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The Outer-connected Domination Number of Sierpiński-like Graphs

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Abstract An outer-connected dominating set in a graph $G = (V, E)$ is a set of vertices $D \subseteq V$ satisfying the condition that, for each vertex $v \notin D$, vertex v is adjacent to some vertex in D and the subgraph induced by $V \setminus D$ is connected. The outer-connected dominating set problem is to find an outer-connected dominating set with the minimum number of vertices which is denoted by $\tilde{\gamma}_c(G)$. In this paper, we determine $\tilde{\gamma}_c(S(n, k))$, $\tilde{\gamma}_c(S^+(n, k))$, $\tilde{\gamma}_c(S^{++}(n, k))$, and $\tilde{\gamma}_c(S_n)$, where $S(n, k)$, $S^+(n, k)$, $S^{++}(n, k)$, and S_n are Sierpiński-like graphs.

Keywords Outer-connected domination · Dominating set · Sierpiński graphs · Extended Sierpiński graphs · Sierpiński-like graphs

1 Introduction

Let $G = (V, E)$ be an undirected graph, where $V(G)$ and $E(G)$ are vertex and edge sets of G respectively. For simplicity, we also use V and E to represent $V(G)$ and $E(G)$, respectively, when only one graph is mentioned. All graphs considered in this paper are simple, i.e., with no loops and multiple edges. For any vertex $v \in V$ and a set $S \subseteq V$, the *open neighborhood* of v in S is the set $N_S(v) = \{u \in S | uv \in E\}$. The

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closed neighborhood of v in S is $N_S[v] = N_S(v) \cup \{v\}$. If $S = V$, then we simply write $N(v)$ and $N[v]$ rather than $N_V(v)$ and $N_V[v]$, respectively.

Definition 1.1 For a graph G, a set $D \subseteq V$ is a *dominating set* if $N[D] = V$. The minimum size of a dominating set is the *domination number*, denoted by $\gamma(G)$. The *domination* problem is to determine a minimum dominating set of a graph G.

Definition 1.2 For a graph G, a dominating set D is an *outer-connected dominating set*, abbreviated as *OCD-set*, if the subgraph induced by $V \backslash D$ is connected. The *outer-connected domination number*, denoted by $\tilde{\gamma}_c(G)$, is the cardinality of a minimum OCD-set. The *outer-connected domination* problem is to determine a minimum OCD-set of a graph G.

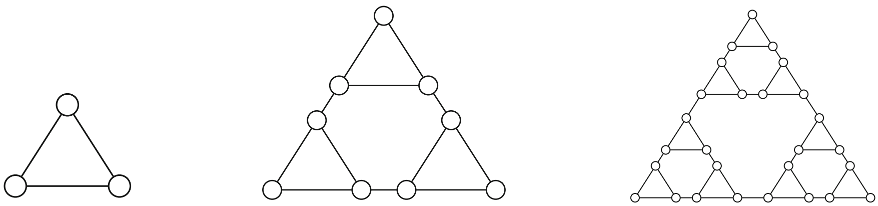
It is clear that $\gamma(G) \leqslant \tilde{\gamma}_c(G)$. The concept of outer-connected domination problem in graphs was introduced in [3] and subsequently studied in [1, 11, 20]. The outer-connected domination problem has been shown to be NP-complete for bipartite graphs [3], doubly chordal graphs and undirected path graphs [20], where a graph G is called an *undirected path graph* if G is the intersection graph of a family of paths of a tree. In [20], MarkKeil and Pradhan proposed a linear time algorithm for computing a minimum OCD-set in proper interval graphs.

The *Sierpiński graph* $S(n, k)$ consists of k copies of $S(n-1, k)$ for $n > 1$, where $S(1, k)$ is a complete graph of k vertices [13]. Graphs very similar to Sierpiński graphs were named *WK-recursive networks* in [25].

For example, $S(1, 3)$, $S(2, 3)$, and $S(3, 3)$ are shown in Fig. 1a, b, and c, respectively. In general, Sierpiński-like graphs include Sierpiński graphs, extended Sierpiński graphs, and Sierpiński gasket graphs. All those Sierpiński-like graphs will be introduced in Section 2.

The results of this paper are as follows:

(1) $\tilde{\gamma}_c(S(n, k)) = k^{n-1}$, for $n \geqslant 1$ and $k \geqslant 3$,
(2) $\tilde{\gamma}_c(S^{+}(n, k)) = k^{n-1}$, for $n \geqslant 1$ and $k \geqslant 3$,
(3) $\tilde{\gamma}_c(S^{++}(n, k)) = k^{n-1} + k^{n-2}$, for $n \geqslant 1$ and $k \geqslant 3$, and
(4) $\tilde{\gamma}_c(S_n) = 3^{n-2}$ if $n \geqslant 3$; otherwise, $\tilde{\gamma}_c(S_n) = n$,



(a) $S(1,3)$ (b) $S(2,3)$ (c) $S(3,3)$

Fig. 1 Sierpiński graphs

where $S(n,k)$ denotes a Sierpiński graph, $S^{+}(n,k)$ and $S^{++}(n,k)$ denote two different extended Sierpiński graphs, and S_n denotes a Sierpiński gasket graph.

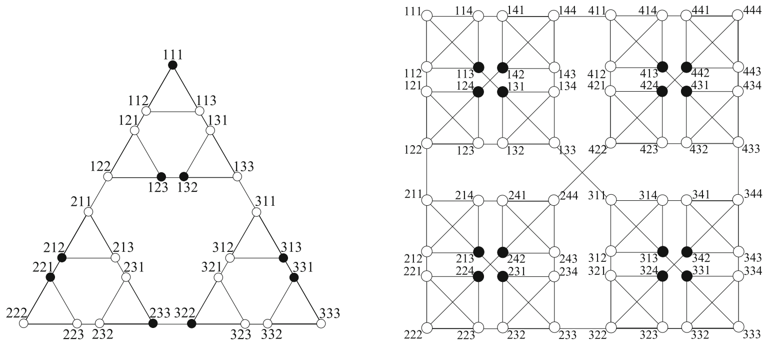
The organization of this paper is as follows. In Section 2, we introduce Sierpiński-like graphs in detail. The outer-connected domination number of Sierpiński graphs and extended Sierpiński graphs are investigated in Sections 3 and 4, respectively. In Section 5, we investigate the outer-connected domination number of Sierpiński gasket graphs.

2 Sierpiński-like Graphs

The definitions of Sierpiński-like graphs are described as follows. The reader is referred to [2, 4, 7, 13, 22, 25] for the details. The vertex set of $S(n,k)$ consists of all n-tuples of integers $1, 2, \ldots, k$, for integers $n \geqslant 1$ and $k \geqslant 3$, namely $V(S(n,k)) = \{1, 2, \ldots, k\}^n$. Accordingly, the label of vertex v, denoted by $\ell(v)$, is $v_1 v_2 \cdots v_n$ in regular expression form. By using a convention on representing regular expressions, we always use w, x, y, and z to denote a substring of $v_1 v_2 \cdots v_n$ and a, b, c, and d to denote a number in $v_1 v_2 \cdots v_n$, i.e., $a, b, c, d \in \{1, 2, \ldots, k\}$. The length of a substring w is denoted by $|w|$. For example, $\ell(v) = wab^{n-h}$, for $1 \leqslant h \leqslant n$, means that the label of v begins with prefix w, then concatenates with number a, and finally ends with $n - hb's$, where b^h is the *Kleene closure* in regular expression. Thus $|w| = h - 1$. For convenience, we also say that $v_1 v_2 \cdots v_n$ is a vertex if $\ell(v) = v_1 v_2 \cdots v_n$.

Two different vertices u and v are adjacent in $S(n,k)$ if and only if $\ell(u) = wab^{n-h}$ and $\ell(v) = wba^{n-h}$ with $a \neq b$ for some $1 \leqslant h \leqslant n$. Note that if $h = 1$, then $w = \epsilon$ which is a null string. Furthermore, if $h = n$, then both b^{n-h} and a^{n-h} are empty. By the definition above, the subgraph of $S(n,k)$ induced by the set of vertices whose labels begin with a is a Sierpiński subgraph $S(n-1,k)$ and is denoted by $S_a(n-1,k)$. Similarly, we also use $S_w(n-|w|,k)$ to denote the subgraph induced by the vertices with prefix w in their labels. When $n - |w| = 1$, it is obvious that $S_w(1,k)$ is a complete graph and we call it a *terminal clique*. A vertex v is an *extreme vertex* of $S_w(n-|w|,k)$ for $0 \leqslant |w| \leqslant n-1$ if v is of the form $wa^{n-|w|}$ for $1 \leqslant a \leqslant k$. Therefore, there are exactly k extreme vertices in every $S_w(n-|w|,k)$. Since the label of an extreme vertex v is a^n in $S(n,k)$, by definition, v has exactly $k-1$ neighbors whose labels are of the form $a^{n-1}b$ with $b \neq a$. Every non-extreme vertex $\ell(v) = wab^{n-h}$ with $a \neq b$ in $S(n,k)$ has exactly k neighbors whose labels are of the form wba^{n-h} and $wab^{n-h-1}c$ with $1 \leqslant c \leqslant k$ and $c \neq b$. Thus the degree of every extreme vertex in $S(n,k)$ is $k-1$ while all other vertices have degree k. Figure 2 depicts $S(3,3)$ and $S(3,4)$. An interesting connection is that $S(n,3)$ for $n \geqslant 1$ is isomorphic to the graphs of the Tower of Hanoi puzzle with n disks [6, 13] and has been extensively studied (see [7, 9] for an overview and the references therein for the details). Lately, Hinz and Parisse determined the average eccentricity of Sierpiński graphs [8]. In [14], S. Klavžar, U. Milutinović and Ciril Petr investigate 1-perfect codes in $S(n,k)$. Parisse studied metric properties of $S(n,k)$[21].

The extended Sierpiński graphs $S^{+}(n,k)$ and $S^{++}(n,k)$ were introduced by Klavžar and Mohar [15]. The graph $S^{+}(n,k)$ is obtained from $S(n,k)$ by adding

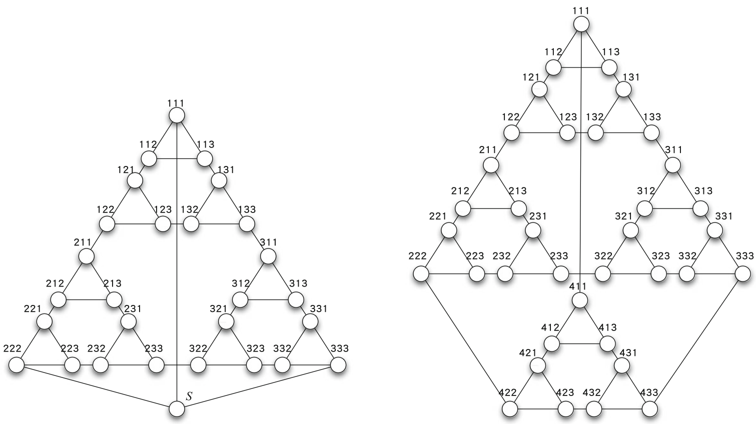


(a) $S(3,3)$ (b) $S(3,4)$

Fig. 2 Labeled Sierpiński graphs

a special vertex, say s, and edges joining s to all extreme vertices of $S(n,k)$ (see Fig. 3a). The graph $S^{++}(n,k)$ is obtained from $S(n,k)$ by adding a new copy of $S(n-1,k)$ which is denoted by $S_{k+1}(n-1,k)$, and joining an extreme vertex a^n in $S(n,k)$ to the vertex ba^{n-1} in the added $S_{k+1}(n-1,k)$ for $1 \leqslant a \leqslant k$, where $b = k+1$ (see Fig. 3b).

The *Sierpiński gasket graph* S_n is a variant of the Sierpiński graph $S(n,3)$. The graph S_n can be obtained from $S(n,3)$ by contracting every edge of $S(n,3)$ that lies in no triangle. For example, compare Figs. 2a and 4. Vertices with labels 112 and 121 in $S(3,3)$ are contracted to the vertex $1(12|21)$ in S_3, where "|" is the *union operation* in regular expression. According to the definition of extreme vertices in $S(n,k)$, the vertices with labels 1^n, 2^n, and 3^n in S_n are also called *extreme vertices*. The labels of other vertices are of the form $w(ab^h|ba^h)$ where $1 \leqslant h \leqslant n-1$, $w \in \{1,2,\ldots,k\}^{n-h-1}$, and a and b are one of the pairs: 1 and 2, 1 and 3, or 2 and



(a) $S^+(3,3)$ (b) $S^{++}(3,3)$

Fig. 3 Extended Sierpiński graphs: $S^+(3,3)$ and $S^{++}(3,3)$

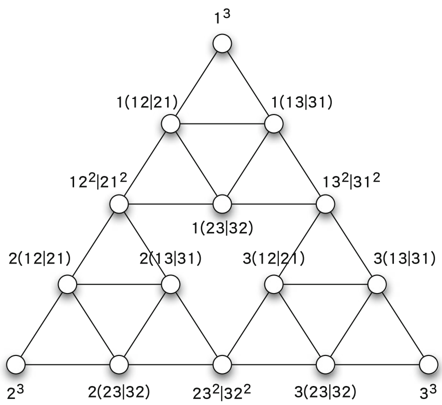


Fig. 4 Sierpiński gasket graph S_3

3. For convenience, we also use wab^h or wba^h to represent the contracted vertex $w(ab^h|ba^h)$. The vertices with labels $12^{n-1}|21^{n-1}$, $13^{n-1}|31^{n-1}$, and $23^{n-1}|32^{n-1}$ are called the *waist vertices* of S_n. The neighbors of the extreme vertex a^n are of the form $a^{n-2}(ab|ba)$ with $a \neq b$. The neighbors of vertex v with label $w(ab^h|ba^h)$ are of the form: $wab^{h-2}(bc|cb)$ and $wba^{h-2}(ad|da)$ for $c \neq b$ and $d \neq a$. The Sierpiński gasket graph S_n also contains 3^x copies of S_{n-x} which are denoted by $S_{n-x,a}$, for $a \in \{1, 2, 3\}^x$, where $S_{n-x,a}$ contains all vertices whose labels begin with a.

Many properties of Sierpiński-like graphs have been studied such as the hamiltonicity in S_n [16, 24] and in $S(n, k)$ [13], the pancyclicity in S_n [24], the efficient domination number in $S(n, k)$ [14], and the coloring number in S_n [16] and in $S(n, k)$ [21]. The vertex-, edge-, and total-colorings on Sierpiński-like graphs have been studied by Jakovac and Klavžar [10]. Lin, Liu, and Wang determined the hub numbers in [18] and global strong defensive alliances in [19] of Sierpiński-like graphs. Moreover, Sierpiński gasket graphs play an important role in dynamic systems and probability [5, 12] as well as in psychology [17, 23].

3 Computing $\tilde{\gamma}_c(S(n, k))$

For a vertex v in $S_w(n - |w|, k)$, a vertex $u \in N(v)$ is an *outer-neighbor* of v if u is not in $V(S_w(n - |w|, k))$. Note that every extreme vertex $v \in S_w(n - |w|, k)$ with $|w| \neq 0$ has exactly one outer-neighbor while a non-extreme vertex has no outer-neighbor. Furthermore, if u is the outer-neighbor of v with respect to $S_w(n - |w|, k)$, then v is also the outer-neighbor of u with respect to $S_{w'}(n - |w'|, k)$, where w' is a prefix of u and $|w'| = |w|$.

Lemma 3.1 *For $n \geqslant 1$ and $k \geqslant 3$, $\tilde{\gamma}_c(S(n, k)) \geqslant k^{n-1}$.*

Proof First, we claim that if D is an OCD-set of $S(n, k)$ and the graph induced by $V(S(n, k)) \setminus D$ is not a terminal clique, then $D \cap V(S_w(1, k)) \neq \emptyset$ for any w with $|w| = n - 1$. Suppose to the contrary that there exists a terminal clique $S_w(1, k)$ for some w such that $D \cap V(S_w(1,k)) = \emptyset$. This implies that all outer-neighbors of the vertices in $S_w(1,k)$ are in D. Let H be the graph induced by $V(S(n,k)) \setminus D$. Since H is not a terminal clique, it contains at least two components and one of them is $S_w(1,k)$. This contradicts that D is an OCD-set of $S(n,k)$ and the claim holds.

Note that there are k^{n-1} terminal cliques in $S(n,k)$. By the claim above, it follows that $\tilde{\gamma}_c(S(n,k)) \geqslant k^{n-1}$. □

By Lemma 3.1, to prove that $\tilde{\gamma}_c(S(n,k)) = k^{n-1}$, it suffices to show that there exists an OCD-set whose cardinality is equal to k^{n-1}. In the following, we describe how to construct such an OCD-set. Hereafter, the plus operation on computing the label of a vertex is always taken modulo k. However, if the resulting value is 0, then we always use k to replace it.

Definition 3.2 Let $v = v_1 \cdots v_n$ be a vertex in Sierpiński-like graphs $S(n,k)$, $S^{+}(n,k)$, $S^{++}(n,k)$ or S_n. Define $f(v) = u_1 \cdots u_{n-1}$ with $u_i \equiv v_{i+1} + v_1 - 1 \pmod{k}$ for $1 \leqslant i \leqslant n-1$ as a folding operation on v. For brevity, let $f^1(v) = f(v)$, $f^2(v) = f(f(v))$, $f^3(v) = f(f^2(v))$, and so on.

Definition 3.3 For k odd, let $F_{\mathrm{o},i}(n,k)$(or simply $F_i(n,k)$) for $1 \leqslant i \leqslant k$ be the set of vertices v with $f^{n-1}(v) = i$, namely $F_i(n,k) = \{v : f^{n-1}(v) = i\}$ for $1 \leqslant i \leqslant k$. When k is even, define $F_{\mathrm{e}}(n,k) = \{v : v_n = v_{n-1} + \frac{k}{2}\}$.

For example, in $S(3,3)$, we have $f(233) = 11$ and $f^2(233) = f(f(233)) = f(11) = 1$. The set $F_1(3,3)$ contains vertices 111, 123, 132, 212, 221, 233, 313, 322, and 331 in $S(3,3)$ (see the black vertices in Fig. 2a). For $S(3,4)$, we have $F_{\mathrm{e}}(3,4) = \{xy : x \in \{1,2,3,4\}, y \in \{13,24,31,42\}\}$ (see the black vertices in Fig. 2b). In Lemmas 3.4-3.7, we show that $F_i(n,k)$, $1 \leqslant i \leqslant k$, is an OCD-set of $S(n,k)$ with k odd, and, in Lemmas 3.8-3.9, we show that $F_{\mathrm{e}}(n,k)$, $1 \leqslant i \leqslant k$, is an OCD-set of $S(n,k)$ with k even.

Lemma 3.4 *Let $v = v_1 \cdots v_n$ be a vertex in $S(n,k)$, for $n \geqslant 1$ and $k \geqslant 3$. Then*

$$f^{n-1}(v) \equiv v_n + \sum_{i=1}^{n-1} 2^{n-i-1} \cdot (v_i - 1) \pmod{k}.$$

Proof Let $v^j = f^j(v)$ for $1 \leqslant j \leqslant n-1$ and $v^j = v_1^j \cdots v_{n-j}^j$. By definition, it is easy to derive that

$$v_1^j \equiv v_{j+1} + 2^{j-1}(v_1 - 1) + 2^{j-2}(v_2 - 1) + \cdots + 2^0(v_j - 1) \pmod{k}. \quad (1)$$

Since $f^{n-1}(v) = v^{n-1}$ and there is exactly one number in v^{n-1}, it follows that $f^{n-1}(v) = v^{n-1} = v_1^{n-1}$. By (1), we have $f^{n-1}(v) \equiv v_n + \sum_{i=1}^{n-1} 2^{n-i-1} \cdot (v_i - 1) \pmod{k}$. This completes the proof. □

Lemma 3.5 *For every terminal clique $S_w(1,k)$ in $S(n,k)$, we have $|V(S_w(1,k)) \cap F_i(n,k)| = 1$, for $1 \leqslant i \leqslant k$, $n \geqslant 1$, $k \geqslant 3$.*

Proof Suppose to the contrary that there are two distinct vertices u and v which are in $V(S_w(1,k)) \cap F_i(n,k)$ for some i with $1 \leqslant i \leqslant k$. Let $u = wu_n$ and $v = wv_n$. By Lemma 3.4, $f^{n-1}(wu_n) \equiv u_n + \sum_{i=1}^{n-1} 2^{n-i-1} \cdot (u_i - 1) \pmod{k}$ and $f^{n-1}(wv_n) \equiv v_n + \sum_{i=1}^{n-1} 2^{n-i-1} \cdot (v_i - 1) \pmod{k}$. By the definition of u and v, it follows that $u_i = v_i$ for $1 \leqslant i \leqslant n-1$. Thus $\sum_{i=1}^{n-1} 2^{n-i-1} \cdot (u_i - 1) = \sum_{i=1}^{n-1} 2^{n-i-1} \cdot (v_i - 1)$. Since both u and v are in $F_i(n,k)$ for some i with $1 \leqslant i \leqslant k$, it follows that $f^{n-1}(wu_n) = f^{n-1}(wv_n) = i$. Thus $u_n \equiv v_n \pmod{k}$. Since both u_n and v_n are smaller than or equal to k, this yields $u = v$, a contradiction. Thus $|V(S_w(1,k)) \cap F_i(n,k)| \leqslant 1$. Since $0 \leqslant f^{n-1}(v) \leqslant k-1$ and there are no two distinct vertices u and v in $V(S_w(1,k))$ such that $f^{n-1}(u) = f^{n-1}(v)$, by the pigeonhole principle, we have $|V(S_w(1,k)) \cap F_i(n,k)| = 1$ for $1 \leqslant i \leqslant k$. This completes the proof. □

Lemma 3.6 *Let D be the set $\{wa^{n-|w|} : 1 \leqslant a \leqslant k\}$ of vertices in $S(n,k)$ for $0 \leqslant |w| \leqslant n-1$. If k is odd, then $|D \cap F_i(n,k)| = 1$ for $1 \leqslant i \leqslant k$, $n \geqslant 1$, $k \geqslant 3$.*

Proof Let u and v be any two distinct vertices in D with $u = wa^{n-|w|}$ and $v = wb^{n-|w|}$. Assume without loss of generality that $a > b$ and $w = w_1 w_2 \cdots w_{|w|}$. To prove that $|D \cap F_i(n,k)| = 1$ for $1 \leqslant i \leqslant k$ when k is odd, it suffices to show that $f^{n-1}(u) \neq f^{n-1}(v)$. By Lemma 3.4,

$$f^{n-1}(u) \equiv a + \sum_{i=1}^{|w|} 2^{n-i-1} \cdot (w_i - 1) + \sum_{i=|w|+1}^{n-1} 2^{n-i-1} \cdot (a-1) \pmod{k}$$

and

$$f^{n-1}(v) \equiv b + \sum_{i=1}^{|w|} 2^{n-i-1} \cdot (w_i - 1) + \sum_{i=|w|+1}^{n-1} 2^{n-i-1} \cdot (b-1) \pmod{k}.$$

Subtracting $f^{n-1}(v)$ from $f^{n-1}(u)$, we can obtain that $f^{n-1}(u) - f^{n-1}(v) \equiv a - b + \sum_{i=|w|+1}^{n-1} 2^{n-i-1} \cdot (a-b) \pmod{k}$. After simplifying, $f^{n-1}(u) - f^{n-1}(v) \equiv 2^{n-|w|-1} \cdot (a-b) \pmod{k}$. Since $\gcd(2^{n-|w|-1}, k) = 1$ and $a - b < k$, this yields $2^{n-|w|-1} \cdot (a-b) \not\equiv 0 \pmod{k}$. This implies that $f^{n-1}(u) \neq f^{n-1}(v)$ and the lemma follows. □

Lemma 3.7 *If k is odd, then $F_i(n,k)$ is an OCD-set of $S(n,k)$ for every $1 \leqslant i \leqslant k$, $n \geqslant 1$, $k \geqslant 3$.*

Proof By Lemma 3.5, we know that $F_i(n,k)$ is a dominating set of $S(n,k)$ for every $1 \leqslant i \leqslant k$. It remains to prove that the subgraph induced by $V(S(n,k)) \setminus F_i(n,k)$ for each $1 \leqslant i \leqslant k$ is connected. We prove it by induction on n.

First, consider the basis step, i.e., $n = 1$. Clearly, $S(1,k)$ is a complete graph and $F_i(1,k)$ for each $1 \leqslant i \leqslant k$ contains exactly one vertex. Thus the basis holds immediately.

Now we consider the induction step for $n > 1$. Since $S(n,k)$ consists of k copies of $S(n-1,k)$, by the inductive hypothesis, the subgraph induced by $V(S_a(n-1,k)) \setminus$

$F_i(n,k)$ for each $1 \leqslant a \leqslant k$ is connected. It remains to prove that those induced subgraphs are connected. Assume that $1a^{n-1} \in F_i(n,k)$ for some $1 \leqslant i \leqslant k$. By Lemma 3.6, all other extreme vertices, say $1b^{n-1}$, of $S_1(n-1,k)$ with $b \neq a$ are not in $F_i(n,k)$. We claim that the outer-neighbor of $1b^{n-1}$, i.e., $b1^{n-1}$ if exists, is not in $F_i(n,k)$. Suppose to the contrary that $b1^{n-1} \in F_i(n,k)$. By Lemma 3.4,

$$\begin{aligned} f^{n-1}\left(b1^{n-1}\right) &= 1 + 2^{n-2} \cdot (b-1) + \sum_{i=2}^{n-1} 2^{n-i-1} \cdot (1-1) \pmod k \\ &= 1 + 2^{n-2} \cdot (b-1) \pmod k \\ &= i \pmod k \end{aligned}$$

and

$$\begin{aligned} f^{n-1}\left(1a^{n-1}\right) &= a + 2^{n-2} \cdot (1-1) + \sum_{i=2}^{n-1} 2^{n-i-1} \cdot (a-1) \pmod k \\ &= a + \sum_{i=2}^{n-1} 2^{n-i-1} \cdot (a-1) \pmod k \\ &= 1 + 2^{n-2} \cdot (a-1) \pmod k \\ &= i \pmod k. \end{aligned}$$

By the derivation above, we have $1 + 2^{n-2} \cdot (a-1) \equiv 1 + 2^{n-2} \cdot (b-1) \pmod k$. Since k is odd and $\gcd(2^{n-2}, k) = 1$, this results in $a = b$, a contradiction. Thus the claim holds. Therefore, $V(S(n,k)) \setminus F_i(n,k)$ induces a connected subgraph. This establishes the proof of the lemma. □

Now we consider the case where k is even in $S(n,k)$.

Lemma 3.8 *If k is even and $k > 0$, then $wa^2 \notin F_e(n,k)$.*

Proof Clearly, the equality $a = a + \frac{k}{2}$ does not hold unless $k = 0$. Thus the lemma follows immediately. □

Lemma 3.9 *If k is even, then $F_e(n,k)$ is an OCD-set of $S(n,k)$, $n \geqslant 1$, $k \geqslant 3$.*

Proof It is obvious that $F_e(n,k)$ is a dominating set of $S(n,k)$ since every terminal clique has a unique vertex in $F_e(n,k)$. All we have to prove is that the subgraph induced by $V(S(n,k)) \setminus F_e(n,k)$ is connected. We consider the following three cases.

Case 1 $n = 1$. Clearly, the subgraph induced by $V(S(n,k)) \setminus F_e(n,k)$ is connected when $n = 1$. □

Case 2 $n = 2$.

Note that, by the definition of $S(n, k)$, it follows that $k \geqslant 4$ when k is even. Accordingly, any vertex in the set $\{v : v_2 = v_1 + 1 \text{ or } v_2 = v_1 - 1\}$ is not in $F_e(2, k)$. Thus, for induced subgraphs $S_a(1, k) \setminus F_e(2, k)$ and $S_{a+1}(1, k) \setminus F_e(2, k)$ for $1 \leqslant a \leqslant k-1$, there exists an edge between vertices $a(a+1)$ and $(a+1)a$. Note that $a(a+1) \in S_a(1, k)$ and $(a+1)a \in S_{a+1}(1, k)$. Therefore, the subgraph induced by $V(S_w(2, k)) \setminus F_e(2, k)$ is connected.

Case 3 $n \geqslant 3$.

By using a similar argument as is Case 2, every subgraph induced by $V(S_w(2, k)) \setminus F_e(n, k)$ is connected, where $|w| = n-2$. For $1 \leqslant |w| \leqslant n-3$, by Lemma 3.8, every extreme vertex in $S_w(n - |w|, k)$ is not in $F_e(n, k)$. Note that the extreme vertices in $S_w(n - |w|, k)$ for $1 \leqslant |w| \leqslant n-3$ are of the form $w'ba^{n-|w|}$ for $1 \leqslant a \leqslant k$, where $w = w'b$. By definition, vertex $w'ba^{n-|w|}$ is adjacent to $w'ab^{n-|w|}$ which is an extreme vertex of $S_w(n - |w|, k)$ with $w = w'a$. Therefore, this implies that the subgraph induced by $V(S_w(n, k)) \setminus F_e(n, k)$ is connected.

Theorem 3.10 $\tilde{\gamma}_c(S(n, k)) = k^{n-1}$, $n \geqslant 1$, $k \geqslant 3$.

Proof For the case where k is odd, by Lemma 3.5 and the number of terminal cliques in $S(n, k)$, it follows that $|F_i(n, k)| = k^{n-1}$ for $1 \leqslant i \leqslant k$. When k is even, by the definition of $F_e(n, k)$, we have $|V(S_w(1, k)) \cap F_e(n, k)| = 1$ and $|F_e(n, k)| = k^{n-1}$. By Lemmas 3.1, 3.7, and 3.9, the theorem follows. □

4 Computing $\tilde{\gamma}_c(S^+(n, k))$ and $\tilde{\gamma}_c(S^{++}(n, k))$

Recall that the graph $S^+(n, k)$ is obtained from $S(n, k)$ by adding a special vertex, say s, and edges joining s to all extreme vertices of $S(n, k)$ (see Fig. 3a) and the graph $S^{++}(n, k)$ is obtained from $S(n, k)$ by adding a new copy of $S(n-1, k)$ which is denoted by $S_{k+1}(n-1, k)$, and joining an extreme vertex a^n in $S(n, k)$ to the vertex ba^{n-1} in the added $S_{k+1}(n-1, k)$ for $1 \leqslant a \leqslant k$, where $b = k+1$ (see Fig. 3b).

Lemma 4.1 *For $n \geqslant 1$ and $k \geqslant 3$, $\tilde{\gamma}_c(S^+(n, k)) \geqslant k^{n-1}$ and $\tilde{\gamma}_c(S^{++}(n, k)) \geqslant k^{n-1} + k^{n-2}$.*

Proof By using a similar argument as in Lemma 3.1, we can show that if D is an OCD-set of $S^+(n, k)$ (respectively, $S^{++}(n, k)$) and the graph induced by $V(S^+(n, k)) \setminus D$ (respectively, $V(S^{++}(n, k)) \setminus D$) is not a terminal clique, then every terminal clique has at least one vertex in D. By definition, $S^+(n, k)$ contains $S(n, k)$ as a subgraph and $S^{++}(n, k)$ contains two disjoint subgraphs $S(n, k)$ and $S_{k+1}(n-1, k)$. Thus $\tilde{\gamma}_c(S^+(n, k)) \geqslant k^{n-1}$ and $\tilde{\gamma}_c(S^{++}(n, k)) \geqslant k^{n-1} + k^{n-2}$. This completes the proof. □

In the following, we show that there exists an OCD-set whose cardinality is exactly equal to the lower bound described in Lemma 4.1 for $S^+(n, k)$ and $S^{++}(n, k)$.

Definition 4.2 Define

$$F^{+}(n,k)=\begin{cases} F_{\mathrm{o},1}(n,k) & \text{if } k \text{ is odd,}\\ F_{\mathrm{e}}(n,k)\cup\{1^{n}\}\setminus\left\{1^{n-1}\left(1+\frac{k}{2}\right)\right\} & \text{if } k \text{ is even,}\end{cases}$$

and

$$F^{++}(n,k)=\begin{cases} F_{\mathrm{o},1}(n,k)\cup F_{\mathrm{o},1}(n-1,k) & \text{if } k \text{ is odd,}\\ F_{\mathrm{e}}(n,k)\cup F_{\mathrm{e}}(n-1,k) & \text{if } k \text{ is even,}\end{cases}$$

where $F_{\mathrm{o},1}(n,k)$ and $F_{\mathrm{e}}(n,k)$ are the OCD-sets of the subgraph $S(n,k)$ of $S^{+}(n,k)$(or $S^{++}(n,k)$) when k is odd and even, respectively, and $F_{\mathrm{o},1}(n-1,k)$ and $F_{\mathrm{e}}(n-1,k)$ are the OCD-sets of the subgraph $S_{k+1}(n-1,k)$ of $S^{++}(n,k)$.

Proposition 4.3 *For $n \geqslant 1$ and $k \geqslant 3$, $\tilde{\gamma}_c(S^{+}(n,k)) = k^{n-1}$.*

Proof By Lemma 4.1, it suffices to show that $F^{+}(n,k)$ is an OCD-set of $S^{+}(n,k)$. By Lemmas 3.7 and 3.9, we know that $F^{+}(n,k)$ is an OCD-set of $S(n,k)$. When k is odd, vertex 1^n is in $F^{+}(n,k)$, namely $F_{\mathrm{o},1}(n,k)$. Since s is adjacent to a^n in $S^{+}(n,k)$, vertex s of $S^{+}(n,k)$ is dominated by vertex 1^n. Note that all vertices a^n for $1 \leqslant a \leqslant k$ are not in $F^{+}(n,k)$ except $a = 1$. This implies that the subgraph induced by $V(S^{+}(n,k)) \setminus F^{+}(n,k)$ is connected. Thus $F^{+}(n,k)$ is an OCD-set of $S^{+}(n,k)$ when k is odd.

For the case where k is even, the set $F^{+}(n,k)$ is equal to $F_{\mathrm{e}}(n,k) \cup \{1^n\} \setminus \left\{1^{n-1}\left(1+\frac{k}{2}\right)\right\}$. Clearly, the neighbors of $1^{n-1}(1+\frac{k}{2})$ in the terminal clique containing $1^{n-1}\left(1+\frac{k}{2}\right)$ are dominated by vertex 1^n. Moreover, vertex s is also dominated by 1^n. It is easy to verify that the subgraph induced by $V(S^{+}(n,k)) \setminus F^{+}(n,k)$ is connected. Thus $F^{+}(n,k)$ is an OCD-set of $S^{+}(n,k)$ when k is even. Note that $|F^{+}(n,k)| = k^{n-1}$. This completes the proof. □

Proposition 4.4 *For $n \geqslant 1$ and $k \geqslant 3$, $\tilde{\gamma}_c(S^{++}(n,k)) = k^{n-1} + k^{n-2}$.*

Proof First we consider the case where k is odd. By Theorem 3.10, we know that $F_{\mathrm{o},1}(n,k)$ and $F_{\mathrm{o},1}(n-1,k)$ in $F^{++}(n,k)$ are OCD-sets of subgraphs $S(n,k)$ and $S_{k+1}(n-1,k)$ of $S^{++}(n,k)$. Thus $F^{++}(n,k)$ is a dominating set of $S^{++}(n,k)$. Note that vertex 2^n is in $V(S(n,k)) \setminus F_{\mathrm{o},1}(n,k)$ and $(k+1)2^{n-1}$ is in $V(S_{k+1}(n-1,k)) \setminus F_{\mathrm{o},1}(n-1,k)$. By definition, vertex 2^n is adjacent to vertex $(k+1)2^{n-1}$. This implies that the subgraph induced by $V(S^{++}(n,k)) \setminus F^{++}(n,k)$ is connected. Thus this case holds.

Now we consider the case where k is even. Clearly, the set $F^{++}(n,k)$ which is equal to $F_{\mathrm{e}}(n,k) \cup F_{\mathrm{e}}(n-1,k)$ is a dominating set of $S^{++}(n,k)$. Note that $|F^{++}(n,k)| = k^{n-1} + k^{n-2}$. Since 1^n and $(k+1)1^{n-1}$ are adjacent and both of them are in the subgraph induced by $V(S^{++}(n,k)) \setminus F^{++}(n,k)$, it follows that $F^{++}(n,k)$ is an OCD-set of $S^{++}(n,k)$. This completes the proof. □

5 Computing $\tilde{\gamma}_c(S_n)$

Recall that every non-extreme vertex in Sierpiński gasket graphs S_n is contracted from two adjacent vertices whose edge lies in no triangle in $S(n, 3)$. The label of every contracted vertex can be expressed as $w(ab^h|ba^h)$ for some $1 \leqslant h \leqslant n-1$ where the possible pairs of a and b are: 1 and 2, 1 and 3, or 2 and 3. It is easy to verify that $\tilde{\gamma}_c(S_1) = 1$, $\tilde{\gamma}_c(S_2) = 2$ and $\tilde{\gamma}_c(S_3) = 3$. Thus we assume that $n \geqslant 3$ in the rest of this section unless stated otherwise.

Theorem 5.1 (Theorem 7 in [24]) *For $n \geqslant 3$, we have $\gamma(S_n) = 3^{n-2}$.*

Since $\tilde{\gamma}_c(S_n) \geqslant \gamma(S_n)$, we have the following corollary.

Corollary 5.2 *For S_n with $n \geqslant 3$, we have $\tilde{\gamma}_c(S_n) \geqslant 3^{n-2}$.*

In the following, we introduce how to find an outer-connected dominating set D_n with cardinality 3^{n-2} for $n \geqslant 3$. Define $D_n = \{v : v_{n-2}v_{n-1}v_n \in \{1(12|21), 2(23|32), 3(13|31)\}\}$. For example, D_4 is depicted in Fig. 5.

Lemma 5.3 *For S_n with $n \geqslant 4$, the set D_n is an OCD-set in S_n with $|D_n| = 3^{n-2}$.*

Proof It is easy to verify that D_4 is an OCD-set in S_4. For S_n with $n > 4$, it is clear that every D_4 of $S_{4,a}$ for $a \in \{1, 2, 3\}^{n-4}$ is also an OCD-set of that $S_{4,a}$, and all extreme vertices of $S_{4,a}$ are not in its corresponding D_4. Thus D_n is the union of all those D_4's which form an OCD-set of S_n. Accordingly, $|D_n| = 3^{n-4} \cdot 3^2 = 3^{n-2}$. This completes the proof. □

Hence, we have our final result as follows:

Theorem 5.4 *For S_n with $n \geqslant 1$,*

$$\tilde{\gamma}_c(S_n) = \begin{cases} n & if \;\; n = 1, 2, \\ 3^{n-2} & if \;\; n \geqslant 3. \end{cases}$$

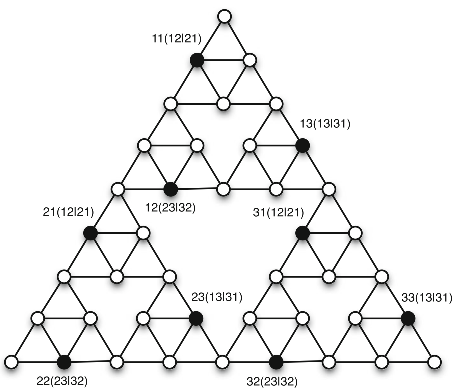


Fig. 5 All black vertices are in D_4

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