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Covering Triangles in Edge-Weighted Graphs

Xujin Chen[1,2] · Zhuo Diao[3] · Xiaodong Hu[1,2] · Zhongzheng Tang[1,2,4]

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Abstract Let $G = (V, E)$ be a simple graph and $\mathbf{w} \in \mathbb{Z}^E_{>0}$ assign each edge $e \in E$ a positive integer weight $w(e)$. A subset of E that intersects every triangle of G is called a triangle cover of $(G, \mathbf{w})$, and its weight is the total weight of its edges. A collection of triangles in G (repetition allowed) is called a triangle packing of $(G, \mathbf{w})$ if each edge $e \in E$ appears in at most $w(e)$ members of the collection. Let $\tau_t(G, \mathbf{w})$ and $\nu_t(G, \mathbf{w})$ denote the minimum weight of a triangle cover and the maximum cardinality of a triangle packing of $(G, \mathbf{w})$, respectively. Generalizing Tuza's conjecture for unit weight, Chapuy et al. conjectured that $\tau_t(G, \mathbf{w})/\nu_t(G, \mathbf{w}) \leq 2$ holds for every simple graph G and every $\mathbf{w} \in \mathbb{Z}^E_{>0}$. In this paper, using a hypergraph approach, we design polynomial-time combinatorial algorithms for finding triangle covers of small weights. These algorithms imply new sufficient conditions for the conjecture of Chapuy et al. More precisely, given $(G, \mathbf{w})$, suppose that all edges of G are covered by the set $\mathscr{T}_G$ consisting of edge sets of triangles in G. Let $|E|_w = \sum_{e\in E} w(e)$ and $|\mathscr{T}_G|_w = \sum_{\{e,f,g\}\in\mathscr{T}_G} w(e)w(f)w(g)$ denote the weighted numbers of edges and triangles in $(G, \mathbf{w})$, respectively. We show that a triangle cover of $(G, \mathbf{w})$ of weight

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✉ Zhuo Diao
diaozhuo@amss.ac.cn

1 Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China

2 School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, China

3 School of Statistics and Mathematics, Central University of Finance and Economics, Beijing, China

4 Department of Computer Science, City University of Hong Kong, HKSAR, Kowlon Tong, China

at most $2\nu_t(G, \mathbf{w})$ can be found in strongly polynomial time if one of the following conditions is satisfied: (i) $\nu_t(G, \mathbf{w})/|\mathscr{T}_G|_w \geq \frac{1}{3}$, (ii) $\nu_t(G, \mathbf{w})/|E|_w \geq \frac{1}{4}$, (iii) $|E|_w/|\mathscr{T}_G|_w \geq 2$.

Keywords Triangle cover · Triangle packing · Linear 3-uniform hypergraphs · Combinatorial algorithms

1 Introduction

Graphs considered in this paper are undirected, finite and may have multiple edges. Given a graph $G = (V, E)$ with vertex set $V(G) = V$ and edge set $E(G) = E$, a *triangle* in G is a 3-vertex complete subgraph. For convenience, we often identify a triangle in G with its edge set. A subset of E is called a *triangle cover* if it intersects each triangle of G. Let $\tau_t(G)$ denote the minimum cardinality of a triangle cover of G, referred to as the *triangle covering number* of G. A set of pairwise edge-disjoint triangles in G is called a *triangle packing* of G. Let $\nu_t(G)$ denote the maximum cardinality of a triangle packing of G, referred to as the *triangle packing number* of G. It is clear that $1 \leq \tau_t(G)/\nu_t(G) \leq 3$ holds for every graph G. Our research is motivated by the following conjecture of Tuza [1], and its weighted generalization of Chapuy et al. [2].

Conjecture 1.1 (Tuza's Conjecture [1]) $\tau_t(G)/\nu_t(G) \leq 2$ holds for every simple graph G.

To the best of our knowledge, the conjecture is still unsolved in general. If it is true, then the upper bound 2 is sharp as shown by K_4 and K_5 – the complete graphs of orders 4 and 5.

Related Work The only known universal upper bound smaller than 3 was given by Haxell [3], who showed that $\tau_t(G)/\nu_t(G) \leq 66/23 = 2.8695\ldots$ holds for all simple graphs G. Haxell's proof [3] implies a polynomial-time algorithm for finding a triangle cover of cardinality at most 66/23 times that of a maximal triangle packing. Other partial results on Tuza's conjecture concern special classes of graphs.

Tuza [4] proved his conjecture holds for planar simple graphs, K_5-free chordal simple graphs, and simple graphs with n vertices and at least $7n^2/16$ edges. The proof for planar graphs [4] gives an elegant polynomial-time algorithm for finding a triangle cover in planar simple graphs with cardinality at most twice that of a maximal triangle packing. The validity of Tuza's conjecture on the class of planar graphs was later generalized by Krivelevich [5] to the class of simple graphs without $K_{3,3}$-subdivision. Haxell and Kohayakawa [6] showed that $\tau_t(G)/\nu_t(G) \leq 2 - \epsilon$ for tripartite simple graphs G, where ϵ is about 0.044. Haxell, Kostochka and Thomassé [7] proved that every K_4-free planar simple graph G satisfies $\tau_t(G)/\nu_t(G) \leq 1.5$.

Regarding the tightness of the conjectured upper bound 2, Tuza [4] noticed that infinitely many simple graphs G attain the conjectured upper bound $\tau_t(G)/\nu_t(G) = 2$. Cui, Haxell and Ma [8] characterized planar simple graphs G satisfying

$\tau_t(G)/\nu_t(G) = 2$; these graphs are edge-disjoint unions of K_4's plus possibly some vertices and edges that are not in triangles. Baron and Kahn [9] proved that Tuza's conjecture is asymptotically tight for dense simple graphs.

Fractional and weighted variants of Conjecture 1.1 were studied in literature. Krivelevich [5] proved two fractional versions of the conjecture: $\tau_t(G) \leq 2\nu_t^*(G)$ and $\tau_t^*(G) \leq 2\nu_t(G)$, where $\tau_t^*(G)$ and $\nu_t^*(G)$ are the values of an optimal fractional triangle cover and an optimal fractional triangle packing of simple graph G, respectively. The result was generalized by Chapuy et al. [2] to the weighted version. Given a simple graph G and a positive integer edge weight function $\mathbf{w} \in \mathbb{Z}_{>0}^{E(G)}$, the *weight* of any subset S of $E(G)$ is the total weight of its edges. A *triangle packing of* $(G, \mathbf{w})$ refers to a collection of triangles in G (repetition allowed) such that each edge $e \in E(G)$ appears in at most $w(e)$ members of the collection. Let $\tau_t(G, \mathbf{w})$ and $\nu_t(G, \mathbf{w})$ denote the minimum weight of a triangle cover and the maximum cardinality of a triangle packing of $(G, \mathbf{w})$, respectively. The values of $\tau_t(G, \mathbf{w})$ and $\nu_t(G, \mathbf{w})$ are often referred to as the *weighted triangle covering number* and *weighted triangle packing number* of G, respectively. Observe that $1 \leq \tau_t(G, \mathbf{w})/\nu_t(G, \mathbf{w}) \leq 3$ holds for every weighted graph $(G, \mathbf{w})$. Chapuy et al. [2] studied the following weighted (version of) Tuza's conjecture:

Conjecture 1.2 ([2]) $\tau_t(G, \mathbf{w})/\nu_t(G, \mathbf{w}) \leq 2$ holds for every simple graph G and every weight function $\mathbf{w} \in \mathbb{Z}_{>0}^{E(G)}$.

The authors [2] showed that $\tau_t(G, \mathbf{w}) \leq 2\nu_t^*(G, \mathbf{w}) - \sqrt{\nu_t^*(G, \mathbf{w})/6} + 1$ and $\tau_t^*(G, \mathbf{w}) \leq 2\nu_t(G, \mathbf{w})$, where $\tau_t^*(G, \mathbf{w})$ and $\nu_t^*(G, \mathbf{w})$ are the (equal) values of an optimal fractional triangle cover and an optimal fractional triangle packing of $(G, \mathbf{w})$, respectively, for which $\tau_t(G, \mathbf{w}) \geq \tau_t^*(G, \mathbf{w}) = \nu_t^*(G, \mathbf{w}) \geq \nu_t(G, \mathbf{w})$ is guaranteed by the linear programming (LP) duality. Their arguments imply an LP-based 2-approximation algorithm for finding a minimum weighted triangle cover.

Our Contributions Along a different line, we establish new sufficient conditions for validity of (weighted) Tuza's conjecture by comparing the (weighted) triangle packing number, the (weighted) number of triangles and the (weighted) number of edges in graphs.

Given a graph G, we use $\mathscr{T}_G = \{E(T) : T \text{ is a triangle in } G\}$ to denote the set consisting of the (edge sets of) triangles in G. Without loss of generality, we focus on the graphs in which every edge is contained in some triangle. These graphs are called *irreducible*.

Theorem 1.3 *Let $G = (V, E)$ be an irreducible graph. Then a triangle cover of G with cardinality at most $2\nu_t(G)$ can be found in polynomial time, which implies $\tau_t(G) \leq 2\nu_t(G)$, if one of the following conditions is satisfied:*

(i) $\nu_t(G)/|\mathscr{T}_G| \geq \frac{1}{3}$,
(ii) $\nu_t(G)/|E| \geq \frac{1}{4}$,
(iii) $|E|/|\mathscr{T}_G| \geq 2$.

The primary idea behind the theorem is simple: any one of conditions (i) – (iii) allows us to remove at most $\nu_t(G)$ edges from G to make the resulting graph G' satisfy $\tau_t(G') = \nu_t(G')$; the removed edges and the edges in a minimum triangle cover of G' form a triangle cover of G with size at most $\nu_t(G) + \nu_t(G') \leq 2\nu_t(G)$. The idea is realized by establishing new results on (linear) 3-uniform hypergraphs (see Section 2); the most important one states that such a hypergraphs could be made acyclic by removing a number of vertices that is no more than a third of the number of its edges. A key observation here is that hypergraph $(E, \mathscr{T}_G)$ is 3-uniform, and it is linear when G is simple.

It is worthwhile pointing out that strengthening Theorem 1.3, our arguments actually establish stronger results for 3-uniform hypergraphs (see Theorem 4.1).

Theoretically, weighted triangle packing and covering in $(G, \mathbf{w})$ amounts to (unweighted) triangle packing and covering in multigraph G_w which is obtained from G by replacing each edge $e \in E(G)$ with a number $w(e)$ of multiple edges.[1] However, from an algorithmic point of view, polynomial-time solvability of triangle packing and covering in G_w does not necessarily imply the same for $(G, \mathbf{w})$. By more careful consideration on edge weights and utilization of unique properties of $(E, \mathscr{T}_G)$, we ensure *strong polynomial-time computation* for weighted graphs.

Theorem 1.4 *Let $G = (V, E)$ be an irreducible simple graph, $\mathbf{w} \in \mathbb{Z}^E_{>0}$, $|E|_w = \sum_{e\in E} w(e)$ and $|\mathscr{T}_G|_w = \sum_{\{e,f,g\}\in\mathscr{T}_G} w(e)w(f)w(g)$. Then a triangle cover of (G, w) with weight at most $2\nu_t(G, \mathbf{w})$ can be found in strongly polynomial time, which implies $\tau_t(G, \mathbf{w}) \leq 2\nu_t(G, \mathbf{w})$, if one of the following conditions is satisfied:*

(i) $\nu_t(G, \mathbf{w})/|\mathscr{T}_G|_w \geq \frac{1}{3}$,
(ii) $\nu_t(G, \mathbf{w})/|E|_w \geq \frac{1}{4}$,
(iii) $|E|_w/|\mathscr{T}_G|_w \geq 2$.

When $\mathbf{w} = \mathbf{1}$, Theorem 1.4 is nothing but Theorem 1.3. To show the quality of conditions (i) – (iii) in Theorems 1.3 and 1.4, we obtain the following result which complements to the constants $\frac{1}{3}$, $\frac{1}{4}$ and 2 in these conditions with $\frac{1}{4}$, $\frac{1}{5}$ and $\frac{3}{2}$, respectively.

Theorem 1.5 *Tuza's conjecture holds for every simple graph (resp. multigraph) if there exists some real $\delta > 0$ such that Tuza's conjecture holds for every irreducible simple graph (resp. multigraph) G satisfying one of the following properties:*

(i') $\nu_t(G)/|\mathscr{T}_G| \geq \frac{1}{4} - \delta$,
(ii') $\nu_t(G)/|E| \geq \frac{1}{5} - \delta$,
(iii') $|E|/|\mathscr{T}_G| \geq \frac{3}{2} - \delta$.

Note that the statement of Theorem 1.5 for multigraphs can be converted to an equivalent weighted version concerning with edge-weighted simple graphs.

[1] Sometimes we use the term "multigraph" to emphasize that the graph under investigation might not be simple.

This paper turns out to be a complete generalization and strengthening of the recent work on unweighted simple graphs [10]. In the present paper, we study triangle packing and covering not only for *general multigraphs*, but also for *weighted* simple graphs from an algorithmic point of view. The strong polynomial-time algorithms we design for the weighted case exhibit nice combinatorial properties. These efficient algorithms strengthen the theoretical equivalence between packing and covering triangles in multigraphs and that in weighted simple graphs. Regarding hypergraph theory, we generalize previous results on linear 3-uniform graphs [10] to the ones without the linearity requirement (see Section 2.1). In addition, we establish a new upper bound on the transversal number of connected linear 3-uniform hypergraphs (Theorem 2.10); the proof implies a *faster* 2-approximation algorithm for finding minimum triangle cover in simple graph G provided $\nu_t(G)/|\mathscr{T}_G| \geq 1/3$ (see Algorithm 2, Corollaries 2.11 and 3.10). Moreover, as an application of condition (ii) in Theorem 1.3, we investigate Tuza's conjecture on the classical Erdős-Rényi random graph $\mathcal{G}(n, p)$, and prove that $\mathbf{Pr}[\tau_t(G)/\nu_t(G) \leq 2] = 1 - o(1)$ provided $G \in \mathcal{G}(n, p)$ and $p > \sqrt{3}/2$ (Theorem 3.14).

The rest of paper is organized as follows. Section 2 proves theoretical and algorithmic results on 3-uniform hypergraphs concerning feedback sets and transversals, which are main technical tools for establishing new sufficient conditions for (weighted) Tuza's conjecture in Section 3. Section 4 concludes the paper with extensions and future research directions.

2 Hypergraphs

This section develops hypergraph tools for studying Tuza's conjecture. The theoretical and algorithmic results are of interest in their own right.

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. For convenience, we use $\|\mathcal{H}\|$ to denote the number $|\mathcal{E}|$ of edges in $\mathcal{H}$. If a hypergraph $\mathcal{H}' = (\mathcal{V}', \mathcal{E}')$ satisfies $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$, we call $\mathcal{H}'$ a *subhypergraph* of $\mathcal{H}$, and write $\mathcal{H}' \subseteq \mathcal{H}$. For each $v \in \mathcal{V}$, the *degree* $d_{\mathcal{H}}(v)$ is the number of edges in $\mathcal{E}$ that contain v. We say v is an *isolated vertex* of $\mathcal{H}$ if $d_{\mathcal{H}}(v) = 0$. Let $k \in \mathbb{Z}_{>0}$ be a positive integer. A hypergraph $\mathcal{H}$ is called *k-regular* if $d_{\mathcal{H}}(u) = k$ for each $u \in \mathcal{V}$, and *k-uniform* if $|e| = k$ for each $e \in \mathcal{E}$. A hypergraph $\mathcal{H}$ is *linear* if $|e \cap f| \leq 1$ for any pair of distinct edges $e, f \in \mathcal{E}$.

A vertex-edge alternating sequence $v_1e_1v_2 \cdots v_ke_kv_{k+1}$ of $\mathcal{H}$ is called a *path* (of *length* k) between v_1 and v_{k+1} if $v_1, v_2, \ldots, v_{k+1} \in \mathcal{V}$ are distinct, $e_1, e_2, \ldots, e_k \in \mathcal{E}$ are distinct, and $\{v_i, v_{i+1}\} \subseteq e_i$ for each $i \in [k] = \{1, \ldots, k\}$. For each $i \in [k]$, edges e_i and e_{i+1} are *consecutive*, where $e_{k+1} = e_1$. We consider each vertex of $\mathcal{H}$ as a path of length 0. A hypergraph $\mathcal{H}$ is said to be *connected* if there is a path between any pair of distinct vertices in $\mathcal{H}$. A maximal connected subhypergraph of $\mathcal{H}$ is called a *component* of $\mathcal{H}$. Obviously, $\mathcal{H}$ is connected if and only if it has only one component.

A vertex-edge alternating sequence $\mathcal{C} = v_1e_1v_2e_2 \cdots v_ke_kv_1$, where $k \geq 2$, is called a *cycle* (of length k) if $v_1, v_2, \ldots, v_k \in \mathcal{V}$ are distinct, $e_1, e_2, \ldots, e_k \in \mathcal{E}$ are distinct, and $\{v_i, v_{i+1}\} \subseteq e_i$ for each $i \in [k]$, where $v_{k+1} = v_1$. We consider the cycle $\mathcal{C}$ as a subhypergraph of $\mathcal{H}$ with vertex set $\cup_{i\in[k]}e_i$ and edge set $\{e_i : i \in [k]\}$. We call vertices $v_1, v_2, \ldots, v_k$ *join vertices* of $\mathcal{C}$, and the other vertices *non-join vertices* of $\mathcal{C}$. For any $\mathcal{S} \subset \mathcal{V}$ (resp. $S \subset \mathcal{E}$), we write $\mathcal{H} \setminus \mathcal{S}$ for the subhypergraph of $\mathcal{H}$ obtained from $\mathcal{H}$ by deleting all vertices in $\mathcal{S}$ and all edges incident with some vertices in $\mathcal{S}$ (resp. deleting all edges in $\mathcal{S}$ and keeping all the vertices). If $\mathcal{S}$ is a singleton set $\{s\}$, we write $\mathcal{H} \setminus s$ instead of $\mathcal{H} \setminus \{s\}$. For any $\mathcal{S} \subseteq 2^{\mathcal{V}}$, the hypergraph $(\mathcal{V}, \mathcal{E} \cup \mathcal{S})$ is often written as $\mathcal{H} \uplus \mathcal{S}$, and as $\mathcal{H} \oplus \mathcal{S}$ if $\mathcal{S} \cap \mathcal{E} = \emptyset$. A connected hypergraph without any cycle is called a *tree*.

This section is divided into three subsections, discussing feedback sets of 3-uniform hypergraphs, weighted hypergraphs, and transversals (i.e., vertex sets covering all edges) of linear 3-uniform hypergraphs, respectively.

2.1 Feedback Sets

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. A vertex (resp. edge) subset of $\mathcal{H}$ is called a *feedback vertex set* or FVS (resp. *feedback edge set* or FES) of $\mathcal{H}$ if it intersects the vertex (resp. edge) set of every cycle of $\mathcal{H}$. A vertex subset of $\mathcal{H}$ is called a *quasi-FVS* of $\mathcal{H}$ if it intersects every cycle of length at least 3 in $\mathcal{H}$. Let $\tau_c^{\mathcal{V}}(\mathcal{H})$, $\tau_c^{\mathcal{E}}(\mathcal{H})$ and $\tau_\circ^{\mathcal{V}}(\mathcal{H})$ denote, respectively, the minimum cardinalities of a FVS, a FES, and a quasi-FVS of $\mathcal{H}$. It is easy to see that $\tau_\circ^{\mathcal{V}}(\mathcal{H}) \le \tau_c^{\mathcal{V}}(\mathcal{H}) \le \tau_c^{\mathcal{E}}(\mathcal{H})$.

Our discussion will frequently use the trivial observation that if no cycle of $\mathcal{H}$ contains any element of some subset $\mathcal{S}$ (with $\mathcal{S} \subset \mathcal{V}$ or $\mathcal{S} \subset \mathcal{E}$), then $\mathcal{H}$ and $\mathcal{H} \setminus \mathcal{S}$ have the same set of quasi-FVS's, and $\tau_\circ^{\mathcal{V}}(\mathcal{H}) = \tau_\circ^{\mathcal{V}}(\mathcal{H} \setminus \mathcal{S})$. The following theorem is one of main contributions of this paper.

Theorem 2.1 *Let $\mathcal{H}$ be a 3-uniform hypergraph. Then $\tau_\circ^{\mathcal{V}}(\mathcal{H}) \le \|\mathcal{H}\|/3$.*

Proof Suppose that the theorem failed. We take a counterexample $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $\tau_\circ^{\mathcal{V}}(\mathcal{H}) > |\mathcal{E}|/3$ such that $\|\mathcal{H}\| = |\mathcal{E}|$ is as small as possible. Obviously $|\mathcal{E}| \ge 3$. Without loss of generality, we can assume that $\mathcal{H}$ has no isolated vertices.

If there exists $e \in \mathcal{E}$ which does not belong to any cycle of $\mathcal{H}$ with length at least 3, then $\tau_\circ^{\mathcal{V}}(\mathcal{H}) = \tau_\circ^{\mathcal{V}}(\mathcal{H} \setminus e)$. The minimality of $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ implies $\tau_\circ^{\mathcal{V}}(\mathcal{H} \setminus e) \le (|\mathcal{E}| - 1)/3$, giving $\tau_\circ^{\mathcal{V}}(\mathcal{H}) < |\mathcal{E}|/3$, a contradiction. So we have

(1) Every edge in $\mathcal{E}$ is contained in some cycle of $\mathcal{H}$ with length at least 3.

If there exists $v \in \mathcal{V}$ with $d_{\mathcal{H}}(v) \ge 3$, then $\tau_\circ^{\mathcal{V}}(\mathcal{H} \setminus v) \le (|\mathcal{E}| - d_{\mathcal{H}}(v))/3 \le (|\mathcal{E}| - 3)/3$, where the first inequality is due to the minimality of $\mathcal{H}$. Given a minimum quasi-FVS $\mathcal{S}$ of $\mathcal{H} \setminus v$, it is clear that $\mathcal{S} \cup \{v\}$ is a quasi-FVS of $\mathcal{H}$ with size $|\mathcal{S}| + 1 = \tau_\circ^{\mathcal{V}}(\mathcal{H} \setminus v) + 1 \le |\mathcal{E}|/3$, a contradiction to $\tau_\circ^{\mathcal{V}}(\mathcal{H}) > |\mathcal{E}|/3$. So we have

(2) $d_{\mathcal{H}}(v) \le 2$ for all $v \in \mathcal{V}$.

If $\mathcal{H}$ has a pair of distinct edges $e = \{t, u, v\}$ and f sharing two common vertices u, v, then from (2) we see that u and v are incident with e and f only. In view of (1), considering a cycle of length at least 3 that goes through e, we deduce that this cycle goes through an edge $g \in \mathcal{E} - \{e, f\}$ which is incident with e at vertex t. See Fig. 1 for an illustration. Let $\mathcal{S}$ be a minimum quasi-FVS of $\mathcal{H}' = \mathcal{H} \setminus \{e, f, g\}$. It follows from (2) that $\mathcal{H} \setminus t \subseteq \mathcal{H} \setminus \{e, g\} = \mathcal{H}' \oplus f$, and in $\mathcal{H} \setminus \{e, g\}$, edge f intersects at

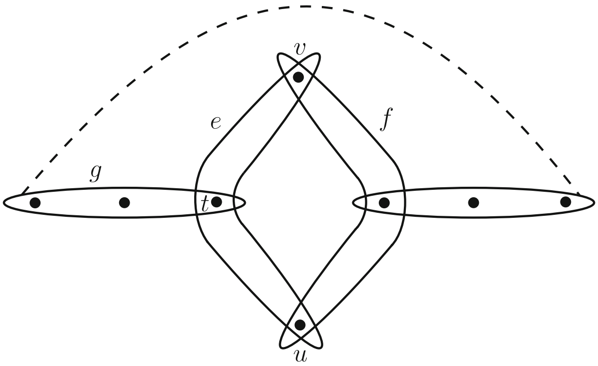


Fig. 1 The case of nonlinearity

most one other edge, which avoids u and v. Therefore f is not contained in any cycle in $\mathcal{H} \setminus \{e, g\}$. Thus $\mathcal{S}$ is a quasi-FVS of $\mathcal{H} \setminus \{e, g\}$, and hence a quasi-FVS of $\mathcal{H} \setminus t$. So $\{t\} \cup \mathcal{S}$ is a quasi-FVS of $\mathcal{H}$. We deduce that $|\mathcal{E}|/3 < \tau_\circ^{\mathcal{V}}(\mathcal{H}) \leq |\{t\} \cup \mathcal{S}| \leq 1 + |\mathcal{S}|$. Therefore $\tau_\circ^{\mathcal{V}}(\mathcal{H}') = |\mathcal{S}| > (|\mathcal{E}| - 3)/3 = \|\mathcal{H}'\|/3$ shows a contradiction to the minimality of $\mathcal{H}$.

Henceforth, we assume $\mathcal{H}$ is linear, and any cycle in $\mathcal{H}$ is of length at least 3. In any subhypergraph $\mathcal{H}'$ of $\mathcal{H}$, all quasi-FVS are FVS, and $\tau_\circ^{\mathcal{V}}(\mathcal{H}') = \tau_c^{\mathcal{V}}(\mathcal{H}')$.

Suppose that there exists $v \in \mathcal{V}$ with $d_{\mathcal{H}}(v) = 1$. Let $e_1 \in \mathcal{E}$ be the unique edge that contains v. Recall from (1) that e_1 is contained in a cycle $\mathcal{C} = v_1 e_1 v_2 e_2 v_3 \cdots e_k v_1$, where $k \geq 3$. By (2), we have $d_{\mathcal{H}}(v_i) = 2$ for all $i \in [k]$. In particular $d_{\mathcal{H}}(v_1) = d_{\mathcal{H}}(v_2) = 2 > d_{\mathcal{H}}(v)$ implies $v \notin \{v_1, v_2\}$, and in turn $v_1, v_2, v \in e_1$ enforces $e_1 = \{v_1, v, v_2\}$. Let $\mathcal{S}$ be a minimum FVS of $\mathcal{H}' = \mathcal{H} \setminus \{e_1, e_2, e_3\}$. It follows from (2) that

$$\mathcal{H} \setminus v_3 \subseteq \mathcal{H} \setminus \{e_2, e_3\} = \mathcal{H}' \oplus e_1,$$

and in $\mathcal{H}' \oplus e_1$, edge e_1 intersects at most one other edge, and therefore is not contained in any cycle. Thus $\mathcal{S}$ is a FVS of $\mathcal{H}' \oplus e_1$, and hence a FVS of $\mathcal{H} \setminus v_3$, implying that $\{v_3\} \cup \mathcal{S}$ is a FVS of $\mathcal{H}$. We deduce that $|\mathcal{E}|/3 < \tau_c^{\mathcal{V}}(\mathcal{H}) \leq |\{v_3\} \cup \mathcal{S}| \leq 1 + |\mathcal{S}|$. Therefore $\tau_c^{\mathcal{V}}(\mathcal{H}') = |\mathcal{S}| > (|\mathcal{E}| - 3)/3 = \|\mathcal{H}'\|/3$ shows a contradiction to the minimality of $\mathcal{H}$. Hence the vertices of $\mathcal{H}$ all have degree at least 2, which together with (2) gives

(3) $\mathcal{H}$ is 2-regular.

Let $\mathcal{C} = (\mathcal{V}_c, \mathcal{E}_c) = v_1 e_1 v_2 e_2 \cdots v_k e_k v_1$ be a shortest cycle in $\mathcal{H}$, where $k \geq 3$. For each $i \in [k]$, suppose that $e_i = \{v_i, u_i, v_{i+1}\}$, where $v_{k+1} = v_1$.

Because $\mathcal{C}$ is a shortest cycle, for each pair of distinct indices $i, j \in [k]$, we have $e_i \cap e_j = \emptyset$ if and only if e_i and e_j are not adjacent in $\mathcal{C}$, i.e., $|i - j| \notin \{1, k-1\}$. This fact along with the linearity of $\mathcal{H}$ says that $v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_k$ are distinct. By (3), each u_i is contained in a unique edge $f_i \in \mathcal{E} \setminus \mathcal{E}_c$, $i \in [k]$. We distinguish among three cases depending on the values of $k \pmod 3$. In each case, we construct a proper subhypergraph $\mathcal{H}'$ of $\mathcal{H}$ with $\|\mathcal{H}'\| < \|\mathcal{H}\|$ and $\tau_c^{\mathcal{V}}(\mathcal{H}') > \|\mathcal{H}'\|/3$ which shows a contradiction to the minimality of $\mathcal{H}$.

CASE 1. $k \equiv 0 \pmod 3$ Let $\mathcal{S}$ be a minimum FVS of $\mathcal{H}' = \mathcal{H} \setminus \mathcal{E}_c$. Setting $\mathcal{V}_* = \{v_i : i \equiv 0 \pmod 3, i \in [k]\}$ and $\mathcal{E}_* = \{e_i : i \equiv 1 \pmod 3, i \in [k]\}$, it follows from (3) that

$$\mathcal{H} \setminus \mathcal{V}_* \subseteq (\mathcal{H} \setminus \mathcal{E}_c) \oplus \mathcal{E}_* = \mathcal{H}' \oplus \mathcal{E}_*,$$

and in $\mathcal{H}' \oplus \mathcal{E}_*$, each edge in $\mathcal{E}_*$ intersects exactly one other edge, and therefore is not contained in any cycle. Thus $(\mathcal{H}' \oplus \mathcal{E}_*) \setminus \mathcal{S}$ is also acyclic, so is $(\mathcal{H} \setminus \mathcal{V}_*) \setminus \mathcal{S}$, saying that $\mathcal{V}_* \cup \mathcal{S}$ is a FVS of $\mathcal{H}$. We deduce that $|\mathcal{E}|/3 < \tau_c^{\mathcal{V}}(\mathcal{H}) \le |\mathcal{V}_* \cup \mathcal{S}| \le k/3 + |\mathcal{S}|$. Therefore $\tau_c^{\mathcal{V}}(\mathcal{H}') = |\mathcal{S}| > (|\mathcal{E}| - k)/3 = \|\mathcal{H}'\|/3$ shows a contradiction.

CASE 2. $k \equiv 1 \pmod 3$ Consider the case where $f_1 \neq f_3$ or $f_2 \neq f_4$. Relabeling the vertices and edges if necessary, we may assume without loss of generality that $f_1 \neq f_3$. Let $\mathcal{S}$ be a minimum FVS of $\mathcal{H}' = \mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1, f_3\})$. Set $\mathcal{V}_* = \emptyset$, $\mathcal{E}_* = \emptyset$ if $k = 4$ and $\mathcal{V}_* = \{v_i : i \equiv 0 \pmod 3, i \in [k] - [3]\}$, $\mathcal{E}_* = \{e_i : i \equiv 1 \pmod 3, i \in [k] - [6]\}$ otherwise. In any case we have $|\mathcal{V}_*| = (k-4)/3$ and

$$\mathcal{H} \setminus (\{u_1, u_3\} \cup \mathcal{V}_*) \subseteq (\mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1, f_3\})) \oplus (\{e_2, e_4\} \cup \mathcal{E}_*) = \mathcal{H}' \oplus (\{e_2, e_4\} \cup \mathcal{E}_*).$$

Note from (3) that in $\mathcal{H}' \oplus (\{e_2, e_4\} \cup \mathcal{E}_*)$, each edge in $\{e_2, e_4\} \cup \mathcal{E}_*$ can intersect at most one other edge, and therefore is not contained in any cycle. Thus $(\mathcal{H}' \oplus (\{e_2, e_4\} \cup \mathcal{E}_*)) \setminus \mathcal{S}$ is also acyclic, so is $(\mathcal{H} \setminus (\{u_1, u_3\} \cup \mathcal{V}_*)) \setminus \mathcal{S}$. Thus $\{u_1, u_3\} \cup \mathcal{V}_* \cup \mathcal{S}$ is a FVS of $\mathcal{H}$, and $|\mathcal{E}|/3 < \tau_c^{\mathcal{V}}(\mathcal{H}) \le |\{u_1, u_3\} \cup \mathcal{V}_* \cup \mathcal{S}| \le 2 + |\mathcal{V}_*| + |\mathcal{S}| = (k+2)/3 + |\mathcal{S}|$. This gives $\tau_c^{\mathcal{V}}(\mathcal{H}') = |\mathcal{S}| > (|\mathcal{E}| - k - 2)/3 = |\mathcal{H}'|/3$, a contradiction.

Consider the case where $f_1 = f_3$ and $f_2 = f_4$. As u_1, u_2, u_3, u_4 are distinct and $|f_1| = |f_2| = 3$, we have $f_1 \neq f_2$. Observe that $u_1 e_1 v_2 e_2 v_3 e_3 u_3 f_3 u_1$ is a cycle in $\mathcal{H}$ of length 4. The minimality of k enforces $k = 4$. Therefore $\mathcal{E}_c \cup \{f_1, f_2\}$ consists of 6 distinct edges. Let $\mathcal{S}$ be a minimum FVS of $\mathcal{H}' = \mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1, f_2\})$. It follows from (3) that

$$\mathcal{H} \setminus \{u_2, u_4\} \subseteq (\mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1, f_2\})) \oplus \{e_1, e_3, f_1\} = \mathcal{H}' \oplus \{e_1, e_3, f_1\}.$$

In $\mathcal{H}' \oplus \{e_1, e_3, f_1\}$, both e_1 and e_3 intersect only one other edge, which is f_1, and any cycle through f_1 must contain e_1 or e_3. It follows that none of e_1, e_3, f_1 is contained in a cycle of $\mathcal{H}' \oplus \{e_1, e_3, f_1\}$. Thus $(\mathcal{H}' \oplus \{e_1, e_3, f_1\}) \setminus \mathcal{S}$ is acyclic, so is $(\mathcal{H} \setminus \{u_2, u_4\}) \setminus \mathcal{S}$, saying that $\{u_2, u_4\} \cup \mathcal{S}$ is a FVS of $\mathcal{H}$. Hence $|\mathcal{E}|/3 < \tau_c^{\mathcal{V}}(\mathcal{H}) \le |\{u_2, u_4\} \cup \mathcal{S}| \le 2 + |\mathcal{S}|$. In turn $\tau_c^{\mathcal{V}}(\mathcal{H}') = |\mathcal{S}| > (|\mathcal{E}| - 6)/3 = \|\mathcal{H}'\|/3$ shows a contradiction.

CASE 3. $k \equiv 2 \pmod 3$ Let $\mathcal{S}$ be a minimum FVS of $\mathcal{H}' = \mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1\})$. Setting $\mathcal{V}_* = \{v_i : i \equiv 1 \pmod 3, i \in [k] - [3]\}$ and $\mathcal{E}_* = \{e_i : i \equiv 2 \pmod 3, i \in [k]\}$, we have $|\mathcal{V}_*| = (k-2)/3$ and

$$\mathcal{H} \setminus (\{u_1\} \cup \mathcal{V}_*) \subseteq (\mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1\})) \oplus \mathcal{E}_* = \mathcal{H}' \oplus \mathcal{E}_*$$

In $\mathcal{H}' \oplus \mathcal{E}_*$, each edge in $\mathcal{E}_*$ intersects at most one other edge, and therefore is not contained in any cycle. Thus $(\mathcal{H}' \oplus \mathcal{E}_*) \setminus \mathcal{S}$ is acyclic, so is $(\mathcal{H} \setminus (\{u_1\} \cup \mathcal{V}_*)) \setminus \mathcal{S}$. Hence $\{u_1\} \cup \mathcal{V}_* \cup \mathcal{S}$ is a FVS of $\mathcal{H}$, yielding $|\mathcal{E}|/3 < \tau_c^{\mathcal{V}}(\mathcal{H}) \le |\{u_1\} \cup \mathcal{V}_* \cup \mathcal{S}| \le 1 + (k-2)/3 + |\mathcal{S}|$ and a contradiction $\tau_c^{\mathcal{V}}(\mathcal{H}') = |\mathcal{S}| > (|\mathcal{E}| - k - 1)/3 = \|\mathcal{H}'\|/3$.

The combination of the above three cases completes the proof. □

Corollary 2.2 *Let $\mathcal{H}$ be a linear 3-uniform hypergraph. Then $\tau_c^{\mathcal{V}}(\mathcal{H}) \le \|\mathcal{H}\|/3$.*

We remark that the upper bound $\|\mathcal{H}\|/3$ in Theorem 2.1 and Corollary 2.2 is best possible. See Fig. 2 for illustrations of five linear 3-uniform hypergraphs attaining the upper bound. It is easy to prove that the maximum degree of every extremal hypergraph (those linear 3-uniform $\mathcal{H}$ with $\tau_c^{\mathcal{V}}(\mathcal{H}) = \|\mathcal{H}\|/3$) is at most three. It would be interesting to characterize all extremal hypergraphs for Corollary 2.2.

The proof of Theorem 2.1 actually gives a recursive combinatorial algorithm for finding in polynomial time a quasi-FVS (resp. FVS) of size at most $\|\mathcal{H}\|/3$ on a 3-uniform (resp. linear 3-uniform) hypergraph $\mathcal{H}$.

Algorithm 1 Quasi-feedback vertex sets of 3-uniform hypergraphs

Input: 3-uniform hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$.
Output: $\textsc{Alg1}(\mathcal{H})$, which is a quasi-FVS of $\mathcal{H}$ with cardinality at most $\|\mathcal{H}\|/3$.

If $|\mathcal{E}| \le 2$ **Then** $\textsc{Alg1}(\mathcal{H}) \leftarrow \emptyset$
  Else If $\exists\, s \in \mathcal{V} \cup \mathcal{E}$ such that s is not contained in any cycle of $\mathcal{H}$
    Then $\textsc{Alg1}(\mathcal{H}) \leftarrow \textsc{Alg1}(\mathcal{H} \setminus s)$
   If $\exists\, s \in \mathcal{V}$ such that $d_{\mathcal{H}}(s) \ge 3$
    Then $\textsc{Alg1}(\mathcal{H}) \leftarrow \{s\} \cup \textsc{Alg1}(\mathcal{H} \setminus s)$
   If $\exists\, e, f \in \mathcal{E}$ such that $|e \cap f| = 2$ and $e - f = \{t\}$
    Then $\textsc{Alg1}(\mathcal{H}) \leftarrow \{t\} \cup \textsc{Alg1}(\mathcal{H} \setminus t)$
   If $\exists\, v \in \mathcal{V}$ such that $d_{\mathcal{H}}(v) = 1$
    Then Let $v_1 e_1 v_2 e_2 v_3 \cdots e_k v_1$ be a cycle of $\mathcal{H}$ such that $e_1 = \{v_1, v_2, v\}$
      $\textsc{Alg1}(\mathcal{H}) \leftarrow \{v_3\} \cup \textsc{Alg1}(\mathcal{H} \setminus \{e_1, e_2, e_3\})$
   Let $(\mathcal{V}_c, \mathcal{E}_c) = v_1 e_1 v_2 e_2 \cdots v_k e_k v_1$ be a shortest cycle in $\mathcal{H}$
   For each $i \in [k]$, let $u_i \in \mathcal{V}_c$, $f_i \in \mathcal{E} \setminus \mathcal{E}_c$ be such that $\{u_i, v_i, v_{i+1}\} = e_i$, $u_i \in f_i$
   If $k \equiv 0 \pmod 3$ **Then** $\textsc{Alg1}(\mathcal{H}) \leftarrow \{v_i : i \equiv 0 \pmod 3, i \in [k]\} \cup \textsc{Alg1}(\mathcal{H} \setminus \mathcal{E}_c)$
   If $k \equiv 1 \pmod 3$
    Then If $f_1 \ne f_3$ or $f_2 \ne f_4$
      Then Relabel vertices and edges if necessary to make $f_1 \ne f_3$
       $\mathcal{V}_* \leftarrow \{v_i : i \equiv 0 \pmod 3, i \in [k] - [3]\}$
       $\textsc{Alg1}(\mathcal{H}) \leftarrow \{u_1, u_3\} \cup \mathcal{V}_* \cup \textsc{Alg1}(\mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1, f_3\}))$
      Else $\textsc{Alg1}(\mathcal{H}) \leftarrow \{u_2, u_4\} \cup \textsc{Alg1}(\mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1, f_2\}))$
   If $k \equiv 2 \pmod 3$
    Then $\textsc{Alg1}(\mathcal{H}) \leftarrow \{u_1\} \cup \{v_i : i \equiv 1 \pmod 3, i \in [k] - [3]\} \cup \textsc{Alg1}(\mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1\}))$
Output $\textsc{Alg1}(\mathcal{H})$

Note that Algorithm 1 never visits isolated vertices (it only scans along the edges of the current hypergraph). The number of iterations performed by the algorithm is

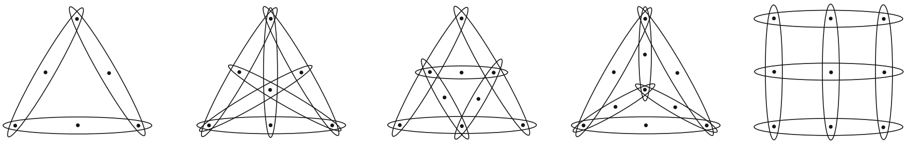


Fig. 2 Extremal linear 3-uniform hypergraphs $\mathcal{H}$ with $\tau_c^{\mathcal{V}}(\mathcal{H}) = \|\mathcal{H}\|/3$.

upper bounded by $|\mathcal{E}|$. Since $\mathcal{H}$ is 3-uniform, the condition in any step is checkable in $O(|\mathcal{E}|^2)$ time. Any cycle in Step 9 or Step 11 can be found in $O(|\mathcal{E}|^2)$ time.[2] Thus Algorithm 1 runs in $O(|\mathcal{E}|^3)$ time.

Corollary 2.3 *Given any linear 3-uniform (resp. 3-uniform) hypergraph $\mathcal{H}$, Algorithm 1 finds in $O(\|\mathcal{H}\|^3)$ time a FVS (resp. quasi-FVS) of $\mathcal{H}$ with size at most $\|\mathcal{H}\|/3$.*

The goal of the next two lemmas is to establish an upper bound on the size of any minimal FES in a 3-uniform hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. Since $\tau_c^{\mathcal{V}}(\mathcal{H}) \leq \tau_c^{\mathcal{E}}(\mathcal{H})$, it follows that both $\tau_c^{\mathcal{V}}(\mathcal{H})$ and $\tau_c^{\mathcal{E}}(\mathcal{H})$ are bounded above by this bound.

Lemma 2.4 *If $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is a connected 3-uniform hypergraph without cycles, then $|\mathcal{V}| = 2|\mathcal{E}| + 1$.*

Proof We prove by induction on $|\mathcal{E}|$. The base case where $|\mathcal{E}| = 0$ is trivial. Inductively, we assume that $|\mathcal{E}| \geq 1$ and the lemma holds for all connected acyclic 3-uniform hypergraph with fewer edges than $\mathcal{H}$. Take arbitrary $e \in \mathcal{E}$. Since $\mathcal{H}$ is connected, acyclic and 3-uniform, $\mathcal{H} \setminus e$ contains exactly three components $\mathcal{H}_i = (\mathcal{V}_i, \mathcal{E}_i)$, $i = 1, 2, 3$. Note that for each $i \in [3]$, hypergraph $\mathcal{H}_i$ with $|\mathcal{E}_i| < |\mathcal{E}|$ is connected, 3-uniform and acyclic. By the induction hypothesis, we have $|\mathcal{V}_i| = 2|\mathcal{E}_i| + 1$ for $i = 1, 2, 3$. It follows that $|\mathcal{V}| = \sum_{i=1}^3 |\mathcal{V}_i| = 2\sum_{i=1}^3 |\mathcal{E}_i| + 3 = 2|\mathcal{E}| + 1$. □

Given any hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, we can easily find a minimal (not necessarily minimum) FES in $O(|\mathcal{E}|^2)$ time: Go through the edges of the trivial FES $\mathcal{E}$ in any order, and remove the edge from the FES if the edge is redundant. The redundancy test can be implemented using Depth First Search.

Lemma 2.5 *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a 3-uniform hypergraph with p components. If $\mathcal{F}$ is a minimal FES of $\mathcal{H}$, then $|\mathcal{F}| \leq 2|\mathcal{E}| - |\mathcal{V}| + p$. In particular, $\tau_c^{\mathcal{E}}(\mathcal{H}) \leq 2|\mathcal{E}| - |\mathcal{V}| + p$.*

[2] A shortest path between any pair of vertices can be found in $O(|\mathcal{E}|)$ time using breadth first search. A shortest cycle can be found by checking all $O(|\mathcal{E}|)$ possibilities.

Proof Suppose that $\mathcal{H} \setminus \mathcal{F}$ contains exactly k components $\mathcal{H}_i = (\mathcal{V}_i, \mathcal{E}_i)$, $i = 1, \ldots, k$. It follows from Lemma 2.4 that $|\mathcal{V}_i| = 2|\mathcal{E}_i| + 1$ for each $i \in [k]$. Thus $|\mathcal{V}| = \sum_{i\in[k]} |\mathcal{V}_i| = 2\sum_{i\in[k]} |\mathcal{E}_i| + k = 2(|\mathcal{E}| - |\mathcal{F}|) + k$, which means $2|\mathcal{F}| = 2|\mathcal{E}| - |\mathcal{V}| + k$. To establish the lemma, it suffices to prove $k \le |\mathcal{F}| + p$.

In case of $|\mathcal{F}| = 0$, we have $\mathcal{F} = \emptyset$ and $k = p = |\mathcal{F}| + p$. In case of $|\mathcal{F}| \ge 1$, suppose that $\mathcal{F} = \{e_1, \ldots, e_{|\mathcal{F}|}\}$. Because $\mathcal{F}$ is a minimal FES of $\mathcal{H}$, for each $i \in [|\mathcal{F}|]$, there is a cycle $\mathcal{C}_i$ in $\mathcal{H} \setminus (\mathcal{F} \setminus \{e_i\})$ such that $e_i \in \mathcal{C}_i$, and $\mathcal{C}_i \setminus e_i$ is a path in $\mathcal{H} \setminus \mathcal{F}$ connecting two of the three vertices in e_i. Considering $\mathcal{H} \setminus \mathcal{F}$ being obtained from $\mathcal{H}$ be removing $e_1, e_2, \ldots, e_{|\mathcal{F}|}$ sequentially, for $i = 1, \ldots, |\mathcal{F}|$, since $|e_i| = 3$, the presence of path $\mathcal{C}_i \setminus e_i$ implies that the removal of e_i can create at most one more component. Therefore we have $k \le p + |\mathcal{F}|$ as desired. □

2.2 Vertex-Weighted Hypergraphs

As will be seen later, we will convert our problem of finding triangle covering and packing numbers to the problem of weighted vertex covering and packing on an acyclic 3-uniform hypergraph. The latter problem has been known to be solvable in strongly polynomial time, for which we recall in this subsection some related results on hypergraph theory and integer programming.

Given a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with n vertices and m edges, and a weight function $\mathbf{w} \in \mathbb{Z}^{\mathcal{V}}_{>0}$, the *weight* (also referred to as *weighted cardinality*) of any subset $\mathcal{S}$ of $\mathcal{V}$ and that of any subset $\mathcal{F}$ of $\mathcal{E}$ are defined as

$$|\mathcal{S}|_w = \sum_{s\in\mathcal{S}} w(s) \text{ and } |\mathcal{F}|_w = \sum_{F\in\mathcal{F}} \prod_{s\in F} w(s),$$

respectively. A *transversal* of $\mathcal{H}$ is a vertex subset of $\mathcal{V}$ that intersects each edge in $\mathcal{E}$. A $\mathbf{w}$-matching of $\mathcal{H}$ is a collection of edges in $\mathcal{H}$ (repetition allowed) such that each vertex $v \in \mathcal{V}$ appears in at most $w(v)$ members of the collection. Let $\tau(\mathcal{H}, \mathbf{w})$ and $\nu(\mathcal{H}, \mathbf{w})$ denote the minimum weight of a transversal and maximum cardinality of a $\mathbf{w}$-matching of $\mathcal{H}$, respectively. It is well known that $\tau(\mathcal{H}, \mathbf{w}) \ge \nu(\mathcal{H}, \mathbf{w})$.

Let $M_{\mathcal{H}}$ be the $\mathcal{V} \times \mathcal{E}$ incidence matrix. From $M_{\mathcal{H}}$, we can construct a bipartite graph $G_{\mathcal{H}}$ with bipartition $\mathcal{V}, \mathcal{E}$ such that there is an edge of $G_{\mathcal{H}}$ between $v \in \mathcal{V}$ and $e \in \mathcal{E}$ if and only if $v \in e$ in $\mathcal{H}$. Suppose that $\mathcal{H}$ is acyclic. It is easy to see that $G_{\mathcal{H}}$ is acyclic. Thus $M = M_{\mathcal{H}}$ falls within the class of *restricted totally unimodular* (RTUM) matrices defined by Yannakakis [11]. As the name indicates, RTUM matrices are all totally unimodular. Hence the total unimodularity and LP duality give the well-known result [12] that

$$\tau(\mathcal{H}, \mathbf{w}) = \min\{\mathbf{w}^T\mathbf{x} : M^T\mathbf{x} \ge \mathbf{1}, x \ge 0\} = \max\{\mathbf{1}^T\mathbf{y} : M\mathbf{y} \le \mathbf{w}, y \ge 0\} = \nu(\mathcal{H}, \mathbf{w}).$$

Moreover, since M is RTUM, both a minimum weighted transversal and a maximum weighted matching of $\mathcal{H}$ can be found in $O(n(m + n\log n)\log n)$ time using

Yanakakis's combinatorial algorithm [11] based on the current best combinatorial algorithms for the b-matching problem and the maximum weighted independent set problem on a bipartite multigraph with n vertices and m edges, where the bipartite b-matching problem can be solved with the minimum-cost flow algorithm in $O(n \log n(m + n \log n))$ time (see Section 21.5 and page 356 of [13]) and the maximum weighted independent set problem can be solved with maximum flow algorithm in $O(nm \log n)$ time (See pages 300-301 of [11]).

Theorem 2.6 ([11, 12]) *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with n non-isolated vertices and m edges, and $\mathbf{w} \in \mathbb{Z}^{\mathcal{V}}_{>0}$. If $\mathcal{H}$ has no cycle, then $\tau(\mathcal{H}, \mathbf{w}) = \nu(\mathcal{H}, \mathbf{w})$, and a minimum weighted transversal and a maximum $\mathbf{w}$-matching of $(\mathcal{H}, \mathbf{w})$ can be found in $O(n(m + n \log n) \log n)$ time.*

In the case of unit weight (i.e., the unweighted case), $\mathbf{1}$-matching of $\mathcal{H}$ is often referred to as a *matching* of $\mathcal{H}$, which is a nonempty set of pairwise disjoint edges in $\mathcal{H}$. As usual, $\tau(\mathcal{H}, \mathbf{1})$ and $\tau(\mathcal{H}, \mathbf{1})$ are abbreviated as $\tau(\mathcal{H})$ and $\nu(\mathcal{H})$, respectively; the symbol $|\cdot|_1$ is simply the traditional $|\cdot|$ representing cardinality. Clearly $\tau(\mathcal{H}) \geq \tau_c^{\mathcal{V}}(\mathcal{H})$.

2.3 Transversals

In this subsection, we will upper bound $\tau(\mathcal{H})$ for connected linear 3-uniform hypergraph $\mathcal{H}$ by $(2\|\mathcal{H}\| + 1)/3$. To this end, we first study some properties of cycles and components in these hypergraphs. Then we establish the upper bound in Theorem 2.10.

A cycle in a hypergraph is *minimal* if every pair of nonconsecutive edges of the cycle is vertex-disjoint. A pair of (minimal) cycles are called *intersecting* if these two cycles have at least one vertex in common. A partition of a set into two parts is called *nontrivial* if neither of the two parts is empty.

Lemma 2.7 *Let $\mathcal{C} = (\mathcal{V}, \mathcal{E})$ be a linear 3-uniform hypergraph that is a cycle. If $\mathcal{C}$ is not minimal, then there exists a pair of intersecting cycles $\mathcal{C}_i = (\mathcal{V}_i, \mathcal{E}_i)$, $i = 1, 2$, in $\mathcal{C}$ such that $|\mathcal{V}_1| < |\mathcal{V}|$, $|\mathcal{V}_2| < |\mathcal{V}|$, $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, $\mathcal{E}_1 \cap \mathcal{E}_2 \neq \emptyset$, and $|\mathcal{E}_1| + |\mathcal{E}_2| \leq |\mathcal{E}| + 2$.*

Proof We may order the vertices and edges of the cycle $\mathcal{C}$ so that $\mathcal{C} = v_1 e_1 v_2 e_2 \cdots v_k e_k v_1$ has two nonconsecutive edges e_1 and e_i that have a vertex v in common, where $3 \leq i \leq k - 1$. Recall that $v_1, v_2, \ldots, v_k$ (resp. $e_1, e_2, \ldots, e_k$) are all distinct. If $v \notin \{v_1, v_2, \ldots, v_k\}$, then $\mathcal{C}$ properly contains cycles $\mathcal{C}_1 = v e_1 v_2 e_2 \cdots v_i v$ with length $i < k = |\mathcal{V}|$ and $\mathcal{C}_2 = v e_i v_{i+1} \cdots v_k e_k v_1 e_1 v$ with length $k - i + 2 < k = |\mathcal{V}|$. It is clear that $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the conclusion of the lemma. It remains to consider the case where either e_1 or e_i consists of three vertices from $\{v_1, v_2, \ldots, v_k\}$. By symmetry, we may assume $e_1 = \{v_1, v_2, v_j\}$ for some $j \in [3, k]$. The linearity of $\mathcal{C}$ enforces $4 \leq j \leq k - 1$, giving $\mathcal{C}_1 = v_j e_1 v_2 e_2 \cdots e_{j-1} v_j$ and $\mathcal{C}_2 = v_j e_j v_{j+1} \cdots v_k v_1 e_1 v_j$ as desired. □

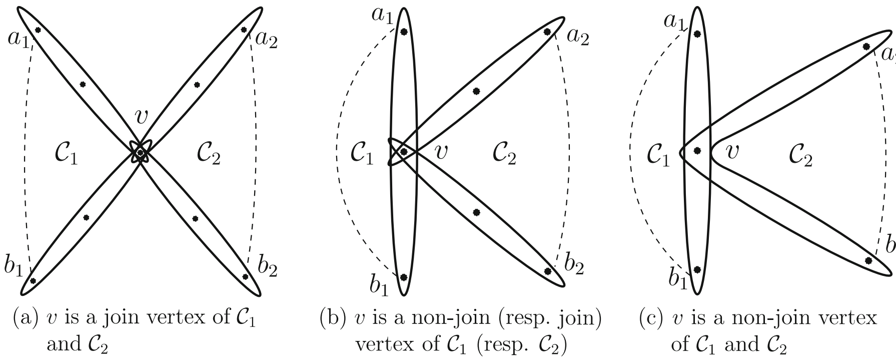


Fig. 3 Two neighbors a_i, b_i $(i = 1, 2)$ of v belong to the same component of $\mathcal{H} \setminus v$

Corollary 2.8 *A linear 3-uniform hypergraph $\mathcal{H}$ has a pair of intersecting cycles if and only if $\mathcal{H}$ has a pair of intersecting minimal cycles.*

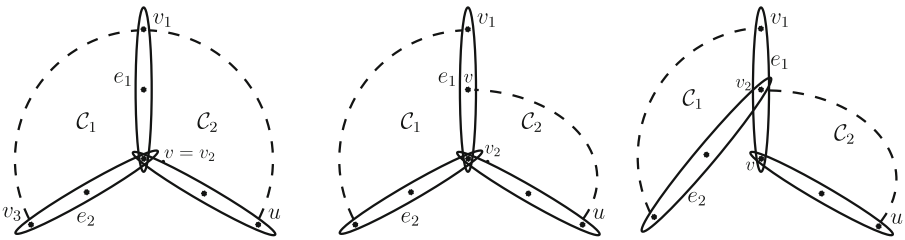
Proof Take $\mathcal{C} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{C}' = (\mathcal{V}', \mathcal{E}')$ to be a pair of intersecting cycles in $\mathcal{H}$ such that $|\mathcal{E}| + |\mathcal{E}'|$ is as small as possible. If one of $\mathcal{C}$ and $\mathcal{C}'$, say $\mathcal{C}$, is not minimal, then by Lemma 2.7, $\mathcal{C}$ and hence $\mathcal{H}$ contain a pair of intersecting cycles $\mathcal{C}_i = (\mathcal{V}_i, \mathcal{E}_i)$, $i = 1, 2$, with $|\mathcal{E}_1| + |\mathcal{E}_2| \le |\mathcal{E}| + 2 < |\mathcal{E}| + |\mathcal{E}'|$, which gives a contradiction to the minimality of $|\mathcal{E}| + |\mathcal{E}'|$. $\square$

Lemma 2.9 *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a connected linear 3-uniform hypergraph. If $\mathcal{H}$ has a pair of intersecting cycles, then there exists a vertex $v \in \mathcal{V}$ such that $\mathcal{H} \setminus v$ has at most $2d_{\mathcal{H}}(v) - 2$ components.*

Proof By Corollary 2.8, there exists a pair of intersecting minimal cycles $\mathcal{C}_1 = v_1 e_1 v_2 e_2 \cdots v_k e_k v_1$ and $\mathcal{C}_2 = u_1 f_1 u_2 f_2 \cdots u_\ell f_\ell u_1$ in $\mathcal{H}$. Since $\mathcal{H}$ is 3-uniform and linear, for any vertex v of $\mathcal{H}$, we observe that v has $2d_{\mathcal{H}}(v)$ neighbors in $\mathcal{H}$. The connectivity of $\mathcal{H}$ implies each component of $\mathcal{H} \setminus v$ has to contain at least one of these $2d_{\mathcal{H}}(v)$ neighbors.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ do not have any common edge, let v be an arbitrarily taken common vertex of $\mathcal{C}_1$ and $\mathcal{C}_2$. For $i = 1, 2$, no matter v is a join vertex of $\mathcal{C}_i$ or not, v is adjacent to exactly two join vertices of $\mathcal{C}_i$, which we denote as a_i, b_i. See Fig. 3 for an illustration. Since $\mathcal{C}_i$ is minimal, v's neighbors a_i and b_i belong to the same component of $\mathcal{H} \setminus v$. Hence $\mathcal{H} \setminus v$ has at most $2d_{\mathcal{H}}(v) - 2$ components.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ have some edge(s) in common, then, as $\mathcal{C}_1$ and $\mathcal{C}_2$ are distinct, we may assume $e_1 \in \mathcal{C}_2$ and $e_2 \notin \mathcal{C}_2$ as illustrated in Fig. 4. In case of v_1, v_2 both being join vertices of $\mathcal{C}_2$ (Fig. 4a), we have $v = v_2$ adjacent to exactly two join vertices of $\mathcal{C}_2$ – one is v_1 and the other written as u is not v_3. So v_1, v_3, u are three distinct neighbors of v which are contained in the same component of $\mathcal{H} \setminus v$, and we are done. In case of only one of v_1 and v_2 being join vertex of $\mathcal{C}_2$ (Fig. 4b), let v be the join vertex of



(a) v_1 and v_2 are join vertices of $\mathcal{C}_2$ (b) Only one of v_1 and v_2 is a join vertex of $\mathcal{C}_2$

Fig. 4 Three neighbors u, v_1, v_i ($i = 2$ or 3) of v belong to the same component of $\mathcal{H} \setminus v$

$\mathcal{C}_2$ contained in $e_1 \setminus \{v_1, v_2\}$ and u be v's neighbor which is a join vertex of $\mathcal{C}_2$ other than v_1 and v_2. Now we see that v_1, v_2, u are three distinct neighbors of v that are contained in the same component of $\mathcal{H} \setminus v$. In either case, we reach the conclusion that $\mathcal{H} \setminus v$ has at most $2d_{\mathcal{H}}(v) - 2$ components. □

Theorem 2.10 *Let $\mathcal{H}$ be a connected linear 3-uniform hypergraph. Then $\tau(\mathcal{H}) \le \frac{2\|\mathcal{H}\|+1}{3}$.*

Proof By contradiction, we take a counterexample $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $\tau(\mathcal{H}) > \frac{2\|\mathcal{H}\|+1}{3}$ such that $\|\mathcal{H}\|$ is minimum.

If $\mathcal{H}$ is acyclic, then $|\mathcal{V}| = 2\|\mathcal{H}\|+1$ by Lemma 2.4 and $\tau(\mathcal{H}) = \nu(\mathcal{H})$ by Theorem 2.6, implying a contradiction $\frac{2\|\mathcal{H}\|+1}{3} < \tau(\mathcal{H}) = \nu(\mathcal{H}) \le \frac{|\mathcal{V}|}{3} = \frac{2\|\mathcal{H}\|+1}{3}$. Thus we have

(1) $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is not acyclic.

Suppose that there exists $v \in \mathcal{V}$ such that $\mathcal{H} \setminus v$ consists of k components $\mathcal{H}_1, \ldots, \mathcal{H}_k$, where $k \le 2d_{\mathcal{H}}(v) - 2$. The minimality of $\|\mathcal{H}\|$ implies $\tau(\mathcal{H}_i) \le \frac{2\|\mathcal{H}_i\|+1}{3}$ for all $i = 1, \ldots, k$. It follows that

$$\tau(\mathcal{H}) \le 1+\tau(\mathcal{H}\setminus v) = 1+\sum_{i=1}^{k}\tau(\mathcal{H}_i) \le 1+\sum_{i=1}^{k}\frac{2\|\mathcal{H}_i\|+1}{3} = \frac{2[\|\mathcal{H}\| - d_{\mathcal{H}}(v)] + k}{3}+1,$$

which along with $k \le 2d_{\mathcal{H}}(v) - 2$ gives $\tau(\mathcal{H}) \le \frac{2\|\mathcal{H}\|+1}{3}$, a contradiction. Therefore for each vertex $v \in \mathcal{V}$, hypergraph $\mathcal{H} \setminus v$ has at least $2d_{\mathcal{H}}(v) - 1$ components. It follows from Lemma 2.9 that $\mathcal{H}$ does not contain any pair of interesting cycles. In turn by Lemma 2.7 we see that all cycles of $\mathcal{H}$ are minimal, and they are connected by trees induced by edges of $\mathcal{H}$ not in any cycle. Let $\mathfrak{C}$ denote the set of cycles in $\mathcal{H}$, and $\mathfrak{T}$ denote the set of components (maximal subtrees) of the hypergraph induced the edges of $\mathcal{E}$ not in any cycle of $\mathcal{H}$. We associate $\mathcal{H}$ with a graph G on vertex set

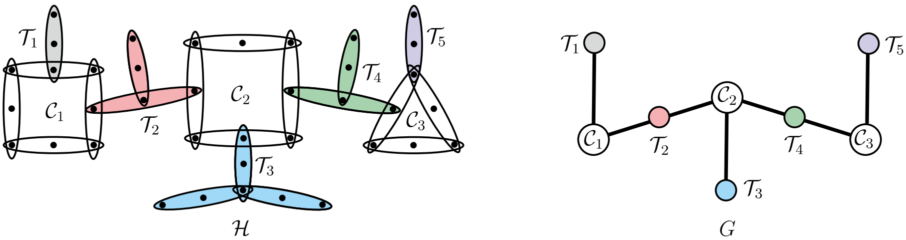


Fig. 5 Hypergraph $\mathcal{H}$ and its associated graph G, where $\mathfrak{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ and $\mathfrak{T} = \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5\}$

$\mathfrak{C} \cup \mathfrak{T}$ in which vertex $\mathcal{C} \in \mathfrak{C}$ and vertex $\mathcal{T} \in \mathfrak{T}$ are joined by an edge if and only if in $\mathcal{H}$ cycle $\mathcal{C}$ and tree $\mathcal{T}$ have some vertex in common. See Fig. 5 for an illustration.

Since all cycles of $\mathcal{H}$ are vertex-disjoint, it is not hard to see that G is a tree. If $\mathcal{H}$ is simply a cycle, then $\tau(\mathcal{H}) = \left\lceil \frac{\|\mathcal{H}\|}{2} \right\rceil$, which shows a contradiction to $\tau(\mathcal{H}) > \frac{2\|\mathcal{H}\|+1}{3}$. Hence

(2) G is a tree of at least two vertices, and it is a bipartite graph with bipartition $(\mathfrak{C}, \mathfrak{T})$.

Consider any nontrivial partition $(\mathcal{E}_1', \mathcal{E}_2')$ of $\mathcal{E}$. For $i = 1, 2$, let $\mathcal{H}_i'$ denote the subhypergraph of $\mathcal{H}$ induced by $\mathcal{E}_i'$. If $\mathcal{H}_1'$ and $\mathcal{H}_2'$ are connected, then the minimality of $\|\mathcal{H}\|$ implies $\tau(\mathcal{H}_i') \le \frac{2\|\mathcal{H}_i'\|+1}{3}$, $i = 1, 2$. Since the union of transversals of $\mathcal{H}_1'$ and $\mathcal{H}_2'$ must be a transversal of $\mathcal{H}$, we have $\frac{2\|\mathcal{H}\|+1}{3} < \tau(\mathcal{H}) \le \tau(\mathcal{H}_1') + \tau(\mathcal{H}_2') \le \frac{2\|\mathcal{H}_1'\|+1}{3} + \frac{2\|\mathcal{H}_2'\|+1}{3} = \frac{2\|\mathcal{H}\|+2}{3}$, which provides the following important property:

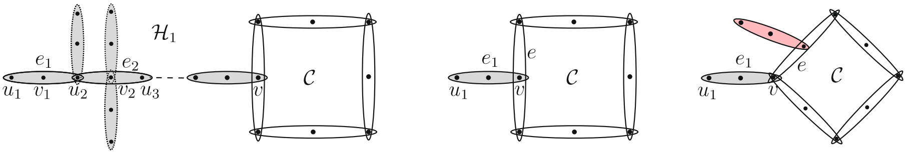
(3) If $\mathcal{H}_1'$ and $\mathcal{H}_2'$ are both connected, then $\tau(\mathcal{H}_i') = \frac{2\|\mathcal{H}_i'\|+1}{3}$ for $i = 1, 2$.

Recalling (2), we consider a leaf of G, which is a minimal cycle in $\mathfrak{C}$ or a tree in $\mathfrak{T}$. We denote it as $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$. As G has at least two vertices, $\mathcal{E}_2 = \mathcal{E} - \mathcal{E}_1$ is nonempty, and it induces a hypergraph $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$. Because $\mathcal{H}_1$ is a leaf of G, both $\mathcal{H}_1$ and $\mathcal{H}_2$ are connected. Thus (3) guarantees that

(4) $\tau(\mathcal{H}_i) = \frac{2\|\mathcal{H}_i\|+1}{3} = \frac{2|\mathcal{E}_i|+1}{3}$ for $i = 1, 2$.

Note that $\mathcal{H}_1 \in \mathfrak{T}$ for otherwise $\mathcal{H}_1 \in \mathfrak{C}$ would be a cycle for which we have $\tau(\mathcal{H}_1) \le \left\lceil \frac{\|\mathcal{H}_1\|}{2} \right\rceil < \frac{2\|\mathcal{H}_1\|+1}{3}$. In G, the leaf $\mathcal{H}_1 \in \mathfrak{T}$ is adjacent to a unique neighbor $\mathcal{C}$, which belongs to $\mathfrak{C}$. By the linearity of $\mathcal{H}$, it is easy to see that tree $\mathcal{H}_1$ and cycle $\mathcal{C}$ intersect at exactly one vertex of $\mathcal{V}$, which we denote as v.

Let $u_1 \in \mathcal{V}_1$ be a vertex in $\mathcal{H}_1$ which is at the maximum distance from v. If u_1 is not adjacent to v in $\mathcal{H}_1$, the unique path between u_1 and v may be written as $u_1 e_1 u_2 e_2 u_3 \cdots v$ (see Fig. 6a for an illustration). Suppose $e_1 = \{u_1, v_1, u_2\}$ and $e_2 = \{u_2, v_2, u_3\}$. Because u_1 is the farthest vertex from v in $\mathcal{H}_1$, it must be the



(a) u_1 and v are not adjacent (b) v is a non-join vertex of $\mathcal{C}$ (c) v is a join vertex of $\mathcal{C}$

Fig. 6 $\mathcal{H}_1$ and its neighboring structures

case that $d_{\mathcal{H}}(u_1) = d_{\mathcal{H}}(v_1) = 1$. Let $\mathcal{E}'_1$ be the set of edges in $\mathcal{H}$ incident with u_2 or v_2. The choice of u_1 implies that all edges in $\mathcal{E}'_1 - \{e_2\}$ are pendant. It follows that $\mathcal{E}'_1$ and $\mathcal{E}'_2 = \mathcal{E} - \mathcal{E}'_1$ induce connected hypergraphs $\mathcal{H}'_1$ and $\mathcal{H}'_2$, respectively. By (3), we have $\tau(\mathcal{H}'_1) = \frac{2|\mathcal{E}'_1|+1}{3}$. Since $\tau(\mathcal{H}'_i)$ is an integer, and $\mathcal{E}'_1$ contains e_1 and e_2 which are distinct, we deduce that $\tau(\mathcal{H}'_1) \geq 3$, which contradicts the fact that $\{u_2, v_2\}$ is a transversal of $\mathcal{H}'_1$. The contradiction implies that u_1 is adjacent to v in $\mathcal{H}$. Furthermore, the choice of u_1 enforces that all vertices in $\mathcal{V}_1 - \{v\}$ are adjacent to v. Hence, $\{v\}$ is a transversal of $\mathcal{H}_1$, which along with (4) enforces the following:

(5) $\mathcal{E}_1$ consists of only one edge e_1.

Note that v is incident with one or two edges in $\mathcal{C}$. We take e to be one of them. See Fig. 6b and c for illustrations. Let hypergraphs $\mathcal{H}'_1$ and $\mathcal{H}'_2$ be induced by $\mathcal{E}'_1 = \{e, e_1\}$ and $\mathcal{E} - \mathcal{E}'_1$ respectively. Clearly,

(6) $\mathcal{H}'_1$ is connected, and $\{v\}$ is a transversal of $\mathcal{H}'_1$.

Note that $\frac{2\|\mathcal{H}'_1\|+1}{3} = \frac{2\times 2+1}{3}$ is not an integer. The combination of (3) and (6) implies that $\mathcal{H}'_2$ is not connected. If v is a non-join vertex of $\mathcal{C}$ (as depicted in Fig. 6b), then it can be easily seen from (5) that $\mathcal{H}'_2$ is connected. The contradiction reduces us to the case where v is a join-vertex of $\mathcal{C}$, as depicted in Fig. 6c. Observe from (5) that $\mathcal{H}'_2$ can have at most two components. So $\mathcal{H}'_2$ consists of exactly two components, written as $\mathcal{H}_3$ and $\mathcal{H}_4$. The minimality of $\mathcal{H}$ guarantees $\tau(\mathcal{H}_i) \leq \frac{2\|\mathcal{H}_i\|+1}{3}$ for $i = 3, 4$. Since the union of minimum transversals of $\mathcal{H}'_1$, $\mathcal{H}_3$, $\mathcal{H}_4$ is a transversal of $\mathcal{H}$, and $\tau(\mathcal{H}'_1) = 1$ by (6), we reach a contradiction $\frac{2\|\mathcal{H}\|+1}{3} < \tau(\mathcal{H}) \leq \tau(\mathcal{H}'_1) + \tau(\mathcal{H}_3) + \tau(\mathcal{H}_4) \leq 1 + \frac{2\|\mathcal{H}_3\|+1}{3} + \frac{2\|\mathcal{H}_4\|+1}{3} = \frac{2\|\mathcal{H}\|+1}{3}$, where the last equation is implied by $\|\mathcal{H}_3\| + \|\mathcal{H}_4\| = \|\mathcal{H}'_2\| = \|\mathcal{H}\| - \|\mathcal{H}'_1\| = \|\mathcal{H}\| - 2$. This completes the proof of the theorem. □

The proof of Theorem 2.10 actually gives a recursive combinatorial algorithm for finding in polynomial time a transversal of size at most $(2\|\mathcal{H}\| + 1)/3$ in a connected linear 3-uniform hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. For any nonempty $\mathcal{F} \subseteq \mathcal{E}$, we use $\mathcal{H}[\mathcal{F}]$ to denote the subhypergraph of $\mathcal{H}$ induced by $\mathcal{F}$.

Algorithm 2 Transversal of connected linear 3-uniform hypergraphs

Input: Connected linear 3-uniform hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$.
Output: $\text{ALG2}(\mathcal{H})$, which is a transversal of $\mathcal{H}$ with cardinality at most $\frac{2\|\mathcal{H}\|+1}{3}$.

1. **If** $\mathcal{H}$ is acyclic,
2. **Then** $\text{ALG2}(\mathcal{H}) \leftarrow$ a minimum transversal computed by Yannakais's algorithm [11]
3. **Else If** there exists $v \in \mathcal{V}$ such that $\mathcal{H} \setminus v$ has k components $\mathcal{H}_1, \ldots, \mathcal{H}_k$ with $k \le 2d_{\mathcal{H}}(v) - 2$
4. **Then** $\text{ALG2}(\mathcal{H}) \leftarrow (\cup_{i \in [k]} \text{ALG2}(\mathcal{H}_i)) \cup \{v\}$
5. **Else** $\mathfrak{C} = \{\text{cycles of } \mathcal{H}\}$
6. $\mathfrak{T} = \{$components of the hypergraph induced by edges not in any cycle of $\mathcal{H}\}$
7. Construct tree G with bipartition $\mathfrak{C} \cup \mathfrak{T}$ associated to $\mathcal{H}$
8. Let $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ be a leaf of G
9. **If** $\mathcal{H} \in \mathfrak{C}$ is a cycle $v_1 e_1 v_2 \cdots v_\ell e_\ell v_1$ in $\mathcal{H}$
10. **Then** $\text{ALG2}(\mathcal{H}) \leftarrow \{v_i : i \equiv 1 \pmod 2\} \cup \text{ALG2}(\mathcal{H}[\mathcal{E} - \mathcal{E}_1])$
11. **Else** $\mathcal{C} \leftarrow$ the unique neighbor of $\mathcal{H}_1$ in G
12. $v \leftarrow$ the unique common vertex of $\mathcal{H}_1$ and $\mathcal{C}$ in $\mathcal{H}$
13. $u_1 \leftarrow$ a farthest vertex from v in $\mathcal{H}_1$
14. **If** u_1 and v are not adjacent in $\mathcal{H}_1$
15. **Then** Let $u_1 e_1 u_2 e_2 u_3 \cdots v$ be the unique path between u_1 and v in $\mathcal{H}_1$
16. Let $v_2 \in \mathcal{V}_1$ be such that $e_2 = \{u_2, v_2, u_3\}$
17. $\text{ALG2}(\mathcal{H}) \leftarrow \{u_2, v_2\} \cup \text{ALG2}(\mathcal{H}[\mathcal{E} -$ {edges of $\mathcal{H}$ incident u_2 or v_2}])
18. **Else If** $|\mathcal{E}_1| \ge 2$ **Then** $\text{ALG2}(\mathcal{H}) \leftarrow \{v\} \cup \text{ALG2}(\mathcal{H}[\mathcal{E} - \mathcal{E}_1])$
19. **Else** $e \leftarrow$ an edge in $\mathcal{C}$ that is incident with v
20. $\mathfrak{S} \leftarrow \{$components of $\mathcal{H}[\mathcal{E} - (\mathcal{E}_1 \cup \{e\})]\}$
21. $\text{ALG2}(\mathcal{H}) \leftarrow \{v\} \cup (\cup_{\mathcal{H}' \in \mathfrak{S}} \text{ALG2}(\mathcal{H}'))$
22. Output $\text{ALG2}(\mathcal{H})$

By Theorem 2.6, the total running of for all iterations of Step 2 is $O(|\mathcal{V}|(|\mathcal{E}| + |\mathcal{V}| \log |\mathcal{V}|) \log |\mathcal{V}|)$. Steps 3 – 4 are implemented at most $|\mathcal{V}|$ times, each taking $O(|\mathcal{V}| \cdot |\mathcal{E}|)$ time. Steps 5 – 21 are implemented at most $|\mathcal{V}|$ times, each taking $O(|\mathcal{E}|)$ time. Thus Algorithm 2 runs in $O(|\mathcal{V}|(|\mathcal{E}| + |\mathcal{V}| \log |\mathcal{V}|) \log |\mathcal{V}|)$ time.

Corollary 2.11 *Given any connected linear 3-uniform hypergraph $\mathcal{H}$ on n vertices and m edges, Algorithm 2 finds in $O(n(m+n\log n)\log n)$ time a transversal of $\mathcal{H}$ with size at most $(2m+1)/3$.*

3 Triangle Packing and Covering

This section establishes several new sufficient conditions for Conjectures 1.1 and 1.2 as well as their algorithmic implications on finding minimum triangle covers. Section 3.1 relates weighted triangle packing and covering in graphs to weighted matching and transversal in triangle hypergraphs, and studies the strong polynomial-time computation of weighted transversals in linear triangle hypergraphs. Section 3.2 deals with graphs of large weighted triangle packing numbers. Section 3.3 investigates irreducible graphs with large weighted numbers of edges.

3.1 Triangle Hypergraphs

To each graph $G=(V,E)$, we associate a hypergraph $\mathcal{H}_G=(E,\mathscr{T}_G)$, referred to as *triangle hypergraph* of G, such that the vertices and edges of $\mathcal{H}_G$ are the edges and triangles of G, respectively. It is easy to see that $\mathcal{H}_G$ is 3-uniform, $\nu(\mathcal{H}_G)=\nu_t(G)$ and $\tau(\mathcal{H}_G)=\tau_t(G)$. Note that the number of non-isolated vertices of $\mathcal{H}_G$ is upper bounded by $3\|\mathcal{H}_G\|=3|\mathscr{T}_G|$, and $|E|\le 3|\mathscr{T}_G|$ if G is irreducible, i.e., $\cup_{T\in\mathscr{T}_G}E(T)=E$.

In order to deal with polynomial-time computability of the weighted triangle covering of weighted graph $(G,\mathbf{w})$ with $G=(V,E)$ being simple and $\mathbf{w}\in\mathbb{Z}^E_{>0}$, we do not work directly on the graph G_w obtained from G by replacing each edge $e\in E$ with a set E_e of $w(e)$ edges of the same ends as e. Before proceeding to discuss the algorithmic details, we make some easy but important observations which will be used implicitly throughout the rest of this paper.

Observation 3.1 The following hold for every edge-weighted graph $(G,\mathbf{w})$.

(i) The weighted triangle covering number $\tau_t(G,\mathbf{w})=\tau(\mathcal{H}_G,\mathbf{w})$ equals $\tau_t(G_w)=\tau(\mathcal{H}_{G_w})$;
(ii) The weighted triangle packing number $\nu_t(G,\mathbf{w})=\nu(\mathcal{H}_G,\mathbf{w})$ equals $\nu_t(G_w)=\nu(\mathcal{H}_{G_w})$;
(iii) The weighted number of triangles $|\mathscr{T}_G|_w=\sum_{\{x,y,z\}\in\mathscr{T}_G}w(x)w(y)w(z)$ equals $|\mathscr{T}_{G_w}|$;
(iv) The weighted number of edges $|E|_w=\sum_{e\in E(G)}w(e)$ equals $|E(G_w)|$.

In our algorithms for finding a FVS in a vertex-weighted triangle hypergraph, at the very beginning we treat each E_e as a single e by introducing to e a weighted degree in hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} \subseteq E$ and $\mathcal{E} \subseteq \mathscr{T}_G$. The *weighted degree* $d^w_{\mathcal{H}}(e)$ of $e \in \mathcal{V}$ in $\mathcal{H}$ is defined as

$$d^w_{\mathcal{H}}(e) = \sum_{f,g:\{e,f,g\}\in\mathcal{E}} w(f)w(g),$$

which can be computed in $O(|V|^2)$ time. It is worth noting that $d^w_{\mathcal{H}_G}(e)$ equals $d_{\mathcal{H}_{G_w}}(f)$ for all $f \in E_e$. A crucial observation on Algorithm 1 is the following.

Observation 3.2 There exists an implementation of Algorithm 1 with input $\mathcal{H}_{G_w}$ such that if an edge $s \in E$ is found and removed from the hypergraph in Step 2 (resp. Step 4), then all edges in E_s are removed one by one in Step 2 (resp. Step 4) of the following recursions.

The correctness of the observation is guaranteed by the fact that all $f \in E_e$ have the same degree in $\mathcal{H}_{G_w}$ and in each subhypergraph of $\mathcal{H}_{G_w}$ produced by Algorithm 1, and the deletion of an $f \in E_e$ does not change the degree of other members of E_e. By virtue of the observation, we can compute a FVS in a vertex-weighted linear 3-uniform hypergraph as in Algorithm 3.

Algorithm 3 Feedback vertex sets of vertex-weighted linear triangle hypergraphs

Input: linear triangle hypergraph $\mathcal{H} = \mathcal{H}_G = (\mathcal{V}, \mathcal{E})$, and $\mathbf{w} \in \mathbb{Z}^{\mathcal{V}}_{>0}$.
Output: $\text{ALG3}(\mathcal{H})$, which is a FVS of $\mathcal{H}_G$ with cardinality at most $\|\mathcal{H}_G\|/3$.

1. **If** $|\mathcal{E}| \leq 2$ **Then** $\text{ALG3}(\mathcal{H}) \leftarrow \emptyset$
2. **Else If** $\exists\, s \in \mathcal{V} \cup \mathcal{E}$ such that s is not contained in any cycle of $\mathcal{H}$
3. **Then** $\text{ALG3}(\mathcal{H}) \leftarrow \text{ALG3}(\mathcal{H} \setminus s)$
4. **If** $\exists\, s \in \mathcal{V}$ such that $d^w_{\mathcal{H}}(s) \geq 3$
5. **Then** $\text{ALG3}(\mathcal{H}) \leftarrow \{s\} \cup \text{ALG3}(\mathcal{H} \setminus s)$
6. $\mathcal{X} \leftarrow \{x \in \mathcal{V} : w(x) \geq 2\}$
7. $\mathcal{V}' \leftarrow \mathcal{V} \cup \{x' : x \in \mathcal{X}\}$,
   $\mathcal{E}' \leftarrow \mathcal{E} \cup \{\{x', y, z\} : \{x, y, z\} \in \mathcal{E} \text{ and} x \in \mathcal{X}\}$, $\mathcal{H}' \leftarrow (\mathcal{V}', \mathcal{E}')$
8. $\mathcal{S} \leftarrow \text{ALG1}(\mathcal{H}')$
9. $\text{ALG3}(\mathcal{H}) \leftarrow \mathcal{S} - \{x' : x \in \mathcal{X}\} - \{x : x \in \mathcal{X} \text{ and } x' \notin \mathcal{S}\}$
10. Output $\text{ALG3}(\mathcal{H})$

Observation 3.3 After recursions at Steps 2-5 of Algorithm 3, we obtain a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $w(e) \in \{1, 2\}$ for all $e \in \mathcal{V}$.

Proof To see it, suppose the contrary: $w(e) \geq 3$ for some $e \in \mathcal{V}$. Let $\{e, f, g\} \in \mathcal{E}$ be a triangle containing e. Then $d^w_{\mathcal{H}}(f) \geq w(e) \geq 3$ says that f should have been removed at Step 5. $\square$

Now let us consider the implementation of Algorithm 1 with input $\mathcal{H}_{G_w}$ stated in Observation 3.2. The hypergraph its Step 6 faces corresponds to the hypergraph $\mathcal{H}$ reached by Step 6 of Algorithm 3. By Observation 3.3, the former hypergraph is exactly the hypergraph $\mathcal{H}'$ constructed at Step 7 of Algorithm 3. (Note that each pair of x, x' in $\mathcal{H}'$ simulates a pair of parallel edges in G_w.) Hence Step 8 of Algorithm 3 simply conducts the computations done by Steps 6 – 21 of Algorithm 1. Furthermore, from the construction of $\mathcal{H}'$, it is straightforward that for any $x \in \mathcal{X}$, if $|\mathcal{S} \cap \{x, x'\}| = 1$ then $\mathcal{S} - \{x, x'\}$ is a quasi-FVS of $\mathcal{H}'$. It follows that the setting in Step 9 of Algorithm 3 does provide a FVS of the linear 3-uniform hypergraph $\mathcal{H}$.

Theorem 3.4 *Given any linear triangle hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and any $\mathbf{w} \in \mathbb{Z}^{\mathcal{V}}_{>0}$, Algorithm 3 finds in $O(\|\mathcal{H}\|^3)$ time a FVS $\mathcal{S}$ of $\mathcal{H}$ whose weight $|\mathcal{S}|_w = \sum_{s\in\mathcal{S}} w(s)$ is at most $\frac{1}{3}|\mathcal{E}|_w = \frac{1}{3}\sum_{\{x,y,z\}\in\mathcal{E}} w(x)w(y)w(z)$.*

Proof Suppose that a simple graph G satisfies $\mathcal{H} = \mathcal{H}_G$ and $\mathcal{S}_w$ is the output of Algorithm 1 with input G_w. It is not hard to check that $\sum_{s\in\mathcal{S}} w(s) = |\mathcal{S}_w|$ and $\sum_{\{x,y,z\}\in\mathcal{E}} w(x)w(y)w(z) = \|\mathcal{H}_{G_w}\|$. The result is instant from Corollary 2.3. □

In case of finding a minimal FES in $\mathcal{H}_{G_w}$, we remove redundant triangles (edges of $\mathcal{H}_{G_w}$) from any given FES. Using a similar idea to Observation 3.2, once a triangle is removed we may remove all the triangles on the same vertex set as the removed triangle. This fact in combination with Lemma 2.5 implies strong polynomial-time computation for a minimal FES $\mathcal{F}$ in $\mathcal{H}_G$ whose weighted cardinality $|\mathcal{F}|_w$ satisfies the following corollary.

Corollary 3.5 *Given any linear triangle hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and any $\mathbf{w} \in \mathbb{Z}^{\mathcal{V}}_{>0}$, if $\mathcal{H}$ consists of p components, then a FES $\mathcal{F}$ of $\mathcal{H}$ with $|\mathcal{F}|_w \le 2|\mathcal{E}|_w - |\mathcal{V}|_w + p$ can be found in $O(|\mathcal{E}|^2)$ time.*

3.2 Graphs with Large Weighted Triangle Packing Numbers

We investigate (weighted) Tuza's conjecture for graphs with large weighted packing numbers, which are firstly compared with the weighted number of triangles, and then with the weighted number of edges.

3.2.1 Comparing with the Weighted Number of Triangles

Theorem 3.6 *Let G be a simple graph and $\mathbf{w} \in \mathbb{Z}^{E(G)}_{>0}$. If real number $c \in (0, 1]$ satisfies $\nu_t(G, \mathbf{w})/|\mathscr{T}_G|_w \ge c$, then a triangle cover of $(G, \mathbf{w})$ with weight at most $\frac{3c+1}{3c}\nu_t(G, \mathbf{w})$ can be found in $O(|\mathscr{T}_G|^3)$ time. Consequently, $\tau_t(G, \mathbf{w})/\nu_t(G, \mathbf{w}) \le \frac{3c+1}{3c}$.*

Proof We consider the triangle hypergraph $\mathcal{H}_G = (E, \mathscr{T}_G)$ of G, which is 3-uniform and linear. By Theorem 3.4, we can find in $O(|\mathscr{T}_G|^3)$ time a FVS $\mathcal{S}$ of

$\mathcal{H}_G$ with $|\mathcal{S}|_w \le |\mathscr{T}_G|_w/3$. Since $\nu(\mathcal{H}_G, \mathbf{w}) = \nu_t(G, \mathbf{w}) \ge c|\mathscr{T}_G|_w$, it follows that $|\mathcal{S}|_w \le \nu(\mathcal{H}_G, \mathbf{w})/(3c)$. As $\mathcal{H}_G \setminus \mathcal{S}$ is acyclic, Theorem 2.6 enables us to find in $O(|\mathscr{T}_G|^2 \log^2 |\mathscr{T}_G|)$ time a minimum weighed transversal $\mathcal{R}$ of $\mathcal{H}_G \setminus \mathcal{S}$ such that $|\mathcal{R}|_w = \tau(\mathcal{H}_G \setminus \mathcal{S}, \mathbf{w}|_{E-\mathcal{S}}) = \nu(\mathcal{H}_G \setminus \mathcal{S}, \mathbf{w}|_{E-\mathcal{S}})$. We observe that $\mathcal{S} \cup \mathcal{R} \subseteq E$ and $G \setminus (\mathcal{S} \cup \mathcal{R})$ is triangle-free. Hence $\mathcal{S} \cup \mathcal{R}$ is a triangle cover of G with weight

$$|\mathcal{S}\cup\mathcal{R}|_w \le \frac{\nu(\mathcal{H}_G, \mathbf{w})}{3c} + \nu(\mathcal{H}_G \setminus \mathcal{S}, \mathbf{w}|_{E-\mathcal{S}}) \le \frac{3c+1}{3c}\nu(\mathcal{H}_G, \mathbf{w}) = \frac{3c+1}{3c}\nu_t(G, \mathbf{w}),$$

which proves the theorem. □

The special case of $c = 1/3$ in the above theorem gives the following result providing a new sufficient condition for the weighted version of Tuza's conjecture.

Corollary 3.7 *If simple graph G and edge weight $\mathbf{w} \in \mathbb{Z}_+^{E(G)}$ satisfy the inequality $\nu_t(G, \mathbf{w})/|\mathscr{T}_G|_w \ge 1/3$, then $\tau_t(G, \mathbf{w})/\nu_t(G, \mathbf{w}) \le 2$.*

The Unweighted Case In the special case of unit weight, the condition $\nu_t(G) \ge |\mathscr{T}_G|/3$ in Corollary 3.7 applies, in some sense, only to the class of large scale sparse simple graphs (which, e.g., does not include complete graphs on four or more vertices). The mapping from the real number c in the condition $\nu_t(G) \ge c|\mathscr{T}_G|$ to the coefficient $\frac{3c+1}{3c}$ in the conclusion $\tau_t(G) \le \frac{3c+1}{3c}\nu_t(G)$ of Theorem 3.6 shows the trade-off between conditions and conclusions. As in Corollary 3.7, $c = \frac{1}{3}$ maps to $\frac{3c+1}{3c} = 2$ hitting the boundary of Tuza's conjecture. It remains to study graphs G with $\nu_t(G)/|\mathscr{T}_G| < \frac{1}{3}$. The next theorem (Theorem 3.8) tells us that actually we only need to take care of graphs G with $\nu_t(G)/|\mathscr{T}_G| \in (\frac{1}{4} - \epsilon, \frac{1}{3})$, where ϵ can be any arbitrarily small positive number. So, in some sense, to solve Tuza's conjecture as well as its weighted generalization, we only have a gap of $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ to be bridged. Interestingly, for $c = \frac{1}{4}$, we have $\frac{3c+1}{3c} = \frac{7}{3} = 2.333\ldots$, which is much better than the best known general bound 2.87 due to Haxell [3]. Only when $c \le \frac{1}{6}$ does $\frac{3c+1}{3c}$ state a trivial bound equal to or greater than 3.

Theorem 3.8 *If there exists some real $\delta > 0$ such that Conjecture 1.1 holds for every simple graph (resp. multigraph) G with $\nu_t(G)/|\mathscr{T}_G| \ge 1/4 - \delta$, then Conjecture 1.1 holds for every simple graph (resp. multigraph).*

Proof If $\delta \ge \frac{1}{4}$, the theorem is trivial. We consider $0 < \delta < \frac{1}{4}$. As the set of rational numbers is dense, we may assume $\delta \in \mathbb{Q}$ and $1/4 - \delta = i/j$ for some $i, j \in \mathbb{Z}_{>0}$. Therefore $i/j < 1/4$ gives $4i + 1 \le j$, i.e., $4 + 1/i \le j/i$. It remains to prove that for any graph G with $\nu_t(G) < (i/j)|\mathscr{T}_G|$ there holds $\tau_t(G) \le 2\nu_t(G)$.

Write k for the positive integer $i|\mathscr{T}_G| - j \cdot \nu_t(G)$. Let G' be the disjoint union of G and k copies of K_4. Clearly, $|\mathscr{T}_{G'}| = |\mathscr{T}_G| + k|\mathscr{T}_{K_4}| = |\mathscr{T}_G| + 4k$, $\tau_t(G') =$

$\tau_t(G) + k \cdot \tau_t(K_4) = \tau_t(G) + 2k$ and $\nu_t(G') = \nu_t(G) + k \cdot \nu_t(K_4) = \nu_t(G) + k$. It follows that

$$\begin{aligned}(i/j)|\mathscr{T}_{G'}| &= (i/j)(|\mathscr{T}_G| + 4k)\\ &= (i/j)((k + j \cdot \nu_t(G))/i + 4k)\\ &= (i/j)(j \cdot \nu_t(G)/i + (4 + 1/i)k)\\ &\le \nu_t(G) + k\\ &= \nu_t(G')\end{aligned}$$

where the inequality is guaranteed by $4 + 1/i \le j/i$. So $\nu_t(G') \ge (1/4 - \delta)|\mathscr{T}_{G'}|$ together with the hypothesis of the theorem implies $\tau_t(G') \le 2\nu_t(G')$, i.e., $\tau_t(G) + 2k \le 2(\nu_t(G) + k)$, giving $\tau_t(G) \le 2\nu_t(G)$ as desired. □

In the proof of the above theorem, the properties of K_4 that $\nu_t(K_4)/|\mathscr{T}_{K_4}| = 1/4$ and $\tau_t(K_4)/\nu_t(K_4) = 2$ plays an important role. It helps to reduce the general (weighted) Tuza's conjecture to the special case where $\nu_t(G) \ge (1/4 - \delta)|\mathscr{T}_G|$. We emphasize that the statement for multigraphs in Theorem 3.8 amounts to an equivalent result for Conjecture 1.2, because Conjecture 1.1 true for multigraphs is equivalent to Conjecture 1.2 true for weighted simple graphs.

More Efficient Computation for Unweighted Simple Graphs Given a graph G and an edge subset $F \subseteq E(G)$, we use $G[F]$ to denote the subgraph G induced by F. We call G *triangle-divisible* if $E(G)$ admits a nontrivial partition (E_1, E_2) such that $(\mathscr{T}_{G[E_1]}, \mathscr{T}_{G[E_2]})$ is a nontrivial partition of $\mathscr{T}_G$, and call G *triangle-indivisible*, otherwise. For example, in Fig. 7, the left graph is triangle-indivisible, but the right graph is not (where partitioning edges into thick ones and bold ones shows its triangle divisibility). The following are straightforward observations.

Observation 3.9 (i) An irreducible nonempty graph $G = (V, E)$ is triangle-indivisible if and only if its associated triangle hypergraph $\mathcal{H}_G = (E, \mathscr{T}_G)$ is connected.
(ii) Conjecture 1.1 is true if the conjectured inequality holds for every triangle-indivisible simple graph.

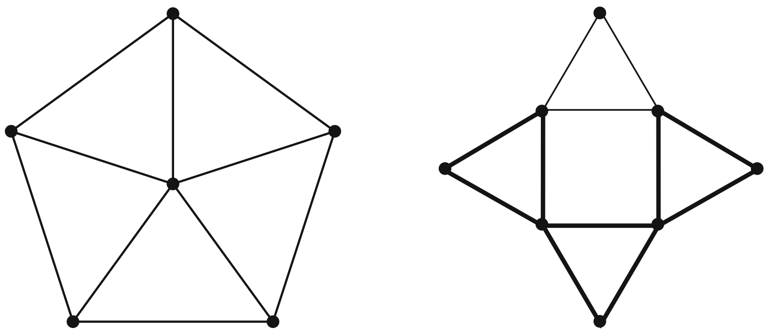


Fig. 7 Examples for triangle (in)divisibility

For simple graphs G, with the help of Algorithm 2 we can improve the $O(|\mathscr{T}_G|^3)$ time complexity stated in Theorem 3.6.

Corollary 3.10 *Let $G = (V, E)$ be a simple triangle-indivisible graph, if $\nu_t(G)/|\mathscr{T}_G| \geq 1/3$, then a triangle cover of G with cardinality at most $2\nu_t(G)$ can be found in $O(|E|(|\mathscr{T}_G| + |E| \log |E|) \log |E|)$ time.*

Proof We consider the triangle hypergraph $\mathcal{H}_G = (E, \mathscr{T}_G)$ of G, which is 3-uniform and linear. Note from Observation 3.9(i) that $\mathcal{H}_G = (E, \mathscr{T}_G)$ is connected. Thus by Corollary 2.11, we obtain in $O(|E|(|\mathscr{T}_G|+|E| \log |E|) \log |E|)$ time a triangle cover of G with cardinality at most $\frac{2|\mathscr{T}_G|+1}{3} \leq 2\nu_t(G) + \frac{1}{3}$. □

3.2.2 Comparing with the Weighted Number of Edges

The sufficient condition that compares the weighted triangle packing number with the weights of edges is based on the fact that every edge-weighted simple graph $(G, \mathbf{w})$ has a bipartite subgraph whose edges have a total weight at least $|E(G)|_w/2$, and such a bipartite subgraph can be found in strongly polynomial time. Since this subgraph does not contain any triangle, we deduce that $\tau_t(G, \mathbf{w}) \leq |E|_w/2$, which implies the following result.

Corollary 3.11 *Let $G = (V, E)$ be a graph with edge weight $\mathbf{w} \in \mathbb{Z}^E_{>0}$. If $\nu_t(G, \mathbf{w})/|E|_w \geq c$ for some $c > 0$, then $\tau_t(G, \mathbf{w})/\nu_t(G, \mathbf{w}) \leq 1/(2c)$. In particular, if $\nu_t(G, \mathbf{w})/|E|_w \geq 1/4$, then $\tau_t(G, \mathbf{w})/\nu_t(G, \mathbf{w}) \leq 2$.*

Thus if $\nu_t(G, \mathbf{w})/|E|_w \geq c$ for some $c > 0$, then a triangle cover of G with weight at most $\nu_t(G, \mathbf{w})/(2c)$ can be found in strongly polynomial time. When restricted to unweighted simple graphs, complementary to Corollary 3.7 whose condition mainly takes care of sparse graphs, the second statement of Corollary 3.11 applies to many dense graphs, including all complete graphs of order at least 3 other than K_5 (see Theorem 2 in [14]).

Similarly to Corollary 3.7 and Theorem 3.8, by which our future investigation space on (weighted) Tuza's conjecture shrinks to interval $(\frac{1}{4} - \epsilon, \frac{1}{3})$ w.r.t. $\nu_t(G, \mathbf{w})/|\mathscr{T}_G|_w$, Corollary 3.11 and the following Theorem 3.12 narrow the interval w.r.t. $\nu_t(G, \mathbf{w})/|E|_w$ to $(\frac{1}{5} - \epsilon, \frac{1}{4})$. Moreover, when taking $c = \frac{1}{5}$ in Corollary 3.11. we obtain $\frac{1}{2c} = 2.5$, still better than Haxell's general bound 2.87 for unweighted simple graphs [3].

Theorem 3.12 *If there exists some real $\delta > 0$ such that Conjecture 1.1 holds for every simple graph (resp. multigraph) G with $\nu_t(G)/|E(G)| \geq 1/5 - \delta$, then Conjecture 1.1 holds for every simple graph (resp. multigraph).*

Proof We use a similar trick to that in proving Theorem 3.8; we add a number of complete graphs on five (instead of four) vertices. We may assume $\delta \in (0, \frac{1}{5}) \cap \mathbb{Q}$

and $1/5 - \delta = i/j$ for some $i, j \in \mathbb{Z}_{>0}$. Therefore $i/j < 1/5$ and the integrality of i, j imply $5 + 1/i \leq j/i$. To prove Tuza's conjecture for each graph G with $\nu_t(G) < (i/j)|E|$, we write $k = i|E| - j \cdot \nu_t(G) \in \mathbb{Z}_{>0}$. Let $G' = (V', E')$ be the disjoint union of G and k copies of K_5's. Then $|E'| = |E| + 10k$, $\tau_t(G') = \tau_t(G) + k \cdot \tau_t(K_5) = \tau_t(G) + 4k$, $\nu_t(G') = \nu_t(G) + k \cdot \nu_t(K_5) = \nu_t(G) + 2k$, and

$$(i/j)|E'| = (i/j)(|E|+10k) = (i/j)(j \cdot \nu_t(G)/i + (10+1/i)k) \leq \nu_t(G) + 2k = \nu_t(G')$$

where the inequality is guaranteed by $10 + 1/i \leq 2j/i$. So $\nu_t(G') \geq (1/5 - \delta)|E'|$ together with the hypothesis the theorem implies $\tau_t(G') \leq 2\nu_t(G')$, i.e., $\tau_t(G) + 4k \leq 2(\nu_t(G) + 2k)$, giving $\tau_t(G) \leq 2\nu_t(G)$ as desired. □

Erdős-Rényi Graphs with High Densities As an application of Corollary 3.11 we investigate Tuza's conjecture on random graphs. Let n be a positive integer and let $p \in [0, 1]$. The Erdős-Rényi random graph model [15] is a probability space over the set $\mathcal{G}(n, p)$ of simple graphs $G = (V, E)$ on the vertex set $V = \{1, \ldots, n\}$, where an edge between vertices i and j is included in E with probability p independently from every other possible edge, i.e.,

$$\mathbf{Pr}[ij \in E] = p \text{ for each pair of distinct } i, j \in V.$$

The $\mathcal{G}(n, p)$ model is often used in the probabilistic method for tackling problems in various areas such as graph theory and combinatorial optimization.

The following result on the triangle packing numbers of complete graphs [16] is useful in deriving a good estimation for the triangle packing numbers of graphs in $\mathcal{G}(n, p)$.

Theorem 3.13 ([16]) $\nu_t(K_n) = |E(K_n)|/3$ *if and only if* $n \equiv 1, 3 \pmod 6$.

Theorem 3.14 *Suppose that* $p > \sqrt{3}/2$ *and* $G = (V, E) \in \mathcal{G}(n, p)$. *Then* $\boldsymbol{Pr}[\nu_t(G) \geq |E|/4] = 1 - o(1)$ *and* $\boldsymbol{Pr}(\tau_t(G) \leq 2\nu_t(G)) = 1 - o(1)$.

Proof Let K_n denote the complete graph on V. For each edge $e \in K_n$, let X_e be the indicator variable satisfying: $X_e = 1$ if $e \in E$ and $X_e = 0$ otherwise. Thus $\mathbf{E}[X_e] = p$, $X = \sum_{e \in K_n} X_e = |E|$, $\mathbf{E}[X] = n(n-1)p/2$. Since $X_e, e \in K_n$, are independent 0-1 variables, by Chernoff Bounds [15], for each $\epsilon \in (0, 1]$, $\mathbf{Pr}[X > (1+\epsilon)\mathbf{E}[X]] \leq exp(-\epsilon^2\mathbf{E}[X]/3) = exp(-\epsilon^2 n(n-1)p/6) = o(1)$. So

$$\mathbf{Pr}[X \leq (1+\epsilon)\mathbf{E}[X]] = \mathbf{Pr}(X \leq (1+\epsilon)n(n-1)p/2) = 1 - o(1).$$

On the other hand, by Theorem 3.13, we can make K_n have an edge-disjoint triangle decomposition by deleting at most three vertices, which implies that $\nu_t(K_n)$ is lower bounded by $k = \lceil (n-3)(n-4)/6 \rceil$. Thus we can take k edge-disjoint triangles $T_1, \ldots, T_k$ from K_n. For each $i \in [k]$, let Y_i be the indicator variable satisfying: $Y_i = 1$ if $T_i \subseteq G$ and $Y_i = 0$ otherwise. Note that $\mathbf{E}[Y_i] = p^3$ for each $i \in [k]$, $\nu_t(G) \geq Y = \sum_{i=1}^k Y_i$ and $\mathbf{E}[Y] = kp^3$. Because $T_1, \ldots, T_k$ are edge-disjoint, $Y_1, \ldots, Y_k$ are independent 0-1 variables. By Chernof f Bounds, for each $\epsilon \in (0, 1)$,

$\mathbf{Pr}[Y < (1-\epsilon)\mathbf{E}[Y]] \le exp(-\epsilon^2\mathbf{E}[Y]/2) \le exp(-\epsilon^2(n-3)(n-4)p^3/12) = o(1)$.Thus

$$\mathbf{Pr}[\nu_t(G) \ge (1-\epsilon)(n-3)(n-4)p^3/6] \ge \mathbf{Pr}[\nu_t(G) \ge (1-\epsilon)kp^3] \ge \mathbf{Pr}[Y \ge (1-\epsilon)\mathbf{E}[Y]] = 1-o(1).$$

Recall that $p > \sqrt{3}/2$. We can take $\epsilon \in (0,1)$ such that $\lim_{n\to\infty} \frac{(1-\epsilon)(n-3)(n-4)p^3/6}{(1+\epsilon)n(n-1)p/8} = \frac{4p^2(1-\epsilon)}{3(1+\epsilon)} > 1$. So for sufficient large n, we have $(1-\epsilon)(n-3)(n-4)p^3/6 > (1+\epsilon)n(n-1)p/8$. Since we have $\nu_t(G) \ge (1-\epsilon)(n-3)(n-4)p^3/6$ with probability $1-o(1)$ and have $|E| = X \le (1+\epsilon)n(n-1)p/2$ with probability $1-o(1)$, we obtain $\nu_t(G) \ge |E|/4$ with probability $1-o(1)$. It follows from Corollary 3.11 that $\mathbf{Pr}(\tau_t(G) \le 2\nu_t(G)) = 1-o(1)$. □

3.3 Graphs with Large Weighted Numbers of Edges

Each graph has a unique maximum irreducible subgraph. Tuza's conjecture, as well as its weighted generalization, is valid for a graph if and only if the conjecture is valid for its maximum irreducible subgraph. In this section, we study sufficient conditions for (weighted) Tuza's conjecture on irreducible graphs that bound the (weighted) number of edges below in terms of the (weighted) number of triangles.

Theorem 3.15 *Let G be an irreducible simple graph and $\mathbf{w} \in \mathbb{Z}_{>0}^{E(G)}$. If $|E|_w/|\mathscr{T}_G|_w \ge 2$, then a triangle cover of $(G,\mathbf{w})$ with weight at most $2\nu_t(G,\mathbf{w})$ can be found in $O(|\mathscr{T}_G|^2\log^2|\mathscr{T}_G|)$ time. Consequently, implies $\tau_t(G,\mathbf{w})/\nu_t(G,\mathbf{w}) \le 2$.*

Proof Recall that the irreducibility of G implies $|E| \le 3|\mathscr{T}_G|$. Suppose that the linear 3-uniform hypergraph $\mathcal{H}_G = (E, \mathscr{T}_G)$ associated to G has exactly p components. By Corollary 3.5, we can find in $O(|\mathscr{T}_G|^2)$ time a minimal FES $\mathcal{F}$ of $\mathcal{H}_G$ such that $|\mathcal{F}|_w \le 2|\mathscr{T}_G|_w - |E|_w + p \le p$. Since G is irreducible, we see that $\mathcal{H}_G$ has no isolated vertices, i.e., every component of $\mathcal{H}_G$ has at least one edge. Thus $\nu(\mathcal{H}_G,\mathbf{w}) \ge p \ge |\mathcal{F}|_w$. For the acyclic hypergraph $\mathcal{H}_G \setminus \mathcal{F}$, by Theorem 2.6 we may find in $O(|\mathscr{T}_G|^2\log^2|\mathscr{T}_G|)$ time a minimum weighted transversal $\mathcal{R}$ of $\mathcal{H}_G \setminus \mathcal{F}$ such that

$$|\mathcal{R}|_w = \tau(\mathcal{H}_G \setminus \mathcal{F}, \mathbf{w}) = \nu(\mathcal{H}_G \setminus \mathcal{F}, \mathbf{w}).$$

Observe that $\mathcal{R} \subseteq E$ and $\mathcal{F} \subseteq \mathscr{T}_G$. For each $F \in \mathcal{F}$, take $e_F \in E$ with $e_F \in F$, and set $\mathcal{S} = \{e_F : F \in \mathcal{F}\}$. It is clear that $\mathcal{R} \cup \mathcal{S}$ is a transversal of $\mathcal{H}$ (i.e., a triangle cover of G) with weight $|\mathcal{R} \cup \mathcal{S}|_w \le \nu(\mathcal{H}_G \setminus \mathcal{F}, \mathbf{w}) + |\mathcal{F}|_w \le 2\nu(\mathcal{H}_G,\mathbf{w}) = 2\nu_t(G,\mathbf{w})$, establishing the theorem. □

We observe that the unweighted simple graphs G which consist of a number of triangles sharing a common edge satisfy $|E(G)| \ge 2|\mathscr{T}_G|$, but $\nu_t(G) < |\mathscr{T}_G|/3$ when $|\mathscr{T}_G| \ge 4$. So in some sense, Theorem 3.15 works a supplement of Corollary 3.7 for unweighted sparse simple graphs.

Along the same line as in the preceeding subsection, regarding weighted Tuza's conjecture on graph G, Theorem 3.15 and the following Theorem 3.16 jointly narrow the interval w.r.t. $|E(G)|_w/|\mathscr{T}_G|_w$ to $(1.5-\epsilon, 2)$ for future study.

Theorem 3.16 *If there exists some real $\delta > 0$ such that Conjecture 1.1 holds for every irreducible simple graph (resp. multigraph) $G = (V, E)$ with $|E|/|\mathscr{T}_G| \geq 3/2 - \delta$, then Conjecture 1.1 holds for every irreducible simple graph (resp. multigraph), and therefore holds for every simple graph (resp. multigraph).*

Proof Again we apply the trick of adding copies of K_4. We may assume $\delta \in (0, 3/2) \cap \mathbb{Q}$ and $3/2-\delta = i/j$ for some $i, j \in \mathbb{Z}_{>0}$. Therefore $2i+1 \leq 3j$, implying $(i/j)(4+1/i) \leq 6$.

For any irreducible graph G with $|E| < (i/j)|\mathscr{T}_G|$, we write $k = i|\mathscr{T}_G| - j|E| \in \mathbb{Z}_{>0}$. Let G' be the disjoint union of G and k copies of K_4. Then G' is irreducible, and

$$(i/j)|\mathscr{T}_{G'}| = (i/j)(|\mathscr{T}_G| + 4k) = (i/j)(j|E|/i + (4+1/i)k) \leq |E| + 6k = |E'|.$$

It follows from the hypothesis of the theorem that $\tau_t(G') \leq 2\nu_t(G')$, i.e., $\tau_t(G) + 2k \leq 2(\nu_t(G) + k)$, giving $\tau_t(G) \leq 2\nu_t(G)$ as desired. □

4 Conclusion

Using tools from hypergraphs, we design strongly polynomial-time algorithms for finding small (weighted) triangle covers in graphs, which particularly imply several sufficient conditions for Tuza's conjecture (Conjecture 1.1) and its weighted version (Conjecture 1.2).

Triangle Packing and Covering In this paper, we have established new sufficient conditions $\nu_t(G, \mathbf{w})/|\mathscr{T}_G|_w \geq 1/3$ and $|E|_w/|\mathscr{T}_G|_w \geq 2$ for weighted Tuza's conjecture on packing and covering triangles in graphs G. We prove the sufficiency by designing polynomial-time combinatorial algorithms for finding a triangle cover of G whose weight (i.e., weighted cardinality) is upper bounded by $2\nu_t(G, \mathbf{w})$. The high level idea of these algorithms is to remove *some edges* from G so that the triangle hypergraph of the remaining graph is *acyclic* (see the proofs of Theorems 2.1 and 3.15), which guarantees that the remaining graph has equal weighted triangle covering number and weighted triangle packing number, and a minimum weighted triangle cover of the remaining graph is computable in strongly polynomial time (see Theorem 2.6). It is well known that the acyclic condition in Theorem 2.6 could be weakened to odd-cycle-freeness [11]. So the lower bound 1/3 and 2 in the sufficient conditions could be (significantly) improved if we can remove edges of a (much) less total weight from G such that the triangle hypergraph of the remaining graph is *odd-cycle free*.

In view of Theorems 3.8, 3.12, and 3.16, the study on the graphs $(G, \mathbf{w})$ satisfying $\nu_t(G, \mathbf{w})/|\mathscr{T}_G|_w \geq 1/4$ or $\nu_t(G, \mathbf{w})/|E|_w \geq 1/5$ or $|E|_w/|\mathscr{T}_G|_w \geq 3/2$ might suggest more insight and foresight for resolving (weighted) Tuza's conjecture. These graphs are critical in the sense that they are standing on the border of the resolution.

Regarding Theorem 3.6, Corollary 3.11, and Theorem 3.15, Gregory J. Puleo made a nice observation that $\nu_t(G)/|\mathscr{T}_G| \le 1/3$, $\nu_t(G)/|E| \le 1/4$, and $|E|/|\mathscr{T}_G| \le 2$ hold for every irreducible extremal graph $G = (V, E)$ – simple graphs satisfying Tuza's conjecture with tight ratio $\tau_t(G)/\nu_t(G) = 2$. So studying extremal graphs might lead to interesting results.

Another intermediate step toward resolving Tuza's conjecture is investigating its validity for the classical Erdős-Rényi random graph model $\mathcal{G}(n, p)$. In this paper, we have shown that Tuza's conjecture holds with high probability for graphs in $\mathcal{G}(n, p)$ when $p > \sqrt{3}/2$. It would be nice to prove the same result for $p \in (0, \sqrt{3}/2]$.

The Generalization to Linear 3-Uniform Hypergraphs Our work has shown very close relations between triangle packing and covering in graphs and edge (resp. cycle) packing and covering in linear 3-uniform hypergraphs. The theoretical and algorithmic results on linear 3-uniform hypergraphs (Corollary 2.3 and Lemma 2.5) are crucial for us to establish sufficient conditions for Tuza's conjecture, and to find in polynomial time a "small" triangle cover under the conditions (see Corollary 3.7 and Theorem 3.15). Recall that, for any simple graph G, its triangle hypergraph $\mathcal{H}_G$ is linear 3-uniform, and Tuza's conjecture is equivalent to $\tau(\mathcal{H}_G) \le 2\nu(\mathcal{H}_G)$. As a natural generalization, one may ask: Does $\tau(\mathcal{H}) \le 2\nu(\mathcal{H})$ hold for all linear 3-uniform hypergraphs $\mathcal{H}$? The answer is negative – as Zbigniew Lonc noticed, the hypergraph of Fano plane is a linear 3-uninform hypergraph with transversal number 3 and matching number 1.

Last but not the least, the arguments in this paper have actually proved the following stronger result on 3-uniform hypergraphs.

Theorem 4.1 *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a linear 3-uniform hypergraph without isolated vertices, and $\mathbf{w} \in \mathbb{Z}_{>0}^{\mathcal{V}}$. Then a transversal of $\mathcal{H}$ with weight at most $2\nu(\mathcal{H}, \mathbf{w})$ can be found in strongly polynomial time, which implies $\tau(\mathcal{H}, \mathbf{w}) \le 2\nu(\mathcal{H}, \mathbf{w})$, if one of the following conditions is satisfied:*

(i) $\nu(\mathcal{H}, \mathbf{w})/|\mathcal{E}|_w \ge 1/3$,
(ii) $|\mathcal{V}|_w/|\mathcal{E}|_w \ge 2$.

Comparing the above result on linear 3-uniform hypergraphs $\mathcal{H}$ with its counterpart on simple graphs presented in Theorem 1.4, one might notice that the condition on the lower bound of $\nu(\mathcal{H}, \mathbf{w})/|\mathcal{V}|_w$ is missing. This reason is that we do not have a nontrivial constant upper bound on $\tau(\mathcal{H}, \mathbf{w})/|\mathcal{V}|_w$. Again, Theorem 4.1 implies the same result for unit-weighted 3-uniform hypergraphs $\mathcal{H}$ which may not be linear.

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