \ldots, a_r\}$ be the set defined in Lemma 11. Then:*

(a) *the limit $\varphi_i^{(\infty)}(q) = \lim_{n\to\infty} \varphi_i(\mathbf{w}|_n; q)$ exists.*
(b) *if $i \notin A(q)$ then*

$$\varphi_i^{(\infty)}(q) = \sum_{\substack{t \in \{0,\ldots,r\}:\\ a_t > i}} \frac{1}{a_t\binom{a_t-1}{t}}.$$

If $i \in A(q)$, say $i = a_s$, then

$$\varphi_i^{(\infty)}(q) = \frac{1}{a_s\binom{a_s-1}{s}} - \sum_{\substack{t \in \{0,\ldots,r\}:\\ a_t > i}} \frac{1}{a_t\binom{a_t-1}{t-1}}.$$

Proof We comment that the convergence of the sums in the theorem is guaranteed by Lemma 5. We again write $A(q) = S = \{a_0, \ldots, a_r\}$.

Suppose first that S is finite; according to Lemma 10, $q \in (w(S^-), w(S)]$. Let j^* be $\max_{j\in S} j$; then for all $n > j^*$, $A|_n(q) = S$, and so Lemma 8 shows that $\varphi_i^{\mathbf{w}|_n}(q) = \varphi_i^{\mathbf{w}|_{j^*}}(q)$. Therefore the limit exists and equals the stated formula, which is the same as the one given by Theorem 1. Thus, we have covered the case where S is finite.

Suppose next that S is infinite; by Lemma 10, $q = w(S)$. Consider first the case in which we can also write $q = w(Q)$ for some finite Q, say $Q = \{q_0, \ldots, q_u\}$ (think again of the case of $q = \frac{1}{2}$, which can be represented by either $\{1\}$ or $\{2, 3, 4, \ldots\}$). Then $S = \{q_0, \ldots, q_{u-1}\} \cup \{q_u + 1, q_u + 2, \ldots, \infty\}$. We now consider several cases.

If $i < q_u$ and $i \notin S$, then $i \notin Q$ and

$$\begin{aligned}\varphi_i^{(\infty)}(q) &= \sum_{\substack{t \in \{0,\ldots,u\}:\\ q_t > i}} \frac{1}{q_t\binom{q_t-1}{t}}\\ &= \sum_{\substack{t \in \{0,\ldots,u-1\}:\\ q_t > i}} \frac{1}{q_t\binom{q_t-1}{t}} + \sum_{\ell=1}^{\infty} \frac{1}{(q_u+\ell)\binom{q_u+\ell-1}{t+\ell-1}},\end{aligned}$$

using Lemma 5. The right-hand side is the expression we gave for $\varphi_i^{(\infty)}(w(S))$.

If $i < q_u$ and $i \in S$, say $i = q_s$, then $i \in Q$ and

$$\begin{aligned}\varphi_i^{(\infty)}(q) &= \frac{1}{i\binom{i-1}{s}} - \sum_{\substack{t \in \{0,\ldots,u\}:\\ q_t > i}} \frac{1}{q_t\binom{q_t-1}{t}}\\ &= \frac{1}{i\binom{i-1}{s}} - \sum_{\substack{t \in \{0,\ldots,u-1\}:\\ q_t > i}} \frac{1}{q_t\binom{q_t-1}{t}} - \sum_{\ell=1}^{\infty} \frac{1}{(q_u+\ell)\binom{q_u+\ell-1}{t+\ell-1}},\end{aligned}$$

using Lemma 5. The right-hand side is the expression we gave for $\varphi_i^{(\infty)}(w(S))$.

If $i = q_u$ then $i \in Q$ and $i \notin S$. In that case

$$\varphi_i^{(\infty)}(q) = \frac{1}{i\binom{i-1}{u}} = \sum_{\ell=1}^{\infty} \frac{1}{(i+\ell)\binom{i+\ell-1}{u+\ell-1}},$$

using Lemma 5. The right-hand side is the expression we gave for $\varphi_i^{(\infty)}(w(S))$.

Finally, if $i > q_u$ then $i \notin Q$ and $i \in S$. Suppose that i is the v-th member in S. In that case

$$\varphi_i^{(\infty)}(q) = 0 = \frac{1}{i\binom{i-1}{v}} - \sum_{\ell=1}^{\infty} \frac{1}{(i+\ell)\binom{i+\ell-1}{v+\ell-1}},$$

using Lemma 5. The right-hand side is the expression we gave for $\varphi_i^{(\infty)}(w(S))$.

It remains to consider the case in which q cannot be written as $q = w(Q)$ for finite Q. In that case, there are infinitely many positive integers n such that $n \in S$ and infinitely many such that $n \notin S$. This implies that for every positive integer n, $q \in (w(S \cap [n]), w(S \cap [n]) + w_n)$, and so $S|_n^-(q) = S \cap [n]$. Lemma 7 shows that $|\varphi_n(q) - \varphi_n(w(S \cap [n]))| \leq \frac{1}{n}$. On the other hand, Theorem 1 readily implies that $\varphi_n(w(S \cap [n]))$ tends to the expression we gave for $\varphi_i^{(\infty)}(w(S))$. We conclude that $\varphi_n(q)$ tends to the same expression. □

We conclude by showing that the limiting functions $\varphi_i^{(\infty)}$ are continuous.

Theorem 4 *Let i be a positive integer. The function $\varphi_i^{(\infty)}$ is continuous on* $(0, w(\infty))$*, and* $\lim_{q \to 0} \varphi_i^{(\infty)}(q) = \lim_{q \to w(\infty)} \varphi_i^{(\infty)}(q) = 0$.

Proof Let $q \in (0, w(\infty))$. We start by showing that $\varphi_i^{(\infty)}$ is continuous from the right at q. Lemma 11 shows that we can find a subset P such that either $q = w(P)$ or $q \in (w(P^-), w(P)]$. If $q < w(P)$ then since $\varphi_i^{(\infty)}$ is constant on $(w(P^-), w(P)]$ according to Theorem 3, clearly $\varphi_i^{(\infty)}$ is continuous from the right at q. Therefore we can assume that $q = w(P)$. Since $q < w(\infty)$, we can further assume that there are infinitely many $n \notin P$.

Suppose that we have a sequence q_j tending to q strictly from the right. For each j we can find a subset P_j such that either $q_j = w(P_j)$ or $q_j \in (w(P_j^-), w(P_j)]$. We can assume that the second case doesn't happen by replacing q_j with $w(P_j^-)$; the new sequence still tends to q strictly from the right. So we can assume that $q_j = w(P_j) > w(P)$. Let $k(j) = \min(P_j \setminus P)$, and let $l(j) > k(j)$ be the smallest index larger than $k(j)$ such that $l(j) \notin P$. Then

$$q_j - q = w(P_j) - w(P) \geq w_{k(j)} - \left(\sum_{t=k(j)+1}^{\infty} w_t - w_{l(j)} \right) \geq w_{l(j)}.$$

As $j \to \infty$, $l(j) \to \infty$ and so $k(j) \to \infty$. Therefore we can assume without loss of generality that $k(j) > i$ for all j. Theorem 3 then implies that

$$|\varphi_i^{(\infty)}(q_j) - \varphi_i^{(\infty)}(q)| \leq \sum_{s=0}^{\infty} \frac{1}{(k(j)+s)\binom{k(j)+s-1}{s}} = \frac{1}{k(j)-1},$$

using Lemma 5. Since $k(j) \to \infty$, $\varphi_i^{(\infty)}(q_j) \to \varphi_i^{(\infty)}(q)$.

We proceed to show that $\varphi_i^{(\infty)}$ is continuous from the left at q. Lemma 11 shows that we can find a subset P such that either $q = w(P)$ or $q \in (w(P^-), w(P)]$. In the second case, since $\varphi_i^{(\infty)}$ is constant on $(w(P^-), w(P)]$ according to Theorem 3, clearly $\varphi_i^{(\infty)}$ is continuous from the left at q. Therefore we can assume that $q = w(P)$. Since $q > 0$, we can further assume that there are infinitely many $n \in P$.

Suppose that we have a sequence q_j tending to q strictly from the left. For each j we can find a subset P_j such that either $q_j = w(P_j)$ or $q_j \in (w(P_j^-), w(P_j)]$, and in both cases $q_j \le w(P_j) < w(P)$. Let $k(j) = \min(P \setminus P_j)$, and let $l(j) > k(j)$ be the smallest index larger than $k(j)$ such that $l(j) \in P$. Then

$$q - q_j \ge w(P) - w(P_j) \ge w_{k(j)} + w_{l(j)} - \sum_{t=k(j)+1}^{\infty} w_t \ge w_{l(j)}.$$

At this point we can prove that $\varphi_i^{(\infty)}(q_j) \to \varphi_i^{(\infty)}(q)$ as in the preceding case.

It remains to show that $\lim_{q\to 0} \varphi_i^{(\infty)}(q) = \lim_{q\to w(\infty)} \varphi_i^{(\infty)}(q) = 0$. We start by showing that $\lim_{q\to 0} \varphi_i^{(\infty)}(q) = 0$. Let q_j be a sequence tending to 0 strictly from the right. As before, we can assume that $q_j = w(P_j)$ for each j. Let $k(j) = \min P_j$. Since $q_j \ge w_{k(j)}$, $k(j) \to \infty$. Therefore we can assume without loss of generality that $k(j) > i$ for all j. Theorem 3 then implies that

$$\varphi_i^{(\infty)}(q_j) \le \sum_{s=0}^{\infty} \frac{1}{(k(j)+s)\binom{k(j)+s-1}{s}} = \frac{1}{k(j)-1},$$

using Lemma 5. Since $k(j) \to \infty$, $\varphi_i^{(\infty)}(q_j) \to 0$.

We finish the proof by showing that $\lim_{q\to w(\infty)} \varphi_i^{(\infty)}(q) = 0$. Let q_j be a sequence tending to M strictly from the left. As before, we can find subsets P_j such that $q_j \le w(P_j)$ and $\varphi_i^{(\infty)}(q_j) = \varphi_i^{(\infty)}(w(P_j))$. Let $k(j)$ be the minimal $k \notin P_j$. Since $q_j \le w(\infty) - w_{k(j)}$, $k(j) \to \infty$. Therefore we can assume without loss of generality that $k(j) > i$ for all j. Theorem 3 implies that

$$\varphi_i^{(\infty)}(q_j) \le \frac{1}{i\binom{i-1}{i-1}} - \sum_{\ell=1}^{k(j)-1-i} \frac{1}{(i+\ell)\binom{i+\ell-1}{i+\ell-2}} = \frac{1}{k(j)-1},$$

using Lemma 5. Since $k(j) \to \infty$, $\varphi_i^{(\infty)}(q_j) \to 0$. □

Summarizing, we can extend the functions $\varphi_i(\mathbf{w}|_n; q)$ to a continuous function $\varphi_i^{(\infty)}$ which agrees with $\varphi_i(\mathbf{w}|_n; q)$ on the points $w(S)$ for $S \subseteq \{1, \dots, n\}$.

When $w_i = 2^{-i}$ the plot of $\varphi^{(\infty)}$ has no flat areas, but when $w_i = d^{-i}$ for $d > 2$, the limiting function is constant on intervals $(w(S^-), w(S)]$. This is reflected in Fig. 1d. These flat areas highlight a curious phenomenon. When $w_1 > \sum_{j=2}^{\infty} w_j$, we have $w(\{2, 3, \dots, \infty\}) < w(\{1\})$, which corresponds to the strict inequality $0.0111\ldots < 0.1$ in binary, or $0.4999\ldots < 0.5$ in decimal. The infinitesimal difference is expanded to an interval $(w(\{1\}^-), w(\{1\})]$ of non-zero width $w_1 - \sum_{j=2}^{\infty} w_j$. When $w_i > \sum_{j=i+1}^{\infty} w_j$ for all i, this phenomenon happens around every dyadic number.

6 Conclusions and Future Work

In this paper we present a series of novel results characterizing the behavior of the Shapley value in WVGs when weights are super-increasing. We derive an explicit formula for the Shapley value in this case, and use it to gain several insights, bounding the gain in value as the quota changes, and explaining our results via the behavior of an interesting fractal function. While our technical results are interesting on their own, they offer some instructive insights on the study of WVGs in the AI lens. In particular, our analysis shows that when weights follow a nice enough pattern, it is easy to infer players' voting power; this is a useful insight if one is interested in strategic manipulation of voting games via merging and weight splitting [1, 31, 32, 52]. These works analyze what happens when two parties (players) decide to merge their weights (in the context of parliaments this would be equivalent to two small parties deciding to vote together on all issues), or a bigger party deciding to split its weights in order to artificially garner more political power. Our work presents an extreme case where analyzing the effects of such manipulative behavior is easy.

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