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Theory of Computing Systems

Congestion, Dilation, and Energy in Radio Networks*

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Abstract. We investigate the problem of path selection in radio networks for a given static set of n sites in two- and three-dimensional space. For static point-to-point communication we define measures for congestion, dilation, and energy consumption that take interferences among communication links into account.

We show that energy-optimal path selection for radio networks can be computed in polynomial time. Then we introduce the diversity $g(V)$ of a set $V \subseteq \mathbb{R}^d$ for any constant d. It can be used to upper bound the number of interfering edges. For real-world applications it can be regarded as $\Theta(\log n)$. A main result is that a c-spanner construction as a communication network allows one to approximate the congestion-optimal path system by a factor of $O(g(V)^2)$.

Furthermore, we show that there are vertex sets where only one of the performance parameters congestion, dilation, and energy can be optimized at a time. We show trade-offs lower bounding congestion $\times$ dilation and dilation $\times$ energy. The trade-off between congestion and dilation increases with switching from two-dimensional to three-dimensional space. For congestion and energy the situation is even worse. It is only possible to find a reasonable approximation for either congestion or energy minimization, while the other parameter is at least a polynomial factor worse than in the optimal network.

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1. Introduction

Radio networks are widely used today. People access voice and data services via mobile phones, bluetooth technology replaces unhandy cables by wireless links and wireless networking is possible via IEEE 802.11 compatible network equipment. Nodes in such networks exchange their data packets usually with fixed base-stations that connect them with a wired backbone. However, in applications such as search and rescue missions or environmental monitoring, no explicit communication infrastructure is available. Since the communication range of the usually mobile nodes is limited, target nodes are not always directly reachable. The data has to be routed over intermediate nodes (multi-hop routing), therefore each node has to have router capabilities. Such networks are called *ad hoc networks*. They impose higher requirements on routing algorithms such as adaptability to dynamic link changes and awareness of the limited energy in mobile nodes while maintaining high throughput and small delays. Methods that enable ad hoc networking can be assessed with three measurable quantities: link congestion, dilation (also known as hop count), and energy consumption. Traditional routing protocols such as AODV, DSDV, and DSR [21] usually choose the path with the lowest hop count. There also exist power-aware routing protocols that use different metrics (e.g., energy consumed per packet, variance in node power level) to choose the best route in order to extend the lifetime of individual nodes or the whole network [25], [26], [3]. The congestion of a route is usually not regarded directly, but some routing protocols choose routes with the shortest route discovery, assuming that the route with the quickest response is less congested (e.g., SSA [5]). All these routing protocols assess the paths that have been found by route discoveries according to a cost function. However, how are the factors congestion, dilation, and energy that are mostly employed in the cost function related to each other?

Due to interfering links, it is not clear how to choose nodes or devices or vertices as communication partners. In our model nodes can adjust their transmission powers in order to change the transmission range. We are looking for the optimal choice of this network for a given static distribution of the nodes in $\mathbb{R}^d$ for $d \in \{2, 3\}$.

In Section 1.1 we introduce our model of radio networks, and define and motivate our notions of congestion, dilation, and energy. Then, in Section 1.2, we shortly present and discuss our results to give the reader a short overview. In Section 2 we relate congestion and dilation to the routing time in radio networks and present upper and lower bounds for the routing time. In Section 3 we present strategies for path selection that provably optimize energy consumption and give an $O(g(V)^2)$-factor approximation of congestion where $g(V)$ is defined as the diversity that describes the number of magnitudes of all node-to-node distances. Two distances d_1, d_2 are in the same magnitude if $\lfloor \log d_1 \rfloor = \lfloor \log d_2 \rfloor$. In Section 4, as a main insight, we can conclude that not any two of these measures can be minimized simultaneously. Trade-offs between two measures are unavoidable. Finally, Section 5 reflects work in the light of an experimental setting, we are working on, and proposes future research directions.

1.1. *Modeling Radio Networks*

We consider a set $V \subseteq \mathbb{R}^d$ of n radio stations (or sites, or vertices, or nodes) for $d \in \{2, 3\}$. In order to transmit a message from a radio station u to a radio station v, u is able to adjust its *transmission radius* to $|u, v| := ||\{u, v\}||_2$, the Euclidean distance between u and v. We say that u establishes the *communication link* $e = \{u, v\}$. Instead of sending the packet directly from u to v, multi-hop communication is also possible by using a path $(u = u_1, \ldots, u_m = v)$ of stations. In order to submit a packet from u to v, the communication links $\{u_i, u_{i+1}\}$, $i = 1, \ldots, m-1$, have to be established.

Consider now a *routing problem* $f : V \times V \to \mathbb{N}$, where $f(u, v)$ packets have to be sent from u to v, for all $u, v \in V$. A collection of paths, one for each packet, forms a *path system* $\mathcal{P}$ for f. In this paper we assume (as in [20]) that each transmission along a link $\{u_i, u_{i+1}\}$ has to be acknowledged, so that the communication from u_{i+1} to u_i also has to be established. Thus the edges of the paths have to be used in both directions. The undirected graph on V defined by the undirected edges of the paths in $\mathcal{P}$ is the *communication network* N defined by $\mathcal{P}$. As already well established and analyzed for wired networks, see, e.g., [17], we define the *dilation* of $\mathcal{P}$ to be the maximum of the lengths of all paths in $\mathcal{P}$. In order to define congestion, we have to look at the specific properties of radio networks. For wired networks, the *load* $\ell(e)$ of an edge $e = \{u, v\}$ of the communication graph G is defined as the number of packets to be forwarded along e. (This load is often called congestion in a wired network, see [17]; we use the notion "load" to distinguish it from our notion of congestion for radio networks, to be described below.)

A major problem in radio networks is the effect of interfering radio signals. If two nodes A and B are in range of a third listening node C, but cannot hear each other, a collision occurs at C if A and B transmit simultaneously. This is the *hidden terminal* problem [2]. Solutions exist that reduce this effect. In the IEEE 802.11 standard, see [13], sender A and receiver C reserve the channel by sending request-to-send (RTS) and clear-to-send (CTS) packets prior to the data communication. Other nodes, also the nodes that cannot hear the sender, hear at least one of these packets and suspend all transmissions until the channel is free. However, this also reduces the network capacity since any node B that hears the RTS of A cannot start a transmission even if C is outside of B's range and thus no collision would occur. This is the *exposed terminal* problem [2].

In our radio model we allow only one radio frequency. Now, if two packets are transmitted at the same time we may experience *radio interference* such that only one or none of them can be received. The area covered by sending and acknowledging a packet from u to v along an edge $e = \{u, v\}$ is $D(e) := D_r(u) \cup D_r(v)$, where $D_r(u)$ denotes a disk with center u and radius $r := |e|$ (see Figure 1). Now, if another packet q has to be sent or received by a site within $D(e)$, the radio interference prevents the successful transmission of q. Since sites adjust their transmission powers for sending packets, interferences may not be symmetric.

As mentioned above, for radio networks we need to reflect the impact of radio interferences on the delay of a packet. Therefore, we define the set of edges interfering with an edge $e = \{u, v\}$ of N as

$$\boldsymbol{Int}(e) := \{e' \in E(N)\backslash\{e\} \mid u \in D(e') \text{ or } v \in D(e')\}.$$

Thus, sending a packet along e is successful only if no other edges from $\boldsymbol{Int}(e)$ send concurrently. We define the *interference number* of a communication link by $|\boldsymbol{Int}(e)|$. The *maximum interference number* of the network is the maximum interference number

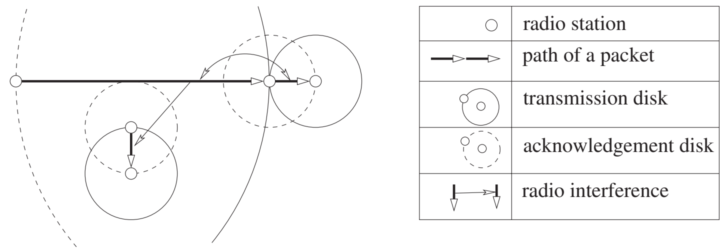

Fig. 1. Stations, packet paths, and induced radio interferences.

of all edges. Now we define the *congestion of the edge e* by

$$C_{\mathcal{P}}(e) := \ell(e) + \sum_{e' \in Int(e)} \ell(e').$$

The *congestion of the path system* $\mathcal{P}$ is defined by

$$C_{\mathcal{P}}(V) := \max_{e \in E_{\mathcal{P}}} \{C_{\mathcal{P}}(e)\}.$$

The variable choice of the transmitter power allows us to reduce the energy consumption, saving on the tight resources of batteries in portable radio stations and reducing interferences. The energy needed to send over a distance of r is given by r^2. It turns out that in practice the energy consumption is $O(r^4)$ or even $O(r^5)$. However, all results besides Theorem 5 can be easily transferred to higher exponents.

We distinguish two energy models reflecting the power consumption by link maintenance and packet transmission. In the first model, called the *unit energy model*, we assume that maintaining a communication link e is proportional to $|e|^2$, where $|e|$ denotes its Euclidean length. So, we completely neglect any impact of power consumption by packet delivery. Therefore, the unit energy *U-Energy* used by radio network N is given by

$$\text{U-Energy}_{\mathcal{P}}(V) := \sum_{e \in E_{\mathcal{P}}(N)} |e|^2.$$

The *flow energy model* reflects the energy actually consumed by transmitting all packets of a routing problem f, if path system $\mathcal{P}$ is used. So, we neglect any power consumption by link maintenance. Here, the power consumption of a communication link e is weighted by its load $\ell(e)$:

$$\text{F-Energy}_{\mathcal{P}}(V) := \sum_{e \in E_{\mathcal{P}}(N)} \ell(e)|e|^2.$$

We subdivide the design of a routing strategy for f into the following three steps:

- *Path selection*: Select a path system $\mathcal{P}$ for f.
- *Interference handling*: Design a strategy, that realizes the transmission of a packet along a link in the presence of interferences.
- *Packet switching*: Decide when and in which order packets are sent along a link.

1.2. *Our Results*

The main topic of this paper is to design path systems and to analyze and compare them with regard to congestion, dilation, and energy. Before we start that, we show in Section 2 how to handle interferences and how to resolve collisions. This will result in lower and upper bounds for the routing time which can be expressed in terms of congestion, dilation, and interference number. Here we adopt methods from [1], where such protocols are designed for the case that V is randomly chosen.

In Section 3 we consider congestion, dilation, and energy in radio networks and try to minimize each of these measures individually. Optimal path systems $\mathcal{P}$ that are optimized for dilation or energy can be found in polynomial time. An optimal network for the dilation is given by the complete network. In Theorem 4 we prove that the unique paths defined by a minimum spanning tree (MST) result in an optimal path system for a radio network with respect to unit energy. It is known that the *Gabriel Graph* $\mathrm{GG}(V)$, introduced in [6], provides energy-optimal paths [18]. Such a Gabriel Graph consists of all edges $\{u, v\}$ such that the open sphere using the line from u to v as diameter does not contain any other node from $V \subseteq \mathbb{R}^d$, $d \in \{2, 3\}$. It is also known that $\mathrm{MST}(V) \subseteq \mathrm{GG}(V)$. In Theorem 5 we show that the shortest paths in a subgraph of the weighted Gabriel Graph (the weight of an edge is given by the square of its length) form an optimal path system for flow energy. Finding optimal solutions for congestion is considerably harder, and to our knowledge nothing non-trivial was known before this work. We begin our considerations with the introduction of the diversity $g(V)$ of a set V that can be used to upper bound the number of interfering edges. After that we present a path selection strategy that gives an $O(g(V)^2)$-approximation for congestion. We use the *Hierarchical Layer Graph* (HL-graph) introduced in [9] to achieve this result. For randomly or "reasonably" distributed nodes, $g(V) = O(\log n)$, which gives an $O(\log^2 n)$-approximation for congestion.

Finally, in Section 4, we study the problem of minimizing combinations of parameters. Here, the main result is that no two of the parameters congestion, dilation, and energy can be minimized at the same time. Tables 1 and 2 give an overview.

2. Upper and Lower Bounds for Routing Time

It is easy to see, and is well known, that both the dilation and the maximum load $\max_{e \in E}\{\ell(e)\}$ lower bound the routing time, even in wired networks. In this section we show that the routing time in radio networks can also be lower bounded in terms of our extended notion of congestion (Theorem 1). We further present an upper bound.

Table 1. Approximation results for logarithmic diversity.

	Congestion	Dilation	Unit Energy	Flow energy
Structure	HL-graph	Complete Network	MST	Gabriel subgraph
Approx.-factor	$O(\log^2 n)$	Optimal	Optimal	Optimal

Table 2. Trade-offs and incompatibilities on network parameters.

	Dilation			Congestion		
Congestion	$C_{\mathcal{P}}(V) \cdot D_{\mathcal{P}}(V)$	$\geq$	$\Omega(W)$	—		
Unit energy	$D_{\mathcal{P}}(V) \cdot \mathrm{UE}_{\mathcal{P}}(V)$	$\geq$	$\Omega(d^2)$	$C_{\mathcal{P}}(V)$ $\mathrm{UE}_{\mathcal{P}}(V)$	$\geq$ $\geq$	$\Omega(n^{1/3} C^*_{\mathcal{P}}(V))$ or $\Omega(n^{1/3} \mathrm{UE}^*_{\mathcal{P}}(V))$
Flow energy	$D_{\mathcal{P}}(V) \cdot \mathrm{FE}_{\mathcal{P}}(V)$	$\geq$	$\Omega(d^2 W)$	$C_{\mathcal{P}}(V)$ $\mathrm{FE}_{\mathcal{P}}(V)$	$\geq$ $\geq$	$\Omega(n^{1/3} C^*_{\mathcal{P}}(V))$ or $\Omega(n^{1/3} \mathrm{FE}^*_{\mathcal{P}}(V))$

Theorem 1. *Consider a radio network N in d-dimensional space ($d \in \{2, 3\}$) and a path system $\mathcal{P}$ for a routing problem f with dilation D and congestion C. Let T be its optimal routing time. Then it holds for $c_2 = 6$ and $c_3 = 20$ that*

$$T \;\geq\; \max\left\{\frac{C}{2c_d}, D\right\} = \Omega(C + D).$$

Proof. Let $e = \{u, v\}$ be an edge with maximum congestion C. Now we try to calculate the number of edges along which successful transmissions can take place simultaneously to e. We partition the d-dimensional space into regions $R_1, \ldots, R_{c_d}$ (see Figure 2).

The main property of these regions is that, for every pair of points $r, s \in R_i$ for all i, the angle between $\overline{ur}$ and $\overline{us}$ is less than or equal to $\pi/3$. Clearly, for two-dimensional space we have $c_2 = 6$. In [12] it has been shown that $c_3 \leq 20$. Similarly we consider the analogous partitioning $R_{c_d+1}, \ldots, R_{2c_d}$ with v as the corner point of angles. Define

$$E_i := \{\{p, q\} \mid (p \in R_i \vee q \in R_i) \wedge \{p, q\} \in \boldsymbol{Int}(e)\}.$$

Note that by a straightforward geometric argument, for two edges $e', e'' \in E_i$, it holds that either $e' \in \boldsymbol{Int}(e'')$ or $e'' \in \boldsymbol{Int}(e')$. Therefore, all transmissions over edges in $E_i \cup \{e\}$ have to be done sequentially. Let $\ell_i := \ell(e) + \sum_{e' \in E_i} \ell(e')$. Then $\sum_{i=1}^{2c_d} \ell_i \geq C$. Hence,

$$T \geq \max_{i \in [2c_d]} \{\ell_i\} \geq \frac{1}{2c_d} \sum_{i=1}^{2c_d} \ell_i \geq \frac{C}{2c_d}. \qquad \square$$

We now turn to upper bounding the routing time. Following the approach of local probabilistic control protocols for the MAC layer (also called *LPC schemes*) given in [1],

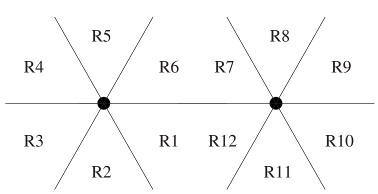

Fig. 2. Partitioning of the two-dimensional space into regions $R_1, \ldots, R_{12}$.

we use the following protocol for handling interferences. If u wants to send a packet (or an acknowledgment) along link e to v, u proceeds as follows. The link e is activated with probability $\varphi(e)$ and so, in each step, it decides with probability $\varphi(e)$ to send a packet. We choose $\varphi(e) := \min\{\frac{1}{2}, \ell(e)/C_{\mathcal{P}}(V)\}$. Then it holds that $\varphi(e) + \sum_{e' \in Int(e)} \varphi(e') \le 1$. We have the following for a transmission between two nodes:

Lemma 1. *The probability of a successful transmission on a link e is at least* $\frac{1}{4}\varphi(e)$. *Therefore, the expected time for a successful transmission is at most* $4/\varphi(e)$. *Further, if u has decided to send a message to v, this transmission attempt has a success probability of at least* $1/4$.

Proof. Note that $1 - p \ge 1/4^p$ for $p \in [0, \frac{1}{2}]$. Let $\boldsymbol{Int}(e)$ be defined as $\{e_1, ..., e_m\}$. Then the following holds:

$$\begin{aligned}\text{Prob[Transmission on link } e \text{ is successful]} &= \varphi(e) \prod_{i=1}^{m} (1 - \varphi(e_i)) \\ &\ge \varphi(e) \prod_{i=1}^{m} 4^{-\varphi(e_i)} = \varphi(e) 4^{-\sum_{i=1}^{m} \varphi(e_i)} \\ &\ge \tfrac{1}{4}\varphi(e).\end{aligned}$$

The bounds for the expected transmission time and the constant success probability follow directly. □

Definition 1 [1, Definition 2.2]. Let the *probabilistic communication graph* (or PCG for short) $G = (V, \tilde{\varphi})$ be defined as a complete directed graph with node set V and edge labels determined by the function $\tilde{\varphi} : V \times V \to [0, 1]$. Every edge e can forward a packet in one time step, but only succeeds in doing this with probability $\tilde{\varphi}(e)$.

The authors of [1] transform the problem of routing in wireless networks to routing in PCGs. Since we have a constant success probability, we can use the same technique to transform the problem of routing in our graphs to routing in PCGs. We make use of the results given in [1].

We adopt the following result, but we need some other notation from [1]. Let the *maximum edge latency* $\tilde{L}$ of a PCG G be defined as the maximum expected time and the *minimum edge latency* $\tilde{l}$ as the minimum expected time needed to successfully transmit a packet along an edge in G. Note that our interference handling guarantees: $\tilde{L} \le \infty$ and $\tilde{l} \ge 2$, since $\frac{1}{2} \ge \varphi(e) \ge c$ for all $e \in E$ and a constant $c \in \mathbb{R}$. Given a collection $\mathcal{P}$ of simple paths in some PCG G, the *PCG-dilation* $\tilde{D}$ of $\mathcal{P}$ is defined as the maximum over all paths in $\mathcal{P}$ of the sum of $1/\tilde{\varphi}(e)$ over all edges e used by it (that is, $\tilde{D}$ denotes the maximum expected time a packet needs to traverse a path in $\mathcal{P}$), and the *PCG-congestion* $\tilde{C}$ of $\mathcal{P}$ is defined as the maximum over all edges e of $1/\tilde{\varphi}(e)$ times the number of paths in $\mathcal{P}$ that cross it (that is, $\tilde{C}$ denotes the maximum expected time spent at an edge e to forward all packets which contain e in their path). The size n of an arbitrary path collection is defined as the number of packets given by the routing problem.

Theorem 2 [1, Theorem 2.12]. *There is an online protocol for sending packets along an arbitrary path collection of size n with PCG-dilation $\tilde{D}$, PCG-congestion $\tilde{C}$, maximum edge latency $\tilde{L}$, and minimum edge latency $\tilde{l}$ in time $O(\tilde{C}+\tilde{D}\log(n\cdot\tilde{L}/\tilde{l}))$ with probability at least $1-n^{-c}$ for any constant c.*

Applying this to our model yields:

Theorem 3. *Consider a radio network $N = (V, E)$ and a path system $\mathcal{P}$ of size n for some routing problem f with maximum interference number I, dilation D, and congestion C. Let T be its optimal routing time, when the path system $\mathcal{P}$ is used. There is an online routing protocol that needs routing time $O(C + D \cdot I \cdot \log(n \cdot I))$, with probability at least $1 - n^{-c}$ for any constant c.*

Proof. By definition we have that $1/\varphi(e) \leq I$ for all $e \in E$. It follows directly that $\tilde{D} \leq D \cdot I$ and by definition we have $\tilde{C} = C$. The maximum edge latency $\tilde{L}$ is given by $\max_{e\in E}\{1/\varphi(e)\} = O(I)$. The minimum edge latency $\tilde{l}$ is at least 2, by definition of $\varphi(e)$. Now we consider the PCG $G = (V, \varphi)$ and use Theorem 2 to complete the proof. □

3. Minimizing Energy and Congestion

In this section we try to optimize congestion, dilation, and energy separately. It is clear that the complete network is the optimal choice for dilation. So we only have to focus on congestion and energy. We show that energy-optimal path selection for radio networks can be computed in polynomial time. In the case of congestion we present an approximation of the congestion-optimal path system.

3.1. *Energy*

The unit energy of a path system for a radio network is defined as the energy consumption necessary to deliver one packet on each communication link. It turns out that the minimum spanning tree (MST(V)) optimizes unit energy, i.e., power consumption for maintaining links while neglecting all additional energy consumption for packet delivery. Note that the hardness results shown in [15] and [4] do not apply because in our model the transmission radii are adjusted for each packet.

Theorem 4. *The unique paths defined by a minimum spanning tree result in an optimal path system for a radio network $N = (V, E)$, $V \subseteq \mathbb{R}^d$ for any d, with respect to the unit energy.*

Proof. Consider the complete graph on V, where each edge g gets weight $|e|^2$. The minimum energy network can be constructed using Prim's or Kruskal's algorithm for a minimum spanning tree. Note that the decisions in this algorithm are based on comparison

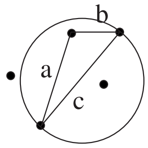

Fig. 3. Communication on an edge c is more expensive with regard to unit energy than communication on the edges a and b ($a^2 + b^2 < c^2$).

of the length of some edges e and e', i.e., $|e| \leq |e'|$. Thus, the minimal network for energy is also the minimum spanning tree for Euclidean distances. □

For the flow energy model, the best network is not necessarily a tree. However, one can compute the minimal flow energy network in polynomial time. In consideration of the flow energy we use the *Gabriel Graph* ($\mathrm{GG}(V)$) introduced in [6]. It consists of all edges $\{u, v\}$ such that the open sphere using the line from u to v as diameter does not contain any other node from $V \subseteq \mathbb{R}^d$, $d \in \{2, 3\}$. It turns out that $\mathrm{MST}(V) \subseteq \mathrm{GG}(V)$. Let $\tilde{\mathrm{GG}}(V)$ denote the weighted version of $\mathrm{GG}(V)$ where each edge e has weight $|e|^2$. The following holds:

Theorem 5. *For a given vertex set V and a routing problem f, the shortest paths between vertices $u, v \in V$ with $f(u, v) \neq 0$ of $\tilde{\mathrm{GG}}(V)$ form an optimal path system for a radio network with respect to the flow energy.*

Proof. By the theorem of Thales, the flow-optimal path between two sites u and v only contains edges of $\mathrm{GG}(V)$ (see Figure 3). Thus, it is a shortest path in $\mathrm{GG}(V)$, where each edge e has weight $|e|^2$, i.e., in $\tilde{\mathrm{GG}}(V)$. By the definition of the flow energy of a path system, the collection of all flow-optimal paths for packets of the routing problem f form a flow-optimal path system. □

Note that a flow-optimal path system can easily be computed in polynomial time by an all-pairs-shortest-paths algorithm. There are situations where edges of the Gabriel Graph can be replaced by less energy-consuming paths, even if no site lies inside the disk described by the edge. In this case the edge of the Gabriel Graph is not part of any energy optimal route.

3.2. *Congestion*

In the following we try to optimize congestion. We begin with the introduction of the diversity $g(V)$ of a set V that can be used to upper bound the number of interfering edges. After this we present a data structure that approximates the congestion-optimal communication network by a factor of $O(g(V)^2)$.

3.2.1. *Diversity of a Vertex Set.* Sometimes the location of the radio stations does not allow any routing without incurring high congestion. Consider a vertex set $V =$

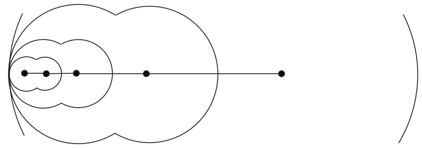

Fig. 4. The high diversity of the vertex set causes many interferences, resulting in high congestion.

$\{v_1, \ldots, v_n\}$ on a line, with distances $|v_i, v_{i+1}| = 2^i$. The edge $\{v_i, v_{i+1}\}$ interferes with all edges $\{v_j, v_{j+1}\}$ for $j \leq i$, see Figure 4. Therefore the interference number of the network is $n-2$. Suppose only v_1 and v_n want to communicate, then the better solution for congestion is to disconnect all interior points and to realize only the edge $\{v_1, v_n\}$. Of course this is not an option when interior nodes need to communicate.

It turns out that a determining parameter for the realization of optimal communication networks for radio networks is the number of magnitudes of distances. Distances have different magnitudes if they differ by more than a factor of 2.

Definition 2. The diversity $g(V)$ of a point set V in Euclidean space is defined by $g(V) := |Q(V)|$, where $Q(V) := \{m \mid \exists u, v \in V : \lfloor \log|u, v| \rfloor = m\}$ denotes the levels of different magnitudes of all distances.

Note that in the scenario of Figure 4 we observe almost maximum diversity of n (and a high interference number). We first show the close connection between the interference number and diversity for vertices on a line.

Theorem 6. *The interference number of a line graph $G = (V, E)$ with edges between direct neighbors, i.e., $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}$ and $E = \{\{v_i, v_{i+1}\} \mid i \in [n-1]\}$, is at most $6 \cdot g(V)$.*

Proof. At most six edges of a length in a distance in the range $[2^q, 2^{q+1})$ can exist with an endpoint within distance 2^{q+1} of a given edge. These are the only edges of this length that possibly can interfere with this edge. Hence, the overall number of interfering edges is at most $6 \cdot |Q(V)|$. $\square$

Because of this relationship between interferences of radio networks and the diversity of points sets, we take a closer look at the possible range of the diversity. It is upper bounded by $\binom{n}{2}$, because it is defined over all possible distances. A divide-and-conquer argument, however, gives an upper bound of $O(n \log n)$. With a randomization technique we will show that the diversity grows at most linear in the number of points. Such a worst case is depicted in Figure 4.

On the other hand, the diversity is at least logarithmic in the number of nodes. Such small diversity can be observed for equidistant points on a line or an $m \times m$ grid.

Theorem 7. *For the diversity $g(V)$ of n points $V \subseteq \mathbb{R}^d$ we observe the following*:

1. $g(V) = \Omega((\log n)/d)$,
2. $g(V) = O(n)$ *for* $d = 1$, *i.e.*, $V \subseteq \mathbb{R}$,
3. $g(V) = O(d(\log d)\, n)$,
4. $g(V) \leq 2 + \log(\max_{u,v\in V}|u, v|/\min_{u,v\in V \wedge u\neq v}|u, v|)$.
5. *For a point set V randomly chosen from* $[0, 1]^d$ *the diversity is at most* $O(\log n) + \frac{1}{2}\log d$ *with probability* $1 - n^{-c}$, *for any constant* $c > 0$.

Proof. Let $r_0 := \min_{u,v\in V, u\neq v} |u, v|$ denote the minimum distance of two different points.

1. $g(V) = \Omega((\log n)/d)$.

   Note that all pairs of points $u \neq v$ have a minimum distance $|u, v| \geq r_0$. For a point u we consider all points W which are at most $2r_0$ distant to u. Now observe that all spheres with a center of W and radius $r_0/2$ do not intersect (yet may be tangent) and are included in the sphere with center u and radius $\frac{5}{2}r_0$ (This results from distance $2r_0$ between center points and u plus $r_0/2$ for the rest of the spheres). Hence, the sum of volumes of all the spheres with radius $r_0/2$ is at most the volume of the larger sphere. Note that the volume of a d-dimensional sphere with radius r is given by $k_d r^d$, where $k_d := \pi^{d/2}/(d/2)!$. This leads to the following inequality:

   $$|W| \cdot k_d \left(\frac{r_0}{2}\right)^d \leq k_d \left(\frac{5r_0}{2}\right)^d .$$

   This yields

   $$|W| \leq 5^d.$$

   Now pick an arbitrary point of V, erase all points of V that are closer than $2r_0$ to this point, and reiterate this step until all points in the resulting set V' have a minimum distance $2r_0$. The above observation implies $|V'| \geq |V|/5^d$, since every point has at most 5^d other points in its $2r_0$ neighborhood.

   Now, we iterate this process on the reduced point set V', with $r_0' = \min_{u,v\in V':u\neq v}|u, v|$, until only one point is left. This process takes at least $\lceil \log_{5^d}|V|\rceil = \Omega((\log n)/d)$ iterations. In every iteration step we find at least one new element of $Q(V)$, since the minimum distance after each reduction of the point set is at least twice as large as before.

   Therefore, the number of rounds of this process gives a lower bound on the diversity of V.
2. For a set $V \subseteq \mathbb{R}$ of n elements we have $g(V) = O(n)$.

   The main difficulty for the proof of this statement arises from side effects due to the rounding in the definition of the diversity. We overcome this problem by randomization, in particular we multiply each number of $V = \{v_1, \ldots, v_n\}$, where $v_1 < v_2 < \cdots < v_n$, by the same factor $k = 2^R$ yielding $v_i' = kv_i$, where R is a continuous uniform random variable over $[0, 1]$. Let $V_R = \{v_1', \ldots, v_n'\}$ be the resulting set of real numbers and define for all $i \in [n]$: $Q_i := Q(\{v'_1, \ldots, v'_i\})$. We will prove, for $i \in \{2, \ldots, n\}$, that

$$\mathbf{E}[|Q_i \setminus Q_{i-1}|] \leq 5. \tag{1}$$

We concentrate on elements in the set S_i defined by

$$S_i := \{m \in \mathbb{Z} \mid \exists j \in [i-1] : m = \lfloor \log(kv_i - kv_j) \rfloor\},$$

because $S_i \setminus S_{i-1} \supseteq Q_i \setminus Q_{i-1}$. We give a short proof that this inclusion is valid:

$$\begin{aligned} m \in Q_i \setminus Q_{i-1} \quad &\Leftrightarrow \quad \exists j \in [i-1] : m = \lfloor \log(kv_i - kv_j) \rfloor \\ &\qquad \wedge \quad \forall \ell \in [i-1], \forall j \in [\ell - 1] : m \neq \lfloor \log(kv_\ell - kv_j) \rfloor \\ &\Rightarrow \quad \exists j \in [i-1] : m = \lfloor \log(kv_i - kv_j) \rfloor \\ &\qquad \wedge \quad \forall j \in [\ell - 2] : m \neq \lfloor \log(kv_{i-1} - kv_j) \rfloor \\ &\Leftrightarrow \quad m \in S_i \setminus S_{i-1}. \end{aligned}$$

For fixed $i \in [2..n]$ let $x := v_i - v_{i-1}$, $y_j := v_{i-1} - v_j$, and $z_j := v_i - v_j$. Analogously, we define $x' := v'_i - v'_{i-1} = kx$, $y'_j := v'_{i-1} - v'_j = ky_j$, and $z'_j := v'_i - v'_j = kz_j$. In this notation we have

$$S_i = \{m \in \mathbb{Z} \mid \exists j \in [i-1] : \ m = \lfloor \log z'_j \rfloor\}.$$

If we observe $y_j \leq 2y_{j+1}$ in an interval $j \in [a..b-1]$, we can conclude that $z_j \leq 2z_{j+1}$, since $x + y_j = z_j$. A further implication is that $y'_j \leq 2y'_{j+1}$ and $z'_j \leq 2z'_{j+1}$. The consequence of $y_j \leq 2y_{j+1}$ is that the rounded logarithms $\lfloor \log y'_j \rfloor \in S_{i-1}$ form a consecutive interval of integer values

$$\{\lfloor \log y'_a \rfloor, \lfloor \log y'_{a+1} \rfloor, \ldots, \lfloor \log y'_b \rfloor\} \ = \ [\lfloor \log y'_a \rfloor .. \lfloor \log y'_b \rfloor] \ \subseteq \ S_{i-1}.$$

Yet, the same is true for S_i:

$$\{\lfloor \log z'_a \rfloor, \lfloor \log z'_{a+1} \rfloor, \ldots, \lfloor \log z'_b \rfloor\} \ = \ [\lfloor \log z'_a \rfloor .. \lfloor \log z'_b \rfloor] \subseteq S_i.$$

If $x \leq y_b$ we have $z'_b = x' + y'_b \leq 2y'_b$. Thus, $\log z'_b \leq 1 + \log y'_b$ and therefore the only contribution of the set $\{\lfloor \log z'_j \rfloor \mid j \in [a..b]\}$ to $S_i \setminus S_{i-1}$ is one element at the most, namely $\lfloor \log z'_b \rfloor$.

Furthermore, the probability that this element occurs decreases proportionally to $1/z_b$ as we will see now. For this, we estimate the probability that $\lfloor \log z'_b \rfloor \neq \lfloor \log y'_b \rfloor$. An equivalent representation of this inequality is

$$\lfloor R + \delta + \log y_b \rfloor \neq \lfloor R + \log y_b \rfloor,$$

where

$$\delta = \log z_b - \log y_b = -\log \frac{z_b - x}{z_b} = -\log \left(1 - \frac{x}{z_b}\right).$$

Clearly, the probability for satisfying this inequality is given by $\max\{\delta, 1\}$, since R is chosen uniformly from $[0, 1]$. Note that for $r \in [0, \frac{1}{2}]$ it holds that $\log(1-r) \geq -2r$. Substituting $r = x/z_b$ we can conclude $\delta \leq 2x/z_b$ if $z_b \geq 2x$.

Putting it all together, we see that if $z_j \leq 2x$ we have

$$\log z'_j \leq 1 + \log z_j \leq 2 + \log x$$

and hence at most the elements $q := \lfloor x \rfloor$, $q+1$, and $q+2$ might be added to $S_i \backslash S_{i-1}$.

For all $z_j > 2x$ we partition $[i-1]$ into intervals $I_k = [a_k..b_k]$ with $a_k \leq b_k < a_{k+1}$ such that for all $j \in I_k \backslash \{b_k\}$ we have $y_i \leq 2y_{i+1}$ and $y_{b_k} > 2y_{a_{k+1}} > 2y_{b_{k+1}}$. The probability for $\lfloor \log z'_{b_k} \rfloor \in S_i \backslash S_{i-1}$ is at most $2x/z_b$. Since $2x/z_{b_1} \leq 1$, and $z_{b_{k+1}} \geq 2z_{b_k}$, the expected number of such elements is bounded by $\sum_{k=1}^{\infty} 2^{k-1} \leq 2$. Since $Q_i \backslash Q_{i-1} \subseteq S_i \backslash S_{i-1}$, this proves (1).

As a consequence of (1) it follows (note that $Q_1 = \emptyset$) that

$$\mathbf{E}[|Q(V_R)|] \;=\; \mathbf{E}\left[\sum_{i=2}^{n} |Q_i \backslash Q_{i-1}|\right] = \sum_{i=2}^{n} \mathbf{E}[|Q_i \backslash Q_{i-1}|] \leq 5(n-1).$$

Now we derandomize: If for the random variable $R \in [0, 1]$ the expected value is bounded by $\mathbf{E}[|Q(V_R)|] \leq 5(n-1)$, than there exists a concrete choice $r \in [0, 1]$ such that $|Q(V_r)| \leq 5(n-1)$.

The relationship between $Q(V_r)$ and the diversity is the following. If $\ell \in Q(V_r)$, then either $\ell \in Q(V)$ or $\ell - 1 \in Q(V)$. Therefore $|Q(V)| \leq 2|Q(V_r)|$, which shows that

$$g(V) = |Q(V)| \leq 2Q(V_r) \leq 10(n-1).$$

3. $g(V) = O(d \log d \cdot n)$.

   For every point u let $u_1, \ldots, u_d \in \mathbb{R}$ denote its coordinates in $\mathbb{R}^d$. We observe, for any points $u, v \in V$, $u \neq v$, that

   $$\frac{|u-v|}{\sqrt{d}} \leq \max_{i \in [d]} |u_i - v_i| \leq |u - v|.$$

   This implies the following:

   $$\lfloor \log |u-v| \rfloor - \tfrac{1}{2} \log d \leq \max_{i \in [d]} \lfloor \log |u_i - v_i| \rfloor \leq \lfloor \log |u-v| \rfloor.$$

   Let m be the number of different values for $\lfloor \log |u_i - v_i| \rfloor$, then the number of different values for $\lfloor \log |u-v| \rfloor$ is bounded by $m(\frac{1}{2} \log d + 1)$.

   Note that for a fixed i the set $\{\lfloor \log |u_i - v_i| \rfloor \mid u, v \in V\}$ describe the diversity levels of a one-dimensional point set and thus has at most $10(n-1)$ elements. Summing over all d dimensions gives $m \leq 10d(n-1)$, bounding the diversity of V by $g(V) \leq 10\, d(n-1)(\frac{1}{2} \log d + 1) = O(nd \log d)$.

4. Note that $\min(Q) > -1 + \log \min_{u,v \in V: u \neq v} |u, v|$ and $\max(Q) \leq \log \max_{u,v \in V} |u, v|$. Therefore

   $$\begin{aligned} g(V) &\leq \max(Q) - \min(Q) + 1 \\ &\leq 2 + \log \max_{u,v \in V} |u, v| - \log \min_{u,v \in V: u \neq v} |u, v| \\ &= 2 + \log \frac{\max_{u,v \in V} |u, v|}{\min_{u,v \in V: u \neq v} |u, v|}\ . \end{aligned}$$

5. The maximum distance between two points in $[0,1]^d$ is $\max_{u,v\in[0,1]^d}|u,v| \le \sqrt{d}$. The probability that two random coordinates u_i, v_i are closer than $1/n^{c+2}$ is at most $1/n^d$. This probability that for a pair $u, v \in V$ of all $n/2$ choices for $|u_i, v_i| \le 1/n^{c+2}$ is bounded by $(1/n^{c+2})(n/2) \le 1/n^c$. Therefore for all $u, v \in V$ we have $|u, v| \ge |u_i, v_i| \ge 1/n^{c+2}$ with probability $1 - n^{-c}$. From the previous item it follows that in this case the diversity is bounded by $1 + \log(n^{c+2}\sqrt{d}) \le 1 + (c+2)\log n + \frac{1}{2}\log d$. □

There are many reasons why for real-world scenarios the diversity can always be assumed to be bounded by $O(\log n)$. To achieve high diversity radio stations must be positioned with high accuracy such that most radio stations are closer than any polynomial fraction of the largest distance. In most other research on mobile radio networks a standard assumption is that the fraction between the largest and smallest distance of radio stations is bounded by a polynomial, which implies logarithmic diversity. A further reason may be that there are not many orders of magnitude between the transmitting range of a radio station and the physical size of the radio stations antenna.

3.2.2. *Approximating Congestion*. To approximate congestion-optimal path systems for radio networks we use the *Hierarchical Layer Graph* (HL-graph) with bounded degree introduced in [9]. Adopting ideas from clustering [7], [8] and generalizing an approach of Adler and Scheideler [1] we present a graph consisting of w layers $L_1, L_2, \ldots, L_w$. The union of all these graphs gives the HL-graph. The lowest layer L_1 contains all vertices V. The vertex set of a higher layer is a subset of the vertex set of a lower layer until in the highest layer there is only one vertex, i.e., $V = V(L_1) \supseteq V(L_2) \supseteq \cdots \supseteq V(L_w) = \{v_0\}$.

The crucial property of these layers is that in each layer L_i vertices obey a minimum distance: $\forall u, v \in V(L_i), |u, v| \ge r_i$. Furthermore, all nodes in the next-lower layer must be covered by this distance: $\forall u \in V(L_i), \exists v \in V(L_{i+1}), |u, v| \le r_{i+1}$. Our construction uses parameters $\alpha \ge \beta > 1$, where for some $r_0 < \min_{u,v\in V}|u, v|$ we use radii $r_i := \beta^i \cdot r_0$ and we define in layer L_i the edge set $E(L_i)$ by $E(L_i) := \{(u, v) \mid u, v \in V(L_i) \wedge |u, v| \le \alpha \cdot r_i\}$.

Note that if $V(L_i) = V(L_j)$ for $i < j$, then all edges of L_i are also in L_j, i.e., $E(L_i) \subseteq E(L_j)$. Hence, we omit the lower layer and consider only layers $i_1 < i_2 < \cdots < i_w$, such that $V(L_{i_j}) \subset V(L_{i_j})$ and all $k \in [i_{j-1} + 1..i_j]$: $V(L_k) = V(L_{i_j})$. The only execption to this rule is the uppermost layer with $|V(L_{i_w})| = 1$, where we choose the minimum i_w with $|V(L_{i_w})| = 1$.

Using this subset of layers $L'_j := L_{i_j}$ we extend the indices of the layers also to negative values, i.e., $i_j \in \mathbb{Z}$. As a side effect, we avoid any dependency between the parameter r_0 and the minimum distance of two points.

It turns out that the number of layers grows linear with the the diversity of the point set.

Lemma 2. *The number of layers of the HL-graph of a point set of n nodes is bounded by* $g(V)(2 + 1/\log\beta) + O(1)$. *If the orders of magnitudes of all distances* $Q(V)$ *form a consecutive interval*, *then the number of layers is bounded by* $g(V)/\log\beta + O(1)$.

Proof. We start with the case that $Q(V)$ is consecutive, i.e., $Q(V) = \{q_0, \ldots, q_{\max}\}$, where $q_{\max} > q_0$. Then there are no vertices $u, v \in V$ with $|u, v| < 2^{q_0}$. For all layers L_i with $r_i < 2^{q_0}$ we have $V(L_i) = V$. Therefore, the first layer of the HL-graph is $i_1 := \lfloor q_0/\log\beta \rfloor$.

For $q_{\max} = \max\{Q(V)\}$ we observe that all vertices $u, v \in V$ satisfy $|u, v| < 2^{q_{\max}+1}$. Hence, in the layer L_i with $r_i \geq 2^{q_{\max}+1}$ there is exactly one vertex. The index of this layer is $i_w = \lceil (q_{\max}+1)/\log\beta \rceil$.

So, we get the following maximum number of layers:

$$\begin{aligned} w = i_w - i_1 + 1 &\leq \left\lceil \frac{q_{\max}+1}{\log\beta} \right\rceil - \left\lfloor \frac{q_0}{\log\beta} \right\rfloor + 1 \\ &= \left\lceil \frac{q_0}{\log\beta} + \frac{g(V)}{\log\beta} \right\rceil - \left\lfloor \frac{q_0}{\log\beta} \right\rfloor + 1 \\ &\leq \left\lceil \frac{g(V)}{\log\beta} \right\rceil + 2. \end{aligned}$$

If $Q(V)$ is non-consecutive, there are m sets $\{q'_i, \ldots, q'_i + \delta_i\} \subseteq \{q_0, \ldots, q_{\max}\}$ such that $\{q'_i, \ldots, q'_i + \delta_i\} \cap Q(V) = \emptyset$. Note that $q_{\max} - q_0 + 1 - \sum_{i=1}^m \delta_i = g(V)$. For all $i \in [m]$ there exists no $u, v \in V$ such that $2^{q'_i} \leq |u, v| < 2^{q'_i+\delta+1}$. Hence, no points $u, v \in V$ exist with distance $\beta^{q'_i/\log\beta} \leq |u, v| < \beta^{(q'_i+\delta_i+1)/\log\beta}$, which implies that layers L_j with $j \in \{1 + \lceil q'_i/\log\beta \rceil, \ldots, \lfloor (q'_i+\delta+1)/\log\beta \rfloor - 1\}$ are omitted in the HL-graph. The number S of all omitted layers can be lower bounded as follows using $m \leq g(V)$:

$$\begin{aligned} S &\geq \sum_{i=1}^m \left\lfloor \frac{q'_i+\delta_i+1}{\log\beta} \right\rfloor - \left\lceil \frac{q'_i}{\log\beta} \right\rceil - 1 \\ &\geq \left\lceil \frac{m + \sum_{i=1}^m \delta_i}{\log\beta} \right\rceil - 2m \\ &\geq \frac{\sum_{i=1}^m \delta_i}{\log\beta} - 2g(V). \end{aligned}$$

Subtracting S from the number of layers in the consecutive case with lowest level q_0 and uppermost level $q_{\max}$ we obtain the following for the number of layers w by using $m \leq g(V)$:

$$\begin{aligned} w &\leq \left\lceil \frac{q_{\max}-q_0}{\log\beta} + 2 \right\rceil - \frac{\sum_{i=1}^m \delta_i}{\log\beta} + 2g(V) \\ &\leq \frac{q_{\max}-q_0}{\log\beta} + 3 - \frac{q_{\max}-q_0+g(V)-1}{\log\beta} + 2g(V) \\ &\leq 2g(V) + \frac{g(V)}{\log\beta} + 3. \end{aligned}$$ □

We will see that the *c-spanner* property has implications for minimizing congestion.

Definition 3. A graph $G = (V, E)$ is a c-spanner, if for all $u, v \in V$ there exists a path p from u to v with $|p| \leq c \cdot |u, v|$.

Theorem 8 [9]. *If $\alpha > 2(\beta/(\beta - 1))$ the HL-graph is a c-spanner for*

$$c = \beta \frac{\alpha(\beta - 1) + 2\beta}{\alpha(\beta - 1) - 2\beta}.$$

Proof. Define a directed tree T on the vertex set $V \times [w]$ as follows. The leafs of T are all pairs $V \times \{1\}$. If $u \in V(L_i)$, then (u, i) is a vertex of T. T consists of the following edges: For $i \geq 1$ if $u \in V(L_i)$, then $\{(u, i - 1), (u, i)\} \in E(T)$. If $u \in V(L_i)\backslash V(L_{i+1})$, then choose arbitrary $v \in V(L_{i+1})$ with $\{u, v\} \in E(L_i)$ and add $\{(u, i), (v, i + 1)\}$ to the edge set of the tree T. Note that the tree has depth w and the root (v_0, w).

Now for two vertices $u, v \in V$ we define a *clamp* of height j, which is a path connecting u and v. The clamp consists of two paths

$$P_u^j := (u, p(u), p^2(u), \ldots, p^j(u)) \quad \text{and} \quad P_v^j := (v, p(v), p^2(v), \ldots, p^j(v))$$

of length $j - 1$, where $p^i(w)$ denotes the ancestor of height i of a vertex w in the tree T. These two paths are connected by the edge $\{p^j(u), p^j(v)\}$.

Lemma 3. *If for vertices u, v the distance is bounded by $|u, v| \leq f_j$, where*

$$f_j = r_j \left(\alpha - 2\frac{\beta - 1/\beta^j}{\beta - 1} \right),$$

then a clamp of height at most j is contained in the HL-graph.

Proof. We have

$$|u, v| \;\leq\; f_j \;=\; \alpha r_j - 2\sum_{i=0}^{j} r_i.$$

Consider the paths $(u, p(u), p^2(u), \ldots, p^j(u))$ and $(v, p(v), p^2(v), \ldots, p^j(v))$. They are contained in the HL-graph, since $|p^i(u), p^{i+1}(u)| \leq r_{i+1}$ and $|p^i(v), p^{i+1}(v)| \leq r_{i+1}$. Further, the edge $\{p^j(u), p^j(v)\}$ is in the HL-graph since $|p^j(u), p^j(v)| \leq |u, v| - 2\sum_{i=0}^{j} r_i \leq \alpha r_j$. Hence, a clamp of height j is contained in G. □

Lemma 4. *A clamp of height j has maximum length g_j, where*

$$g_j = r_j \left(\alpha + 2\frac{\beta - 1/\beta^j}{\beta - 1} \right).$$

Proof. Recall that the length of the paths P_u^j and P_v^j is bounded by $2\sum_{i=1}^{j} r_i$ and the edge $\{p^j(u), p^j(v)\}$ has length of at most αr_j. This gives

$$||C|| \leq \alpha r_j + 2\sum_{i=0}^{j} r_i = g_j. \qquad \square$$

For any pair of vertices u, v with $f_{j-1} < |u, v| \leq f_j$ there is clamp of height j and length g_j. Hence, the stretch factor is bounded by $c = g_j / f_{j-1}$:

$$\begin{aligned} c &= \frac{g_j}{f_{j-1}} \\ &= \frac{g_j}{f_j} \cdot \frac{f_j}{f_{j-1}} \\ &= \frac{g_j}{f_j} \cdot \beta \cdot \frac{\alpha(\beta-1) - 2\beta + 2\beta^{-j}}{\alpha(\beta-1) - 2\beta + 2\beta^{-j+1}} \\ &\leq \beta \cdot \frac{g_j}{f_j} \\ &= \beta \cdot \frac{\alpha(\beta-1) + 2\beta - 2\beta^{-j}}{\alpha(\beta-1) - 2\beta + 2\beta^{-j}} \\ &\leq \beta \cdot \frac{\alpha(\beta-1) + 2\beta}{\alpha(\beta-1) - 2\beta}. \qquad \square \end{aligned}$$

Lemma 5. *For any finite point set $V \subset \mathbb{R}^d$ and every layer L_i of an HL-graph with parameters $\alpha \geq \beta > 1$ we have the following*:

1. *For any $u \in \mathbb{R}^d$, the number of points $v \in V(L_i)$ with $|u - v| \leq cr_i$ is at most $(2c+1)^d$.*
2. *The degree of the subgraph L_i is at most $(2\alpha+1)^d$.*
3. *The interference number of L_i is bounded by $(2\alpha+1)^{2d}$.*

Proof. Recall that $k_d := \pi^{d/2}/(d/2)!$ where $k_d r^d$ is the volume of a d-dimensional sphere with radius r.

1. For all $u, v \in V(L_i)$ we have $|u, v| \geq r_i$. Hence, all spheres with radii $r_i/2$ and center points $u \in V(L_i)$ do not intersect. If $|u, v| \leq cr_i$, then the sphere with center u and radius $r_i/2$ lies inside a sphere with center v and radius $(c + \frac{1}{2})r_i$. Let m be the number of the smaller spheres inside this larger one. Then it follows that
$$mk_d \left(\frac{r_i}{2}\right)^d \leq k_d((c+\tfrac{1}{2}))r_i)^d \quad \Longrightarrow \quad m \leq (2c+1)^d.$$
2. This follows by combining the preceding with the fact that edge $\{u, v\} \in E(L_i)$ if $u, v \in V(L_i)$ and $|u, v| \leq \alpha r_i$.
3. Two edges each of length αr_i can only interfere if their endpoints have at most distance αr_i. For each layer the number of such points of the same layer is bounded

by $(2\alpha+1)^d$. Every one of these points is adjacent to at most $(2\alpha+1)^d$ edges of L_i. Hence, the number of interferences in L_i is bounded by $(2\alpha+1)^{2d}$. □

Theorem 9. *For a vertex set V with diversity $g(V)$ the degree of the HL-graph is at most $g(V)(2+1/\log\beta)(2\alpha+1)^d + O(1)$. The interference number is at most $g(V)(2+1/\log\beta)(2\alpha+1)^{2d} + O(1)$. If $Q(V)$ is consecutive, the degree is bounded by $g(V)((2\alpha+1)^d/\log\beta) + O(1)$ and the interference number is at most $g(V)$ $((2\alpha+1)^{2d}/\log\beta) + O(1)$.*

Proof. This theorem combines Lemmas 4 and 5. For each point we observe that there are at most $(2\alpha+1)^d$ vertices in a layer L_i with distance αr_i. Each of these nodes has at most $(2\alpha+1)^d$ edges of at most length αr_i. Therefore, a vertex may suffer under at most $(2\alpha+1)^{2d}$ interferences per layer. Summing up over all layers this observation proves the upper bound of the interference number of the HL-graph. □

A typical feature of radio communication is that transmitting information blocks a region for other transmissions. We formalize this observation and define the capacity of a region following a similar approach presented in [11]. Let $A(R)$ denote the area of a geometric region R.

Definition 4. Given a network $G = (V, E)$ and a load $\ell : E \to \mathbb{R}_0^+$ we define the interference function of an edge $e \in E$ and a point $x \in \mathbb{R}^d$ as

$$f_\ell(e, x) := \begin{cases} \ell(e), & \text{if} \quad x \in D(e), \\ 0, & \text{elsewhere.} \end{cases}$$

The communication load $\kappa_{G,\ell}$ of a point x and a bounded geometric region $R \subseteq \mathbb{R}^+$ for a given network G and load ℓ is defined as follows:

$$\kappa_{G,\ell}(x) := \sum_{e\in E(G)} f_\ell(e, x) \quad \text{and} \quad \kappa_{G,\ell}(R) := \int_R \kappa_{G,\ell}(x)\, dx.$$

An equivalent description of the communication load can be done by partitioning the region R in elementary regions, where the same subset of edges interfere. For such an elementary region R' we observe

$$\kappa_{G,\ell}(R') = \sum_{e\in E(G):R'\subseteq D(e)} \ell(e) \cdot A(R'),$$

where $A(R')$ denotes the volume of R' (or area for the plane).

For a non-elementary bounded region R we consider a partitioning into elementary regions $R_1, \dots, R_m$ and get $\kappa_{G,\ell}(R) := \sum_{i=1}^m \kappa_{G,\ell}(R_i)$.

The following lemma will help to understand the relationship between the communication load of an elementary area and the congestion.

Lemma 6. *For a graph $G = (V, E)$ with $V \subset \mathbb{R}^d$, load ℓ, and a point $x \in \mathbb{R}^d$ it holds for $d \in \{2, 3\}$ that*

$$\kappa_{G,\ell}(x) \le c_d \cdot \max_{e \in E(G)} \sum_{e' \in Int(e)} \ell(e'),$$

where $c_2 = 6$ and $c_3 = 20$.

Proof. For the point x we partition the space into c_d disjoint subspaces $A_1, \ldots, A_{c_d}$ such that for all $u, v \in A_i$, $|u, x| \le |v, x|$, then $|u, v| \le |v, x|$. Then the angle between $\overline{xu}$ and $\overline{xv}$ is at most $\pi/3$. Clearly, for two dimensions the optimal choice is $c_d = 6$, which resembles six sectors centered at x. For three dimensions, one can show that $c_3 = 20$ cones starting at x suffice. For this, one has to cover the surface of a sphere with disks whose diameter equals the radius of the sphere. In [12] it is shown that that 20 such disks cover a sphere.

Now choose for each subspace A_i a vertex $u_i \in A_i$ that minimizes the distance $|x, u_i|$ (if the subspace is not empty). For every edge $\{v, w\}$ with $x \in D(\{v, w\})$ we show that there exists a vertex u_i with $u_i \in D(\{v, w\})$.

Assume without loss of generality that $x \in D_{|v,w|}(v)$ and let u_i be in the subspace where v lies. Since $|u_i, x| \le |x, v|$ we have $|u_i, v| \le |x, v| \le |v, w|$. Therefore we have

$$\begin{aligned} \sum_{e \in E(G): x \in D(e)} \ell(e) &\le \sum_{i=1}^{c_d} \sum_{e \in E(G): u_i \in D(e)} \ell(e) \\ &\le c_d \cdot \max_{u \in V(G)} \sum_{e \in E(G): u_i \in D(e)} \ell(e) \\ &\le c_d \cdot \max_{e \in E(G)} \sum_{e' \in Int(e)} \ell(e). \end{aligned}$$ □

This definition implies the following relationship between capacity, area, and congestion.

Lemma 7. *Let R be a bounded region of volume $A(R)$ and let C be the congestion of a path system $\mathcal{P}$. Then, the communication load of R is bounded by $\kappa(R) \le c_d \cdot A(R) \cdot C$, where $c_2 = 6$ and $c_3 = 20$.*

Proof.

$$\begin{aligned} \kappa_{G,\ell}(R) &= \int_R \kappa_{G,\ell}(z)\, dz \\ &= \int_R \sum_{e \in E(G)} \sum_{e' : z \in D(e)} \ell(e')\, dz \\ &\le \int_R c_d \cdot C\, dz \\ &= c_d \cdot C \cdot A(R). \end{aligned}$$ □

Every edge e with load $\ell(e)$ has a certain impact on the capacity of the area covered by the radio network. The following lemma claims that an edge e with load $\ell(e)$ induces at least communication load $k_d \ell(e)|e|^d$ into a region R with $D(e) \subseteq R$.

Lemma 8. *Consider an edge e in a region R, i.e., $D(e) \subseteq R$. Let K be the communication load of R without any load on e and let K' be the communication load of R with load $\ell(e)$ on e without any load change on the other edges. Then we observe that*

$$K' - K \;\geq\; k_d \ell(e)|e|^d.$$

Proof. The proof follows from the definition of $f_\ell(e, x)$ and the fact that the volume of $D(e)$ for $e = \{u, v\}$ is at least $V(D_{|u,v|}(u)) = k_d r^d$. □

Lemma 9. *Let C^* be the congestion of the congestion-optimal path system $\mathcal{P}^*$ for a vertex set V. Then every c-spanner N can host a path system $\mathcal{P}'$ such that the induced load $\ell(e)$ in N is bounded by $\ell(e) \leq c' g(V)\, C^*$ for a positive constant c'.*

Proof. Given a path p of the path system $\mathcal{P}^*$, we replace every edge $e = \{u, v\}$ that does not exist in the c-spanner N with a path p from u to v in N such that the new route lies completely inside a disk $D_c(e)$ of radius $(c - \frac{1}{2})|u, v|$ and center $\frac{1}{2}(u + v)$.

For the path system $\mathcal{P}^*$ there may have been interferences between e and other edges. For simplicity we underestimate the area where e can interfere with other communication by the disk $D_1(e)$ with center $\frac{1}{2}(u + v)$ and radius $\frac{1}{2}|u, v|$ (see Figure 5).

We want to describe the impact of rerouting all edges in $E(N^*)$ to a specific edge $e_0 \in E(N)$ in the c-spanner N. If this edge $e_0 = \{u_0, v_0\} \in E(N)$ transmits the traffic of a detour for an edge $e = \{u, v\} \in E(N^*)$ of length at most $c|e|$, then the distance

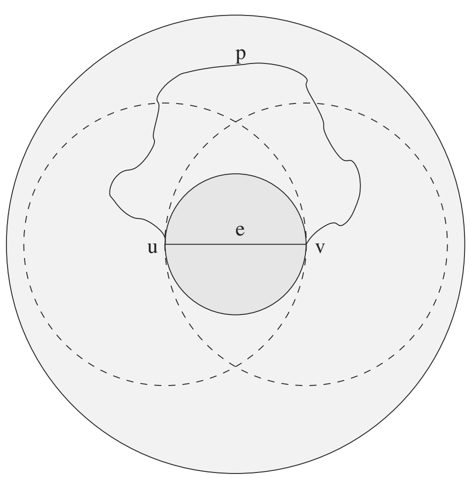

Fig. 5. The edge e interferes with other edges (at least) within the central disk. Its information is rerouted on p, lying completely within the outer disk with radius $(c - \frac{1}{2})|e|$.

between u as well as v and any point of the tour is at most $c|e|$. Hence, also for the center $z_0 := \frac{1}{2}(u_0 + v_0)$ of e_0 and $z := \frac{1}{2}(u + v)$ we observe $|z_0, z| \le c|e|$.

Now consider the edge set $E_{i,e_0} \subseteq E(N^*)$ of edges e with length $|e| \in [2^i, 2^{i+1})$ for $i \in \mathbb{Z}$ which reroute their traffic to e_0. Their center points are located inside a sphere with radius $c2^{i+1}$ and center z_0. The region where e interferes has been defined by $D(e)$. $D(e)$ has volume (resp. area) of at least $k_d 2^{di}$ and lies completely inside the sphere D with radius $2^{i+1}(c+1)$ and center z_0. The volume (resp. area) of D is $k_d 2^{d(i+1)}(c+1)^d$.

Lemma 8 shows that every edge e reduces the capacity in D by at least $k_d \ell(e) 2^{di}$. Because of Lemma 7, the overall capacity of C is at most

$$\kappa_{G,\ell}(D) \ \le\ c_d k_d 2^{d(i+1)}(c+1)^d C^*.$$

This implies the following:

$$\sum_{e \in E_{i,e_0}} \ell(e) k_d 2^{di} \le c_d k_d 2^{d(i+1)}(c+1)^d C^*$$
$$\Longrightarrow \sum_{e \in E_{i,e_0}} \ell(e) \le c_d 2^d (c+1)^d C^*.$$

There are at most $g(V)$ non-empty sets E_{i,e_0}. This implies for the sum of loads $\ell(e)$ of the set $E_{e_0} := \bigcup_i E_{i,e_0} \subseteq E(N^*)$ that

$$\sum_{e \in E_{e_0}} \ell(e) \le c_d 2^d (c+1)^d g(V) C^*. \qquad \square$$

Theorem 10. *Let $\mathcal{P}^*$ be the congestion-optimal path system for the vertex set V. Then the HL-graph contains a path system $\mathcal{P}$ with congestion $O(g(V)^2 C_{\mathcal{P}^*}(V))$.*

Proof. From Theorem 8 we know that the HL-graph is a c-spanner with

$$c = \beta \frac{\alpha(\beta-1)+2\beta}{\alpha(\beta-1)-2\beta} \qquad \text{if} \quad \alpha > 2\beta\frac{\beta}{\beta-1}.$$

Therefore we can use Lemma 9 to show that there exists a routing such that the load of an edge e is bounded by $\ell(e) \le 2^d(c+1)^d g(V) C_{\mathcal{P}^*}(V)$. Theorem 9 shows that the interference number of the network is bounded by $O(g(V))$. So, this implies that $C_{\mathcal{P}}(V) = O(g(V)^2 C_{\mathcal{P}^*}(V))$. $\square$

In practical scenarios the diversity can be seen as a logarithmic term (e.g., because the ratio between the longest and shortest distance is polynomial in the number of vertices, or the points are chosen according to some uniform probability distribution). In these cases the HL-graph provides an $O((\log n)^2)$-approximation for congestion.

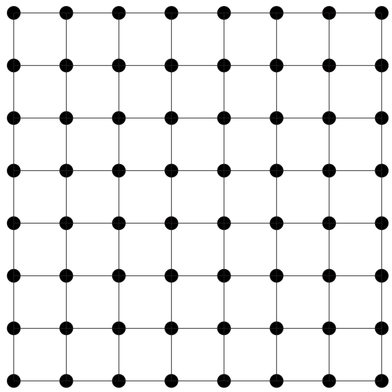

Fig. 6. The grid G_n.

4. Trade-Offs

We have seen efficient ways for selecting paths to optimize energy and approximate congestion. One might wonder whether an algorithm can compute a path system for a radio network optimizing congestion, dilation, and energy at the same time. It turns out that this is not the case.

4.1. *Congestion versus Dilation*

For a vertex set G_n given by a $\sqrt{n} \times \sqrt{n}$ grid with unit grid distance, the best choice to minimize congestion is to connect grid points only to their neighbors given the demand $f(u, v) = W/n^2$ for all pairs of vertices (Figure 6). Then the congestion is $O(W/\sqrt{n})$ and the dilation is given by $O(\sqrt{n})$. In [11] it is shown that such a congestion is best possible in a radio network. A fast realization is given by a tree featuring a hop-distance of $O(\log n)$ and congestion $O(W \log n)$. (Such a tree-construction for the cost-distance problem is presented in [24].) In both cases we observe $C_{\mathcal{P}}(G_n) D_{\mathcal{P}}(G_n) \geq \Omega(W)$. This also is true for any other path selection.

In three-dimensional space we place the vertices on a $\sqrt[3]{n} \times \sqrt[3]{n} \times \sqrt[3]{n}$ grid and experience minimal congestion of $O(Wn^{-2/3})$ with dilation $O(n^{1/3})$. It shows that in three dimensions this trade-off increases.

Theorem 11. *Given the grid vertex set G_n in d-dimensional space ($d \in \{2, 3\}$) with traffic W, then for every path system $\mathcal{P}$ the following trade-off between dilation $D_{\mathcal{P}}(G_n)$ and congestion $C_{\mathcal{P}}(G_n)$ exists*:

$$C_{\mathcal{P}}(G_n) \cdot (D_{\mathcal{P}}(G_n))^{d-1} \geq \Omega(W).$$

Proof. For $n = (3p)^d$ we partition the two-dimensional grid into three $p \times 3p$ rectangle-shaped vertex sets V_1, V_2, V_3, such that V_1 contains all left vertices, V_3 all right vertices, and V_2 the vertices in the middle. Similarly, we partition the thee-dimensional grid into three $p \times 3p \times 3p$ cubicle-shaped vertex sets V_1, V_2, V_3.

In both cases G denotes the complete graph with vertex set G_n and $\mathcal{P}$ denotes a path system for the demand f. We concentrate on one-ninth of the demand starting at V_1 heading for vertices in V_3. Let $D \leq 3p$ be the dilation of the network and let $p_{i,j}$ denote the route from vertex v_i to vertex v_j. Let $\ell(p_{i,j}) = f(u_i, u_j)$ denote the information flow on path $p_{i,j}$.

Consider two vertices $v_i \in V_1$ and $v_j \in V_3$. Then the path $p_{i,j}$ has at most $D_{\mathcal{P}}(G)$ edges. The induced communication load $\kappa_{G,\ell}(p_{i,j})$ of the path $p_{i,j}$ is at least $\kappa_{G,\ell}(p_{i,j}) \geq k_d \ell(p_{i,j}) \sum_{e \in p_{i,j}} |e|^d = k_d(W/n^2) \sum_{e \in p_{i,j}} |e|^d$. This term is minimized if the path uses the maximum possible number $D_{\mathcal{P}}(G)$ of edges and all edges have equal length of $|e| = |u_i, v_j|/D_{\mathcal{P}}(G)$. Since $|u_i, v_j| \geq \frac{1}{3}\sqrt[d]{n}$, this implies $\kappa_{G,\ell}(p_{i,j}) \geq k_d W/3^d n D_{\mathcal{P}}(G)^{d-1}$.

All points with non-zero communication load reside in a square S with edge length $(2\sqrt{d}+1)\sqrt[d]{n}$. (This size is caused by the edge connecting the nodes on the diagonal of the grid.) Lemma 7 states that the communication load of S with area $A(S) = (2\sqrt{d}+1)^d n$ is bounded by

$$\kappa(G, \ell)(S) \leq c_d \cdot C_{\mathcal{P}}(G) A(S) = c_d \cdot C_{\mathcal{P}}(G)(2\sqrt{d}+1)^d n.$$

The sum of the communication load induced by all paths $p_{i,f}$ cannot extend the communication load of S:

$$\sum_{v_i \in V_1} \sum_{v_j \in V_3} \kappa_{G,\ell}(p_{i,j}) \leq \kappa_{G,\ell}(S).$$

Combining the inequalities we get

$$\frac{n^2}{9} \frac{k_d W}{3^d n D_{\mathcal{P}}(G)^{d-1}} \leq \sum_{v_i \in V_1} \sum_{v_j \in V_3} \kappa_{G,\ell}(p_{i,j}) \leq \kappa_{G,\ell}(S) \leq c_d \cdot C_{\mathcal{P}}(G)(2\sqrt{d}+1)^d n.$$

This states the claim, since

$$c_d \cdot C_{\mathcal{P}}(G)(D_{\mathcal{P}}(G))^{d-1} \geq \frac{k_d W}{9 c_d (6\sqrt{d}+3)^d}. \qquad \square$$

4.2. *Dilation versus Energy*

The simplest location of sites is the line vertex set L_n as investigated in [15], see Figure 7. Here all vertices $L_n = \{v_1, \ldots, v_n\}$ are placed on a line with equal distances $|v_i, v_{i+1}| = \Delta/(n-1)$. Only the leftmost and the rightmost nodes want to exchange messages, i.e., $f(v_1, v_n) = W$ and $f(v, w) = 0$ for all other pairs (v, w). The energy-optimal network for unit and flow energy is the path $(v_1, v_2, \ldots, v_n)$, given the unit energy U-Energy$_{\mathcal{P}}(L_n) = (n-1)(\Delta^2/(n-1)^2) = \Delta^2/(n-1)$, the flow energy F-Energy$_{\mathcal{P}}(L_n) = \Delta^2 W/(n-1)$, and the dilation n.

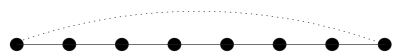

Fig. 7. The line L_n.

The fastest network realizes only the edge $\{v_1, v_n\}$ with hop-distance 1 and unit energy Δ^2 (and flow energy $W\Delta^2$). There are path systems that can give a compromise between these extremes. However, it turns out that the product of dilation and energy cannot be decreased:

Theorem 12. *Given the vertex set L_n with diameter Δ then for every path system $\mathcal{P}$ the following trade-offs between dilation D and unit energy* U-Energy (*resp. flow energy* F-Energy) *exist*:

$$D_{\mathcal{P}}(L_n) \cdot \text{U-Energy}_{\mathcal{P}}(L_n) \geq \Omega(\Delta^2),$$
$$D_{\mathcal{P}}(L_n) \cdot \text{F-Energy}_{\mathcal{P}}(L_n) \geq \Omega(\Delta^2 W).$$

Proof. Let m be the length (i.e., number of edges) of the longest path of $cal\,P$ (without the loss of generality we assume that there are only edges with non-zero information flow $\ell(e) > 0$). For the unit energy model we can assume that there is only a path p from v_1 to v_n (because introducing more edges needs additional energy without decreasing the dilation). We have to minimize $\text{U-Energy}_{\mathcal{P}}(p) := \sum_{i=1}^{m}(L_i)^2$ defined by the edge lengths $L_1, \ldots, L_m$, where $\sum_{i=1}^{m} L_i = \Delta$. Clearly, the energy sum is minimal for $L_1 = L_2 = \cdots = L_m = \Delta/m$ giving $\text{U-Energy}_{\mathcal{P}}(p) \cdot D_{\mathcal{P}}(p) \geq \Delta^2$.

The bound for the flow energy follows analogously. □

4.3. *The Incompatibility of Congestion and Energy*

We will show that for some vertex sets congestion and energy are incompatible. This is the worst occurrence of a trade-off situation since there is no possible compromise between energy and congestion.

The vertex set $U_{\alpha,n}$ for $\alpha \in [0, \frac{1}{2}]$ consists of two horizontal parallel line graphs L_{n^α}. Neighbored (and opposing) vertices have distance Δ/n^α. There is only demand W/n^α between the vertical pairs of opposing vertices of the line graphs. The rest of the $n - n^{-\alpha}$ vertices are equidistantly placed between the vertices of each line graph and the leftmost vertical pair of vertices (see Figure 8).

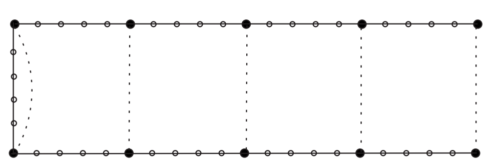

Fig. 8. Vertex set U_α.

The minimum spanning tree consists of n vertices where all edges have length $\Theta(\Delta/n)$. This results in a total unit energy of

$$\text{U-Energy}_{\text{MST}}(U_{\alpha,n}) = O(\Delta^2 n^{-1})$$

and congestion

$$C_{\text{MST}}(U_{\alpha,n}) = O(W).$$

The flow energy of the (same) minimum network is given by

$$\text{F-Energy}_{\text{MST}}(U_{\alpha,n}) = O(W\Delta^2 n^{-1}).$$

The congestion optimal path system $\mathcal{P}'$ connects only vertices with non-zero demand. Its congestion is

$$C_{\mathcal{P}'}(U_{\alpha,n}) = O(Wn^{-\alpha})$$

and its unit energy is

$$\text{U-Energy}_{\mathcal{P}'}(U_{\alpha,n}) = O(\Delta^2 n^{-\alpha}).$$

The flow energy is given by

$$\text{F-Energy}_{\mathcal{P}'}(U_{\alpha,n}) = O(Wn^{-\alpha}\Delta^2).$$

Lemma 10. *For $\alpha \in [0, \frac{1}{2})$ and the vertex set $U_{\alpha,n}$ with diameter Δ, let $x \in \{0, \dots, n^\alpha\}$ be the number of edges of length at least $\Delta n^{-\alpha}$ of a path system for the radio network and let $r \in [0, W]$ be the information flow on these edges. Then we have for the congestion C, unit energy and flow energy*:

$$\text{U-Energy}_{\mathcal{P}}(U_{\alpha,n}) \geq \max\left\{\frac{\Delta^2}{4n}, \frac{x\Delta^2}{n^{2\alpha}}\right\}, \tag{2}$$

$$C_{\mathcal{P}}(U_{\alpha,n}) \geq \frac{W}{x+1}, \tag{3}$$

$$\text{F-Energy}_{\mathcal{P}}(U_{\alpha,n}) \geq \max\left\{W\frac{\Delta^2}{4n}, r\frac{\Delta^2}{n^{2\alpha}}\right\}, \tag{4}$$

$$C_{\mathcal{P}}(U_{\alpha,n}) \geq \max\left\{\frac{r}{12n^\alpha}, W - r\right\}. \tag{5}$$

Proof. The energy consumption of the minimum unit energy network is given by the U-shaped path. The minimum hop-distance between, for half of the communication partners, is at least $n/2$. Hence, the minimum energy is at least $\Delta^2/4n$. For x, edges of length δ/n^α exist, then the unit energy cost of theses edges alone is $x\Delta^2/n^{2\alpha}$.

The lower bound for the flow energy follows analogously.

For simplicity we call an edge with minimum length $\Delta n^{-\alpha}$ a long edge. Note that every long edge $\{u, v\}$ that connects two points on the same horizontal line or one of the leftmost vertical pair and a horizontal line does not reduce the congestion of any point that lies between u and v according to the minimum unit energy network. So, let r denote the number of edges connecting nodes of the lower row with the upper row. Using these r edges the minimum cut of the path system between the upper row and the lower row is $r + 1$. Hence, the minimum load on every edge of this cut is $W/(r + 1)$. If we optimistically assume that these edges do not interfere we obtain the lower bound.

Now, let r denote the number of messages that are delivered on long edges connecting the lower with the upper row. Now consider the rectangular region R between the rows. The communication load of each of these long edges e induced into this region R of area $A(R) = \Delta^2/n^\alpha$ is at least $(\Delta^2/2n^{2\alpha})\ell(e)$. Therefore the communication load induced by all these edges is at least $r(\Delta^2/2n^{2\alpha})$ and at most $c_2 C_\mathcal{P}(U_{\alpha,n})A(R) = 6(\Delta^2/n^\alpha)C_\mathcal{P}(U_{\alpha,n})$. This implies

$$C_\mathcal{P}(U_{\alpha,n}) \geq \frac{r}{12n^\alpha}.$$

The residual $W - r$ packets need to be routed between the shorter edges of the leftmost vertices. Even without counting radio interferences at least congestion W_r will occur. □

Theorem 13. *There exists a vertex set V with a path system minimizing congestion to C^*, and another path system optimizing unit energy by* U-Energy* *and minimal flow energy by* F-Energy* *such that we have for any path system $\mathcal{P}$ on this vertex set V,*

$$\begin{aligned}
&C_\mathcal{P}(V) \geq \Omega(n^{1/3}C^*) \quad \textit{or} \\
&\text{U-Energy}_\mathcal{P}(V) \geq \Omega(n^{1/3}\text{U-Energy}^*), \\
&C_\mathcal{P}(V) \geq \Omega(n^{1/3}C^*) \quad \textit{or} \\
&\text{F-Energy}_\mathcal{P}(V) \geq \Omega(n^{1/3}\text{F-Energy}^*).
\end{aligned}$$

Proof. The claim follows directly by Lemma 10 using the graph $V = U_{1/3,n}$. □

Hence, there is no hope that routing in wireless networks can optimize more than one parameter at a time. The wireless network designer has to decide in favor of small congestion or low energy consumption.

5. Open Problems and Further Work

An interesting topic that remains open for further research is mobility. In this work we investigated static point-to-point communication for a given static set of radio stations in two- and three-dimensional space. We concentrated on the measures congestion, dilation, and energy. Besides this work we have started to consider scenarios in which nodes are allowed to move. In [23] we investigate distributed algorithms for mobile ad hoc networks for moving radio stations with adjustable transmission power in a worst-case scenario. We consider two models to find a reasonable restriction on the worst-case mobility. In the pedestrian model we assume a maximum speed $v_{\max}$ of the radio stations, while in the vehicular model we assume a maximum acceleration $a_{\max}$ of the points. For both models we present distributed algorithms based on a grid clustering technique and a high-dimensional representation of the dynamical start situation which construct a mobile hierarchical layer graph with low congestion, low interference number, low energy-consumption, and low degree. Further, we present solutions for dynamic position information management under both models.

Besides the standard model of omni-directional communication we are investigating a sector model where the sender and receiver can focus signals (e.g., infrared). Such sector communication is a special case of so-called space multiplexing techniques to increase the network capacity (e.g., by using directional antennas [16]). The techniques of the results shown here can be easily transferred to such a model [9]. Besides computer simulations [27], [22], we are currently setting up a testbed consisting of a group of mobile robots that can communicate in sectors. We have developed an infrared-light-based hardware module that allows us to submit data in a eight directions with separate transmission powers [10]. It can be used as an extension module for the mobile mini-robot Khepera [19], [14]. Thus, realistic scenarios for ad hoc networks can be reproduced by performing experiments with these mini-robots. This enables us to validate our communication strategies under practical conditions. Such a network is technically more complicated, but our goal is to show that it is possible to set up a geometric spanner graph as a communication network. Notably, we show that such geometric spanners always provide good solutions for congestion minimization in radio networks.

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