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The Hub Number of Sierpiński-Like Graphs

Chien-Hung Lin · Jia-Jie Liu · Yue-Li Wang · William Chung-Kung Yen

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Abstract A set $Q \subseteq V$ is a hub set of a graph $G = (V, E)$ if, for every pair of vertices $u, v \in V \setminus Q$, there exists a path from u to v such that all intermediate vertices are in Q. The hub number of G is the minimum size of a hub set in G. This paper derives the hub numbers of Sierpiński-like graphs including: Sierpiński graphs, extended Sierpiński graphs, and Sierpiński gasket graphs. Meanwhile, the corresponding minimum hub sets are also obtained.

Keywords Dominating sets · Hub numbers · Sierpiński graphs · Extended Sierpiński graphs · Sierpiński-like graphs

1 Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Sets $V(G)$ and $E(G)$ are simply written as V and E, respectively, when it is clear from context. In [16], Walsh defined a hub set as follows: A set $Q \subseteq V$ is a *hub set* of G if, for every pair of vertices $u, v \in V \setminus Q$, there exists a path from u to v such that all intermediate vertices are in Q. Such a path is called a *Q-path*, denoted $\mathrm{hp}(u, v; Q)$. The *hub number* of G, denoted $h(G)$, is the minimum size of a hub set in G. A set of vertices is called a *connected set* if the subgraph of G induced by S is connected. A hub set Q is a *connected hub set* if Q is a connected set. The *connected*

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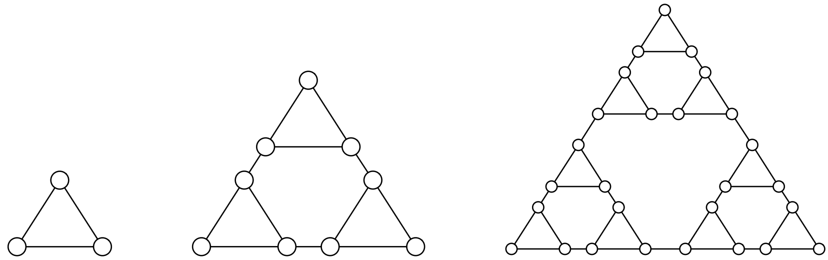
C.-H. Lin · Y.-L. Wang (✉)
Department of Information Management, National Taiwan University of Science and Technology, Taipei, Taiwan
e-mail: ylwang@cs.ntust.edu.tw

J.-J. Liu · W.C.-K. Yen
Department of Information Management, Shih Hsin University, Taipei, Taiwan



(a) $S(1,3)$ (b) $S(2,3)$ (c) $S(3,3)$

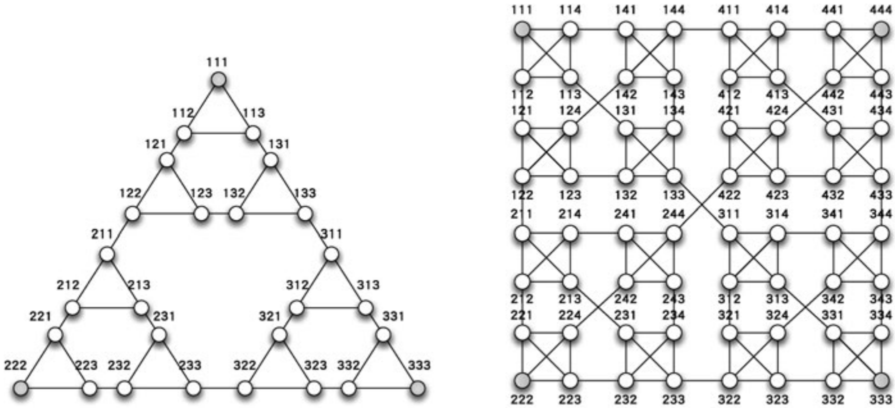
Fig. 1 Sierpiński graphs

hub number of G, denoted $h_c(G)$, is the minimum size of a connected hub set in G. For brevity, we use $Q_h(G)$ (respectively, $Q_{ch}(G)$) to stand for a minimum (respectively, minimum connected) hub set of G. Walsh proved that determining whether a graph G has a (connected) hub set of size k is NP-complete [16]. The problem of finding the hub number of a graph can be applied to a network, e.g., a rapid transit system (RTS), so that every vertex in the network in which no hub is allocated is adjacent to a vertex with a hub [16]. This system can be viewed as that the distance between any two vertices is at most two if the time spent in RTS can be neglected.

A *dominating set* in a graph G is a subset D of V such that every vertex in $V \setminus D$ has at least one adjacent vertex in D. A *connected dominating set* D is a dominating set such that the subgraph of G induced by D is connected. The minimum size of a dominating (respectively, connected dominating) set in G, denoted $\gamma(G)$ (respectively, $\gamma_c(G)$), is called the *domination number* (respectively, *connected domination number*) of G. In [16], Walsh also proved that $\gamma(G) \leqslant h(G)+1$ and $h_c(G) \leqslant \gamma_c(G)$. Later, Grauman et al. showed that $\gamma_c(G) \leqslant h(G) + 1$ and obtain the consecutive inequality $h(G) \leqslant h_c(G) \leqslant \gamma_c(G) \leqslant h(G) + 1$ [1].

In this paper, we are concerned with the hub numbers of Sierpiński-like graphs. The *Sierpiński graph* $S(n,k)$ consists of k copies of $S(n-1,k)$ for $n > 1$ where $S(1,k)$ is the complete graph of k vertices [8]. For example, $S(1,3)$, $S(2,3)$, and $S(3,3)$ are shown in Figs. 1(a), (b), and (c), respectively. Formal definitions of Sierpiński-like graphs including: Sierpiński graphs, extended Sierpiński graphs, and Sierpiński gasket graphs will be introduced in Sect. 2. The hub number is a newly introduced graph invariant and it is known that its determination is NP-hard. Hence it makes sense to determine this invariant exactly for some interesting and non-trivial graph classes. The classes of graphs considered in this paper certainly qualify into this category.

The organization of this paper is as follows. In Sect. 2, we introduce Sierpiński-like graphs in detail. In Sect. 3, we construct a connected set from a Sierpiński graph $S(n,k)$. Then, we prove that the constructed set is a minimum hub set of $S(n,k)$. Accordingly, we also prove that $h(S(n,k)) = h_c(S(n,k))$. The hub numbers of extended Sierpiński graphs and Sierpiński gasket graphs are discussed in Sects. 4 and 5, respectively. We also prove that the hub number is equal to the connected hub number for these Sierpiński-like graphs. Finally, concluding remarks are given in Sect. 6.



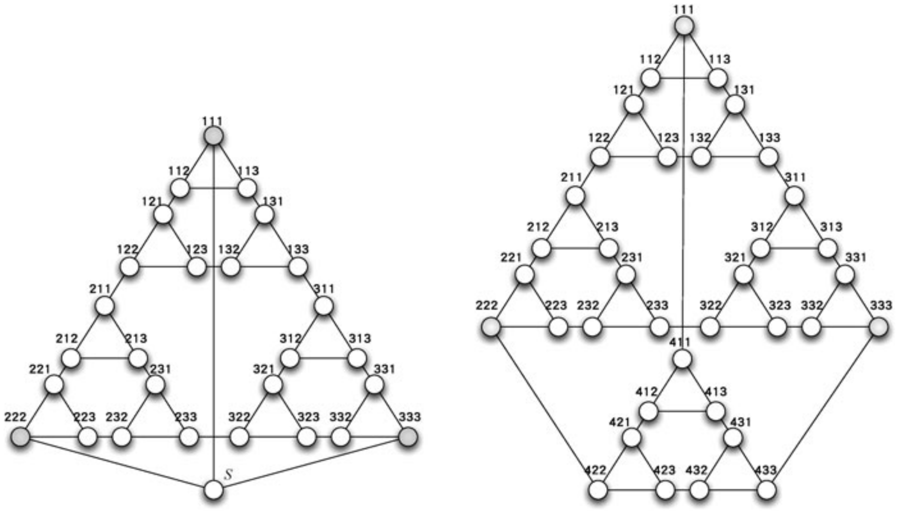
(a) $S(3,3)$ (b) $S(3,4)$

Fig. 2 Labeled Sierpiński graphs

2 Sierpiński-Like Graphs

A formal definition of Sierpiński graphs is described as follows. The reader is referred to [4, 8, 13] for the details. The vertex set of $S(n,k)$ consists of all n-tuples of integers $1,2,\dots,k$ for some integers $n,k \geqslant 1$, that is, $V(S(n,k)) = \{1,2,\dots,k\}^n$. For simplicity, we use $\ell(v)$ to denote the label of v. Thus, if the label of v is $\langle v_1, v_2, \dots, v_n\rangle$, then $\ell(v) = \langle v_1, v_2, \dots, v_n\rangle$, or in the regular expression form $\ell(v) = v_1 v_2 \dots v_n$. By using a convention on representing regular expression, we always use w, x, y, and z to denote a substring of $v_1 v_2 \dots v_n$ and a, b, c, and d to denote a number in $v_1 v_2 \dots v_n$, i.e., $a,b,c,d \in \{1,2,\dots,k\}$. For example, $\ell(v) = wab^{n-h}$, for $1 \leqslant h \leqslant n$, means that the label of v begins with a prefix w, then concatenates with a number a, and finally ends with $n-h$ $b's$, i.e., the *Kleene closure* in regular expression. For convenience, we also use the label form to represent a vertex. This means that if $\ell(v) = wab^{n-h}$, then we also say that wab^{n-h} is a vertex.

Two different vertices u and v are adjacent in $S(n,k)$ if and only if $\ell(u) = wab^{n-h}$ and $\ell(v) = wba^{n-h}$ with $a \neq b$, and for some $1 \leqslant h \leqslant n$. Note that if $h = 1$, then $w = \emptyset$. Further, if $h = n$, then both of b^{n-h} and a^{n-h} are empty. By the above definition, the subgraph of $S(n,k)$ induced by the vertices whose labels begin with a is a Sierpiński subgraph $S(n-1,k)$ and we use $S_a(n-1,k)$ to stand for these subgraphs. Vertex $v \in V(S(n,k))$ is an *extreme vertex* if $\ell(v) = a^n$. Therefore, there are exactly k extreme vertices in $S(n,k)$. Since the label of an extreme vertex v is a^n, by definition, v has only $k-1$ neighbors whose labels are of the form $a^{n-1}b$ with $b \neq a$. Every non-extreme vertex v with $\ell(v) = wab^{n-h}$ has exactly k neighbors whose labels are of the form wba^{n-h} and $wab^{n-h-1}c$ with $1 \leqslant c \leqslant k$ and $c \neq b$. We use $N(v)$ to denote the *open neighborhood* of v, i.e., all adjacent vertices of v. The *closed neighborhood* $N[v] = N(v) \cup \{v\}$. Thus, the degree of every extreme vertex, say v, is $|N(v)| = k-1$ while all other vertices have degree k. Figure 2 depicts $S(3,3)$ and $S(3,4)$ with labels. An interesting connection is that $S(n,3)$, for $n \geqslant 1$, is isomorphic to the graphs of the Tower of Hanoi puzzle with n disks [2, 8] and has



(a) $S^+(3,3)$ (b) $S^{++}(3,3)$

Fig. 3 Extended Sierpiński graphs: $S^+(3,3)$ and $S^{++}(3,3)$

been extensively studied (see [4] for an overview and the references therein for the details).

The extended Sierpiński graphs $S^+(n,k)$ and $S^{++}(n,k)$ were introduced by Klavžar and Mohar [9]. The graph $S^+(n,k)$ is obtained from $S(n,k)$ by adding a *special vertex*, say s, and edges joining s to all extreme vertices of $S(n,k)$ (see Fig. 3(a)). The graph $S^{++}(n,k)$ is obtained from $S(n,k)$ by adding a new copy of $S(n-1,k)$ which is denoted by $S_{k+1}(n,k)$, and joining extreme vertex a^n in $S(n,k)$ to extreme vertex ba^{n-1} in the added $S(n-1,k)$, for $a=1,2,\dots,k$, where $b=k+1$ (see Fig. 3(b)). The vertex-, edge-, and total-colorings on $S(n,k)$, $S^+(n,k)$, and $S^{++}(n,k)$ have been studied by Jakovac and Klavžar [5].

The *Sierpiński gasket graph* S_n is a variant of Sierpiński graph $S(n,3)$. S_n can be obtained from $S(n,3)$ by contracting every edge of $S(n,3)$ that lies in no triangle. For example, see Figs. 2(a) and 4. Vertices $\langle 1,1,2\rangle$ and $\langle 1,2,1\rangle$ in $S(3,3)$ are contracted to be a vertex in S_3 in which we use the regular expression $1(12|21)$ to denote the label of the resulting vertex where $|$ is the *union operation* in regular expression. According to the definition of extreme vertices in $S(n,k)$, the vertices with labels 1^n, 2^n, and 3^n in S_n are also called *extreme vertices*. The labels of other vertices are of the form $w(ab^h|ba^h)$ where $1\leqslant h\leqslant n-1$, $w\in\{1,2,\dots,k\}^{n-h-1}$, and a and b are one of the pairs: 1 and 2, 1 and 3, or 2 and 3. The vertices with labels $12^{n-1}|21^{n-1}$, $13^{n-1}|31^{n-1}$, and $23^{n-1}|32^{n-1}$ are called the *waist vertices* of S_n. The neighbors of the extreme vertex a^n are of the form $a^{n-2}(ab|ba)$ with $a\neq b$. The neighbors of vertex v with label $w(ab^h|ba^h)$ are of the form: $wab^{h-2}(bc|cb)$ and $wba^{h-2}(ad|da)$ for $c\neq b$ and $d\neq a$. The Sierpiński gasket graph S_n also contains three copies of S_{n-1} which are denoted by $S_{n-1,a}$, for $1\leqslant a\leqslant 3$, where $S_{n-1,a}$ contains the extreme vertex a^n.

Many properties of Sierpiński gasket graphs have been studied such as hamiltonicity [7, 15], pancyclicity [15], domination number [8, 10], chromatic number

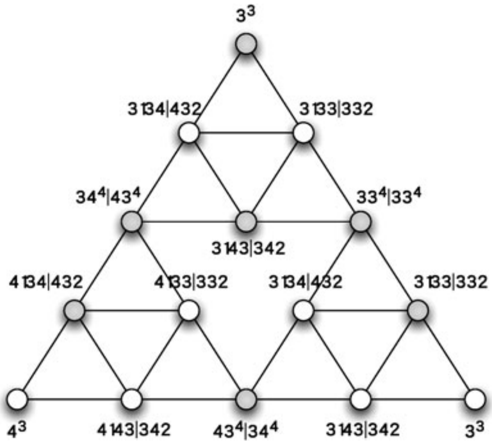


Fig. 4 Sierpiński gasket graph S_3

[7, 12, 15], total chromatic number [5, 15]. Moreover, Sierpiński gasket graphs play an important role in dynamic systems and probability [3, 6] as well as in psychology [11, 14].

3 The Hub Number of $S(n, k)$

In this section, we consider the hub number of $S(n,k)$. Let $Q_{S(n,k)} = \{v | v \in V(S(n,k)) \setminus \{1^n\},\ \ell(v) = w1a^m | wa1^m,\ a \in \{2,3,\dots,k\},\ 1 \leqslant m \leqslant n-1\}$ and $Q_{S_a(n,k)} = \{v | v \in V(S_a(n,k)) \setminus \{a1^{n-1}\},\ \ell(v) = w1b^m | wb1^m,\ b \in \{2,3,\dots,k\},\ 1 \leqslant m \leqslant n-2\}$ for $n \geqslant 2$. Note that, in the definitions of $Q_{S(n,k)}$ and $Q_{S_a(n,k)}$, w is any prefix in the labels of $S(n,k)$ and $S_a(n,k)$, respectively. For example, see Fig. 2(a). $Q_{S_1(3,3)} = \{121, 112, 113, 131\}$, $Q_{S_2(3,3)} = \{221, 212, 213, 231\}$, $Q_{S_3(3,3)} = \{321, 312, 313, 331\}$, and $Q_{S(3,3)} = Q_{S_1(3,3)} \cup Q_{S_2(3,3)} \cup Q_{S_3(3,3)} \cup \{211, 122, 133, 311\}$.

Lemma 1 [16] *Graph G is a complete graph if and only if $h(G) = h_c(G) = 0$.*

Lemma 2 $|Q_{S(n,k)}| = 2(k^{n-1} - 1)$ *for $n \geqslant 1$.*

Proof It is obvious that $|Q_{S(1,k)}| = 0 = 2(k^{1-1} - 1)$ for $n = 1$. It remains to consider the case where $n \geqslant 2$. By the definition of $Q_{S(n,k)}$, for every $m \in \{1, 2, \dots, n-1\}$, there are k^{n-m-1} different combinations of w and, in $\ell(v) = w1a^m$ or $wa1^m$, a has $k-1$ possible values. Thus, for every possible value of m, there are $2(k-1)k^{n-m-1}$ corresponding elements of the forms: $w1a^m$ and $wa1^m$ in $Q_{S(n,k)}$. Therefore,

$$
\begin{aligned}
|Q_{S(n,k)}| &= \sum_{m=1}^{n-1} 2(k-1)k^{n-m-1} \\
&= 2(k^{n-1} - 1).
\end{aligned}
$$

This completes the proof of this lemma. □

Corollary 3 $|Q_{S_i(n,k)}| = 2(k^{n-2} - 1)$ *for* $1 \leqslant i \leqslant k$ *and* $n \geqslant 3$.

Lemma 4 *The subgraph of* $S(n,k)$ *induced by the vertices in* $Q_{S(n,k)}$ *is connected for* $n \geqslant 2$.

Proof We shall prove this lemma by mathematical induction on n. For the basis step, i.e., $n = 2$, $Q_{S(2,k)}$ is the vertex set $\{12, 13, \ldots, 1k, 21, 31, \ldots, k1\}$. By definition, there is an edge between $1a$ and $a1$ for $a \neq 1$. Furthermore, $1a$ and $1b$ for $1 \leqslant a, b \leqslant k$ and $a \neq b$ are also adjacent. Thus, $Q_{S(2,k)}$ is connected and the basis step holds.

By the induction hypothesis, every $S_a(n,k)$, for $a = 1, 2, \ldots, k$, has a corresponding set $Q_{S_a(n,k)}$ which is connected. Moreover, vertex $b1^{n-1}$ is in $Q_{S(n,k)}$ for $b = 2, 3, \ldots, k$. By definition, vertex $b1^{n-1}$, for $b = 2, 3, \ldots, k$, is adjacent with vertex $1b^{n-1}$ which is also in $Q_{S(n,k)}$. Since $b1^{n-1}$ is adjacent with $b1^{n-2}b$ which is in $Q_{S_b(n,k)}$ and $1b^{n-1}$ is adjacent with $1b^{n-2}1$ which is in $Q_{S_1(n,k)}$, $Q_{S(n,k)}$ is connected. Therefore, the lemma follows. □

Lemma 5 $Q_{S(n,k)}$ *is a connected hub set of* $S(n,k)$ *for* $n \geqslant 2$.

Proof By Lemma 4, $Q_{S(n,k)}$ is a connected set. All we have to prove is that $Q_{S(n,k)}$ is a hub set. By definition, every $S(n,k)$ contains k copies of $S(n-1,k)$, every $S(n-1,k)$ contains k copies of $S(n-2,k)$, and so on. Consequently, $S(n,k)$ contains k^{n-1} copies $S(1,k)$ which is a complete graph of k vertices. Clearly, every vertex belongs to some $S(1,k)$ and every $S(1,k)$ has exactly one vertex whose label ends with a for $a = 1, 2, \ldots, k$. By the definition of $Q_{S(n,k)}$, every vertex whose label ends with 1 is in $Q_{S(n,k)}$ except 1^n. Thus, every vertex with label not ended with 1 is adjacent with a vertex whose label ends with 1 in their corresponding $S(1,k)$ except the one containing 1^n. For the $S(1,k)$ containing 1^n, by definition, all of its other vertices are in $Q_{S(n,k)}$. This reveals that every vertex in $V(S(n,k)) \setminus Q_{S(n,k)}$ is adjacent with a vertex in $Q_{S(n,k)}$. Therefore, for every pair of vertices $u, v \in V(S(n,k)) \setminus Q_{S(n,k)}$, there exists a $\mathrm{hp}(u, v; Q_{S(n,k)})$. This completes the proof. □

Corollary 6 *For every* $S(n,k)$ *with* $n \geqslant 2$, $h(S(n,k)) \leqslant h_c(S(n,k)) \leqslant 2(k^{n-1} - 1)$.

Proposition 7 *For any two vertices* $u, v \in V$ *in graph* $G = (V, E)$, *if* $N[u] \subseteq N[v]$, *then at most one of* u *and* v *can be in* $Q_h(G)$.

Proof Suppose to the contrary that $Q_h(G)$ contains both of u and v. It is obvious that $Q_h(G) \setminus \{u\}$ is still a hub set. This contradicts that $Q_h(G)$ is a minimum hub set. □

Lemma 8 *For any* $S(n,k)$ *with* $n, k \geqslant 1$, $h(S(n,k)) \geqslant 2(k^{n-1} - 1)$.

Proof We prove this lemma by induction on n. Since $S(1,k)$ is a complete graph of k vertices, by Lemma 1, $h(S(1,k)) = 0 = 2(k^{1-1} - 1)$. Thus, the basis step holds immediately.

Now we consider the induction step. Let $Q_h(S(n,k))$ be a minimum hub set of $S(n,k)$. Clearly, $Q_h(S(n,k)) \cap S_i(n,k)$ is a hub set of $S_i(n,k)$ and $|Q_h(S(n,k)) \cap$

$S_i(n,k)| \geqslant |Q_h(S_i(n,k))|$ for $i = 1, 2, \ldots, k$. By Proposition 7, any extreme vertex of $S_i(n,k)$ and its neighbors cannot be in $Q_h(S_i(n,k))$ simultaneously. Since there are k copies of $S(n-1,k)$ in $S(n,k)$, at least $2(k-1)$ vertices (namely, $k-1$ edges) must be added to $\bigcup_{i=1}^{k} Q_h(S_i(n,k))$ to ensure that there is a hub path $\mathrm{hp}(u,v; Q_h(S(n,k)))$ between any two vertices $u, v \in V(S(n,k)) \setminus Q_h(S(n,k))$. Therefore, by induction hypothesis, we can have the following derivation.

$$\begin{aligned}
|Q_h(S(n,k))| &\geqslant 2(k-1) + \sum_{i=1}^{k} |Q_h(S_i(n,k))| \\
&\geqslant 2(k-1) + \sum_{i=1}^{k} 2(k^{n-2}-1) \\
&= 2(k-1) + 2k(k^{n-2}-1) \\
&= 2(k^{n-1}-1).
\end{aligned}$$

This concludes the proof of this lemma. □

Combining the results in Corollary 6 and Lemmas 5 and 8, we obtain the following theorem.

Theorem 9 *For $S(n,k)$ with $n \geqslant 1$, $h_c(S(n,k)) = h(S(n,k)) = 2(k^{n-1}-1)$.*

4 The Hub Numbers of $S^+(n,k)$ and $S^{++}(n,k)$

Since $S^+(n,k)$ is obtained from $S(n,k)$ by adding a new vertex s and edges joining s with all extreme vertices in $S(n,k)$, $S^+(n,k)$ also contains $S_i(n,k)$ as a subgraph, for $i = 1, 2, \ldots, k$. For convenience, we assume that $\ell(s) = 0^n$. Let $Q_{S^+(n,k)} = \{v | v \in V(S^+(n,k)),\ \ell(v) = w1i^m | wi1^m,\ i \in \{2,3,\ldots,k\},\ 1 \leqslant m \leqslant n-2\} \cup \{i^n, 0 \leqslant i \leqslant k\}$ for $n \geqslant 2$. See Fig. 5 for an illustration of $Q_{S^+(3,4)}$ which consists of all the gray vertices 11(2|3|4), 1(2|3|4)1, ..., 31(2|3|4), 3(2|3|4)1 and black vertices 000, 111, ..., 444.

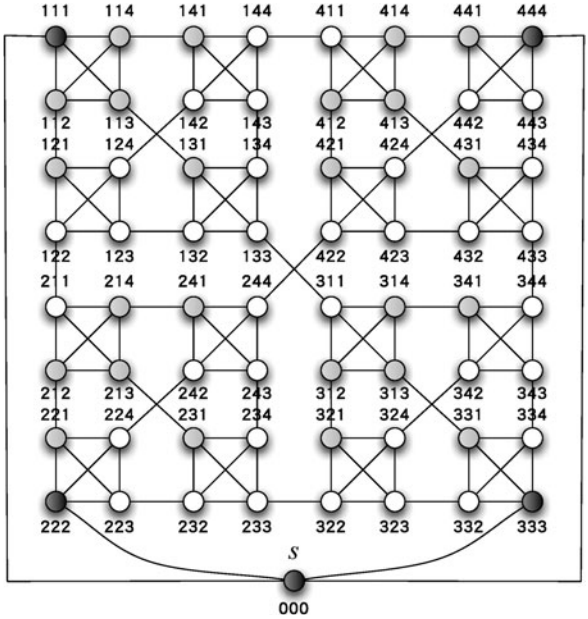
Lemma 10 $|Q_{S^+(n,k)}| = 2k^{n-1} - k + 1$ *for $n \geqslant 2$.*

Proof By comparing the constructions of $Q_{S^+(n,k)}$ with $Q_{S(n,k)}$, vertices with labels $1a^{n-1}$ and $a1^{n-1}$ for $a = 2, 3, \ldots, k$ are removed from $Q_{S(n,k)}$, then vertices with labels b^n for $b = 0, 1, \ldots, k$ are added to the resulting set in order to constitute $Q_{S^+(n,k)}$. Thus,

$$\begin{aligned}
|Q_{S^+(n,k)}| &= |Q_h(S(n,k))| - 2(k-1) + k + 1 \\
&= 2(k^{n-1}-1) - k + 3 \\
&= 2k^{n-1} - k + 1.
\end{aligned}$$

This completes the proof. □

Fig. 5 $Q_{S^+(3,4)}$



Lemma 11 *The subgraph of* $S^+(n,k)$ *induced by the vertices in* $Q_{S^+(n,k)}$ *is connected for* $n \geqslant 2$.

Proof By the definition of $Q_{S^+(n,k)}$, the vertices in $Q_{S_i(n,k)}$ defined in Lemma 4, for $i = 1, 2, \ldots, k$ and $n \geqslant 3$, are also in $Q_{S^+(n,k)}$. Thus, by Lemma 4, the subgraph of $Q_{S^+(n,k)}$ induced by the vertices in $Q_{S_i(n,k)}$ is connected for $i = 1, 2, \ldots, k$ and $n \geqslant 3$. It is clear that the subgraph of $Q_{S^+(n,k)}$ induced by the vertices in $\{0^n, 1^n, \ldots, k^n\}$ is connected. Furthermore, a^n is adjacent with $a^{n-1}1$ in $S_a(n,k)$ and both of them are in $Q_{S^+(n,k)}$ for $a = 2, 3, \ldots, k$. For $S_1(n,k)$, 1^n is adjacent with $1^{n-1}a$ for $a = 2, 3, \ldots, k$ and all of them are in $Q_{S^+(n,k)}$. Hence, the subgraph of $S^+(n,k)$ induced by the vertices in $Q_{S^+(n,k)}$ is connected for $n \geqslant 3$.

To complete the proof, it remains to consider the case where $n = 2$. In $S^+(2,k)$, $Q_{S^+(2,k)}$ is the vertex set $\{0^2, 1^2, \ldots, k^2\}$. By definition, i^2 is adjacent with 0^2 and every vertex in $S_i(1,k)$ is adjacent with i^2, for $i = 1, 2, \ldots, k$. Thus the lemma follows. □

Lemma 12 $Q_{S^+(n,k)}$ *is a connected hub set of* $S^+(n,k)$ *for* $n \geqslant 2$.

Proof By using a similar reasoning as in Lemma 5, this lemma can be proved. □

Corollary 13 *For* $S^+(n,k)$ *with* $n \geqslant 2$, $h(S^+(n,k)) \leqslant h_c(S^+(n,k)) \leqslant 2k^{n-1} - k + 1$.

Lemma 14 *For* $S^+(n,k)$ *with* $n \geqslant 2$, $h(S^+(n,k)) \geqslant 2k^{n-1} - k + 1$.

Proof We can use an analogous way in Lemma 8 to prove this lemma. The only difference occurs at connecting $Q_h(S_i(n,k))$ for $i = 1, 2, \ldots, k$ to form $Q_h(S^+(n,k))$

which can be done by adding k edges (or $k+1$ vertices, namely the vertex set $\{0^n, 1^n, \ldots, k^n\}$). Therefore, by Corollary 3, we can have the following derivation.

$$\begin{aligned}|Q_h(S^+(n,k))| &\geqslant k+1+\sum_{i=1}^{k}|Q_h(S_i(n,k))| \\ &\geqslant k+1+\sum_{i=1}^{k}2(k^{n-2}-1) \\ &= k+1+2k(k^{n-2}-1) \\ &= 2k^{n-1}-k+1.\end{aligned}$$

This completes the proof of this lemma. □

Combining the results in Corollary 13 and Lemmas 12 and 14, we obtain the following theorem.

Theorem 15 *For $S(n,k)$ with $n \geqslant 2$, $h_c(S^+(n,k)) = h(S^+(n,k)) = 2k^{n-1} - k + 1$.*

By definition, $S^{++}(n,k)$ has one more copy of $S(n-1,k)$ than $S(n,k)$. By using an analogous way, we can determine $h(S^{++}(n,k))$ and $h_c(S^{++}(n,k))$ for $S^{++}(n,k)$. The corresponding hub set of $S^{++}(n,k)$, denoted $Q_{S^{++}(n,k)}$, is the set $\{v|v \in V(S^{++}(n,k)),\ \ell(v) = w1i^m|wi1^m|(k+1)1^{n-1},\ i \in \{2,3,\ldots,k\},\ 1 \leqslant m \leqslant n-1\}$. Note that the leading number of w is in the range from 1 to $k+1$ while $k+1$ cannot appear in other positions of w.

Lemma 16 *For $S^{++}(n,k)$ with $n \geqslant 2$, $|Q_{S^{++}(n,k)}| = 2(k^{n-1} + k^{n-2} - 1)$.*

Proof Comparing $Q_{S^{++}(n,k)}$ with $Q_{S(n,k)}$, we can find that all vertices in $Q_{S(n,k)}$ are also in $Q_{S^{++}(n,k)}$. Moreover, $Q_{S^{++}(n,k)}$ contains the set of vertices in $Q_{S_{k+1}(n,k)}$ as well as vertices 1^n and $(k+1)1^{n-1}$. Therefore,

$$\begin{aligned}|Q_{S^{++}(n,k)}| &= |Q_{S(n,k)}| + |Q_{S_{k+1}(n,k)}| + 2 \\ &= 2(k^{n-1}-1) + 2(k^{n-2}-1) + 2 \\ &= 2(k^{n-1}+k^{n-2}-1).\end{aligned}$$

This establishes the lemma. □

Lemma 17 *$Q_{S^{++}(n,k)}$ is a connected hub set of $S^{++}(n,k)$ for $n \geqslant 2$.*

Proof By Lemma 5, the subgraphs of $S^{++}(n,k)$ induced by the vertices in $Q_{S(n,k)}$ and $Q_{S_{k+1}(n,k)}$, respectively, are connected. It is easy to check that the whole set becomes connected after adding vertices 1^n and $(k+1)1^{n-1}$. □

Lemma 18 *For $S^{++}(n,k)$ with $n,k \geqslant 1$, $h(S^{++}(n,k)) \geqslant 2(k^{n-1} + k^{n-2} - 1)$.*

Proof By using a similar reasoning as in Lemma 8, this lemma follows. □

Combining the results in Lemmas 16–18, we obtain the following theorem.

Theorem 19 *For $S^{++}(n,k)$ with $n \geqslant 2$, $h_c(S^{++}(n,k)) = h(S^{++}(n,k)) = 2(k^{n-1} + k^{n-2} - 1)$.*

5 The Hub Number of Sierpiński Gasket Graphs

By an observation on the labels of Sierpiński gasket graphs S_n, all vertices are contracted vertices except extreme vertices. Moreover, the label of every contracted vertex can be expressed as $w(ab^h|ba^h)$ for some $1 \leqslant h \leqslant n-1$ where the possible value-pairs of a and b are: 1 and 2, 1 and 3, or 2 and 3. To determine whether a vertex is in a hub set or not, we define a reduction operation on $\ell(v)$ as follows. For $\ell(v) = v_1v_2\ldots v_n$ with $n > 2$, the *reduction* of $\ell(v)$, denoted $r(\ell(v))$, is defined as follows:

$$r(\ell(v)) = \begin{cases} v_2v_3\ldots v_n & \text{if } v_1 = 1,\\ v_2+1(\text{mod } 3)\ v_3+1(\text{mod } 3)\ \cdots\ v_n+1(\text{mod } 3) & \text{if } v_1 = 2,\\ v_2-1(\text{mod } 3)\ v_3-1(\text{mod } 3)\ \cdots\ v_n-1(\text{mod } 3) & \text{if } v_1 = 3. \end{cases}$$

Note that, in the above operation, the addition or subtraction is congruent to modulo 3, i.e., $0 \equiv 3(\text{mod } 3)$. Furthermore, if $\ell(v) = w(ab^{n-1}|ba^{n-1})$, then $r(\ell(v)) = (r(wab^{n-1})|r(wba^{n-1}))$. A label $\ell(v)$ after taking the reduction operation m times is represented as $r^m(\ell(v))$. For example, if $\ell(v) = 31(12|21)$, then $r^2(\ell(v)) = r^2(31(12|21)) = r^2(3112)|r^2(3121) = r(331)|r(313) = 23|32$.

By inspection, we can find that the minimum hub set of S_2 contains either any two adjacent waist vertices or an extreme vertex i^2 and the waist vertex having no i in its label. Note that the former hub set is connected. Since $1 \leqslant i \leqslant 3$, the latter hub set can be $\{11, (23|32)\}$, $\{22, (13|31)\}$, or $\{33, (12|21)\}$. For clarity, we use the set $\{11, (23|32)\}$ as the *fundamental set* in our discussion and call $(23|32)$ and i^2, for $1 \leqslant i \leqslant 3$, the *fundamental labels*. Thus, we have the following proposition.

Proposition 20 $h(S_2) = h_c(S_2) = 2$.

For ease of readability, we introduce how to construct a minimum hub set Q for S_3 by using the fundamental set $\{11, (23|32)\}$ of S_2. It is easy to check that the set $Q = \{111, (23^2|32^2), 2(12|21), (12^2|21^2), 1(23|32), (13^2|31^2), 3(13|31)\}$ is a minimum hub set in S_3 (see the vertices with gray color in Fig. 4). The construction of Q is explained as follows. We can find that the label of each vertex in $S_{n-1,2}$ (respectively, $S_{n-1,3}$) can be mapped to that of a corresponding vertex in $S_{n-1,1}$ after $S_{n-1,2}$ (respectively, $S_{n-1,3}$) is rotated $4\pi/3$ (respectively, $2\pi/3$) radians in clockwise. That is what the reduction operation wants to do. We can see that after applying the reduction operation on the pair of vertices $23^2|32^2$ and $2(12|21)$ (respectively, $23^2|32^2$ and $3(13|31)$) in $S_{n-1,2}$ (respectively, $S_{n-1,3}$), this yields $r(23^2|32^2) = 11$ and $r(2(12|21)) = (23|32)$ (respectively, $r(23^2|32^2) = 11$

and $r(3(13|31)) = (23|32))$ which constitute the fundamental set of S_2. After including another two waist vertices $12^2|21^2$ and $13^2|31^2$ of S_3, Q is obtained. By applying the reduction operation on the labels of these two waist vertices, we can derive $r(12^2|21^2) = 22$ and $r(13^2|31^2) = 33$. That is the reason why we say that 22 and 33 are also fundamental labels.

In Propositions 21–23 and Lemma 24, we assume that $Q_{S_n} = \{v|v \in V(S_n) \setminus \{2^n, 3^n\}, r^{n-2}(v)$ is a fundamental label$\}$. The reader can see that, by setting $n = 3$, Q_{S_n} is exactly the set mentioned in the previous paragraph. Before showing that, actually, Q_{S_n} is a minimum hub set of S_n, we use the following propositions to introduce some properties of Q_{S_n}.

Proposition 21 *All $v \in Q_{S_n}$ with $r^{n-2}(v) \neq 11$ form a path of length $2(3^{n-2} - 1)$ for $n \geqslant 3$.*

Proof For $n \geqslant 3$, we can view that S_n is composed of 3^{n-2} $S_2's$ where each S_2 contains all of the vertices having the same $n - 2$ leading numbers in their labels. Clearly, performing the reduction operation on the label of each vertex in S_2 according to their $n - 2$ leading numbers, there is exactly one vertex with label 23|32 in each S_2. Similarly, the labels of all waist vertices will be reduced to 11, 22, and 33. After including all waist vertices with reduced labels 22 and 33 of each S_2, the three vertices with label 23|32 of the $S_2's$ in an S_3 will form a hexagonal path of length 4 (see the path passing through vertices 2(12|21), $12^2|21^2$, 1(23|32), $13^2|31^2$, and 3(13|31) in Fig. 4), three hexagonal paths of length 4 in the $S_3's$ of an S_4 will form a hexagonal path of length 16, and finally a hexagonal path of length $2(3^{n-2} - 1)$ will be formed in S_n. □

We use $P_{Q_{S_n}}$ to denote the hexagonal path described in Proposition 21.

Proposition 22 *For every $v \in Q_{S_n}$ with $r^{n-2}(v) = 11$, no neighbor of v is in Q_{S_n}. Furthermore, the number of vertices with reduced label 11 is $(3^{n-2} + 1)/2$.*

Proof By using a similar reasoning as Proposition 21, we can find that the labels of all neighbors of each vertex with reduced label 11 are either 12|21 or 13|31. By the definition of Q_{S_n}, these neighbors are not in Q_{S_n}. □

Proposition 23 *For every $v \in V(S_n) \setminus Q_{S_n}$, there is a neighbor of v is in $P_{Q_{S_n}}$.*

Lemma 24 *Q_{S_n} is a hub set of S_n and $|Q_{S_n}| = (5 \cdot 3^{n-2} - 1)/2$.*

Proof By Propositions 21–23, this lemma follows directly. □

Corollary 25 *For S_n with $n \geqslant 2$, $h(S_n) \leqslant (5 \cdot 3^{n-2} - 1)/2$.*

A path $P_3 = (u, v, w)$ of length 2 in S_n is called a *waist* P_3 if v is a waist vertex, and u and w are in different S_{n-1} of S_n.

Lemma 26 *At least two waist vertices must be in $Q_h(S_n)$ for $n \geqslant 3$ and each of them is contained in a waist P_3.*

Proof For the purpose of contradiction, we assume that less than two waist vertices are in $Q_h(S_n)$. Without loss of generality, we may assume that waist vertices $12^{n-1}|21^{n-1}$ and $13^{n-1}|31^{n-1}$ are not in $Q_h(S_n)$. By Proposition 7, for every extreme vertex i^n, at least one vertex in $N[i^n]$ is not in $Q_h(S_n)$. Accordingly, there exist vertices u, v, and w which are in $N[1^n]$, $N[2^n]$, and $N[3^n]$, respectively, and all of them are not in $Q_h(S_n)$. Clearly, any path from u to v (respectively, w) must pass either through $12^{n-1}|21^{n-1}$ or $13^{n-1}|31^{n-1}$. However, waist vertices $12^{n-1}|21^{n-1}$ and $13^{n-1}|31^{n-1}$ are not in $Q_h(S_n)$. This implies that $\mathrm{hp}(u, v; Q_h(S_n))$ (respectively, $\mathrm{hp}(u, w; Q_h(S_n))$) does not exist, a contradiction. Therefore, at least two waist vertices must be in $Q_h(S_n)$ and two of the three waist vertices are in waist paths. This completes the proof. □

Lemma 27 $h(S_n) \geqslant (5 \cdot 3^{n-2} - 1)/2$, *for* $n \geqslant 2$.

Proof By Lemma 26, we assume that waist vertices $12^{n-1}|21^{n-1}$ and $13^{n-1}|31^{n-1}$ are in $Q_h(S_n)$ and each of them is in a waist P_3. Let $Q_i = Q_h(S_n) \cap S_{n-1,i}$ for $1 \leqslant i \leqslant 3$. Clearly, by Proposition 7 and the existence of waist P_3's, $|Q_1| \geqslant h(S_{n-1}) + 2$ and $|Q_i| \geqslant h(S_{n-1}) + 1$ for $i = 2$ and 3. If there are exactly two waist vertices in $Q_h(S_n)$, then $|Q_h(S_n)| = |Q_1| + |Q_2| + |Q_3| - 2$; otherwise, $|Q_h(S_n)| = |Q_1| + |Q_2| + |Q_3| - 3$. Since the latter equation yields a smaller lower bound, we have the following derivation when $n \geqslant 3$:

$$\begin{aligned} h(S_n) &= |Q_h(S_n)| \\ &= |Q_1| + |Q_2| + |Q_3| - 3 \\ &\geqslant h(S_{n-1}) + 2 + h(S_{n-1}) + 1 + h(S_{n-1}) + 1 - 3 \\ &\geqslant 3h(S_{n-1}) + 1. \end{aligned}$$

By Proposition 20, the initial condition of the above recurrence formula is $h(S_2) = 2$. After solving this recurrence formula, this yields $h(S_n) \geqslant (5 \cdot 3^{n-2} - 1)/2$, for $n \geqslant 2$. □

By Propositions 22 and 23 and Lemma 24, we can replace every $v \in Q_{S_n}$ with one of its neighbor if $r^{n-2}(v) = 11$. The resulting set will be a connected hub set. Then, combining the results in Corollary 25 and Lemma 27, we can obtain the following theorem.

Theorem 28 *For S_n with $n \geqslant 2$, $h(S_n) = h_c(S_n) = (5 \cdot 3^{n-2} - 1)/2$.*

6 Concluding Remarks

In this paper, we prove that the hub number is equal to the connected hub number for Sierpiński-like graphs. In particular, we also construct a minimum connected hub set for each of them. In our proposed constructions, every vertex only needs to examine its own label to determine whether it is in a hub set or not.

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