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Theory of Computing Systems

Exact Complexity of the Winner Problem for Young Elections*

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Abstract. In 1977 Young proposed a voting scheme that extends the Condorcet Principle based on the fewest possible number of voters whose removal yields a Condorcet winner. We prove that both the winner and the ranking problem for Young elections is complete for $P_{\|}^{NP}$, the class of problems solvable in polynomial time by parallel access to NP. Analogous results for Lewis Carroll's 1876 voting scheme were recently established by Hemaspaandra et al. In contrast, we prove that the winner and ranking problems in Fishburn's homogeneous variant of Carroll's voting scheme can be solved efficiently by linear programming.

1. Introduction

More than a decade ago, Bartholdi et al. [BTT1]–[BTT3] initiated the study of electoral systems with respect to their computational properties. In particular, they proved NP hardness lower bounds [BTT2] for determining the winner in the voting schemes proposed by Dodgson (more commonly known by his pen name, Lewis Carroll) and by Kemeny, and they studied complexity issues related to the problem of manipulating elections by strategic voting [BTT1], [BTT3]. Since then, a number of related results and improvements of their results have been obtained. Hemaspaandra et al. [HHR]

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classified both the winner and the ranking problem for Dodgson elections by proving them complete for $P_{\|}^{NP}$, the class of problems solvable in polynomial time by parallel access to an NP oracle. E. Hemaspaandra (as cited in [HH]) and Spakowski and Vogel [SV2] obtained the analogous result for Kemeny elections; a joint paper by E. Hemaspaandra, Spakowski, and Vogel is in preparation. For many further results and the state of the art regarding computational politics, we refer to the survey [HH].

In this paper we study complexity issues related to Young and Dodgson elections. In 1977 Young proposed a voting scheme that extends the Condorcet Principle based on the fewest possible number of voters whose removal makes a given candidate c the Condorcet winner, i.e., c defeats all other candidates by a strict majority of the votes. We prove that both the winner and the ranking problem for Young elections is complete for $P_{\|}^{NP}$. To this end, we give a reduction from the problem `Maximum Set Packing Compare`, which we also prove is $P_{\|}^{NP}$-complete.

In Section 3 we study a homogeneous variant of Dodgson elections that was introduced by Fishburn [F]. In contrast to the above-mentioned result of Hemaspaandra et al. [HHR], we show that both the winner and the ranking problem for Fishburn's homogeneous Dodgson elections can be solved efficiently by a linear program that is based on an integer linear program of Bartholdi et al. [BTT2].

2. Complexity of the Winner Problem for Young Elections

2.1. *Some Background from Social Choice Theory*

We first give some background from social choice theory. Let C be the set of all candidates (or alternatives). We assume that each voter has strict preferences over the candidates. Formally, the preference order of each voter is strict (i.e., irreflexive and antisymmetric), transitive, and complete (i.e., all candidates are ranked by each voter). An election is given by a *preference profile*, a pair $\langle C, V \rangle$ such that C is a set of candidates and V is the multiset of the voters' preference orders on C. Note that distinct voters may have the same preferences over the candidates. A *voting scheme* (or *social choice function*, SCF for short) is a rule for how to determine the winner(s) of an election; i.e., an SCF maps any given preference profile to society's aggregate *choice set*, the set of candidates who have won the election. For any SCF f and any preference profile $\langle C, V \rangle$, $f(\langle C, V \rangle)$ denotes the set of winning candidates. For example, the *majority rule* says that a candidate A *defeats* a candidate B if and only if A is preferred to B by a strict majority of the voters. According to the majority rule, an election is won by a candidate who defeats every other candidate. Such a candidate is called the *Condorcet winner*.

In 1785 Marie-Jean-Antoine-Nicolas de Caritat, the Marquis de Condorcet, noted in his seminal essay [C] that whenever there are at least three candidates, say A, B, and C, the majority rule may yield cycles. His example consists of the following three voters:

$$A > B > C,$$
$$B > C > A,$$
$$C > A > B.$$

Thus, A defeats B and B defeats C, and yet C defeats A. That is, even though each individual voter has a rational (i.e., transitive or noncyclic) preference order, society

may behave irrationally and Condorcet winners do not always exist. This observation is known as the Condorcet Paradox. The *Condorcet Principle* says that for each preference profile, the winner of the election is to be determined by the majority rule. An SCF is said to be a *Condorcet SCF* if and only if it respects the Condorcet Principle in the sense that the Condorcet winner is elected whenever one exists. Note that Condorcet winners are uniquely determined if they exist.

Many Condorcet SCFs have been proposed in the social choice literature; for an overview of the most central ones, we refer to the work of Fishburn [F]. They extend the Condorcet Principle in a way that avoids the troubling feature of the majority rule. In this paper we focus on only two such Condorcet SCFs, the Dodgson voting scheme [D] and the Young voting scheme [Y].

In 1876 Charles L. Dodgson (better known by his pen name, Lewis Carroll) proposed a voting scheme [D] that suggests that we remain most faithful to the Condorcet Principle if the election is won by any candidate who is "closest" to being a Condorcet winner. To define "closeness," each candidate c in a given election $\langle C, V\rangle$ is assigned a score, denoted $\mathrm{DodgsonScore}(C, c, V)$, which is the smallest number of sequential interchanges of adjacent candidates in the voters' preferences that are needed to make c a Condorcet winner. Here, one interchange means that, in (any) one of the voters, two adjacent candidates are switched. A *Dodgson winner* is any candidate with the minimum Dodgson score. Using Dodgson scores, one can also tell who of two given candidates is ranked better according to the Dodgson SCF.

Young's approach to extending the Condorcet Principle is reminiscent of Dodgson's approach in that it is also based on altered profiles. Unlike Dodgson, however, Young [Y] suggests that we remain most faithful to the Condorcet Principle if the election is won by any candidate who is made a Condorcet winner by *removing the fewest possible number of voters*, instead of doing the fewest possible number of switches in the voters' preferences. For each candidate c in a given preference profile $\langle C, V\rangle$, define $\mathrm{YoungScore}(C, c, V)$ to be the size of a largest submultiset of V for which c is a Condorcet winner. A *Young winner* is any candidate with the maximum Young score.

Homogeneous variants of these voting schemes will be defined in Section 3.

2.2. *Complexity Issues Related to Voting Schemes*

To study computational complexity issues related to Dodgson's voting scheme, Bartholdi et al. [BTT2] defined the following decision problems:

`Dodgson Winner`

Instance: A preference profile $\langle C, V\rangle$ and a designated candidate $c \in C$.
Question: Is c a Dodgson winner of the election? That is, is it true that for all $d \in C$,

$$\mathrm{DodgsonScore}(C, c, V) \leq \mathrm{DodgsonScore}(C, d, V)?$$

`Dodgson Ranking`

Instance: A preference profile $\langle C, V\rangle$ and two designated candidates $c, d \in C$.
Question: Does c tie-or-defeat d in the election? That is, is it true that

$$\mathrm{DodgsonScore}(C, c, V) \leq \mathrm{DodgsonScore}(C, d, V)?$$

Bartholdi et al. [BTT2] established an NP-hardness lower bound for both these problems. Their result was optimally improved by Hemaspaandra et al. [HHR] who proved that `Dodgson Winner` and `Dodgson Ranking` are complete for $P_{\|}^{NP}$, the class of problems solvable in polynomial time with parallel (i.e., truth-table) access to an NP oracle.

As above, we define the corresponding decision problems for Young elections as follows:

`Young Winner`

Instance: A preference profile $\langle C, V\rangle$ and a designated candidate $c \in C$.
Question: Is c a Young winner of the election? That is, is it true that for all $d \in C$,

$$\text{YoungScore}(C, c, V) \geq \text{YoungScore}(C, d, V)?$$

`Young Ranking`

Instance: A preference profile $\langle C, V\rangle$ and two designated candidates $c, d \in C$.
Question: Does c tie-or-defeat d in the election? That is, is it true that

$$\text{YoungScore}(C, c, V) \geq \text{YoungScore}(C, d, V)?$$

2.3. *Hardness of Determining Young Winners*

The main result in this section is that the problems `Young Winner` and `Young Ranking` are complete for $P_{\|}^{NP}$. In Theorem 2.3 below, we give a reduction from the problem `Maximum Set Packing Compare` that is defined below. For a given collection $\mathcal{S}$ of sets, let $\kappa(\mathcal{S})$ be the maximum number of pairwise disjoint sets in $\mathcal{S}$.

`Maximum Set Packing Compare`

Instance: Two sets B_1 and B_2 and two collections $\mathcal{S}_1$ and $\mathcal{S}_2$ of finite, nonempty sets such that, for $i \in \{1, 2\}$, each $S \in \mathcal{S}_i$ is a subset of B_i.
Question: Does it hold that $\kappa(\mathcal{S}_1) \geq \kappa(\mathcal{S}_2)$?

To prove that `Maximum Set Packing Compare` is $P_{\|}^{NP}$-complete, we give a reduction from the problem `Independence Number Compare`, which has also been used in [HRS]. To define the problem, let G be an undirected, simple graph. An *independent set of* G is any subset I of the vertex set of G such that no two vertices in I are adjacent. For any graph G, let $\alpha(G)$ be the *independence number of* G, i.e., the size of a maximum independent set of G.

`Independence Number Compare`

Instance: Two graphs G_1 and G_2.
Question: Does it hold that $\alpha(G_1) \geq \alpha(G_2)$?

Without loss of generality, we may assume that G_1 and G_2 contain no isolated vertices.

Using the techniques of Wagner [W], it can be shown that the problem Independence Number Compare is $P_{\|}^{NP}$-complete; see Theorem 12 of [SV1] for an explicit proof of this result.

Proposition 2.1 (see [W] and [SV1]). Independence Number Compare *is* $P_{\|}^{NP}$*-complete.*

Lemma 2.2. Maximum Set Packing Compare *is* $P_{\|}^{NP}$*-complete.*

Proof. We give a polynomial-time many-one reduction from the problem Independence Number Compare to the problem Maximum Set Packing Compare. Let G_1 and G_2 be two given graphs. For $i \in \{1, 2\}$, define B_i to be the set of edges of G_i, and define $\mathcal{S}_i$ so as to contain exactly $\|V(G_i)\|$ sets: For each vertex v of G_i, add to $\mathcal{S}_i$ the set of edges incident to v. Thus, for each $i \in \{1, 2\}$, we have $\alpha(G_i) = \kappa(\mathcal{S}_i)$, which proves the lemma. □

Now, we prove the main result of this section.

Theorem 2.3. Young Ranking *is* $P_{\|}^{NP}$*-complete.*

Proof. It is easy to see that Young Ranking and Young Winner are in $P_{\|}^{NP}$. To prove the $P_{\|}^{NP}$ lower bound, we give a polynomial-time many-one reduction from the problem Maximum Set Packing Compare. Let $B_1 = \{x_1, x_2, \ldots, x_m\}$ and $B_2 = \{y_1, y_2, \ldots, y_n\}$ be two given sets, and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be given collections of subsets of B_1 and B_2, respectively. Recall that $\kappa(\mathcal{S}_i)$, for $i \in \{1, 2\}$, is the maximum number of pairwise disjoint sets in $\mathcal{S}_i$; without loss of generality, we may assume that $\kappa(\mathcal{S}_i) > 2$.

We define a preference profile $\langle C, V \rangle$ such that c and d are designated candidates in C, and it holds that

$$\text{YoungScore}(C, c, V) = 2 \cdot \kappa(\mathcal{S}_1) + 1; \tag{2.1}$$

$$\text{YoungScore}(C, d, V) = 2 \cdot \kappa(\mathcal{S}_2) + 1. \tag{2.2}$$

Define the set C of candidates as follows:

- create the two designated candidates c and d;
- for each element x_i of B_1, create a candidate x_i;
- for each element y_i of B_2, create a candidate y_i;
- create two auxiliary candidates, a and b.

Define the set V of voters as follows:

- **Voters representing $\mathcal{S}_1$:** For each set $E \in \mathcal{S}_1$, create a single voter v_E as follows:
  — Enumerate E as $\{e_1, e_2, \ldots, e_{\|E\|}\}$ (renaming the candidates e_i chosen from $\{x_1, x_2, \ldots, x_m\}$ for notational convenience), and enumerate its complement $\bar{E} = B_1 - E$ as $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{m-\|E\|}\}$.

— To make the preference orders easier to parse, we use

"$\overrightarrow{E}$" to represent the text string "$e_1 > e_2 > \cdots > e_{\|E\|}$";
"$\overrightarrow{\bar{E}}$" to represent the text string "$\bar{e}_1 > \bar{e}_2 > \cdots > \bar{e}_{m-\|E\|}$";
"$\overrightarrow{B_1}$" to represent the text string "$x_1 > x_2 > \cdots > x_m$";
"$\overrightarrow{B_2}$" to represent the text string "$y_1 > y_2 > \cdots > y_n$".

— Create one voter v_E with preference order

$$\overrightarrow{E} > a > c > \overrightarrow{\bar{E}} > \overrightarrow{B_2} > b > d. \tag{2.3}$$

- Additionally, create two voters with preference order

$$c > \overrightarrow{B_1} > a > \overrightarrow{B_2} > b > d, \tag{2.4}$$

and create $\|\mathcal{S}_1\| - 1$ voters with preference order

$$\overrightarrow{B_1} > c > a > \overrightarrow{B_2} > b > d. \tag{2.5}$$

- **Voters representing $\mathcal{S}_2$:** The case of $\mathcal{S}_2$ is treated analogously with the roles of respectively $\mathcal{S}_1$, B_1, x_i, c, a, E, e_j, and $\bar{e}_k$ interchanged with $\mathcal{S}_2$, B_2, y_i, d, b, F, f_j, and $\bar{f}_k$. More precisely, for each set $F \in \mathcal{S}_2$, create a single voter v_F as follows:
  — Enumerate F as $\{f_1, f_2, \ldots, f_{\|F\|}\}$ (renaming the candidates f_j chosen from $\{y_1, y_2, \ldots, y_n\}$ for notational convenience), and enumerate its complement $\bar{F} = B_1 - F$ as $\{\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{n-\|F\|}\}$.
  — To make the preference orders easier to parse, we use

"$\overrightarrow{F}$" to represent the text string "$f_1 > f_2 > \cdots > f_{\|F\|}$";
"$\overrightarrow{\bar{F}}$" to represent the text string "$\bar{f}_1 > \bar{f}_2 > \cdots > \bar{f}_{n-\|F\|}$".

  — Create one voter v_F with preference order

$$\overrightarrow{F} > b > d > \overrightarrow{\bar{F}} > \overrightarrow{B_1} > a > c. \tag{2.6}$$

- Additionally, create two voters with preference order

$$d > \overrightarrow{B_2} > b > \overrightarrow{B_1} > a > c, \tag{2.7}$$

and create $\|\mathcal{S}_2\| - 1$ voters with preference order

$$\overrightarrow{B_2} > d > b > \overrightarrow{B_1} > a > c. \tag{2.8}$$

We now prove (2.1): $\text{YoungScore}(C, c, V) = 2 \cdot \kappa(\mathcal{S}_1) + 1$.

Let $E_1, E_2, \ldots, E_{\kappa(\mathcal{S}_1)} \in \mathcal{S}_1$ be $\kappa(\mathcal{S}_1)$ pairwise disjoint subsets of B_1. Consider the following submultiset $\hat{V}$ of the voters V. $\hat{V}$ consists of:

- every voter v_{E_i} corresponding to the set E_i, where $1 \leq i \leq \kappa(\mathcal{S}_1)$;
- the two voters given in (2.4);
- $\kappa(\mathcal{S}_1) - 1$ voters of the form given in (2.5).

Then $\|\hat{V}\| = 2 \cdot \kappa(\mathcal{S}_1) + 1$. Note that a strict majority of the voters in $\hat{V}$ prefer c over any other candidate, and thus c is a Condorcet winner in $\langle C, \hat{V}\rangle$. Hence,

$$\text{YoungScore}(C, c, V) \geq 2 \cdot \kappa(\mathcal{S}_1) + 1.$$

Conversely, to prove that $\text{YoungScore}(C, c, V) \leq 2 \cdot \kappa(\mathcal{S}_1) + 1$, we need the following lemma.

Lemma 2.4. *For any λ with $3 < \lambda \leq \|\mathcal{S}_1\| + 1$, let V_λ be any submultiset of V such that V_λ contains exactly λ voters of the form* (2.4) *or* (2.5) *and c is a Condorcet winner in $\langle C, V_\lambda\rangle$. Then V_λ contains exactly $\lambda - 1$ voters of the form* (2.3) *and no voters of the form* (2.6), (2.7), *or* (2.8). *Moreover, the $\lambda - 1$ voters of the form* (2.3) *in V_λ represent pairwise disjoint sets from $\mathcal{S}_1$.*

Proof of Lemma 2.4. Let V_λ for fixed λ be given as above. Consider the submultiset of V_λ that consists of the λ voters of the form (2.4) or (2.5). Every candidate x_i, $1 \leq i \leq m$, is preferred to c by the at least $\lambda - 2$ voters of the form (2.5). Since c is a Condorcet winner in $\langle C, V_\lambda\rangle$, there exist, for every x_i, at least $\lambda - 1 > 2$ voters in V_λ who prefer c to x_i. By construction, these voters must be of the form (2.3) or (2.4). Since there are at most two voters of the form (2.4), there exists at least one voter of the form (2.3), say $\tilde{v}$. Since the voters of the form (2.3) represent $\mathcal{S}_1$, which contains only nonempty sets, there exists some candidate x_j who is preferred to c by $\tilde{v}$. In particular, c must outpoll x_j in $\langle C, V_\lambda\rangle$ and thus needs more than $(\lambda - 2) + 1$ votes of the form (2.3) or (2.4). There are at most two voters of the form (2.4); hence, c must be preferred by at least $\lambda - 2$ voters of the form (2.3) that are distinct from $\tilde{v}$. Summing up, V_λ contains at least $\lambda - 1$ voters of the form (2.3).

On the other hand, since c is a Condorcet winner in $\langle C, V_\lambda\rangle$, c must in particular outpoll a, who is not preferred to c by the λ voters of the form (2.4) or (2.5) and who is preferred to c by all other voters. Hence, V_λ may contain at most $\lambda - 1$ voters of the form (2.3), (2.6), (2.7), or (2.8). It follows that V_λ contains exactly $\lambda - 1$ voters of the form (2.3) and no voters of the form (2.6), (2.7), or (2.8).

For a contradiction, suppose that there is a candidate x_j who is preferred to c by more than one voter of the form (2.3) in V_λ. Then

- c is preferred to x_j by at most two voters of the form (2.4) and by at most $(\lambda - 1) - 2 = \lambda - 3$ voters of the form (2.3);
- x_j is preferred to c by at least $\lambda - 2$ voters of the form (2.5) and by at least two voters of the form (2.3).

Since c thus has at most $\lambda - 1$ votes and x_j has at least λ votes in V_λ, c is not a Condorcet winner in $\langle C, V_\lambda \rangle$, a contradiction. Thus, every candidate x_i, $1 \leq i \leq m$, is preferred to c by at most one voter of the form (2.3) in V_λ, which means that the $\lambda - 1$ voters of the form (2.3) in V_λ represent pairwise disjoint sets from $\mathcal{S}_1$. □

To continue the proof of Theorem 2.3, let $k = \text{YoungScore}(C, c, V)$. Let $\hat{V} \subseteq V$ be a submultiset of size k such that c is a Condorcet winner in $\langle C, \hat{V} \rangle$. Suppose that there are exactly $\lambda \leq \|\mathcal{S}_1\| + 1$ voters of the form (2.4) or (2.5) in $\hat{V}$. Since c, the Condorcet winner of $\langle C, \hat{V} \rangle$, must in particular outpoll a, we have $\lambda \geq \lceil (k+1)/2 \rceil$. By our assumption that $\kappa(\mathcal{S}_1) > 2$, it follows from $k \geq 2 \cdot \kappa(\mathcal{S}_1) + 1$ that $\lambda > 3$. Lemma 2.4 then implies that there are exactly $\lambda - 1$ voters of the form (2.3) in $\hat{V}$, which represent pairwise disjoint sets from $\mathcal{S}_1$, and $\hat{V}$ contains no voters of the form (2.6), (2.7), or (2.8). Hence, $k = 2 \cdot \lambda - 1$ is odd, and $(k-1)/2 = \lambda - 1 \leq \kappa(\mathcal{S}_1)$, which proves (2.1).

Equation (2.2) can be proven analogously. Thus, we have

$$\kappa(\mathcal{S}_1) \geq \kappa(\mathcal{S}_2) \quad \text{if and only if} \quad \text{YoungScore}(C, c, V) \geq \text{YoungScore}(C, d, V).$$

This completes the proof of Theorem 2.3. □

Theorem 2.5. `Young Winner` *is* $\mathrm{P}_{\|}^{\mathrm{NP}}$*-complete.*

Proof. To prove the theorem, we modify the reduction from Theorem 2.3 to a reduction from the problem `Maximum Set Packing Compare` to the problem `Young Winner` as follows. Let $\langle C, V \rangle$ be the preference profile constructed in the proof of Theorem 2.3 with the designated candidates c and d. We alter this profile such that all other candidates do worse than c and d.

From $\langle C, V \rangle$, we construct a new preference profile $\langle D, W \rangle$. To define the new set D of candidates, replace every candidate $g \in C$ except c and d by $\|V\|$ candidates $g^1, g^2, \ldots, g^{\|V\|}$.

To define the new voter set W, replace each occurrence of candidate g in the ith voter of V by the text string

$$g^{i \bmod \|V\|} > g^{i+1 \bmod \|V\|} > g^{i+2 \bmod \|V\|} > \cdots > g^{i+\|V\|-1 \bmod \|V\|}.$$

Let V' be any submultiset of V, and let W' be the submultiset of W corresponding to V'. It is easy to see that c is a Condorcet winner in V' if and only if c is a Condorcet winner in W'. Thus, the change from $\langle C, V \rangle$ to $\langle D, W \rangle$ does not alter the Young score of c and d. On the other hand, the Young score of any other candidate now is at most 1. Thus, there is no candidate h with $\text{YoungScore}(C, h, V) > \text{YoungScore}(C, c, V)$ or $\text{YoungScore}(C, h, V) > \text{YoungScore}(C, d, V)$. Hence, $\kappa(\mathcal{S}_1) \geq \kappa(\mathcal{S}_2)$ if and only if c is a winner of the election $\langle D, W \rangle$. □

3. Homogeneous Young and Dodgson Voting Schemes

Social choice theorists have studied many "reasonable" properties that any "fair" election procedure arguably should satisfy, including very natural properties such as nondictatorship, monotonicity, the Pareto Principle, and independence of irrelevant alternatives. One of the most notable results in this regard is Arrow's famous Impossibility Theorem [A] stating that the just-mentioned four properties are logically inconsistent, and thus no "fair" voting scheme can exist.

In this section we are concerned with another quite natural property, the homogeneity of voting schemes (see [F] and [Y]).

Definition 3.1. A voting scheme f is said to be *homogeneous* if and only if for each preference profile $\langle C, V\rangle$ and for all positive integers q, it holds that

$$f(\langle C, V\rangle) = f(\langle C, qV\rangle),$$

where qV denotes V replicated q times.

Homogeneity means that splitting each voter $v \in V$ into q voters, each of whom has the same preference order as v, yields exactly the same choice set of winning candidates.

Fishburn [F] showed that neither the Dodgson nor the Young voting schemes are homogeneous. For the Dodgson SCF, he presented a counterexample with seven voters and eight candidates; for the Young SCF, he modified a preference profile constructed by Young with thirty-seven voters and five candidates. Fishburn [F] provided the following limit device in order to define homogeneous variants of the Dodgson and Young SCFs. For example, the Dodgson scheme can be made homogeneous by defining from the function DodgsonScore for each preference profile $\langle C, V\rangle$ and designated candidate $c \in C$ the function

$$\text{DodgsonScore}^*(C, c, V) = \lim_{q\to\infty} \frac{\text{DodgsonScore}(C, c, qV)}{q}.$$

The resulting SCF is denoted by Dodgson* SCF, and the corresponding winner and ranking problems are denoted by `Dodgson`* `Winner` and `Dodgson`* `Ranking`.

Example 3.2 [F]. We provide here Fishburn's example [F] showing that the original Dodgson voting scheme is not homogeneous. Consider the preference profile $\langle C, V\rangle$, where C consists of the eight candidates $a_1, a_2, \ldots, a_7$, and c, and V consists of the following preference orders:

$$a_1 > a_2 > a_3 > a_4 > c > a_5 > a_6 > a_7,$$
$$a_7 > a_1 > a_2 > a_3 > c > a_4 > a_5 > a_6,$$
$$a_6 > a_7 > a_1 > a_2 > c > a_3 > a_4 > a_5,$$
$$a_5 > a_6 > a_7 > a_1 > c > a_2 > a_3 > a_4,$$
$$a_4 > a_5 > a_6 > a_7 > c > a_1 > a_2 > a_3,$$

$$a_3 > a_4 > a_5 > a_6 > c > a_7 > a_1 > a_2,$$
$$a_2 > a_3 > a_4 > a_5 > c > a_6 > a_7 > a_1.$$

One can verify that DodgsonScore$(C, c, V) = 7$ and DodgsonScore$(C, a_i, V) = 6$, for each i. Thus, according to the original Dodgson scheme, the choice set of winning candidates in $\langle C, V\rangle$ is $\{a_i \mid 1 \leq i \leq 7\}$. However, DodgsonScore$^*(C, c, V) = 3.5$ and DodgsonScore$^*(C, a_i, V) = 4.5$, for each i, which implies that, according to the original Dodgson scheme and for a large enough q, the choice set of winning candidates in $\langle C, qV\rangle$ is $\{c\}$. Hence, the original Dodgson voting scheme is not homogeneous.

Analogously, the Young voting scheme defined in Section 2.2 can be made homogeneous by defining YoungScore*. Remarkably, Young [Y] showed that the corresponding problem Young* Winner can be solved by a linear program. Hence, the problem Young* Winner is efficiently solvable, since the problem Linear Programming can be decided in polynomial time [H], see also [K]. Inspired by Young's work, we establish an analogous result for the problems Dodgson* Winner and Dodgson* Ranking below. Theorem 3.3 should be contrasted with the known result [HHR] that Dodgson Winner and Dodgson Ranking are complete for $\mathrm{P}_{\|}^{\mathrm{NP}}$.

Theorem 3.3. Dodgson* Winner *and* Dodgson* Ranking *can be solved in polynomial time.*

Proof. Bartholdi et al. [BTT2] provided an integer linear program for determining the Dodgson score of a given candidate c. They noted that if the number of candidates is fixed, then the winner problem for Dodgson elections (in the inhomogeneous case defined in Section 2.2) can be solved in polynomial time using the algorithm of Lenstra [L].

Based on their integer linear program, we provide a linear program for computing DodgsonScore$^*(C, c, V)$ for a given preference profile $\langle C, V\rangle$ and a given candidate c. Since Linear Programming is polynomial-time solvable [H], it follows that the problems Dodgson* Winner and Dodgson* Ranking can be solved in polynomial time, even if the number of candidates is not prespecified.

Let a profile $\langle C, V\rangle$ and a candidate $c \in C$ be given, and let $V = \{v_1, v_2, \ldots, v_n\}$. Our linear program has the variables $x_{i,j}$, and constants $e_{i,j,k}$, and w_k, where $1 \leq i \leq n$, $1 \leq j \leq \|C\| - 1$, and $k \in C - \{c\}$. The constants are obtained from the profile $\langle C, V\rangle$ as follows:

- For given i, j, and k, set $e_{i,j,k} = 1$ if the result of moving c upwards by j positions in the preference order of voter v_i is that c gains one additional vote against candidate k, and set $e_{i,j,k} = 0$ otherwise.
- For any candidate k other than c, the constant w_k gives the number of voters who prefer c over k.

DodgsonScore$^*(C, c, V)$ is the value of the linear program

$$\min \sum_{i,j} j \cdot x_{i,j} \tag{3.1}$$

subject to the constraints:

(1) $\sum_j x_{i,j} = 1$ for each voter v_i;
(2) $\sum_{i,j} e_{i,j,k} \cdot x_{i,j} + w_k > n/2$ for each candidate $k \in C - \{c\}$;
(3) $0 \leq x_{i,j} \leq 1$ for each i and j.

The variables and constraints can be interpreted as follows:

1. For given i and j, the variable $x_{i,j}$ is a rational number in the interval [0, 1] (by the set of constraints (3)) that gives the percentage $v_{i,j}^q/q$, where q is the least common multiple of the denominators in all $x_{i,j}$, and $v_{i,j}^q$ is the number of voters among the q replicants of voter v_i in which c is moved upwards by j positions.
2. The set of constraints (2) ensures that c becomes a Condorcet winner.
3. The set of constraints (1) ensures that $v_{i,j}^q$, summed over all possible positions j, equals the number q of all replicants of voter v_i.

The objective is to minimize the number of switches needed to make c a Condorcet winner. For the homogeneous case of Dodgson elections, the linear program (3.1) tells us how many times we have to replicate each voter v_i (namely, q times) and in how many of the replicants of each voter v_i the given candidate c has to be moved upwards by how many positions in order to achieve this objective. □

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