Theory Comput Syst (2009) 45: 917–925
DOI 10.1007/s00224-009-9176-8

Nash Stability in Additively Separable Hedonic Games and Community Structures

Martin Olsen

Published online: 27 January 2009

Abstract We prove that the problem of deciding whether a Nash stable partition exists in an Additively Separable Hedonic Game is NP-complete. We also show that the problem of deciding whether a *non trivial* Nash stable partition exists in an Additively Separable Hedonic Game with *non-negative* and *symmetric* preferences is NP-complete. We motivate our study of the computational complexity by linking Nash stable partitions in Additively Separable Hedonic Games to community structures in networks. Our results formally justify that computing community structures in general is hard.

Keywords Additively separable hedonic games · NP-completeness · Nash stability · Community structures

1 Introduction

In a *Coalition Formation Game* a set of players splits up in coalitions so that each player belongs to exactly one coalition. Each player prefers certain partitions[1] of the players to other partitions. If all players are satisfied with the partition in some formalized sense—or not able to move—the partition is said to be *stable*. A stable partition is called an *equilibrium*. For an overview of the field of *Coalition Formation Games* we refer to the report [10] by Hajdukova.

A given notion of stability can have limitations in terms of computability. For some types of games it might be impossible to effectively compute equilibriums on

[1] A partition of a set N is a collection of non empty disjoint subsets of N with union N.

The research is partly sponsored by the company Cofman (www.cofman.com).

M. Olsen (✉)
MADALGO, Department of Computer Science, University of Aarhus, Aabogade 34, 8200 Aarhus N, Denmark
e-mail: mo@madalgo.au.dk

a computing device under the assumption NP $\neq$ P. If a real world system is modeled using Coalition Formation Games and equilibriums with such limitations you should not expect to be able to calculate the equilibriums using a computer if the model is large. It is also an interesting question whether a real system is able to find an equilibrium if a computer can not find it effectively. This is the motivation for analyzing the computational complexity for a given notion of stability as also pointed out by Daskalakis and Papadimitriou in [6] and Chen and Rudra in [5]. In this paper we prove limitations for the notion of *Nash stability* in *Additively Separable Hedonic Games*.

A community structure of a network can be loosely defined as a partition of the nodes into groups so that there are many connections between nodes belonging to the same group and few connections between nodes belonging to different groups. We will link community structures to equilibriums so that the limitations proven in this paper of the stability concepts formally indicate that computing community structures is hard.

1.1 Results and Outline of the Paper

We formally introduce hedonic games and the related stability concepts in Sect. 2 which also includes a discussion of related work. In Sect. 3 we restrict our attention to Additively Separable Hedonic Games and show that the problem of deciding whether a Nash stable partition exists in such a game is NP-complete. In Sect. 4 we relate the field of detection of community structures to Nash stable partitions in Additively Separable Hedonic Games and argue that community structures in networks can be viewed as Nash stable partitions. This motivates looking at the computational complexity of computing equilibriums in games with symmetric and positive preferences which is the subject of Sect. 5. In this section we show that the problem of deciding whether a *non trivial* Nash stable partition exists in an Additively Separable Hedonic Game with *non-negative* and *symmetric* preferences is NP-complete. This result also applies to individually stable partitions since individually stable partitions are Nash stable and vice versa in such games.

2 Hedonic Games

The players of a hedonic game form coalitions so that each player belongs to exactly one coalition and the players only care about which other players team up with them. In order to define the game we specify which coalitions player i prefers to be a member of for each player i:

Definition 1 A *hedonic game* is a pair $(N, \preceq)$ where $N = \{1, 2, \dots, n\}$ is the set of *players* and $\preceq\ = (\preceq_1, \preceq_2, \dots, \preceq_n)$ is the *preference profile* specifying for each player $i \in N$ a reflexive, complete and transitive *preference relation* $\preceq_i$ on the set $N_i = \{S \subseteq N : i \in S\}$.

In an *additively separable* hedonic game we are given a function $v_i : N \to \mathbb{R}$ for each player $i \in N$ where $v_i(j)$ is the *payoff* of player i for belonging to the same coalition as player j:

Definition 2 A hedonic game $(N, \preceq)$ is *additively separable* if there exists a *utility function* $v_i : N \to \mathbb{R}$ for each $i \in N$ such that

$$\forall S, T \in N_i : T \preceq_i S \quad \Longleftrightarrow \quad \sum_{j \in T} v_i(j) \le \sum_{j \in S} v_i(j).$$

Changing the value $v_i(i)$ has no effect on $\preceq_i$ so we assume $v_i(i) = 0$.

We will now present an example of an Additively Separable Hedonic Game. We will use biological terminology metaphorically to ease the understanding for the game. The game does *not* represent a serious attempt to model a biological system.

Example 1 The buffalo-parasite-game. Assume that there are two buffaloes b_1 and b_2 in an area with n waterholes $w_1, w_2, \dots, w_n$. Each waterhole w_i has a capacity $c(w_i)$ specifying how much water a buffalo can drink from that hole per year. There are also two parasites p_1 and p_2 in the area. The only possible host for p_1 is b_1 and b_1 must drink a lot of water if p_1 is sitting on its back. The same goes for p_2 and b_2. Now assume that b_1 and b_2 are enemies and that a buffalo must drink water corresponding to half the total capacity C of the waterholes if it is the host of a parasite. This system can be viewed as an Additively Separable Hedonic Game depicted as a weighted directed graph in Fig. 1 where the weight of edge (i, j) is $v_i(j)$— if there is no edge (i, j) then $v_i(j) = 0$. We have added two edges (b_1, b_2) and (b_2, b_1) with capacity $-C - 1$ to model that b_1 and b_2 are enemies. Please note that the waterholes are also players in the game. The waterholes do not care which coalitions they belong to.

2.1 Stability Concepts

In this paper we will focus on one type of stability: *Nash stability*. A partition Π of N is *Nash stable* if it is impossible to find a player p and a coalition $T \in \Pi \cup \{\emptyset\}$ such that p strictly prefers $T \cup \{p\}$ to the coalition of p in Π—in which case p would be better off by joining T:

Definition 3 The partition $\Pi = \{S_1, S_2, \dots, S_K\}$ of N is Nash stable if and only if

$$\forall i \in N,\ \forall S_k \in \Pi \cup \{\emptyset\} : S_k \cup \{i\} \preceq_i S_\Pi(i), \tag{1}$$

where $S_\Pi(i)$ denotes the set in the partition Π that i belongs to.

Now consider the buffalo-parasite-game from Example 1. A partition of the players is not Nash stable if b_1 is not the host of p_1—in this case p_1 would be strictly better off by joining $S_\Pi(b_1)$. This fact can be expressed more formally: $S_\Pi(b_1) \cup \{p_1\} \succ_{p_1} S_\Pi(p_1)$ if $S_\Pi(p_1) \neq S_\Pi(b_1)$. In this game a Nash stable partition of the players exists if and only if we can split the waterholes in two groups with the same capacity. We will formally show and use this fact in Sect. 3.

We will briefly mention the three other main stability concepts for hedonic games: *individual stability*, *contractual individual stability* and *core stability*. A partition Π is individually stable if it is impossible to find a player p and a coalition $T \in \Pi \cup \{\emptyset\}$

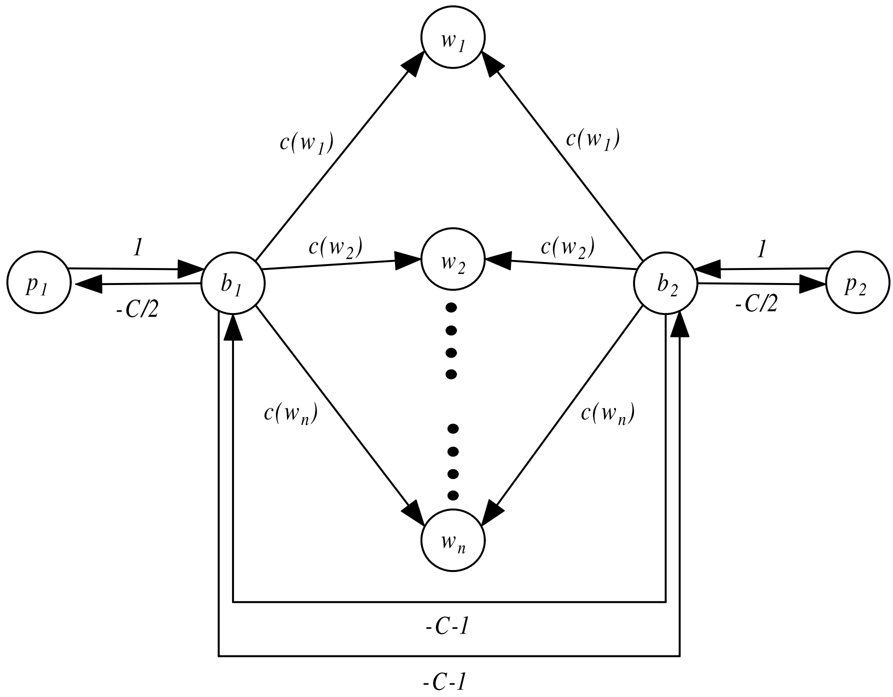

Fig. 1 An example of an Additively Separable Hedonic Game: The buffalo-parasite-game

such that (1) p is better off by joining T and (2) No player in T would be worse off if p joined T. A partition Π is contractually individually stable if we can not find a player p and a coalition $T \in \Pi \cup \{\emptyset\}$ satisfying (1) and (2) above and the following condition: (3) No player in $S_\Pi(p)$ would be worse off if p left $S_\Pi(p)$. This shows that Nash stability implies individual stability and that individual stability implies contractual individual stability.

The concepts of Nash stability and core stability are on the other hand independent in the sense that none of the concepts imply the other one [10]. A partition Π is core stable if no $X \subseteq N$ exists such that all players in X strictly prefer X to their coalition in Π. We refer to [10] for more details.

2.2 Related Work

Sung and Dimitrov [13] show that the problem of deciding whether a *given* partition is core stable in an Additively Separable Hedonic Game is co-NP complete—the corresponding problem concerning Nash stability is clearly solvable in polynomial time. Cechlarova and Hajdukova [3, 4] study the problem of computing core stable partitions in hedonic games where the players compare the best (or worst) members in two coalitions when evaluating the coalitions. Actually different variants of core stability are considered by Cechlarova and Hajdukova.

Ballester has shown in [1] that the problem of deciding whether a Nash stable partition exists in a hedonic game with *arbitrary* preferences is NP-complete. On the other hand Bogomolnaia and Jackson show in [11] that a Nash stable partition exists in every Additively Separable Hedonic Game with *symmetric* preferences. The preferences are symmetric if $\forall i, j \in N : v_i(j) = v_j(i)$. If v_{ij} is the common value for $v_i(j)$ and $v_j(i)$ in a symmetric game then Bogomolnaia and Jackson show that any partition Π maximizing $f(\Pi) = \sum_{S\in\Pi} \sum_{i,j\in S} v_{ij}$ is Nash stable.

Burani and Zwicker introduces the concept of *descending separable* preferences in [2]. Burani and Zwicker show that descending separable preferences guarantees the existence of a Nash stable partition. They also show that descending separable preferences do not imply and are not implied by additively separable preferences.

The detection of community structure in networks has been subject to a great deal of research [7, 12]. Newman and Girvan [12] present a class of *divisive* algorithms for detecting community structure in networks. An algorithm in this class iteratively removes the edge with the highest score of some *betweenness* measure. The betweenness measure is recalculated after each edge removal. One way of measuring the betweenness is to count the number of shortest paths that runs through an edge. A so-called *modularity measure* is used to calculate the quality of the current partition each time a new group of nodes is isolated by the edge removal procedure.

As opposed to Newman and Girvan [12] a formal definition of a *community* appears in [8] by Flake et al. The web graph and the CiteSeer graph are examples of networks that are processed in [8]. Using the terminology from coalition formation games a *community* is a subset of players $C \subseteq N$ in an additively separable game with symmetric preferences such that $\forall i \in C : \sum_{j\in C} v_{ij} \geq \sum_{j\in N-C} v_{ij}$. In other words each player in C gets at least half the total possible payoff by belonging to C. Flake et al. show that the problem of deciding whether it is possible to partition N into k communities is NP-complete. Such a partition is Nash stable but a Nash stable partition is not necessarily a partition into communities. The proof techniques used in this paper are similar to those used in [8].

3 Restricting to Additively Separable Games

In this section we restrict our attention to Additively Separable Hedonic Games compared to Ballester [1]. Compared to Bogomolnaia and Jackson [11], we also allow asymmetric preferences. Informally we show that things are complicated even when looking at Additively Separable Hedonic Games. With an intuitively clear proof based on the buffalo-parasite-game from Example 1 we show that the problem of deciding whether a Nash stable partition exists in a hedonic game remains NP-complete when restricting to additively separable preferences. We will now formally define the problem:

Definition 4 The ASH-NASH problem:

- *Instance*: A set $N = \{1, 2, \dots, n\}$ and a function $v_i : N \to \mathbb{R}$ such that $v_i(i) = 0$ for each $i \in N$.
- *Question*: Does a partition Π of N exist such that

$$\forall i \in N,\ \forall S_k \in \Pi \cup \{\emptyset\} : \sum_{j\in S_\Pi(i)} v_i(j) \geq \sum_{j\in S_k\cup\{i\}} v_i(j)? \tag{2}$$

We are going to prove that this problem is intractable.

Theorem 1 *ASH-NASH is NP-complete.*

Proof It is easy to check in polynomial time that Π is a partition satisfying (2) thus ASH-NASH is in NP.

We will transform an instance of the NP-complete problem PARTITION [9] into an instance of ASH-NASH in polynomial time such that the answers to the questions posed in the two instances are identical—if such a transformation exists we will write PARTITION $\propto$ ASH-NASH following the notation in [9]. This means that we can solve the NP-complete problem PARTITION in polynomial time if we can solve ASH-NASH in polynomial time thus ASH-NASH is NP-complete since it is a member of NP. The rest of the proof explains the details of the transformation.

An instance[2] of PARTITION is a finite set $W = \{w_1, w_2, \ldots, w_n\}$ and a capacity $c(w) \in \mathbb{Z}^+$ for each $w \in W$. The question is whether a subset $W' \subset W$ exists such that $\sum_{w \in W'} c(w) = \frac{C}{2}$ where $C = \sum_{w \in W} c(w)$.

Now suppose we are given an instance of PARTITION. The PARTITION instance is transformed into the buffalo-parasite-game from Example 1 in linear time. All we have to do to translate this as an ASH-NASH instance is to perform a simple numbering of the players in the game.

Now we only have to show that a Nash stable partition of the game in Fig. 1 exists if and only if W' exists. This can be seen from the following argument:

- The partition $\Pi = \{\{b_1, p_1\} \cup W', \{b_2, p_2\} \cup W - W'\}$ is Nash stable if W' exists.
- Now assume that a Nash stable partition Π exists and define $W_1 = S_\Pi(b_1) \cap W$ and $W_2 = S_\Pi(b_2) \cap W$. The set $S_\Pi(b_1)$ must contain p_1. Due to the stability we can conclude that $\sum_{w \in W_1} c(w) \geq \frac{C}{2}$—otherwise b_1 would be better off by its own. By a symmetric argument we have $\sum_{w \in W_2} c(w) \geq \frac{C}{2}$. The two nodes b_1 and b_2 are not in the same coalition so the two sets W_1 and W_2 are disjoint, so we can conclude that $\sum_{w \in W_1} c(w) = \sum_{w \in W_2} c(w) = \frac{C}{2}$. We can take $W' = W_1$ or $W' = W_2$. □

4 Community Structures as Nash Stable Partitions

In this section we relate community structures in networks and Nash stable partitions in Additively Separable Hedonic Games. It seems natural to *define* a *community structure* of N as a partition Π of N such that for any $C \in \Pi$ we have that all members of C feel more related to the members of C compared to any other set in the partition. This is just a less formal way of stating (1)—the property defining a Nash stable partition in a hedonic game.

Suppose we are given a set N and a number $v_{ij} \in \mathbb{R}^+ \cup \{0\}$ for each pair of nodes $\{i, j\}$ in N modeling the strength of the connection between i and j. As an example we could be given an undirected and unweighted graph $G(N, E)$ and let $v_{ij} = 1$ if $\{i, j\} \in E$ and 0 otherwise. If we adopt the definition above of a community structure

[2] The objects constituting an instance in [9] are renamed to match the game from Example 1.

then we essentially have an Additively Separable Hedonic Game with *non-negative* and *symmetric* preferences with community structures appearing as Nash stable partitions. That community structures appear in this way seems to be a reasonable assumption based on visual inspection of the communities identified by Newman and Girvan in [12].

If for example the members of N form a clique where all the connections have identical strength then the trivial partition $\Pi = \{N\}$ is the only Nash stable partition. In this case there would not be any non trivial community structure which sounds intuitively reasonable. On the other hand, let us assume that two disjoint communities S and T of players exist as defined in [8]. If we collapse these communities to two players s and t then we can effectively calculate the s-t minimum cut in the underlying graph for the game. This cut defines a non trivial Nash stable partition. As noted in Sect. 2.2 then a partition of communities following the definition in [8] would certainly be a community structure—but the converse is not always true. The definition of a community structure suggested above can thus be seen as a sort of generalization of the definition of a community in [8].

We will denote a non trivial Nash stable partition as an *inefficient equilibrium*—if the numbers v_{ij} are seen as payoffs then it is optimal for all members of the network to cooperate. In the next section we will prove that inefficient equilibriums generally are hard to compute. To be more specific we will prove that the problem of deciding whether they exist is NP-complete. This result formally indicates that computing community structures is a hard job.

5 Non-negative and Symmetric Preferences

As in the proof of Theorem 1 we need a known NP-complete problem in the proof of the theorem of this section. The "base" problem of the proof in this section is the *EQUAL CARDINALITY PARTITION* problem:

Definition 5 The EQUAL CARDINALITY PARTITION problem:

- *Instance*: A finite set $W = \{w_1, w_2, \dots, w_n\}$ and a capacity $c(w) \in \mathbb{Z}^+$ for each $w \in W$.
- *Question*: Does a non trivial partition $\{W_1, \dots, W_k\}$ of W exist such that $|W_i| = |W_j|$ and $\sum_{w \in W_i} c(w) = \sum_{w \in W_j} c(w)$ for all $1 \leq i, j \leq k$?

EQUAL CARDINALITY PARTITION is closely related to the balanced version of PARTITION where we are looking for a set $W' \subset W$ such that $\sum_{w \in W'} c(w) = \frac{C}{2}$ and $|W'| = \frac{|W|}{2}$. The balanced version of PARTITION is known to be NP-complete [9]. An instance of the balanced version of PARTITION is transformed into an equivalent instance of EQUAL CARDINALITY PARTITION by adding two more elements to the set W—both with capacity $C + 1$. This shows that EQUAL CARDINALITY PARTITION is NP-complete since it is easily seen to belong to NP.

We will now formally define the problem of deciding whether a non trivial Nash stable partition exists in an additively separable hedonic game with non-negative and symmetric preferences:

Definition 6 The INEFFICIENT EQUILIBRIUM problem:

- *Instance*: A set $N = \{1, 2, \ldots, n\}$ and a function $v_i : N \to \mathbb{R}^+ \cup \{0\}$ such that $v_i(i) = 0$ for each $i \in N$ and $v_i(j) = v_j(i)$ for each $i, j \in N$.
- *Question*: Does a non trivial partition Π of N exist such that

$$\forall i \in N,\ \forall S_k \in \Pi \cup \{\emptyset\}: \sum_{j \in S_\Pi(i)} v_i(j) \geq \sum_{j \in S_k \cup \{i\}} v_i(j)?$$

Theorem 2 *INEFFICIENT EQUILIBRIUM is NP-complete.*

Proof We will show that EQUAL CARDINALITY PARTITION $\propto$ INEFFICIENT EQUILIBRIUM. By the same line of reasoning as in the proof of Theorem 1 we conclude that INEFFICIENT EQUILIBRIUM is NP-complete since INEFFICIENT EQUILIBRIUM is easily seen to belong to NP.

We will now show how to transform an instance of EQUAL CARDINALITY PARTITION into an equivalent instance of INEFFICIENT EQUILIBRIUM. All the members of W are players in the instance of INEFFICIENT EQUILIBRIUM and the payoff for w_i and w_j for cooperating is $c(w_i) + c(w_j) + C$. For each player w_i we also add a player z_i. Player z_i only gets a strictly positive payoff by cooperating with w_i—in this case the payoff is $2c(w_i) + C$. Figure 2 depicts a part of the INEFFICIENT EQUILIBRIUM instance as an undirected weighted graph. The members of W are fully connected but z_i is only connected to w_i in the graph.

We will now prove that the two instances are equivalent:

- Suppose that we have a non trivial Nash stable partition Π of the players in Fig. 2. For $S_k \in \Pi$ we define $W_k = S_k \cap W$. The player z_i cooperates with w_i— otherwise Π would not be stable. The total payoff of $w_i \in W_k$ is $|W_k|(C + c(w_i)) + \sum_{w \in W_k} c(w)$.
  - $|W_i| = |W_j|$: If $|W_i| < |W_j|$ then all the players in W_i would be strictly better off by joining W_j. This contradicts that Π is stable.
  - $\sum_{w \in W_i} c(w) = \sum_{w \in W_j} c(w)$: Now assume $\sum_{w \in W_i} c(w) < \sum_{w \in W_j} c(w)$. Once again the players in W_i would be strictly better off by joining W_j since $|W_i| = |W_j|$. Yet another contradiction.

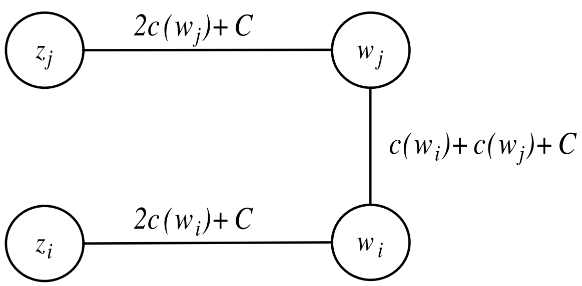

Fig. 2 A part of a game with positive and symmetric preferences

– Suppose that we have a non trivial partition of W into sets with equal cardinality and capacity. For a set W_i in this partition let S_i be the union of W_i and the corresponding z-members. The set of S_i's is easily seen to be a non trivial Nash stable partition of the game in Fig. 2. □

6 Conclusion and Future Work

We have shown that deciding whether a Nash stable partition exists in an Additively Separable Hedonic Game is an NP-complete problem. Moreover, deciding whether non trivial Nash stable partitions exist in additively separable games with non-negative and symmetric preferences is also shown to be NP-complete. We have also shown how community structures in networks are related to Nash stable partitions in Additively Separable Hedonic Games. The results on the computational complexity formally justify that detection of community structures is hard. There is still more work to do with respect to the computational complexity of equilibriums in Additively Separable Hedonic Games since Nash stability is only one of several concepts of stability for such games.

Acknowledgements The author of this paper would like to thank Peter Bro Miltersen and Gerth S. Brodal from University of Aarhus and an anonymous referee for valuable comments and constructive criticism.

References

1. Ballester, C.: NP-completeness in hedonic games. Games Econ. Behav. **49**(1), 1–30 (2004)
2. Burani, N., Zwicker, W.S.: Coalition formation games with separable preferences. Math. Soc. Sci. **45**(1), 27–52 (2003)
3. Cechlárová, K., Hajduková, J.: Computational complexity of stable partitions with b-preferences. Int. J. Game Theory **31**(3), 353–364 (2002)
4. Cechlárová, K., Hajduková, J.: Stable partitions with w-preferences. Discrete Appl. Math. **138**(3), 333–347 (2004)
5. Chen, N., Rudra, A.: Walrasian equilibrium: Hardness, approximations and tractable instances. In: WINE, pp. 141–150 (2005)
6. Daskalakis, K., Papadimitriou, C.H.: The complexity of games on highly regular graphs. In: ESA, pp. 71–82 (2005)
7. Duch, J., Danon, L., Díaz-Guilera, A., Arenas, A.: Comparing community structure identification. J. Stat. Mech. Theory Exp. **2005**(09), 09008 (2005)
8. Flake, G., Tarjan, R., Tsioutsiouliklis, K.: Graph clustering and minimum cut trees. Internet Math. **1**(4), 385–408 (2004)
9. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. Freeman, New York (1979)
10. Hajdukova, J.: On coalition formation games. Technical Report, Institute of Mathematics, PJ Safarik University (2004)
11. Jackson, M.O., Bogomolnaia, A.: The stability of hedonic coalition structures. Games Econ. Behav. **38**(2), 201–230 (2002)
12. Newman, M.E.J., Girvan, M.: Finding and evaluating community structure in networks. Phys. Rev. E **69**, 026113 (2004)
13. Sung, S.C., Dimitrov, D.: On core membership testing for hedonic coalition formation games. Oper. Res. Lett. **35**(2), 155–158 (2007)