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Non-abelian Sylow subgroups of finite groups of even order

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1. Introduction

The purpose of this paper is to prove:

Main Theorem. *Every non-abelian Sylow subgroup of a finite group of even order contains a non-trivial element which commutes with an involution.*

Our main theorem announced in [4] is closely related to the prime graphs of finite groups. Let G be a finite group and $\Gamma(G)$ the prime graph of G. $\Gamma(G)$ is the graph such that the vertex set is the set of prime divisors of $|G|$, denoted by $\pi(G)$, and two distinct vertices p and r are joined by an edge if and only if there exists an element of order pr in G. Let $n(\Gamma(G))$ be the number of connected components of $\Gamma(G)$ and $d_G(p, r)$ the distance between two vertices p and r of $\Gamma(G)$, that is, the length of the shortest path between p and r. We define $d_G(p, r) = \infty$ if there is no path between p and r. It has been proved that $n(\Gamma(G)) \leq 6$ in [17], [12], [15].

Theorem 1. *Let G be a finite group of even order and p be a prime divisor of $|G|$. If $d_G(2, p) \geq 2$, then Sylow p-subgroups of G are abelian.*

Theorem 1 is a restatement of our main theorem in terms of the prime graph $\Gamma(G)$ of G.

Corollary 1. *Let G be a finite group of even order and p be a prime divisor of $|G|$. If Δ is a connected component of $\Gamma(G) - \{p\}$ not containing 2, then Sylow r-subgroups of G are abelian for $r \in \Delta$.*

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There is a certain relation between a subgraph $\Gamma(G) - \{p\}$ of $\Gamma(G)$ and Brauer characters of p-modular representations of G (See [3]). As for the distance $d_G(2, p)$ we have:

Theorem 2. *Let G be a finite group of even order and p be an odd prime divisor of $|G|$. Then we have:*

$$d_G(2, p) = \begin{cases} 1 \\ 2 \\ 3 \\ \infty \end{cases}$$

Corollary 2. *Let G be a finite non-abelian simple group and p be an odd prime divisor of $|G|$. Then we have:*

$$d_G(2, p) = \begin{cases} 1 \\ 2 \\ \infty \end{cases}$$

We have a theorem of Gruenberg-Kegel type (See Lemma 5).

Theorem 3. *Let G be a finite group of even order. Suppose that $d_G(2, p) \geq 2$ for some prime $p \in \pi(G)$. Then one of the following holds:*

(i) G is solvable,
(ii) G has a chain of normal subgroups $G \supseteq H \supseteq K$ such that G/H and K are solvable and H/K is a non-abelian simple group.

The significance of the prime graphs of finite groups can be found in [2], [7], [8], [9], [10], [11], [18], [19]. Theorems 1, 2 and 3 are the consequences of the classification of finite simple groups (See [2], [9], [12]) which at least for the moment is not established beyond any doubt although the proof has been announced and advertised since 1981.

2. Preliminaries

The purpose of this section is to provide several lemmas which will be applied in the proof of our theorems. The following lemma is straightforward by the definition. However it is important when we apply mathematical induction.

Lemma 1. *Let G be a finite group and H a subgroup of G.*

(i) If $p, r \in \pi(H)$, then $d_G(p, r) \leq d_H(p, r)$.
(ii) If $H \lhd G$ and $p, r \in \pi(G/H)$, then $d_G(p, r) \leq d_{G/H}(p, r)$.

Proof. (i) Straightforward. (ii) Suppose that $d_{G/H}(p,r) = 1$. Then there exists an element xH of order pr in G/H. It follows that $x^{pr} \in H$ and some power of x is of order pr in G. This yields $d_G(p,r) = 1$. Repeat this argument and we have $d_G(p,r) \leq d_{G/H}(p,r)$.

Lemma 2. *Suppose that $N \lhd G$.*

(i) If $d_G(p,r) = 1$ for $p, r \in \pi(G) - \pi(N)$, then $d_{G/N}(p,r) = 1$.
(ii) Suppose that $d_G(p,r) = 2$ for $p, r \in \pi(G) - \pi(N)$, i.e., there exists $q \in \pi(G)$ such that $d_G(p,q) = 1$ and $d_G(q,r) = 1$. If $q \in \pi(G)-\pi(N)$, then $d_{G/N}(p,r) = 2$.

Proof. (i) There exists a element $g \in G$ such that $o(g) = pr$. Then gN has order pr in G/N. (ii) follows from (i).

The following is also important.

Lemma 3. *Let G be a non-solvable group, and π a connected component of $\Gamma(G)$ not containing 2. Then G contains a nilpotent Hall π-subgroup.*

Proof. See [17].

Lemma 4. *Let G be a finite non-solvable almost simple group such that $N \subseteq G \subseteq Aut(N)$ for a non-abelian simple group N. Let p be an odd prime in $\pi(G/N)$. Then $d_G(2, p) = 1$.*

Proof. If N is an alternating group or one of 26 sporadic simple groups, then $(G : N) = 1, 2$ or 4. We can assume that N is a simple group of Lie type. The outer automorphism group is a semidirect product (in this order) of group of order d (diagonal automorphisms), f (field automorphisms) and g (graph automorphisms modulo field automorphisms) except that for $B_2(2^f)$, $G_2(3^f)$, $F_4(2^f)$ the extraordinary graph automorphism squares to the generating field automorphism. Let $x \in G - N$ with $x^p \in N$. If p divides d or g, then we see that p divides $|N|$ and that $d_N(2, p) = 1$. This yields that if $d_N(2, p) \geq 2$ or p does not divide $|N|$, then p divides $f/(f, dg)$. In this case, there exists a field automorphism y such that $N\langle x\rangle \simeq N\langle y\rangle$. Since field automorphisms centralize a group of Lie type over the prime field, we have $d_G(2, p) = 1$. This completes the proof (See [14]).

A finite group G is said to be a 2-Frobenius group if and only if there exists a chain $G \supset H \supset K \supset 1$ of normal subgroups of G such that H and G/K are Frobenius groups with Frobenius kernels K and H/K, respectively. Since H/K is cyclic of odd order, a 2-Frobenius group is always solvable. Next lemma essentially due to Gruenberg and Kegel (See [11], [17]) is fundamental important when we study finite groups with disconnected prime graphs.

Lemma 5. *Let G be a finite group with $n(\Gamma(G)) \geq 2$. Then we have two possibilities.*

(i) *G is a Frobenius group or a 2-Frobenius group.*
(ii) *G has a chain $G \supseteq M \supseteq N \supseteq 1$ of normal subgroups such that N is a nilpotent π-group, M/N is a non-abelian simple group and G/M is a solvable π-group where π is the connected component of $\Gamma(G)$ containing 2.*

Proof. Suppose that G is neither a Frobenius group nor a 2-Frobenius group. By Gruenberg-Kegel's theorem [17, Theorem A], G has a chain $G \supseteq M \supseteq N \supseteq 1$ of normal subgroups such that N is a nilpotent π-subgroup, M/N is a non-abelian simple group with $n(\Gamma(G)) \geq 2$ and G/M is a π-separable group. Since $n(\Gamma(G)) \geq 2$, we have $C_{G/N}(M/N) = \bar{1}$. This yields that G/N is an almost simple group with $n(\Gamma(G)) \geq 2$. Since Schreier conjuecture holds true, we see that G/M is a solvable π-group by Lemma 4.

By Lemma 5 if G is a solvable group with $n(\Gamma(G)) \geq 2$, then G is a Frobenius group or a 2-Frobenius group, and $n(\Gamma(G)) = 2$.

Lemma 6. *Let G be a finite group and N a normal subgroup of G. Suppose that G/N is isomorphic to the direct product of more than one non-abelian simple group. Then $d_G(2, p) = 1$ for $p \in \pi(G) - \{2\}$.*

Proof. By our assumption $\Gamma(G/N)$ is a complete graph and $2 \in \pi(G/N)$. We can assume that there exists a prime p in $\pi(N) - \pi(G/N)$. Let $P \in Syl_p(G)$. It follows that $G = N_G(P)N$ by the Frattini argument. Let $Q \in Syl_q(N_G(P))$ where $q \in \pi(G/N)$. If $d_G(p, q) \neq 1$, then PQ is a Frobenius group with kernel P. Thus Q is cyclic or a generalized quaternion group. This is not the case since G/N is isomorphic to the direct product of more than one non-abelian simple group. Now $d_G(p, q) = 1$ for all $q \in \pi(G/N)$.

Lemma 7. *Let G be a finite group and N a minimal characteristic subgroup. Suppose that N is isomorphic to a direct product of more than one non-abelian simple group. Then $d_G(2, p) = 1$ for $p \in \pi(G) - \{2\}$.*

Proof. By our assumption $\Gamma(N)$ is a complete graph and $2 \in \pi(N)$. Since $C_G(N)$ is also a characteristic subgroup, $C_G(N) \cap N = 1$. This yields that $d_{C_G(N)N}(2, t) = 1$ for $t \in \pi(C_G(N)N) - \{2\}$. Let $N = S_1 \times \cdots \times S_r$ where $S_i \simeq S$, a non-abelian simple group and $r \geq 2$. G acts transitively on $\{S_1, \ldots, S_r\}$ by conjugation. Let C_1 be the stabilizer of S_1. Then $C_1/C_G(S_1)$ is isomorphic to a subgroup of $Aut(S_1)$. Let $p \in \pi(C_1/C_G(S_1)) - \pi(S)$. It follows that $d_{C_1}(2, p) = 1$ by Lemma 4. Assume that there exists $q \in \pi(G) - \pi(C_1)$. Let y be a q-element of G and x an involution in S_1. Since the involution $xx^y x^{y^2} \cdots x^{y^{q-1}}$ commutes with y, $d_G(2, q) = 1$. This complete the proof.

Lemma 8. *Let G be a finite group. Suppose that G has a chain of normal subgroups $H_1 \supseteq H_2 \supseteq K_1 \supseteq K_2$ such that G/H_1, H_2/K_1 and K_2 are solvable, and H_1/H_2 and K_1/K_2 are non-abelian simple groups. Then $d_G(2, p) = 1$ for $p \in \pi(G) - \{2\}$.*

Proof. Put $\bar{G} = G/K_2$, $\bar{H}_i = H_i/K_2$ for $i = 1, 2$ and $\bar{K}_1 = K_1/K_2$. Since $\bar{K}_1$ is a non-abelian simple group and the Schreier conjecture holds true, $\bar{G}/\bar{K}_1 C_{\bar{G}}(\bar{K}_1)$ is solvable. By Lemma 4, we have $d_{\bar{G}}(2, p) = 1$ for $p \in \pi(\bar{G}/\bar{K}_1 C_{\bar{G}}(\bar{K}_1))$. Since $C_{\bar{G}}(\bar{K}_1) \cap \bar{K}_1 = \bar{1}$ and $C_{\bar{G}}(\bar{K}_1)$ is of even order, $d_{\bar{G}}(2, p) = 1$ for $p \in \pi(\bar{G}) - \{2\}$. Let $Q \in Syl_q(K_2)$ for odd prime $q \in \pi(K_2)$. Since Sylow 2-subgroups of $N_G(Q)$ are neither cyclic nor generalized quaternion, we have $d_G(2, q) = 1$. The proof is complete.

3. A minimal counter example to Theorem 1

The purpose of this section is to prove that a counter example of possible minimal order to Theorem 1 is a non-abelian simple group.

Proposition 1. *Let G be a solvable group of even order and p be a prime divisor of $|G|$. If $d_G(2, p) \geq 2$, then Sylow p-subgroups of G are abelian.*

Proof. There exists a Hall $\{2, p\}$-subgroup H of G. Note that a Sylow p-subgroup of H is a Sylow p-subgroup of G and $n(\Gamma(H)) = 2$. By Lemma 5, H is a Frobenius group or a 2-Frobenius group. If H is a Frobenius group, then $H = NL$, where N is Frobenius kernel of H and L is a Frobenius complement of H. If p divides $|N|$, then N is a Sylow p-subgroup of H and L is a 2-group. It follows that N has a fixed point free automorphism of order 2. This yields that N is abelian. If p divides $|L|$, then L is a Sylow p-subgroup of H. Since L is a Frobenius complement, L is cyclic. If H is a 2-Frobenius group, H has normal subgroups M and N such that M is a Frobenius group with Frobenius kernel N and H/N is a Frobenius group with Frobenius kernel M/N. It follows that M/N is isomorphic to a Sylow p-subgroup of H. This implies that Sylow p-subgroups of H are cyclic. This completes the proof.

Proposition 2. *Let G be a minimal counter example to Theorem 1. Then G is a non-solvable simple group.*

Proof. By Proposition 1, G is non-solvable. Let P be a non-abelian Sylow p-subgroup of G such that $d_G(2, p) \geq 2$. Take N a minimal normal subgroup of G. Suppose that $N \simeq Z_t \times \cdots \times Z_t$ for some prime t. Since G/N is non-solvable, $|G/N|$ is divisible by 2. If $t \neq p$, then $\bar{P} \in Syl_p(G/N)$ is abelian by the choice of G because $d_{G/N}(2, p) \geq d_G(2, p) \geq 2$. Since $\bar{P} \simeq P$, P is abelian, a contradiction. We have $t = p$. If p does not divide $|G/N|$, then N is a Sylow p-subgroup of G, a contradiction because N is abelian. This implies that p divides $|G/N|$. Take Q a Sylow 2-subgroup of G. Then NQ is a Frobenius group with Frobenius kernel N. Since G is non-solvable, Q must be a generalized quaternion group. By Brauer-Suzuki's theorem [13, pp.102], $|Z(G/O(G))| = 2$. Namely, we have $d_G(2, r) \leq 1$ for any prime r dividing $|G/O(G)|$. This yields that $P \subseteq O(G)$. By the Frattini argument, we have $G = N_G(P)O(G)$. This implies that $|N_G(P)|$ is divisible by 2. Take $Q \in Syl_2(N_G(P))$. Then PQ is a Frobenius group with Frobenius kernel P. This is a contradiction since P has a fixed point free automorphism of order 2.

We have $N \simeq S \times \cdots \times S$ (n times) for a non-abelian simple group S. Suppose that p does not divide N. Take $Q \in Syl_2(N)$. By the Frattini argument, we have $G = N_G(Q)N$. There exists an element $g \in G$ such that $P^g \subset N_G(Q)$. Since QP^g is a Frobenius group with Frobenius kernel Q, we see that P^g is cyclic, a contradiction. This yields that p divides $|N|$ and $n = 1$, *i.e.*, N is a non-abelian simple group. Suppose that p does not divide $|G/N|$. Then $P \in Syl_p(N)$. If N is a proper subgroup of G, then P is abelian by the choice of G since $d_N(2,\ p) \geq d_G(2,\ p) \geq 2$. This yields that $G = N$ is non-abelian simple. Suppose that p divides $|G/N|$. Then both 2 and p do not divide $|C_G(N)|$ since 2 and p divide $|N|$. This yields that $2p$ divides $|G/C_G(N)|$. If $C_G(N) \neq 1$, then $\bar{P} \in Syl_p(G/C_G(N))$ is abelian by the choice of G since $d_{G/C_G(N)}(2,\ p) \geq d_G(2,\ p) \geq 2$. This is a contradiction. This yields that $C_G(N) = 1$. Hence we have $N \subseteq G \subseteq Aut(N)$ for some non-abelian simple group N.

By Lemma 4 we see that a counter example of possible minimal order to Theorem 1 is a non-abelian simple group.

4. A minimal counter example to Theorem 2

The purpose of this section is to prove that a counter example of possible minimal order to Theorem 2 is a non-abelian simple group.

Proposition 3. *Let G be a solvable group. Then $d_G(p, q) \leq 3$ or ∞ for any $p, q \in \pi(G)$.*

Proof. Suppose that $d_G(p, q) = 4$ for some $p,\ q \in \pi(G)$. There exists r, s, $t \in \pi(G)$ with $d_G(p, r) = 1$, $d_G(p, s) = 2$, $d_G(p, t) = 3$, $d_G(r, s) = 1$, $d_G(r, t) = 2$, $d_G(r, q) = 3$, $d_G(s, t) = 1$, $d_G(s, q) = 2$ and $d_G(t, q) = 1$. Take Hall $\{p, s, q\}$-subgroup H of G. Then we have $n(\Gamma(H)) = 3$. This contradicts Lemma 5.

Proposition 4. *Let G be a minimal counter example to Theorem 2. Then G is a non-solvable simple group.*

Proof. By Proposition 3, G is non-solvable. There exist primes q, r, s, $p \in \pi(G)$ with $d_G(2, q) = 1$, $d_G(2, r) = 2$, $d_G(2, s) = 3$, $d_G(2, p) = 4$, $d_G(q, r) = 1$, $d_G(q, s) = 2$, $d_G(q, p) = 3$, $d_G(r, s) = 1$, $d_G(r, p) = 2$ and $d_G(s, p) = 1$. Take a minimal normal subgroup N of G. Suppose that $N \simeq Z_t \times \cdots \times Z_t$ (n times) for some prime t. If $d_G(2, t) \geq 2$, then NU is a Frobenius group with kernel N, where $U \in Syl_2(G)$. Since G is non-solvable, U is a generalized quaternion group. By Brauer-Suzuki's theorem, we have $|Z(G/O(G))| = 2$ and therefore Sylow r-subgroups, Sylow s-subgroups and Sylow p-subgroups are contained in $O(G)$. Take H a Hall $\{r, s, p\}$-subgroup of $O(G)$. By the Frattini argument, $N_G(H)$ contains a Sylow 2-subgroup U^g $(g \in G)$ of G. Then HU^g is a Frobenius group with kernel H. This is a contradiction since H is nilpotent and $d_H(r, p) \geq 2$. If $d_G(2, t) = 1$ or $t = 2$, then $d_{G/N}(r, s) = 1$, $d_{G/N}(r, p) = 2$ and $d_{G/N}(s, p) = 1$ by Lemma 2. By the choice of G, $d_{G/N}(2, p) = \infty$. This yields that $\{r, s, p\}$ is contained in a component π of $\Gamma(G/N)$ not containing 2. By Lemma 3, there exists a nilpotent Hall π-subgroup of G/N, a contradiction since $d_{G/N}(r, p) = 2$.

We may assume that $N \simeq S \times \cdots \times S$ (n times) for some non-abelian simple group S. Suppose that $n \geq 2$. Then r, s and p do not divide $|N|$. We have $d_{G/N}(r, s) = 1$, $d_{G/N}(r, p) = 2$ and $d_{G/N}(s, p) = 1$ by Lemma 2. If G/N is non-solvable, then we have a contradiction by the choice of G and Lemma 3. We may assume that G/N is solvable. By the Frattini argument, we have $G = N_G(U)N$ for $U \in Syl_2(N)$. Since $N_G(U)/N_G(U) \cap N$ is solvable and r, s and p do not divide $|N|$, $N_G(U)$ contains a Hall $\{r, s, p\}$-subgroup H of G. Then UH is a Frobenius group with kernel U, a contradiction since $d_H(r, p) \geq 2$. This yields that $n = 1$.

Suppose that $C_G(N) \neq 1$. We see that r, s and p do not divide $|C_G(N)|$. We can get a contradiction by an argument similar to the above paragraph. This implies that G is almost simple.

By Lemma 4 we see that a counter example of possible minimal order to Theorem 2 is a non-abelian simple group.

5. Simple groups

The purpose of this section is to verify Theorems 1 and 2 for finite non-abelian simple groups.

(1) Let G be one of the 26 sporadic simple groups. If $d_G(2, p) \geq 2$, then $d_G(2, p) = 2$ or $d_G(2, p) = \infty$ by [5]. Furthermore $d_G(2, p) = 2$ if and only if $p = 5$, $G = M_{23}$ or $p = 13$, $G = F_3$ or $p = 29$, $G = F_1$. In the cases Sylow p-subgroups of G are abelian. Thus Theorems 1 and 2 hold true for G.

(2) Let G be the alternating group on n-letters. If $8 \geq n \geq 5$, then Sylow subgroups of odd order are abelian and it is easy to verify Theorem 2. Assume that $n \geq 9$ and $p \in \pi(G)$. If $p \leq n - 4$, then $d_G(2, p) = 1$. If $p = n - 3$, then Sylow p-subgroups of G are cyclic of order p and $d_G(2, p) = 2$. If $p \geq n - 2$, then Sylow p-subgroups of G are cyclic of order p and $d_G(2, p) = \infty$. Thus Theorems 1 and 2 hold true for the alternating groups.

For a positive integer k let $\pi(k)$ be the set of all prime divisors of k. Let $\pi_0 = \{p \in \pi(G) | d_G(2, p) \leq 1\}$. Then we do not have to think about primes in π_0 in order to verify Theorems 1 and 2 for the simple groups of Lie type.

The centralizers of involutions in simple groups of Lie type can be found in [1], [17], [12] and [15].

(3) Let $G = PSL(n, q)$. Then $|G| = q^{n(n-1)/2} \prod_{i=1}^{n-1} (q^{i+1} - 1)d^{-1}$, $d = (n, q-1)$.

Suppose that $q \equiv 0 \pmod 2$. Let I_j be the $j \times j$ identity matrix. Put

$$t'_k = \begin{pmatrix} I_k & 0 & 0 \\ 0 & I_{n-2k} & 0 \\ I_k & 0 & I_k \end{pmatrix}$$

Then $t'_k (r \geq k \geq 1)$ where $r = [n/2]$, are representatives of the conjugacy classes of involutions in $SL(n, q)$. The centralizer of t'_k in $SL(n, q)$ is the set of all matrices of the form

$$\begin{pmatrix} A & 0 & 0 \\ H & B & 0 \\ K & L & A \end{pmatrix}$$

where $(detA)^2 detB = 1$ and A is an $k \times k$ nonsingular matrix. Denote t_k the homomorphic image of t'_k in $PSL(n, q)$. Then t_k $(r \geq k \geq 1)$ are representatives of the conjugacy classes of involutions in $PSL(n, q)$. Let $C_k = C_G(t_k)$. Then

$$\pi(C_k) = \pi\Big(2 \prod_{i=1}^{n-2k} (q^i - 1)/(q-1)d\Big)$$

and

$$\pi_0 = \pi\Big(\prod_{k=1}^{r} |C_k|\Big) = \pi\Big(2\prod_{i=1}^{n-2}(q^i - 1)\Big)$$

Suppose that $n \geq 4$. Then the only factor of $|G|$ to be considered is $(q^{n-1} - 1)(q^n - 1)$. There are maximal tori $T(A_{n-2})$ of order $(q^{n-1} - 1)d^{-1}$ and $T(A_{n-1})$ of order $(q^n - 1)/(q-1)d$. Let $p \in \pi(T(X)) - \pi_0$ where $X = A_{n-1}$ or A_{n-2}. Note that $(q^n - 1, q^{n-1} - 1) = q - 1$ and $\pi(q-1) \subseteq \pi_0$. Let P be a Sylow p-subgroup of $T(X)$. Then P is a Sylow p-subgroup of G. Since P is abelian, Theorems 1 and 2 hold true for $G = PSL(n, q)$, $n \geq 4$.

Suppose that $n = 3$. Then Sylow subgroups of G of odd order are abelian. We have verified Theorem 1 for $G = PSL(3, q)$. Since $|G| = q^3(q^2 - 1)(q^3 - 1)d^{-1}$ and there are three classes of maximal tori of orders

$$(q-1)^2 d^{-1}, \quad (q^2 - 1)d^{-1}, \quad (q^2 + q + 1)d^{-1},$$

we have verified Theorem 2.

It is trivial that Theorems 1 and 2 hold true for $n = 2$.

Suppose that $q \equiv 1 \pmod 2$. If $n = 2$, then Sylow subgroups of odd order are abelian. Assume $n > 2$. Let t be an element in G such that the inverse image in $SL(n, q)$ is:

$$\begin{pmatrix} -I_2 & 0 \\ 0 & I_{n-2} \end{pmatrix}.$$

It is evident that $C_G(t)$ contains a subgroup isomorphic to

$$(SL_{n-2}(q) \times SL_2(q))/Z(SL_n(q)).$$

Assume that $d_G(2, p) \geq 2$. Then p divides $(q^n - 1)(q^{n-1} - 1)$ and p does not divide $q(q^{n-1-m} - 1)$ for $1 \leq m \leq n-2$. Now we can apply the same arguments as the case $q \equiv 0 \pmod 2$.

(4) Let $G = PSU(n, q)$. Then $|G| = q^{n(n-1)/2} \prod_{i=1}^{n-1} (q^{i+1} - (-1)^{i+1}) d^{-1}$, $d = (n, q+1)$.

Suppose that $q \equiv 0 \pmod 2$. Let $r = [n/2]$ be the number of conjugacy classes of involutions and t_k $(r \geq k \geq 1)$ the representatives of the conjugacy classes of involutions. Put $C_k = C_G(t_k)$ $(r \geq k \geq 1)$. Then

$$\pi(C_k) = \pi\Big(2 \prod_{i=1}^{k} (q^i - (-1)^i) \prod_{i=1}^{n-2k} (q^i - (-1)^i)/(q+1)d\Big).$$

Suppose that $n \geq 4$. Since $\pi_0 = \pi(\prod_{k=1}^{r} |C_k|) = \pi(2 \prod_{i=1}^{n-2} (q^i - (-1)^i))$, we have to think about the foctors $(q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})$ of $|G|$. There are maximal tori $T({}^2\mathrm{A}_{n-2})$ of order $(q^{n-1} - (-1)^{n-1})/d$ and $T({}^2\mathrm{A}_{n-1})$ of order $(q^n - (-1)^n)/(q+1)d$ in G. It follows that

$$\pi\big(T({}^2\mathrm{A}_{n-2})\big) \cap \pi\big(T({}^2\mathrm{A}_{n-1})\big) \subseteq \pi(q+1) \subseteq \pi_0.$$

and Sylow p-subgroups of G are abelian for $p \in \pi(G) - \pi_0$.

Suppose that $n = 3$. Note that Sylow subgroups of odd order of G are abelian. Since G contains a maximal torus of order $(q^2 - 1)d^{-1}$, it follows that $d_G(2, r) = 2$ for $r \in \pi(q-1)$ and $d_G(2, r) = \infty$ for $r \in \pi((q^2 - q + 1)d^{-1})$.

This verifies Theorems 1 and 2 for $PSU(n, q)$ for $q \equiv 0 \pmod 2$.

By the similar arguments Theorems 1 and 2 hold true for $q \equiv 1 \pmod 2$.

(5) Let $G = PS_p(2n, q)$, $q \equiv 0 \pmod 2$. Then $|G| = q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1)$. There are $r = n + [n/2]$ conjugacy classes of involutions. Let $t_k (r \geq k \geq 1)$ be their representatives and put $C_k = C_G(t_k)$. It follows that $\pi_0 = \pi(\prod_{k=1}^{r} |C_k|) = \pi(2 \prod_{i=1}^{n-1} (q^{2i} - 1))$. The factors of $|G|$ to be considered are

$(q^n+1)(q^n-1)$. There are maximal tori $T(C_n)$ and $T(A_{n-1})$ of orders q^n+1 and q^n-1, respectively. It follows that Theorems 1 and 2 hold true in the case.

Suppose that $q \equiv 1 \pmod 2$. Let V be a $2n$-dimensional vector space over the field of q-elements. Let P be a parabolic subspace of V. Then there exists an involution t in $S_p(V)$ of the form:

$$\begin{pmatrix} -I_P & 0 \\ 0 & I_{P^\perp} \end{pmatrix}$$

where I_P (*resp.* $I_{P^\perp}$) is the identity transformation on P (*resp.* $P^\perp$). It follows that $C_{S_p(V)}(t)$ contains a subgroup isomorphic to $S_p(2,q) \times S_p(2n-2,q)$. Hence if $d_G(2,p) \geq 2$, p divides $(q^{2n}-1)$ and p does not divides $(q^{2s}-1)q$ for $1 \leq s \leq l-1$. Since there exist tori $T(C_n)$ and $T(A_{n-1})$ whose orders are $(q^n+1)/2$ and $(q^n-1)/2$, respectively, Sylow p-subgroups of G are abelian for $d_G(2,p) \geq 2$. It follows that Theorems 1 and 2 hold true in the case.

(6) Let $G = P\Omega_{2n}(+1,q)$, $q \equiv 0 \pmod 2$. Then $|G| = q^{n(n-1)}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)d^{-1}$ where $d=(4,q^n-1)$. Let $t_k (r \geq k \geq 1)$ be the representatives of the conjugacy classes of involutions and $C_k = C_G(t_k) (r \geq k \geq 1)$. It follows that $r = n+(-1)^n$ and $\pi_0 = \pi(\prod_{k=1}^{r}|C_k|) = \pi(2\prod_{i=1}^{n-2}(q^{2i}-1))$. Thus the factors of $|G|$ which we have to think about are $(q^{n-1}-1)(q^{n-1}+1)(q^n-1)$. There are maximal tori $T(A_{n-1})$, $T(D_n)$ and $T(A_{n-2})$ of orders q^n-1, $(q^{n-1}+1)(q+1)$ and $(q^{n-1}-1)(q-1)$ respectively. Noting that $(q^{n-1}-1, q^n-1) = q-1$ and

$$\pi(q^n-1) \cap \pi(q^{n-1}+1) \subseteq \pi(q+1) \subseteq \pi_0,$$

we have Theorems 1 and 2.

Similar arguments can be applied to the case $q \equiv 1 \pmod 2$.

Let $G = P\Omega_{2n}(-1,q)$, $q \equiv 0 \pmod 2$. Then $|G| = q^{n(n-1)}(q^n+1)\prod_{i=1}^{n-1}(q^{2i}-1)d^{-1}$ where $d = (4, q^n+1)$ and $\pi_0 = \pi(2\prod_{i=1}^{n-2}(q^{2i}-1))$. The factors of $|G|$ which we have to think about are $(q^{n-1}-1)(q^{n-1}+1)(q^n+1)$. There are maximal tori $T(C_n)$, $T({}^2D_n)$ and $T({}^2A_{n-2})$ of orders q^n+1, $(q^{n-1}+1)(q-1)$ and $(q^{n-1}-1)(q+1)$ respectively. By the same way as above Theorems 1 and 2 hold true.

Similar arguments can be applied to the case $q \equiv 1 \pmod 2$.

(7) Let $G = P\Omega_{2n+1}(q)$, $q \equiv 1 \pmod 2$. Since

$$\pi(\Omega_{2n}(+1,q)) \cup \pi(\Omega_{2n}(-1,q)) = \pi(\Omega_{2n+1}(q))$$

and G contains subgroups isomorphic to $\Omega_{2n}(+1,q)$ and $\Omega_{2n}(-1,q)$, (6) implies that Theorems 1 and 2 hold true for $P\Omega_{2n+1}(q)$.

(8) Let $G = {}^2B_2(q)$, $q = 2^{2m+1}$. Then the centralizer of any involution is a 2-group and Sylow subgroups of odd order are abelian. Theorems 1 and 2 hold true.

(9) Let $G = G_2(q)$. Then $|G| = q^6(q^6-1)(q^2-1)$. Suppose that $q \equiv 0 \pmod 2$. There are two conjugacy classes of involutions and $\pi_0 = \pi(2(q^2-1))$. There are maximal tori $T(A_2)$ and $T(G_2)$ of orders q^2+q+1 and q^2-q+1. The results follow immediately. For $q \equiv 1 \pmod 2$ since $\pi_0 = \pi(2(q^2-1))$ we can verify Theorems 1 and 2 by the same way.

(10) Let $G = {}^2G_2(q)$, $q = 3^{2m+1}$. It follows that $|G| = (q^3+1)q^3(q-1)$. Since the centralizer of an involution contains $PSL(2,q)$, $\pi(q(q^2-1)) \subseteq \pi_0$. G contains cyclic subgroups of order $q^2+\sqrt{3q}+1$ and $q^2-\sqrt{3q}+1$ which are self-centralizing. Thus $d_G(2,p) \geq 2$ yields $d_G(2,p) = \infty$ for $p \in \pi(G)$. Theorems 1 and 2 hold true.

(11) Let $G = {}^3D_4(q)$. It follows that $|G| = q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$. Suppose that $q \equiv 0 \pmod 2$. Since the centralizer of an involution involves $SL(2,q^3)$, $\pi(2(q^6-1)) \subseteq \pi_0$. Thus the only factor of $|G|$ we have to think about is q^4-q^2+1. There is a maximal torus of order q^4-q^2+1 which is an isolated subgroup in G (See [12], [15]). It follows that $d_G(2,p) = 1$ or ∞. Theorems 1 and 2 hold true for ${}^3D_4(q)$.

For $q \equiv 1 \pmod 2$ we can verify Theorems 1 and 2 by the same way.

(12) Let $G = F_4(q)$. It follows that $|G| = q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$. Suppose that $q \equiv 0 \pmod 2$. There is an involution t such that $|C_G(t)| = q^{24}(q^2-1)(q^4-1)(q^6-1)$. Thus $\pi_0 \supseteq \pi((q^6-1)(q^2+1))$. The only factor of $|G|$ to be considered is $(q^4+1)(q^4-q^2+1)$. There are isolated tori of orders (q^4+1) and (q^4-q^2+1) in G. It follows that $d_G(2,p) = 1$ or ∞ for $p \in \pi(G)$. Theorems 1 and 2 hold true for $F_4(q)$, $q \equiv 0 \pmod 2$.

For $q \equiv 1 \pmod 2$ we can verify Theorems 1 and 2 by the same way.

(13) Let $G = {}^2F_4(q)$, $q = 2^{2n+1}$. Then $|G| = q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$ and $\pi_0 = \pi((2(q^4-1))$. For $r \in \pi(q^2-q+1)$, $d_G(3,r) = 1$ since an element of order 3 in G centralizes $SU(3,q)$. $PSU(3,q)$ has a maximal torus of order $(q^2-q+1)/3$. Thus $d_G(2,r) = 2$ for $r \in \pi(q^2-q+1) - \pi_0$. For $r \in \pi(q^4-q^2+1)$, $d_G(2,r) = \infty$ and Sylow r-subgroups of G are abelian since q^4-q^2+1 is the product of orders of two maximal tori of coprime order.

Let $G = {}^2F_4(2)'$. Then $|G| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$. It follows that $d_G(2,3) = d_G(2,5) = 1$ and $d_G(2,13) = \infty$. Theorems 1 and 2 hold true for G.

(14) Let $G = E_6(q)$. It follows that $|G| = q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)d^{-1}$ where $d = (3, q-1)$.

Suppose that $q \equiv 0 \pmod 2$. Then $\pi_0 = \pi(2\prod_{i=1}^{6}(q^i-1)d^{-1})$. Thus the factors of $|G|$ to be considered are $(q^4-q^2+1)(q^4+1)(q^6+q^3+1)d^{-1}$. There are maximal tori $T(D_5)$, $T(E_6)$ and $T(E_6(a_1))$ of orders $(q^4+1)(q^2-1)d^{-1}$, $(q^4-q^2+1)(q^2+q+1)d^{-1}$ and $(q^6+q^3+1)d^{-1}$, respectively. Note that

$$\pi(q^4+1)\cap\pi(q^4-q^2+1)\subseteq\pi_0$$

$$\pi(q^4+1)\cap\pi(q^6+q^2+1)\subseteq\pi_0$$

$$\pi(q^4-q^2+1)\cap\pi(q^4-q^2+1)\subseteq\pi_0.$$

Theorems 1 and 2 follow immediately.

By the same way as above Theorems 1 and 2 hold true for $q\equiv 1 \pmod 2$.

(15) Let $G={}^2E_6(q)$. It is $|G|=q^{36}(q^{12}-1)(q^9+1)(q^8-1)\,(q^6-1)(q^5+1)(q^2-1)d^{-1}$ where $d=(3,q+1)$

Suppose that $q\equiv 0 \pmod 2$. It follows that $\pi_0=\pi(2(q^4-1)\,(q^5+1)(q^6-1))$. Thus the factors of $|G|$ to be considered are $(q^4-q^2+1)(q^4+1)\,(q^6-q^3+1)d^{-1}$. There are maximal tori $T({}^2D_5)$, $T({}^2E_6)$ and $T({}^2E_6(a_1))$ of orders $(q^4+1)(q^2-1)d^{-1}$, $(q^4-q^2+1)(q^2-q+1)d^{-1}$ and $(q^6-q^3+1)d^{-1}$, respectively. Since

$$\pi(q^4+1)\cap\pi(q^4-q^2+1)\subseteq\pi_0$$

$$\pi(q^4+1)\cap\pi(q^6-q^2+1)\subseteq\pi_0$$

$$\pi(q^4-q^2+1)\cap\pi(q^4-q^2+1)\subseteq\pi_0,$$

the results follow immediately.

By the same way Theorems 1 and 2 hold true for $q\equiv 1 \pmod 2$.

(16) Let $G=E_7(q)$. It follows that $|G|=q^{63}(q^{18}-1)(q^{14}-1)\,(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1)$.

Suppose that $q\equiv 0 \pmod 2$. It follows that $\pi_0=\pi(2(\prod_{i=1}^{6}(q^{2i}-1))$. Thus the only factors of $|G|$ to be considered is

$$(q^7-1)(q^6+q^3+1)(q^6-q^3+1)(q^7+1)/(q^2-1).$$

There are maximal tori $T(A_6)$, $T(E_6(a_1))$, $T(E_7)$ and $T(E_7(a_1))$ of orders q^7-1, $(q^6+q^3+1)(q-1)$, $(q^6-q^3+1)(q+1)$ and q^7+1 respectively. If $q=2$, then

$$|G|=2^{63}\cdot 3^{11}\cdot 5^2\cdot 7^3\cdot 11\cdot 13\cdot 17\cdot 19\cdot 31\cdot 43\cdot 73\cdot 127.$$

Since $d_G(2,3)=d_G(2,7)=1$ and $d_G(2,19)=d_G(2,43)=2$, Theorems 1 and 2 follow for $E_7(2)$. Assume that $q>2$. We note that

$$\pi(q^6+q^3+1)\cap\pi(q^7+1)\subseteq\pi_0$$

$$\pi(q^6+q^3+1)\cap\pi(q^7-1)\subseteq\pi_0$$

$$\pi(q^6-q^3+1)\cap\pi(q^7-1/q-1)\subseteq\pi_0$$

$$\pi(q^6-q^3+1)\cap\pi(q^7+1/q+1)\subseteq\pi_0.$$

Let X be one of the admissible diagrams A_6, $E_6(a_1)$, E_7, $E_7(a_1)$ and $p\in\pi(T(X))-\pi_0$. Let P be a Sylow p-subgroup of $T(X)$. Then P is abelian and P is a Sylow p-subgroup of G. This verifies Theorems 1 and 2 for $G=E_7(q)$, $q\equiv 0 \pmod 2$.

For $q\equiv 1 \pmod 2$ we can verify Theorems 1 and 2 by the same way.

(17) Let $G=E_8(q)$, $q\equiv 1 \pmod 2$. There are two classes of involutions in G and $d_G(2,p)=1$ for p which divides

$$q(q^2-1)(q^4-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1).$$

There are maximal tori $T(E_8)$, $T(E_8(a_1))$ and $T(E_8(a_5))$ of orders $(q^{10}-q^5+1)/(q^2-q+1)$, $(q^{12}+1)/(q^4+1)$ and $(q^{10}+q^5+1)/(q^2+q+1)$ respectively. They are cyclic. It follows that if $d_G(2,p)\geq 2$, then p divides

$$|T(E_8)||T(E_8(a_1))||T(E_8(a_5))|.$$

This implies that Sylow p-subgroups of G are abelian if $d_G(2,p)\geq 2$.

For $q\equiv 0 \pmod 2$ we can verify Theorems 1 and 2 by the same way.

We have verified Theorems 1 and 2 for all the non-abelian simple groups. This completes the proofs of Theorems 1 and 2.

The observations above yield the following:

Theorem 4. *Let G be a simple group of Lie type and T a maximal torus. Let $p\in\pi(T)-\pi_0$, where $\pi_0=\{p\in\pi(G)|d_G(2,p)\leq 1\}$. Then T contains a Sylow p-subgroup of G.*

Theorem 4 is a corollary of Theorem 1. Actually we proved Theorem 4 for specified tori of G when we gave the proof of Theorem 1.

Remark 1. Suzuki [16] determined the structure of (CIT)-groups. A (CIT)-group is a finite group of even order in which the centralizer of every involution is a 2-group. His theorem implies that if p is an odd prime, then a Sylow p-subgroup of a (CIT)-group is always abelian. This means that if a finite group G of even order contains a non-abelian Sylow p-subgroup for odd prime p, then G is not a (CIT)-group. Suzuki's theorem, however, appears not to give us any information as to whether any non-abelian Sylow p-subgroup of a finite group of even order always contains a non-trivial element which commutes with an involution. Our main theorem guarantees the existence of such a non-trivial element in any non-abelian Sylow p-subgroup. Thus our maim theorem is a far reaching generalization of [16].

Remark 2. Let G be a finite group of even order. In terms of the prime graph $\Gamma(G)$ of G Suzuki [16] proved that if $d_G(2, p) = \infty$ for all $p \in \pi(G) - \{2\}$, then Sylow p-subgroups of G are abelian.

Remark 3. There is an example for $d_G(2, p) = 3$. Let $G = S_z(2^9) : 3$, the extension of Suzuki's simple group $S_z(2^9)$ by the field automorphism of order 3. Then $d_G(2, 37) = d_G(2, 109) = d_G(2, 73) = 3$ and $d_G(37, 109) = d_G(37, 73) = d_G(73, 109) = 4$.

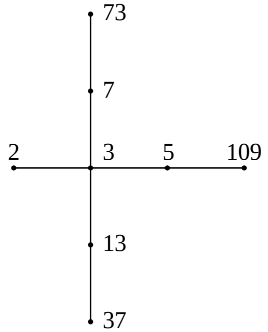

Fig. 1. $\Gamma(Sz(2^9) : 3)$

Also there is a solvable group X with $d_X(2, p) = 3$ for some $p \in \pi(X)$.

6. The proof of Theorem 3

We will give the proof of Theorem 3. Let G be a finite group of even order such that $d_G(2, p) \geq 2$ for some prime $p \in \pi(G)$. Let

$$G = G_0 \supset G_1 \supset \cdots \supset G_{s-1} \supset G_s = 1$$

be a series of characteristic subgroups of G such that G_i is a maximal characteristic subgroup of G_{i-1} for $i = 1, \ldots, s$. Since G_{i-1}/G_i is a minimal characteristic subgroup in G/G_i, Lemmas 6 and 7 yield that G_{i-1}/G_i is a non-abelian simple group or an abelian group. If the chain has two non-abelian simple factors, we have a contradiction by Lemma 8. If the chain has more than two non-abelian simple factors, then there exists i such that G_{i-1}/G_i is a non-abelian simple group and G/G_i has two non-abelian simple factors. Lemma 8 yields $d_{G/G_i}(2, p) = 1$ for $p \in \pi(G/G_i) - \{2\}$. The group G_{i-1} has more than two non-abelian simple factors. By the induction on the number of non-abelian simple factors of the chain, we can complete the proof.

Remark 4. By Fisman [6] we can know the simple factor H/K of G.

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