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A Riemann Roch Theorem for infinite genus Riemann surfaces

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Abstract. In this article we prove a Riemann Roch Theorem for a class of holomorphic line bundles over Riemann surfaces of infinite genus. The theorem shows that the space of holomorphic sections satisfying a pointwise asymptotic growth condition has finite dimension and it provides a formula for this dimension. The gluing functions describing the surface and the transition functions defining the line bundle have to satisfy some asymptotic bounds. The theorem applies to holomorphic line bundles associated to divisors of infinite degree that assign one point to every handle on the surface. Applications of this Riemann Roch Theorem to the description of the Kadomcev Petviashvilli flow were provided in the author's doctoral thesis.

1 Introduction

Riemann surfaces of infinite genus arise naturally as spectral varieties of various ordinary and partial linear differential equations in mathematical physics such as Hill's equation ([7], [8]) and the heat equation with a periodic potential, see [5], [2] and [1], Part III. There is a function theory of these Riemann surfaces, culminating in infinite genus analogies of classical theorems for compact Riemann surfaces, such as the Riemann Vanishing Theorem and the Torelli Theorem ([1], Part II). This theory has various applications in the examination of the Korteweg de Vries flow and the Kadomcev Petviashvilli flow ([1], Part IV).

This article provides a contribution to this theory, the proof of a Riemann Roch Theorem being applicable to the naturally defined Bloch bundles over the surfaces mentioned above.

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One can build the Riemann surfaces of infinite genus under consideration in the following way (see also [5] and [1], Part III): Take a complex plane with infinitely many pairs of disjoint disks removed (coordinates $z_0 = 1/z_\infty$). The holes should only accumulate for $|z_0| \to \infty$. Then we glue in handles, in coordinates described by $z_j z_{-j} = t_j$, $|z_{\pm j}| <$ const, to join corresponding holes together. The Riemann Roch Theorem requires some asymptotic conditions on the width t_j of the handles, on the distribution of the holes and on the coordinate changes (Sect. 2.2). Finally a compact piece of the resulting surface may be changed in an arbitrary way.

In applications, the handles arise from the perturbation of double point singularities $z_j z_{-j} = 0$. On the one hand the handles are usually asymptotically well–controlled up to error terms with known bounds, since the curve under consideration is asymptotically close to an unperturbed curve, which is a copy of the complex plane with infinitely many transverse self–intersections. On the other hand, the compact piece is a domain where perturbation theory only provides poor information.

The classical Riemann Roch Theorem states

$$\mathrm{r}(D) - \mathrm{i}(-D) = 1 - \mathrm{genus}(X) + \deg(D),$$

$\mathrm{r}(D)$ denoting the number of linearly independent meromorphic functions on a compact Riemann surface X that are multiples of the divisor D, $\mathrm{i}(-D)$ denoting the number of linearly independent meromorphic 1–forms on X that are multiples of the divisor $-D$. Naively viewed, on a (noncompact) surface of infinite genus with a divisor of infinite degree, both sides of the Riemann Roch Formula become indeterminate: "$\infty - \infty = 1 - \infty + \infty$". However, when one imposes asymptotic growth conditions at infinity for the meromorphic functions and 1-forms allowed, one gets a version of the Riemann Roch Theorem for some infinite genus Riemann surfaces.

Here is a typical example on which the theorem applies: choose one point q_j in every handle, close to its center, outside the domain of z_0, and fix an integer ν, called "the order at infinity". Let $\mathcal{M}$ denote the space of meromorphic functions f on the surface with poles at most of first order allowed only at the q_j and with $f/z_\infty^{-\nu}$ being bounded near infinity in the domain of z_0. Similarly let $\mathcal{N}$ denote the space of holomorphic 1–forms ω on the surface with zeroes prescribed at the q_j and with $\omega/(z_\infty^\nu dz_\infty)$ being bounded near infinity in the domain of z_0. Then the Riemann Roch Theorem states that $\mathcal{M}$ and $\mathcal{N}$ have finite dimension and it determines $\dim \mathcal{M} - \dim \mathcal{N}$.

It is technically more convenient to work with sections in holomorphic line bundles rather than meromorphic functions for several reasons: First, the applications to the natural dual Bloch bundle of a spectral variety and to theta bundles are described in a uniform way in the language of bundles. Second, the symmetry between meromorphic functions and holomorphic 1–forms, which plays an important role in the proof, is seen more clearly using bundles. Third, the theorem also applies to divisors with more than one point per handle, if only the "net degree" per handle is +1, such as having two points with multiplicity $+1$ and one point with multiplicity -1 per handle. This case arises when one examines the Jacobi group operation. It is included naturally in the line bundle version of the Riemann Roch Theorem. The presence of "net degree" +1 of the divisor per handle is reflected by having nontrivial line bundle gluing functions (winding number 1) in the handles, see hypothesis (L1) in Sect. 2.2.

The key to the proof of the theorem consists in showing that the Cauchy Riemann operator $\overline{\partial}$ is a Fredholm operator between Hilbert spaces defined by weighted L^2 and Sobolev norms. This is done by constructing a quasi-inverse as an integral operator with a kernel that approximates the Cauchy kernel. The interpolation of the Cauchy kernel through handles is one important step; here the nontriviality of the line bundle transition functions is essential. The compact piece, which is much less under control, may be ignored first, later we examine the change of the index of $\overline{\partial}$ when gluing the compact piece in. Finally we have to examine the relation between L^2 bounds and pointwise bounds at infinity for holomorphic sections.

There exist some other Riemann Roch type theorems for infinite genus Riemann surfaces: An early version for infinite genus surfaces but divisors of finite degree was described by [14]; a version for infinite genus hyperelliptic surfaces is included in [15], and recently a version for half form bundles with Möbius functions as coordinate gluing functions was given by [17]; however, these results do not apply to the mentioned natural bundles.

The motivation for this work were applications of the Riemann Roch Theorem to the inverse spectral theory of the 1+1 dimensional heat equation and to the examination of the Kadomcev Petviashvilli flow. These applications are described in the author's doctoral thesis [10] and will also be described in a forthcoming article. This work grew out of part of the author's thesis under supervision of Horst Knörrer at the Swiss Federal Institute of Technology (ETH), Zürich.

2 Statement and hypotheses of the theorem

2.1 Riemann Roch Theorem for infinite genus Riemann surfaces

Let a structure $(X, L, U_0, z_0, \psi_{0A})$ be given, where X is an infinite genus Riemann surface, L is a holomorphic line bundle over X, $z_0 : U_0 \to \mathbb{C}$ is a coordinate function defined on an open subset $U_0 \subseteq X$ and ψ_{0A} is a holomorphic basis section in L defined over U_0. ψ_{0A} will serve as a reference section to measure the asymptotic behaviour of sections. U_0 is called the "regular piece". Let $\check{L}^{1,0} = \check{L} \otimes \mathcal{O}^{1,0}$ denote the holomorphic line bundle of 1–forms of type 1,0 with coefficients in the dual of L. We set $z_\infty := 1/z_0$.

We add one additional point ∞ to X, the resulting set (which is *not* a Riemann surface) is called X^∞. Below we shall specify a topology on X^∞ for a class of infinite genus Riemann surfaces X; it has the property that the complement of every neighbourhood of ∞ only has connected components being relatively compact in X.

To provide a quick impression of the theorem, the assertion is stated first. The precise technical hypotheses are described in the next section.

Theorem 2.1 (Riemann Roch Theorem for infinite genus surfaces) *Assume that $(X, L, U_0, z_0, \psi_{0A})$ satisfies the hypotheses (X1-6) and (L1-2) below. Take $\nu \in \mathbb{Z}$, called the order at infinity. Let $\mathcal{M}$ denote the space of all global holomorphic sections f in L with $f/(z_\infty^{-\nu}\psi_{0A})$ being bounded in a neighbourhood of ∞. Similarly let $\mathcal{N}$ denote the space of all global holomorphic sections ω in $\check{L}^{1,0}$ with $\omega/(z_\infty^{\nu}\psi_{0A}^{-1} \otimes dz_\infty)$ being bounded in a neighbourhood of ∞. Then $\mathcal{M}$ and $\mathcal{N}$ are both finite dimensional complex vector spaces and the following relation holds:*

Riemann Roch Formula:

$$\dim \mathcal{M} - \dim \mathcal{N} = 1 + \nu - \operatorname{codeg}(L, \psi_{0A})$$

The integer $\operatorname{codeg}(L, \psi_{0A})$, called the codegree of (L, ψ_{0A}), which is our substitute for the ill-defined expression "genus(X) − degree(L)", will be defined in Sect. 2.4.

2.2 Hypotheses on the Riemann surface and the line bundle

Here are the hypotheses of the Theorem:

Hypotheses for the surface.

(X1) **Pieces of the surface.** $X \setminus U_0$ only has compact connected components. These components are covered by relatively compact open sets $U_j = U_{-j}$, $j \in J$, called the "handles", and one additional relatively compact open set U_c, called the "compact piece". J denotes a countable index set with a fixpoint free involution $- : J \to J$. The $U_j, U_{j'}$, $j, j' \in J$ are disjoint for $\pm j \neq j'$ and disjoint from the compact piece. $U_j \cap U_0$ consists of two connected components $V_{\pm j}$.

(X2) **Handle coordinates.** Handles may be "nondegenerate" or "degenerate". In the nondegenerate case U_j is connected and there are two coordinate functions $z_{\pm j}$ on U_j related by $z_j z_{-j} = t_j$, $0 < t_j < 1$ being constants. In the degenerate case U_j consists of two simply connected components $D_{\pm j} \supseteq V_{\pm j}$, and we have coordinate functions $z_{\pm j}$ on $D_{\pm j}$, their range containing the unit disk in $\mathbb{C}$. Here we set $z_{\pm j} = 0$ on $D_{\mp j}$ and define $t_j = 0$.
Hence in both cases $z_j z_{-j} = t_j$. We assume that in both cases the range $(z_j, z_{-j})[U_j]$ contains the "model handle" $\{(z_1, z_2) \in \mathbb{C}^2$:

$z_1 z_2 = t_j;\ |z_1|, |z_2| < 1\}$ and is contained in $\{(z_1, z_2) \in \mathbb{C}^2 \colon z_1 z_2 = t_j;\ |z_1|, |z_2| < \epsilon_1\}$ for some constant ϵ_1. The range $z_j[V_j]$ should contain some annulus $\Delta_1(0) \setminus \Delta_\epsilon(0)$ and be contained in some annulus $\Delta_{\epsilon_1}(0) \setminus \Delta_{\epsilon_2}(0)$, the radii not depending on j, $\epsilon_2^2 > \sup_j t_j$. Here $\Delta_r(s)$ denotes the open disk in $\mathbb{C}$ with radius r centered at s.

A degenerate handle may be viewed as two disjoint disks arising from the blow up of a double point singularity.

By shrinking V_j if necessary, we may assume further that for $x, y \in V_j$ there is a path in $z_j[V_j]$ of length $O(|z_j(x) - z_j(y)|)$ joining $z_j(x)$ and $z_j(y)$; this is obvious when $z_j[V_j]$ is an annulus.

(X3) **Coordinate on the regular piece.** There is a coordinate function z_0 on U_0. There are pairwise disjoint disks $\Delta_{r_j}(s_j)$, $j \in J$, containing $z_0[V_j]$. Their centers $\{s_j : j \in J\}$ are a discrete set in $\mathbb{C}$. A set $M \subseteq X$ is relatively compact if and only if M meets only finitely many handles and $z_0[M \cap U_0]$ is bounded.

Sometimes it will be inconvenient to have

$$0 \in \overline{z_0[U_0]} \quad \text{or} \quad 0 \in \overline{\Delta_{r_j}(s_j)}. \tag{1}$$

This can be avoided by shrinking the regular piece, enlarging the compact piece at the same time, or including some handles to the compact piece, removing their indices from J.

(X4) **Coordinate transitions.** The derivative dz_j/dz_0 of the coordinate transition has a holomorphic logarithm over V_j. There is a bounded family of weights $o_j > 0$, $j \in J$, and there are scaling constants $c_j \in \mathbb{C}^*$ such that

$$\sup_{\substack{x,y \in V_j \\ x \neq y}} \left| \frac{z_j(x) - z_j(y)}{z_0(x) - z_0(y)} - c_j \right| \leq |c_j| O(o_j),$$

$$\sup_{x \in V_j} \left| \frac{dz_j}{dz_0}(x) - c_j \right| \leq |c_j| O(o_j)$$

and $\sup_{V_j} \left| \frac{d^2 z_0}{dz_j^2} \right| \leq |c_j|^{-1} O(o_j)$.

(X5) **Parabolicity.** For all $N \subset\subset X$ and $\epsilon > 0$ there is $\chi \in \mathcal{C}_c^\infty(X, [0,1])$ with $\chi | N = 1$ and $\int_X d\chi \wedge * d\chi < \epsilon$.

In fact this is a consequence of an assumption that an exhaustion function h of finite charge on X exists, say with normalisation $\int_X |d * dh| \leq 1$. To see this, let $g : \mathbb{R} \to [0,1]$ be a smooth decreasing function with $|g'| < \epsilon$ and with $\chi := g \circ h$ equal 1 on N and $\chi \in \mathcal{C}_c^\infty(X, [0,1])$. Then $\int_X d\chi \wedge * d\chi \leq \epsilon \, |\int_X d\chi \wedge * dh| = \epsilon \, |\int_X \chi \, d * dh| \leq \epsilon \int_X |d * dh| \leq \epsilon$. (For details concerning these exhaustion functions of finite charge see [1], part I.)

(X6) **Bounds on the family of weights.** $\alpha_j := |c_j| |s_j|^2$ should be bounded from below by a positive constant. We further assume that $t_j \leq O(o_j^2)$

and $\sum_{j\in J}(\alpha_j o_j)^2 < \infty$. Further, $o_j\alpha_j^2$, $|c_j/c_{-j}|$, and $|s_j/s_{-j}|$ $(j \in J)$ should be bounded from above, and $\limsup_j |r_j/s_j| < 1$.

Hypotheses for the line bundle.

(L1) **Bundle gluing functions in the handles.** For nondegenerate handles we assume that there are holomorphic basis sections ψ_{jA} in L over the handles $U_j = U_{-j}$ with the following properties: On the j^{th} handle U_j

$$\psi_{-jA} = -\frac{Q_j}{z_j}\psi_{jA}$$

holds, with constants Q_j satisfying

$$Q_j Q_{-j} = t_j, \quad |Q_j| \le O(o_j),$$

and $|Q_j/Q_{-j}|$ being bounded from above and below by positive constants. For degenerate handles we assume that ψ_{jA} is a basis section over D_j, but it is extended by 0 to D_{-j}.

(L2) **Bundle gluing functions between handles and the regular piece.** On V_j the bundle transition function ψ_{jA}/ψ_{0A} has a continuous logarithm. It satisfies the following asymptotic bound:

$$\sup_{V_j}\left|\left(\frac{g_{jA}\psi_{0A}}{\psi_{jA}}\right)^{\pm 1} - 1\right| \le O(o_j)\ , \quad \sup_{V_j}\left|\frac{d}{dz_j}\frac{g_{jA}\psi_{0A}}{\psi_{jA}}\right| \le O(o_j).$$

Here $g_{jA} : V_j \to \mathbb{C}^*$ should be holomorphic and satisfy the conditions Ext-Disk(g_{jA}) and Ext-Disk($1/g_{jA}$), defined by:

Ext-Disk(f) $:\Longleftrightarrow$ *f is a holomorphic (resp. meromorphic) function at least defined over V_j. $f \circ z_0^{-1}$, i. e. f written in the coordinate of the regular sheet, extends to a holomorphic (resp. meromorphic) function on a simply connected domain in $\mathbb{C}$. If f is only meromorphic, the extension should not have any poles outside the original domain $z_0[V_j]$.*

Intuitively Ext-Disk(f) states that when the inner part of the j^{th} handle is removed and the remaining hole near V_j is filled with a disk then f extends to the disk. We assume

$$\sup_j \frac{\sup_{V_j}|g_{jA}|}{\inf_{V_j}|g_{jA}|} < \infty \quad \text{and} \quad \sup_j \frac{\sup_{V_j}|g_{jA}|}{\inf_{V_{-j}}|g_{-jA}|} < \infty\ . \qquad (2)$$

2.3 Example: Line bundles associated to divisors of infinite degree

Assume that the surface X satisfies (X1-6).

Here is an easy but typical example for line bundles L over X that satisfy the hypotheses (L1-2); it is a line bundle, which has a global section ψ_{0A}, with one zero in each handle; the space of sections of this line bundle is isomorphic to all meromorphic functions with a pole at the zeroes of the global section:

Let $J' \subseteq J$ denote an index set listing every handle only once. Choose one point $q_j = q_{-j} \in U_j \setminus \overline{U_0}$, $j \in J'$, associated to every handle, with $|z_j(q_j)/\sqrt{t_j}|$ being bounded from above and from below by positive constants. Let $Q_j := z_j(q_j)$ denote the coordinates of these points. We examine the divisor $D = \sum_{j \in J'} q_j$ of infinite degree. Intuitively, every handle (outside the compact piece) of the surface X has precisely one point of the divisor associated to it, located not too far from its center. Let $\mathcal{O}(D)$ denote the line bundle associated to the divisor D, i. e. sections in $\mathcal{O}(D)$ are meromorphic functions over X at most with poles of first order at the q_j. In the regular piece U_0 we prescribe the constant basis section $\psi_{0A} := 1$.

This example is realized in terms of gluing operations by choosing all $g_{jA} = 1$, and the transition functions are given by

$$\frac{\psi_{0A}}{\psi_{jA}} = 1 - \frac{Q_j}{z_j} \,, \qquad \frac{\psi_{jA}}{\psi_{0A}} = 1 + \frac{Q_j}{z_j - Q_j} \,; \tag{3}$$

so we have set

$$\psi_{jA} = \frac{z_j}{z_j - Q_j} = -\frac{Q_{-j}}{z_{-j} - Q_{-j}}$$

over U_j. Then obviously (L1-2) are satisfied; to see this one uses (3) and inserts $|Q_j| \leq \text{const}\sqrt{t_j} \leq O(o_j)$ by (X6) and $\inf_{j \in J} \inf_{y \in V_j} |z_j(y) - Q_j| > 0$.

The presence of one point of the divisor in every handle is reflected by the nontriviality of the bundle transition functions in the handles.

Some side remarks on other examples: When $(L_1, \psi_{0A}^{(1)})$, $(L_2, \psi_{0A}^{(2)})$, $(L_3, \psi_{0A}^{(3)})$ satisfy the hypotheses (L1-2), then $(L_1 \otimes L_2 \otimes \check{L}_3, \psi_{0A}^{(1)}\psi_{0A}^{(2)}(\psi_{0A}^{(3)})^{-1})$ satisfies (L1-2) too. This shows that the class of line bundles satisfying (L1-2) is large enough to allow a Jacobi group operation. Thinking of $L_k = \mathcal{O}(D_k)$ with divisors D_k as above, the theorem is also applicable to divisors with a bounded number of points per handle having "net degree" $+1$ per handle.

2.4 Definitions

The statement of the Riemann Roch Theorem requires the notion of functions to be bounded near infinity. The codegree remains to be defined, too. Here are the relevant definitions:

Definition 2.2 (Point at infinity; topology of the enlarged surface) *We endow $X^\infty = X \cup \{\infty\}$ with the following topology:*
$U \subseteq X^\infty$ is open $:\Longleftrightarrow$

- *if $\infty \notin U$ and U is open in X;*
- *or if $\infty \in U$, $U \setminus \{\infty\}$ is open in X and there is a compact set $K \subset X$ and there are radii $r_1 > r_2 > 0$ with $z_j[U_j \cap U_0] \supseteq \{z \in \mathbb{C} \mid r_1 < |z| < r_2\}$ for all except possibly a finite number of handle indices j, with the following property:*

$$\forall x \in X \setminus U : x \in K \text{ or } \exists j \in J : (x \in U_j \text{ and } |z_j(x)| < r_1 \text{ and } |z_{-j}(x)| < r_1)\ .$$

Intuitively this means that U contains all points except possibly a compact piece and the "inner part" of the handles U_j which are separated from the outside of U_j by annuli with ratios of radii bounded (uniformly in j) from below.

Definition 2.3 (Codegree) *Let L be a line bundle over X that satisfies the hypotheses (L1-2). Associate to X a* compact *Riemann surface X_{bc} by the following procedure:*

- *Remove the inner part $U_j \setminus U_0$ of all handles except possibly a finite number of them;*
- *glue disks to the remaining holes with the identity map as coordinate transition from z_0 to the disk coordinate; this means that the regular sheet is extended to include the "filled" holes;*
- *insert one new point ∞ at infinity of the regular sheet, taking $z_\infty = 1/z_0$ as coordinate near ∞.*

We extend the basis section ψ_{0A} of L over the regular sheet in a trivial manner to the filled holes, identifying $\psi_{0A}|V_j$ with the section $1|V_j$ in the trivial line bundle over the disk.

Similarly, over an open neighbourhood U_∞ of ∞, we glue the trivial line bundle $U_\infty \times \mathbb{C}$ to L over $X_{bc} \setminus \{\infty\}$ by identifying the unit section 1 over $U_\infty \setminus \{\infty\}$ with ψ_{0A} in L over $U_\infty \setminus \{\infty\}$. The resulting line bundle is called $L_{bc}(\psi_{0A})$ or – when there is no risk of confusion, which basis section was taken near ∞ – simply L_{bc}. We define the codegree of L by:

$$\operatorname{codeg}(L, \psi_{0A}) := \operatorname{genus}(X_{bc}) - \deg(L_{bc}).$$

Here $\deg(L_{bc})$ denotes the Chern number of L_{bc}, i. e. the degree of any divisor defining the isomorphism class of L_{bc}.

To be sure that the the codegree is well defined, i. e. does not depend on the choice of the handles which are included in X_{bc}, if only these are sufficiently many, we observe the following:

- On the one hand, for every additional nondegenerate handle U_j that is included in X_{bc}, the genus of X_{bc} increases by 1, while degenerate handles do not change the genus.

- On the other hand, using the hypotheses (L1-2) for the transition functions, we see that the Chern number of L_{bc} also increases by 1 when an additional nondegenerate handle U_j is included in X_{bc}, while degenerate handles do not change the Chern number. The increase of the Chern number by 1 reflects that we have the winding number 1 of the transition function $\psi_{jA}/\psi_{-jA} = \text{const} \cdot z_j$, as described by (L1), around 0 over the cycle $a_j = \{x \in U_j \,:\, |z_j(x)|^2 = t_j\}$.

2.5 Symmetry with respect to dualisation

As basis sections in $\check{L}^{1,0}$ we specify $\psi_{jB} = \psi_{jA}^{-1} \otimes dz_j$, $\psi_{0B} = \psi_{0A}^{-1} \otimes dz_0$. Using $dz_{-j} = -(t_j/z_j^2)\, dz_j$ and $Q_j Q_{-j} = t_j$, we get the dualised versions of the hypotheses (L1), (L2):

(L1$^\vee$) $\psi_{-jB} = \dfrac{Q_{-j}}{z_j} \psi_{jB}$,

(L2$^\vee$) $\dfrac{\psi_{jB}}{\psi_{0B}} = (1 + O(o_j)) g_{jB}$ with $g_{jB} = c_j / g_{jA}$, and

$$\sup_{V_j} \left| \frac{d}{dz_j} \frac{g_{jB} \psi_{0B}}{\psi_{jB}} \right| = \sup_{V_j} \left| \frac{d}{dz_j} \left(\frac{\psi_{jA}}{g_{jA} \psi_{0A}} \cdot c_j \frac{dz_0}{dz_j} \right) \right| \overset{(L2),(X4)}{=} O(o_j)\,.$$

Using (X4) and the bound of $|c_j / c_{-j}|$ in (X6) we have as well:

$$\sup_j \frac{\sup_{V_j} |g_{jB}|}{\inf_{V_{\pm j}} |g_{\pm jB}|} < \infty.$$

Ext-Disk($g_{jB}^{\pm 1}$) holds, too. Hence the situation of the Riemann Roch Theorem is indeed completely symmetric with respect to the exchange $L \leftrightarrow \check{L}^{1,0}$, $A \leftrightarrow B$.

3 Proof of the Theorem

The proof of the Riemann Roch Theorem consists of several steps: In Sect. 3.1 we show that it suffices to consider the order $\nu = -1$ at infinity. Next we concentrate on the Fredholm theory of the Cauchy Riemann operator. To get appropriate weighted L^2 and Sobolev norms on the space of holomorphic sections, a volume form on X and Hermitian metrics on L and related bundles are introduced in Sect. 3.2.1, but their explicit construction will be postponed to Sect. 3.3. We establish hypotheses on an integral kernel which approximates the Cauchy kernel in Sect. 3.2.2. They will guarantee that the corresponding integral operator is a Fredholm operator, and we obtain conditions for the Cauchy Riemann operator to be invertible

(Sect. 3.2.3). These conditions turn out to be fulfilled when a sufficiently large compact piece is removed and is replaced by a disk. The explicit construction of the integral kernel is given in Sect. 3.3.2. This requires the introduction of an interpolation operator through handles, which allows us to interpolate the Cauchy kernel on the regular sheet almost holomorphically through handles. The change of the index of the Cauchy Riemann operator when the compact piece is removed and replaced by a simpler piece is examined in Sect. 3.4. It allows us by cutting and gluing operations to split the infinite genus Riemann surface that we started with into simpler pieces. As the last step in the proof of the theorem, we derive pointwise asymptotic bounds near infinity from L^2 bounds in Sect. 3.5.

3.1 Reducing the order at infinity to $\nu = -1$

The proof of the Riemann Roch Theorem is most easily given for the case $\nu = -1$. The following observation shows that we can reduce the general case to this special situation:

On the left hand side of the Riemann Roch Formula, the replacement of ν by -1 is compensated by the following redefinition of the basis sections:

$$\psi_{0A}^{\sim} = \psi_{0A} \cdot z_0^{\nu+1} , \tag{4}$$

$$\psi_{0B}^{\sim} = \psi_{0B} \cdot z_0^{-(\nu+1)} . \tag{5}$$

We assume that the case (1) in (X3) is excluded. Then one uses the bounds on $|r_j/s_j|$ and on $|s_j/s_{-j}|$ in (X6) to see

$$\sup_j \sup_{x,y\in U_0\cap U_j} \left|\frac{z_0(x)}{z_0(y)}\right| < \infty,$$

which implies the bound (2) for

$$g_{jA}^{\sim} := z_0^{-(\nu+1)} g_{jA} = \frac{\psi_{jA}}{\psi_{0A}^{\sim}} \cdot (1 + O(o_j)).$$

Therefore $g_{jA}^{\sim}$ still fulfills (L2).

The right hand side of the Riemann Roch Formula is reduced to the case $\nu = -1$ by the following lemma:

Lemma 3.1 (Rescaling at infinity)

a) *Let $\mathcal{O}((\nu+1)\infty)$ denote the holomorphic line bundle over X_{bc} associated to the divisor that assigns order $\nu + 1$ to ∞ but 0 to all other points. Then $L_{bc}(\psi_{0A}^{\sim}) \cong L_{bc}(\psi_{0A}) \otimes \mathcal{O}((\nu + 1)\infty)$.*

b) $-\mathrm{codeg}(L, z_0^{\nu+1}\psi_{0A}) = 1 + \nu - \mathrm{codeg}(L, \psi_{0A})$.

Proof:

a) This can be read off immediately from the transition maps for the bundles $L_{bc}(\psi_{0A}^{\sim})$, $L_{bc}(\psi_{0A})$ and $\mathcal{O}((\nu+1)\infty)$ near ∞. Each row in the following tabular shows which sections are glued together:

line bundle	(section over the regular sheet)	(section over U_∞ in the trivial line bundle)
$L_{bc}(\psi_{0A}^{\sim})$	$(z_0^{\nu+1}\psi_{0A}$ in $L)$	(unit section 1)
$L_{bc}(\psi_{0A}^{\sim})$	$(\psi_{0A}$ in $L)$	(section $z_\infty^{\nu+1}$)
$L_{bc}(\psi_{0A})$	$(\psi_{0A}$ in $L)$	(unit section 1)
$\mathcal{O}((\nu+1)\infty)$	(unit section 1 in the trivial line bundle)	(section $z_\infty^{\nu+1}$)

b) This is an immediate consequence of a), when we use

$$\deg\left[L_{bc}(\psi_{0A}) \otimes \mathcal{O}((\nu+1)\infty)\right] = \nu + 1 + \deg L_{bc}(\psi_{0A}) \ .$$

□

For the rest of the proof of the Riemann Roch Theorem, we assume that $\nu = -1$, writing ψ_{0A} instead of $\psi_{0A}^{\sim}$.

3.2 Fredholm theory for the Cauchy–Riemann operator

In this section we establish quite general functional analytic considerations.

3.2.1 Basic notions

We endow L with a Hermitian metric $|\cdot|_A$. We fix a volume form Ω on X with a finite total volume: $\int_X \Omega < \infty$; a specific choice of Ω will be given later. Associated with L, the following line bundles, endowed with Hermitian metrics, will be important for us: the bundle of complex valued $p+q$–forms over X of the type (p,q) is denoted by $\mathcal{E}^{p,q}$, and for any line bundle F we set $F^{p,q} = F \otimes \mathcal{E}^{p,q}$.

Line bundle	$\mathcal{E}^{1,1}$	$\mathcal{E}^{1,0}$	$\mathcal{E}^{0,1}$	L	$\check{L}^{1,0}$	$\check{L}^{1,1}$	$L^{0,1}$
Hermitian metric	$\lvert\cdot\rvert_\Omega$	$\lvert\cdot\rvert_2$	$\lvert\cdot\rvert_2$	$\lvert\cdot\rvert_A$	$\lvert\cdot\rvert_B$	$\lvert\cdot\rvert_{\check{A}}$	$\lvert\cdot\rvert_{\check{B}}$
Norm on global sections	—	$\lVert\cdot\rVert_2$	$\lVert\cdot\rVert_2$	$\lVert\cdot\rVert_A$	$\lVert\cdot\rVert_B$	$\lVert\cdot\rVert_{\check{A}}$	$\lVert\cdot\rVert_{\check{B}}$
L^2 – Hilbert space	—	$L^2(X, \mathcal{E}^{1,0})$	$L^2(X, \mathcal{E}^{0,1})$	A	B	$\check{A}$	$\check{B}$

The norms on global sections are obtained by integrating the squared Hermitian metrics with respect to the volume form, e. g.

$$\|f\|_A^2 = \int_X |f|_A^2 \Omega$$
$$A = \{f \mid f \text{ is a global section in } L,\ \|f\|_A < \infty\} .$$

Specific choices for the Hermitian metrics will be given in a later section; the metrics should be mutually compatible in the following sense:

- $|\Omega|_\Omega = 1$;
- $|\alpha|_2 = |\bar{\alpha}|_2$, $|\alpha\beta|_\Omega = |\alpha|_2|\beta|_2$, for $\alpha \in \mathcal{E}_x^{1,0}$, $\beta \in \mathcal{E}_x^{0,1}$, $\alpha\beta := \alpha \wedge \beta \in \mathcal{E}_x^{1,1}$;
- $|\alpha|_A|\beta|_{\check{A}} = |\alpha\beta|_\Omega$ for $\alpha \in L_x$, $\beta \in \check{L}_x^{1,1}$, $\alpha\beta \in \mathcal{E}_x^{1,1}$;
- $|\alpha|_B|\beta|_{\check{B}} = |\alpha\beta|_\Omega$ for $\alpha \in \check{L}_x^{1,0}$, $\beta \in L_x^{0,1}$, $\alpha\beta \in \mathcal{E}_x^{1,1}$;
- $|\alpha|_A|\beta|_B = |\alpha\beta|_2$ for $\alpha \in L_x$, $\beta \in \check{L}_x^{1,0}$, $\alpha\beta \in \mathcal{E}_x^{1,0}$.

By these compatibility relations for the various Hermitian metrics, we may view $(A, \check{A})$ and also $(B, \check{B})$ as dual pairs of Hilbert spaces via the bilinear forms

$$(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle := \int_X \alpha\beta \in \mathbb{C} .$$

Finally, for $f \in \mathcal{C}_c^\infty(X, L)$, let $\|f\|_V^2 := \|f\|_A^2 + \left\|\overline{\partial} f\right\|_{\check{B}}^2$, with $\overline{\partial} : \mathcal{C}_c^\infty(X, L) \to \mathcal{C}_c^\infty(X, L^{0,1})$ denoting the Cauchy–Riemann operator. Let V be the completion of $\mathcal{C}_c^\infty(X, L)$ with respect to the norm $\|\cdot\|_V$. Then the inclusion map $i : \mathcal{C}_c^\infty(X, L) \overset{\subseteq}{\longrightarrow} A$ and the Cauchy–Riemann operator $\overline{\partial}$ obviously extend to bounded linear operators $i_V : V \to A$ and $\overline{\partial}_V : V \to \check{B}$.

Lemma 3.2 (Properties of the inclusion $V \to A$)

i) For every $f \in V$, $\|f\|_V^2 = \|i_V f\|_A^2 + \left\|\overline{\partial}_V f\right\|_{\check{B}}^2$.
ii) i_V is injective; its image is dense in A.

Proof:

i) This is obvious for $f \in \mathcal{C}_c^\infty(X, L)$, which is dense in V, and both sides are continuous functions of $f \in V$.

ii) Let $f \in \ker(i_V)$. Choose $\mathcal{C}_c^\infty(X, L) \ni f_n \overset{n\to\infty}{\longrightarrow} f$ in V. Then $\overline{\partial} f_n \overset{n\to\infty}{\longrightarrow} \overline{\partial}_V f$ in $\check{B}$, therefore we have for all test sections $g \in \mathcal{C}_c^\infty(X, \check{L}^{1,0}) \subseteq B$:

$$\left\langle g, \overline{\partial}_V f \right\rangle \overset{n\to\infty}{\longleftarrow} \int_X g\, \overline{\partial} f_n = \int_X f_n\, \overline{\partial} g = \left\langle \overline{\partial} g,\ f_n \right\rangle .$$

Here, Stokes theorem is applicable since g is compactly supported. Now $\overline{\partial} g \in \mathcal{C}_c^\infty(X, \check{L}^{1,1}) \subseteq \check{A}$ implies $\left|\left\langle \overline{\partial} g,\ f_n \right\rangle\right| \le \left\|\overline{\partial} g\right\|_{\check{A}} \|f_n\|_A \overset{n\to\infty}{\longrightarrow} 0$, where we used $f_n = i_V f_n \overset{n\to\infty}{\longrightarrow} i_V f = 0$ in A. Consequently $\langle g, \overline{\partial}_V f\rangle = 0$, and therefore $\overline{\partial}_V f = 0$, since $g \in \mathcal{C}^\infty_c(X, \check{L}^{1,0}) \overset{\text{dense}}{\subseteq} B$. Together with $i_V f = 0$ we get $f = 0$ by i). In addition, $i_V V \supseteq \mathcal{C}^\infty_c(X, L) \overset{\text{dense}}{\subseteq} A$, therefore $i_V V$ is dense in A. □

We therefore view V as a dense subset of A via i_V. The relation $\int g\overline{\partial} f = \int f\overline{\partial} g$ for $g \in \mathcal{C}^\infty_c(X, \check{L}^{1,0})$, $f \in \mathcal{C}^\infty_c(X, L)$ together with $g \in \mathcal{C}^\infty_c(X, \check{L}^{1,0}) \overset{\text{dense}}{\subseteq} B$ imply:

Lemma 3.3 *The dual operator* $\overline{\partial}_V^\vee : B \to \check{V}$ *of* $\overline{\partial}_V : V \to \check{B}$ *equals the bounded extension of the Cauchy–Riemann operator* $\overline{\partial} : \mathcal{C}^\infty_c(X, \check{L}^{1,0}) \to \mathcal{C}^\infty_c(X, \check{L}^{1,1})$. □

It may be helpful to keep the following diagram in mind:

$$\begin{array}{ccccc} & \text{over } L & & \text{over } L^{0,1} & \\ A & \overset{\supseteq}{\longleftarrow} & V & \overset{\overline{\partial}_V}{\longrightarrow} & \check{B} \\ \times & & \times & & \times \\ \check{A} & \overset{\subseteq}{\longrightarrow} & \check{V} & \overset{\overline{\partial}_V^\vee}{\longleftarrow} & B \\ & \text{over } \check{L}^{1,1} & & \text{over } \check{L}^{1,0} & \end{array}$$

3.2.2 Hypotheses on the integral kernel

Our next goal is to establish sufficient conditions for $\overline{\partial}_V$ to be a Fredholm operator. As a quasiinverse, we use an integral operator $g \mapsto \int_{y\in X} g(y)K(\cdot, y)$. We first state some hypotheses on the integral kernel K; in a second step, we have to check that an integral kernel satisfying these hypotheses exists.

Hypotheses K

(K1) K is a section in the external tensor product bundle $L \otimes_{X\times X} \check{L}^{1,0}$ (fibre $L_x \otimes \check{L}^{1,0}_y$ over $(x, y) \in X \times X$). K is defined and $\mathcal{C}^\infty$ outside the diagonal.

(K2) **(K approximates the Cauchy kernel.)** X can be covered with open sets U which are domains of coordinates $z : U \to \mathbb{C}$, and there are basis sections $\omega \circ z \in \mathcal{O}(U, L)$ so that

$$K - \frac{1}{2\pi i}\frac{1}{z_1 - z_2}\omega(z_1) \otimes \omega^{-1}(z_2)\, dz_2$$

has a $\mathcal{C}^\infty$–extension to $U \times U$ (including the diagonal). Here $z_1, z_2 : U \times U \to \mathbb{C}$ denote the coordinates on $U \times U$, and $\omega^{-1} \in \mathcal{O}(U, \check{L})$ is the basis section dual to ω.

To state the last hypotheses (K3-4) for K, we define Hermitian metrics on the external tensor products:

$$
\begin{aligned}
|\cdot|_{AB}: &\quad L \otimes_{X\times X} \check{L}^{1,0} \to \mathbb{R}\,,\\
|\cdot|_{\check{B}B}: &\quad L^{0,1} \otimes_{X\times X} \check{L}^{1,0} \to \mathbb{R}\,,\\
|\cdot|_{A\check{A}}: &\quad L \otimes_{X\times X} \check{L}^{1,1} \to \mathbb{R}
\end{aligned}
$$

by

$$
\begin{aligned}
|\alpha\otimes\beta|_{AB} &= |\alpha|_A|\beta|_B\,, \quad (\alpha\in L_x,\ \beta\in \check{L}_y^{1,0})\,,\\
|\alpha\otimes\beta|_{\check{B}B} &= |\alpha|_{\check{B}}|\beta|_B\,, \quad (\alpha\in L_x^{0,1},\ \beta\in \check{L}_y^{1,0})\,,\\
|\alpha\otimes\beta|_{A\check{A}} &= |\alpha|_A|\beta|_{\check{A}}\,, \quad (\alpha\in L_x,\ \beta\in \check{L}_y^{1,1})\,.
\end{aligned}
$$

Then we suppose

(K3) **(Both L^∞–L^1–norms of K are finite.)**

$$
\begin{aligned}
k_1 &:= \sup_{x\in X}\int_{y\in X} |K(x,y)|_{AB}\Omega_y < \infty\,,\\
k_2 &:= \sup_{y\in X}\int_{x\in X} |K(x,y)|_{AB}\Omega_x < \infty\,.
\end{aligned}
$$

(K4) **(Finiteness of Hilbert–Schmidt norms)**

$$
\begin{aligned}
h_B^2 &:= \iint_{x\neq y} |\overline{\partial}_x K(x,y)|^2_{\check{B}B}\Omega_x\Omega_y < \infty\,,\\
h_A^2 &:= \iint_{x\neq y} |\overline{\partial}_y K(x,y)|^2_{A\check{A}}\Omega_x\Omega_y < \infty\,.
\end{aligned}
$$

3.2.3 Consequences of the hypotheses K

As a first consequence, we get

Lemma 3.4 *$\mathcal{K}_c : \mathcal{C}_c^\infty(X, L^{0,1}) \to \mathcal{C}^\infty(X, L)$, $g \mapsto \int_{y\in X} g(y)K(\cdot, y)$ is well defined.*

Proof: $g(y) \in L_y^{0,1}$, $K(x, y) \in L_x \otimes \check{L}_y^{1,0}$ implies $g(y)K(x, y) \in L_x \otimes \mathcal{E}_y^{1,1}$, therefore

$$
\int_{y\in X} g(y)K(x,y) \in L_x
$$

is well defined. To check that it depends smoothly on x, it suffices to assume that supp g is contained in the domain U of a coordinate, using a partition of unity. The integral splits for $x \in U$ into an integral with a $\mathcal{C}^\infty$–kernel and a convolution integral with the Cauchy kernel because of (K2); and for $x \notin \operatorname{supp} g$, the integral kernel $K(x, y)$, $y \in \operatorname{supp} g$, is $\mathcal{C}^\infty$. □

The next lemma follows from the standard argument that shows L^2–operator norms being bounded by L^∞–L^1–norms:

Lemma 3.5 (L^2–operator norms)

a) *Let F be a $L \otimes_{X\times X} \check{L}^{1,0}$–valued (resp. $L \otimes_{X\times X} \check{L}^{1,1}$–valued) integral kernel with finite L^∞–L^1–norms, i. e.*

$$c_1 = \sup_x \int_y |F(x, y)|_{AC} \Omega_y < \infty \ ,$$
$$c_2 = \sup_y \int_x |F(x, y)|_{AC} \Omega_x < \infty$$

with the notation $C = B$ (resp. $C = \check{A}$). Let $\mathcal{F} : g \mapsto \int_y F(\cdot, y)g(y)$ be the corresponding integral operator, g being a square integrable section in $L^{0,1}$ (resp. L). Then the L^2–operator norm of $\mathcal{F}$ is bounded by $\sqrt{c_1 c_2}$.

b) *$\mathcal{K}_c$ extends to a bounded linear map $\mathcal{K}_A : \check{B} \to A$. Its operator norm is bounded by $\sqrt{k_1 k_2}$.* □

Definition 3.6 (Deviation of K from holomorphy)

i) *Let $H_B \in \mathcal{C}^\infty(X \times X, L^{0,1} \otimes \check{L}^{1,0})$ be the section $H_B(x, y) := \overline{\partial}_x K(x, y)$ for $x \neq y$, using (K2) extended to the diagonal.*

ii) *Similarly let $H_A \in \mathcal{C}^\infty(X \times X, L \otimes \check{L}^{1,1})$ be given by $H_A(x, y) := -\overline{\partial}_y K(x, y)$.*

The hypothesis (K4) bounds the Hilbert–Schmidt norm of these two integral kernels. Therefore we get

Lemma 3.7 (Hilbert–Schmidt property) *The maps $\mathcal{H}_B : \check{B} \to \check{B}$, $g \mapsto \int_y g(y)H_B(\cdot, y)$ and $\mathcal{H}_A : A \to A$, $g \mapsto \int_y g(y)H_A(\cdot, y)$ are well defined Hilbert–Schmidt operators with the Hilbert–Schmidt norm $\|\mathcal{H}_B\|_{\text{HS}} = h_B$, $\|\mathcal{H}_A\|_{\text{HS}} = h_A$.*

Proof/Reference: $g(y) \in L_y^{0,1}$, $H_B(x, y) \in L_x^{0,1} \otimes \check{L}_y^{1,0}$ implies $g(y)H_B(x, y) \in L_x^{0,1} \otimes \mathcal{E}_y^{1,1}$ and $f(y) \in L_y$, $H_A(x, y) \in L_x \otimes \check{L}_y^{1,1}$ implies $f(y)H_A(x, y) \in L_x \otimes \mathcal{E}_y^{1,1}$. The Hilbert–Schmidt property of $\mathcal{H}_A$ and $\mathcal{H}_B$ follows in case of trivial line bundles and weight functions 1 for the Hermitian metrics e. g. from [13], Theorem VI.23, and the proof is similar for arbitrary line bundles. □

Next we shall see that $\mathcal{K}_c$ is a quasiinverse of $\overline{\partial}$ up to a Hilbert–Schmidt operator.

Lemma 3.8 (A parametrix for the Cauchy–Riemann operator – $\mathcal{C}^\infty$-version)

i) For $g \in \mathcal{C}_c^\infty(X, L^{0,1})$, $f = \mathcal{K}_c g \in \mathcal{C}^\infty(X, L)$, *we have*

$$\overline{\partial} f = g + \mathcal{H}_B g \in \mathcal{C}_c^\infty(X, L^{0,1}) \subseteq \check{B} \ .$$

ii) For $f \in \mathcal{C}_c^\infty(X, L)$, $g = \overline{\partial} f \in \mathcal{C}_c^\infty(X, L^{0,1})$, *there holds*

$$\mathcal{K}_c g = f + \mathcal{H}_A f \in \mathcal{C}^\infty(X, L) \cap A \ .$$

Proof:

i) We may cut g into several pieces using a partition of unity; therefore we assume w. l. o. g. that supp g is contained in the domain of a coordinate function $x : U \to \mathbb{C}$. To calculate $\overline{\partial} f(x)$ we have to distinguish two cases:

Case 1: If $x \notin \operatorname{supp} g$, we differentiate under the integral and get

$$\overline{\partial} f(x) = \int_{y \in U} g(y) \overline{\partial}_x K(x, y) = \mathcal{H}_B g(x) = g(x) + \mathcal{H}_B g(x) \ .$$

Case 2: If $x \in U$, we use that

$$K - \frac{1}{2\pi i} \frac{1}{z_1 - z_2} \omega(z_1) \otimes \omega^{-1}(z_2)\, dz_2$$

extends to a $\mathcal{C}^\infty$–section K' over $U \times U$ by (K2), and $H_B(x, y) = \overline{\partial}_x K'(x, y)$. Therefore

$$\overline{\partial} f(x) = \overline{\partial}_x \int_{y \in U} g(y) \frac{1}{2\pi i} \frac{1}{z(x) - z(y)} \omega(z(x)) \otimes \omega^{-1}(z(y))\, dz(y) + \mathcal{H}_B g(x)$$

holds. The first summand equals $g(x)$ since the Cauchy kernel is a fundamental solution of the Cauchy–Riemann operator.

ii) For every test section $h \in \mathcal{C}_c^\infty(X, \check{L}^{1,1})$ we have

$$\begin{aligned} \langle h, \mathcal{K}_c g \rangle &= \int_x h(x) \int_y (\overline{\partial} f)(y) K(x, y) \\ &= \int_y (\overline{\partial} f)(y) \int_x h(x) K(x, y) \\ &\overset{\text{Stokes}}{=} - \int_y f(y) \overline{\partial}_y \int_x h(x) K(x, y) \ . \end{aligned}$$

Similarly as in i) we see that

$$\overline{\partial}_y \int_x h(x) K(x, y) = -h(y) - \int_x h(x) H_A(x, y) \ .$$

(The first minus sign arises from the reversed roles of $z(x)$, $z(y)$ in the Cauchy kernel.) Therefore we get

$$\langle h, \mathcal{K}_c g\rangle = \langle h, f\rangle + \int_y f(y)\int_x h(x)H_A(x,y) = \langle h, f\rangle + \langle h, \mathcal{H}_A f\rangle \ ,$$

which implies the assertion, since $h \in \mathcal{C}_c^\infty(X, \check{L}^{1,1}) \overset{\text{dense}}{\subseteq} \check{A}$. □

Corollary 3.9 (Bounds for $\overline{\partial}\mathcal{K}_c$ – $\mathcal{C}^\infty$-version) *For $g \in \mathcal{C}_c^\infty(X, \check{L}^{0,1})$, $f := \mathcal{K}_c g \in \mathcal{C}^\infty(X, L) \cap A$ we have $\left\|\overline{\partial} f\right\|_{\check{B}} \le (1 + h_B)\,\|g\|_{\check{B}}$.*

Proof: $\left\|\overline{\partial} f\right\|_{\check{B}} = \|g + \mathcal{H}_B g\|_{\check{B}} \le \|g\|_{\check{B}} + \|\mathcal{H}_B\|_{\mathrm{HS}}\,\|g\|_{\check{B}}$. □

Even though we know for $g \in \mathcal{C}_c^\infty(X, L^{0,1})$ both $\mathcal{K}_c g \in \mathcal{C}^\infty(X, L) \cap A$ and $\overline{\partial}\mathcal{K}_c g \in \check{B}$, we have not yet shown $\mathcal{K}_c g \in V$. To ensure this, we have to approximate $\mathcal{K}_c g$ by sections with compact support:

Lemma 3.10 (Extension of the integral operator to $\check{B}$)
i) $g \in \mathcal{C}_c^\infty(X, L^{0,1})$ implies $\mathcal{K}_c g \in V$.
ii) $\mathcal{K}_c : \mathcal{C}_c^\infty(X, L^{0,1}) \to V$ extends to a bounded linear map $\mathcal{K} : \check{B} \to V$.

Proof:
i) For $f = \mathcal{K}_c g$, we only have to show:

$$\forall \epsilon > 0\ \exists h \in \mathcal{C}_c^\infty(X, L):\ \|f - h\|_A^2 + \left\|\overline{\partial}(f - h)\right\|_{\check{B}}^2 \le \epsilon\ .$$

Choose $N \subset\subset X$ large enough so that

$$\int_{N^c} |f|_A^2 \Omega \le \frac{\epsilon}{2}\ , \quad \int_{N^c} |\overline{\partial} f|_{\check{B}}^2 \Omega \le \frac{\epsilon}{8}\ , \tag{6}$$

and by (X5) a $\chi : X \to [0,1]$, $\chi|N = 1$, $\chi \in \mathcal{C}_c^\infty(X)$ in such a way that

$$\left\|\overline{\partial}\chi\right\|_2^2 \le \frac{\epsilon}{R}\ , \tag{7}$$

where $R := 8k_1^2 \sup_{y\in X} |g(y)|_{\check{B}}^2 \overset{(K3)}{<} \infty$. We use a cutoff version $h := \chi f$ of f. Then $\|f - h\|_A^2 = \|(1-\chi) f\|_A^2 \le \int_{N^c} |f|_A^2 \Omega \le \frac{\epsilon}{2}$ and

$$\begin{aligned}
\left\|\overline{\partial}(f - h)\right\|_{\check{B}}^2 &= \int_{N^c} \left|(-\overline{\partial}\chi) f + (1-\chi)\overline{\partial} f\right|_{\check{B}}^2 \Omega \\
&\le 2\int_{N^c} \left(|\overline{\partial}\chi|_2^2 |f|_A^2 + |\overline{\partial} f|_{\check{B}}^2\right)\Omega \\
&\overset{(6)}{\le} 2 \sup_{x\in N^c} |f(x)|_A^2 \int_{N^c} |\overline{\partial}\chi|_2^2 \Omega + \frac{\epsilon}{4} \\
&\overset{(7)}{\le} \frac{2\epsilon}{R} \sup_{x\in N^c} |f(x)|_A^2 + \frac{\epsilon}{4}\ .
\end{aligned} \tag{8}$$

We estimate:

$$\sup_{x\in N^c} |f(x)|_A \le \sup_{x\in X} \left| \int_{y\in X} g(y)K(x,y) \right|_A$$
$$\le \sup_{x\in X} \int_{y\in X} |g(y)|_{\check{B}} |K(x,y)|_{AB} \Omega_y \le k_1 \sup_{y\in X} |g(y)|_{\check{B}} = \left(\frac{R}{8}\right)^{\frac{1}{2}} .$$

If we insert this in (8), we get $\left\|\overline{\partial}(f-h)\right\|_{\check{B}}^2 \le \frac{\epsilon}{4}+\frac{\epsilon}{4}$, therefore $\|f-h\|_A^2 + \left\|\overline{\partial}(f-h)\right\|_{\check{B}}^2 \le \epsilon$.

ii) For $g \in \mathcal{C}_c^\infty(X, L^{0,1})$ we get $\mathcal{K}_c g \in \mathcal{C}^\infty(X,L)\cap V$, therefore $\overline{\partial}\mathcal{K}_c g = \overline{\partial}_V \mathcal{K}_c g$ and

$$\|\mathcal{K}_c g\|_V^2 = \|\mathcal{K}_c g\|_A^2 + \left\|\overline{\partial}\mathcal{K}_c g\right\|_{\check{B}}^2 \overset{3.9, 3.5}{\le} k_1 k_2 \, \|g\|_{\check{B}}^2 + (1+h_B)^2 \, \|g\|_{\check{B}}^2 .$$

The assertion follows from $\mathcal{C}_c^\infty(X, L^{0,1}) \overset{\text{dense}}{\subseteq} \check{B}$. □

Lemma 3.8 and Lemma 3.10 imply

Corollary 3.11 $\overline{\partial}_V \mathcal{K} = \mathrm{id}_{\check{B}} + \mathcal{H}_B$. □

Lemma 3.12 (Restriction to V) *$\mathcal{H}_A : A \to A$ has a bounded restriction $\mathcal{H}_V : V \to V$, its operator norm satisfies $\|\mathcal{H}_V\| \le h$, where $h := \max\{h_A, h_B\}$.*

Proof: It suffices to show: $\forall f \in \mathcal{C}_c^\infty(X,L) : \mathcal{H}_A f \in V$, $\|\mathcal{H}_A f\|_V \le h \, \|f\|_V$. To see this, we calculate

$$\mathcal{H}_A f \overset{3.8}{=} \mathcal{K}_c \overline{\partial} f - f = \mathcal{K}\overline{\partial}_V f - f \in V ,$$

because of $\overline{\partial}_V : V \to \check{B}$ and $\mathcal{K} : \check{B} \to V$. Moreover $\|\mathcal{H}_A f\|_V^2 \overset{3.2}{=} \|\mathcal{H}_A f\|_A^2 + \left\|\overline{\partial}_V \mathcal{H}_A f\right\|_{\check{B}}^2$. But $\|\mathcal{H}_A f\|_A \overset{3.7}{\le} h_A \, \|f\|_A$ and

$$\begin{aligned} \left\|\overline{\partial}_V \mathcal{H}_A f\right\|_{\check{B}}^2 &\overset{3.8\text{ii)}}{=} \left\|\overline{\partial}_V (\mathcal{K}\overline{\partial}_V - \mathrm{id}_V) f\right\|_{\check{B}} \\ &= \left\|(\overline{\partial}_V \mathcal{K} - \mathrm{id}_{\check{B}})\overline{\partial}_V f\right\|_{\check{B}} \\ &\overset{3.11}{=} \left\|\mathcal{H}_B \overline{\partial}_V f\right\|_{\check{B}} \overset{3.7}{\le} h_B \left\|\overline{\partial}_V f\right\|_{\check{B}} \end{aligned}$$

together lead to $\|\mathcal{H}_A f\|_V^2 \le h_A^2 \, \|f\|_A^2 + h_B^2 \left\|\overline{\partial}_V f\right\|_{\check{B}}^2 \overset{3.2}{\le} h^2 \, \|f\|_V^2$. □

Lemma 3.13 $\mathcal{K}\overline{\partial}_V = \mathrm{id}_V + \mathcal{H}_V$.

Proof: Both sides are bounded operators $V \to V$, and for $f \in \mathcal{C}_c^\infty(X, L) \overset{\text{dense}}{\subseteq} V$ we get $\mathcal{K}\overline{\partial}_V f = f + \mathcal{H}_V f$ by Lemma 3.8 ii). □

Lemma 3.14 *$\mathcal{H}_V : V \to V$ is a compact operator.*

Proof: Let (f_n) be a bounded sequence in V.

$$\overline{\partial}_V \mathcal{H}_V = \overline{\partial}_V \mathcal{K} \overline{\partial}_V - \overline{\partial}_V = \mathcal{H}_B \overline{\partial}_V$$

is a compact operator since $\overline{\partial}_V$ is bounded and $\mathcal{H}_B$ is compact as a consequence of Lemma 3.7. We choose a subsequence (again called (f_n)) such that $(\overline{\partial}_V \mathcal{H}_V f_n)$ converges in $\check{B}$. (f_n), viewed as a sequence in $A \supseteq V$, is bounded and $\mathcal{H}_A : A \to A$ is a compact operator by Lemma 3.7. Hence for some subsequence (still called (f_n)), $(\mathcal{H}_A f_n)$ converges in A. Using

$$\|\mathcal{H}_V(f_n - f_m)\|_V^2 = \|\mathcal{H}_A(f_n - f_m)\|_A^2 + \left\|\overline{\partial}_V \mathcal{H}_V(f_n - f_m)\right\|_{\check{B}}^2 ,$$

we find out that $(\mathcal{H}_V f_n)$ is a Cauchy sequence in V, therefore convergent. □

Summarising the above we end up with

Theorem 3.15 (Fredholm property for the Cauchy–Riemann operator) *The hypotheses* K *imply that $\overline{\partial}_V : V \to \check{B}$ is a Fredholm operator. It has a quasiinverse $\mathcal{K} : \check{B} \to V$ such that $\mathcal{H}_V = \mathcal{K}\overline{\partial}_V - \mathrm{id}_V$ has the operator norm $\|\mathcal{H}_V\| \le \max\{h_A, h_B\}$ and is a compact operator, $\mathcal{H}_B = \overline{\partial}_V \mathcal{K} - \mathrm{id}_{\check{B}}$ has the operator norm $\|\mathcal{H}_B\| = h_B$ and is a Hilbert–Schmidt operator.* □

Corollary 3.16 (Invertibility of the Cauchy–Riemann operator) *If in addition $h_A < 1$ and $h_B < 1$, then $\overline{\partial}_V : V \to \check{B}$ has a bounded inverse.*

Proof: The additional hypotheses guarantee the convergence of the von Neumann series

$$(\mathcal{K}\overline{\partial}_V)^{-1} = \sum_{n=0}^{\infty} (-\mathcal{H}_V)^n \quad \text{and} \quad (\overline{\partial}_V \mathcal{K})^{-1} = \sum_{n=0}^{\infty} (-\mathcal{H}_B)^n \ ;$$

in this case $\overline{\partial}_V(\mathcal{K}(\overline{\partial}_V \mathcal{K})^{-1}) = \mathrm{id}_{\check{B}}$ and $((\mathcal{K}\overline{\partial}_V)^{-1}\mathcal{K})\overline{\partial}_V = \mathrm{id}_V$. □

Theorem 3.15 and Lemma 3.3 immediately imply:

Corollary 3.17 (Dualised Cauchy–Riemann operator)

i) $\overline{\partial} : \mathcal{C}_c^\infty(X, \check{L}^{1,0}) \to \mathcal{C}_c^\infty(X, \check{L}^{1,1})$ *extends to a bounded operator* $\overline{\partial}_B : B \to \check{V}$.

ii) $\overline{\partial}_B$ *is the dual operator to* $\overline{\partial}_V$*. In particular, it is a Fredholm operator.* □

As a consequence, we remark:

Proposition 3.18 (Serre duality)

$$\operatorname{index}(\overline{\partial}_B : B \to \check{V}) = -\operatorname{index}(\overline{\partial}_V : V \to \check{B}) .$$

There are canonical nondegenerate bilinear forms

$$(\mathcal{O}(X, L) \cap V) \times \operatorname{coker} \overline{\partial}_B \to \mathbb{C} ,$$
$$(\mathcal{O}(X, \check{L}^{1,0}) \cap B) \times \operatorname{coker} \overline{\partial}_V \to \mathbb{C} .$$

Proof: By Weyl's lemma, $\ker \overline{\partial}_V \subseteq \mathcal{O}(X, L)$ and $\ker \overline{\partial}_B \subseteq \mathcal{O}(X, \check{L}^{1,0})$. Since $\overline{\partial}_B$ and $\overline{\partial}_V$ are Fredholm operators, range $\overline{\partial}_B \subseteq \check{V}$ is closed and range $\overline{\partial}_V$ is closed too, so $\mathcal{O}(X, L) \cap V = \ker \overline{\partial}_V \overset{\sim}{=} (\operatorname{coker} \overline{\partial}_B)^\vee$ and $\mathcal{O}(X, \check{L}^{1,0}) \cap B = \ker \overline{\partial}_B \overset{\sim}{=} (\operatorname{coker} \overline{\partial}_V)^\vee$. □

This formulation of Serre duality does not look symmetric under the exchange $L \leftrightarrow \check{L}^{1,0}$, $A \leftrightarrow B$, since we have a Sobolev space V on one side, but a L^2–Hilbert space B on the other side, and we have not proved $\mathcal{O}(X, L) \cap V = \mathcal{O}(X, L) \cap A$ in the preceding. Note that this equation states that every holomorphic $f \in A$ can be approximated in V by sections with compact support. It will turn out to be a surprising consequence of the Riemann–Roch Theorem.

3.3 Construction of a integral kernel

3.3.1 Specification of Hermitian metrics and a volume form

We now specify the Hermitian metrics $|\cdot|_A$ and $|\cdot|_B$ and the volume form Ω. A change by bounded factors of the weights chosen is irrelevant for our purposes, so we even define Ω, $|\cdot|_A$ and $|\cdot|_B$ only modulo an equivalence relation $\sim$, where $f \sim g$ means that $\frac{f}{g}$ and $\frac{g}{f}$ are uniformly bounded by a constant not depending on the index of a handle.

The definitions are given separately for the various pieces of the surface. Of course, we can patch the pieces together by using a partition of unity. We have to ensure the compatibility relations stated in Sect. 3.2.1.

- On the compact piece, choose any continuous volume form Ω and continuous Hermitian metrics $|\cdot|_A$, $|\cdot|_B$ compatible with Ω. Two choices surely agree modulo $\sim$ on compact sets.
- On the regular sheet, we take, with the abbreviation $d^2z := d\operatorname{Re} z \wedge d\operatorname{Im} z$,

$$\Omega \sim (1 + |z_0|^4)^{-1} d^2 z_0 \overset{\substack{\text{outside} \\ \text{a compact} \\ \text{piece}}}{\sim} |z_0|^{-4}\, d^2 z_0 .$$

This choice is motivated by the standard volume form on the Riemann sphere. The choice of Ω determines the Hermitian metric on one–forms:

$$|dz_0|_2 \sim (1+|z_0|^4)^{\frac{1}{2}} \overset{\substack{\text{outside}\\ \text{a compact}\\ \text{piece}}}{\sim} |z_0|^2 \,.$$

We further take

$$|\psi_{0A}|_A \sim (1+|z_0|^4)^{\frac{1}{4}} \overset{\substack{\text{outside}\\ \text{a compact}\\ \text{piece}}}{\sim} |z_0| \,,$$
$$|\psi_{0B}|_B \sim (1+|z_0|^4)^{\frac{1}{4}} \overset{\substack{\text{outside}\\ \text{a compact}\\ \text{piece}}}{\sim} |z_0| \,.$$

- On the j^{th} handle, we take for nondegenerate handles

$$|\psi_{jA}|_A \sim \alpha_{jA}\min\left\{1, \frac{|z_j|}{|Q_j|}\right\} \,, \tag{9}$$
$$|\psi_{jB}|_B \sim \alpha_{jB}\min\left\{1, \frac{|z_j|}{|Q_{-j}|}\right\} \tag{10}$$

with constants

$$\alpha_{jA} = |g_{jA}(P_j)s_j| \overset{(X6),(2)}{\sim} \alpha_{-jA} \,, \tag{11}$$
$$\alpha_{jB} = |g_{jB}(P_j)s_j| \sim \alpha_{-jB} \,, \tag{12}$$

with any point P_j in V_j. They are related by $\alpha_{jA}\alpha_{jB} \overset{(L2^\vee)}{=} |c_j||s_j|^2 = \alpha_j$. The bounds (2) in (L2) guarantee that the definition of α_{jA} and α_{jB} depends only up to bounded factors on the choice of P_j. To define Ω consistently with $|\cdot|_A$ and $|\cdot|_B$ we now have to take

$$\Omega \sim \alpha_j^{-2}\min\left\{1, \frac{|z_j|}{|Q_j|}\right\}^{-4} d^2z_j, \quad |dz_j|_2 \sim \alpha_j\min\left\{1, \frac{|z_j|}{|Q_j|}\right\}^2$$

since we need $|\psi_{jA}|_A^2|\psi_{jB}|_B^2 = |dz_j|_2^2 \sim |d^2z_j|_\Omega$. We used $|Q_j| \sim |Q_{-j}|$ from (L1). For degenerate handles, we simply take $|\psi_{jA}|_A \sim \alpha_{jA}$, $|\psi_{jA}|_A \sim \alpha_{jB}$, $\Omega \sim \alpha_j^{-2}\, d^2z_j$ on D_j.

This finishes the choice of the volume form and the Hermitian metrics.

Remarks: The definition is indeed symmetric under the reversion of the handles $j \leftrightarrow -j$. The different expressions for Ω, $|\cdot|_A$, $|\cdot|_B$ given in the regular sheet and the handles coincide in the common domain $U_0 \cap U_j$ up to a bounded factor. This is a consequence of the bounds on the bundle transition functions described in (L1-2).

3.3.2 Construction of the integral kernel K

In this section we assume that the compact piece is empty; it will be glued to the regular sheet in a later section. K will be defined by patching several pieces together:

$$K(x, y) = \sum_{a,b\in\{0,h\}} \chi_a(x)\chi_b(y)K_{ab}(x, y)$$

with appropriately chosen K_{ab}. Here we choose once and for all a $\mathcal{C}^\infty$–partition of unity $(\chi_0, \chi_h = \sum_{j\in J'} \chi_j)$ compatible with the open cover $(U_0, U_h = \bigcup_{j\in J} U_j)$; $J' \subseteq J$ denotes an index set listing every handle only once. More precisely, $\chi_j = \chi_{-j}$, supp $\chi_j \subseteq U_j$, is chosen as $\chi_j = \rho(|z_j|)$ on V_j with a $\mathcal{C}^\infty$–function $\rho : \mathbb{R}^+ \to [0, 1]$ that equals 0 resp. 1 for small resp. large arguments. We remark for repeated future use:

Lemma 3.19 (A bound for the cutoff functions)

$$\sup_{V_j} |d\chi_j|_2 \le \sup |\rho'| \sup_{V_j} |dz_j|_2 \le O(\alpha_j)\ . \qquad \square$$

We now construct the pieces K_{ab} of K:

K_{00} is simply defined to be the Cauchy kernel

$$K_{00}(x, y) := \frac{1}{2\pi i}\frac{\psi_{0A}(x)\psi_{0B}(y)}{z_0(x) - z_0(y)} = \frac{1}{2\pi i}\frac{\psi_{0A}(x)}{\psi_{0A}(y)}\frac{dz_0(y)}{z_0(x) - z_0(y)}\ .$$

We also define a Cauchy kernel in the handles (symmetric under $j \leftrightarrow -j$): For $x, y \in U_j$ (nondegenerate case) or $x, y \in D_j$ (degenerate case) we set

$$C_h(x, y) := \frac{1}{2\pi i}\frac{\psi_{jA}(x)}{\psi_{jA}(y)}\frac{dz_j(y)}{z_j(x) - z_j(y)}$$

For x, y in different handles or (in the degenerate case) $x \in D_j$, $y \in D_{-j}$ we take $C_h(x, y) := 0$. Unfortunately, C_h is not close enough to K_{00} to be suitable directly as K_{hh}, mainly for two reasons:

- In general, g_{jA} is not a constant, giving rise to error terms larger than $O(o_j)$ from the basis change $\psi_{0A} \rightsquigarrow \psi_{jA}$.
- $|K_{00}(x, y)|_{AB}$ is not close to zero for $x \in V_j$, $y \in V_{-j}$, while $|C_h(x, y)|_{AB}$ is very small there.

However, C_h plays an essential role in the definition of the following "operation of interpolation through handles" I, which will allow us to interpolate K_{00} between V_j and V_{-j}:

Definition 3.20 (Interpolation through handles) *For meromorphic sections f in L over $V_j \cup V_{-j} = U_0 \cap U_j$ with at most a finite number of* *poles of order at most* 1 *and* $\int_{U_0 \cap U_j} |f|_A \Omega < \infty$ *we define the "interpolated section" If in L over the whole handle U_j by*

$$If(x) = f(x)\chi_0(x) - \int_{z \in V_j \cup V_{-j}} f(z) C_h(x,z) \wedge d\chi_j(z) + 2\pi i \sum_{z \in V_j \cup V_{-j}} \chi_j(z)(\operatorname{res}_z f) C_h(x,z) \ .$$

The analogous definition is taken to define If for sections f over $\check{L}^{1,0}$, with reversed roles of $A \leftrightarrow B$. The residue of f is a well defined element of $(\check{L}^{1,0}_z)^\vee = L_z \otimes (\mathcal{E}^{1,0}_z)^\vee$, so $(\operatorname{res}_z f) C_h(x,z) \in L_x$, as required.

The following lemma states that C_h has finite L^∞–L^1–norms, which shows us that If is well defined.

Lemma 3.21 (Bounds for C_h) *The expressions*

$$\alpha_j \sup_{x \in U_j} \int_{y \in U_j} |C_h(x,y)|_{AB} \Omega_y \quad \text{and} \quad \alpha_j \sup_{y \in U_j} \int_{x \in U_j} |C_h(x,y)|_{AB} \Omega_x$$

are uniformly bounded as functions of j. In the case of degenerate handles, U_j may be replaced by D_j.

We only give the *Proof* for nondegenerate handles. For degenerate handles it is in fact simpler, since we can examine D_j and D_{-j} separately.

Because of symmetry, we may only consider the first expression. Fix $x \in U_j$. Then

$$\epsilon_1 \ge |z_j(x)| \ge \sqrt{t_j} \quad \text{or} \quad \epsilon_1 \ge |z_{-j}(x)| \ge \sqrt{t_j} \ ,$$

by the symmetry $j \leftrightarrow -j$ we may assume the first case. We also split the y–integral into two pieces

$$W_+ = \{y \in U_j : \epsilon_1 \ge |z_j(y)| \ge \sqrt{t_j}\} \quad \text{and}$$
$$W_- = \{y \in U_j : \epsilon_1 \ge |z_{-j}(y)| > \sqrt{t_j}\} \ :$$

The first piece is best estimated in the coordinate z_j:

$$\int_{W_+} |C_h(x,y)|_{AB} \Omega_y \sim \int_{W_+} \frac{|\psi_{jA}(x)|_A |\psi_{jB}(y)|_B}{|z_j(x) - z_j(y)|} \alpha_j^{-2}\, d^2 z_j(y) \le O(\alpha_j^{-1}) \int_0^{2\epsilon_1} \frac{2\pi r\, dr}{r} \sim \alpha_j^{-1} \ ,$$

where we used polar coordinates around $z_j(x)$ and

$$|\psi_{jA}(x)|_A \sim \alpha_{jA} \ , \quad |\psi_{jB}(y)|_B \sim \alpha_{jB} \ .$$

The second piece is better estimated in the coordinate z_{-j}. The calculation is almost the same: using polar coordinates again and because of $|\psi_{-jA}(x)|_A \leq O(\alpha_{jA})$, $|\psi_{-jB}(y)|_B \sim \alpha_{jB}$ we get

$$\int_{W_-} |C_h(x,y)|_{AB}\Omega_y \sim \int_{W_-} \frac{|\psi_{-jA}(x)|_A |\psi_{-jB}(y)|_B}{|z_{-j}(x) - z_{-j}(y)|} \alpha_j^{-2}\, d^2 z_{-j}(y)$$

$$\leq O(\alpha_j^{-1}) \int_0^{2\epsilon_1} \frac{2\pi r\, dr}{r} \sim \alpha_j^{-1} .$$

Summing up the two pieces, the assertion follows. □

As a consequence of the preceding lemma, we get:

Lemma 3.22 (Bounds for the interpolation operator)

a) **(L^1–bound.)** *There is a constant M, independent of j, such that*

$$\int_{U_j} |If|_A \Omega \leq M \left(\int_{U_j \cap U_0} |f|_A \Omega + \alpha_j^{-1} \sum_{a \in U_j \cap U_0} |\mathrm{res}_a f|_{B,\mathrm{dual}} \right) .$$

Here $|\cdot|_{B,\mathrm{dual}}$ denotes the Hermitian metric on $(\check{L}^{1,0})^\vee$ dual to $|\cdot|_B$.

b) **(L^∞–L^1–bound for parameter dependent sections.)** *For a sufficiently large constant $M > 0$ (independent of j) the following holds: assume that f is a holomorphic section in $L \otimes_{X \times X} \check{L}^{1,0}$ over $V_{\pm j} \times U$, $U \subseteq X$ open. Then for any positive measure $\tilde{\Omega}$ on U:*

$$\sup_{x \in U_j} \int_{y \in U} |I_1 f(x,y)|_{AB} \tilde{\Omega}(y) \leq M \sup_{x \in V_j \cup V_{-j}} \int_{y \in U} |f(x,y)|_{AB} \tilde{\Omega}(y) .$$

Here I_1 means that the interpolation operator is applied to the first argument with fixed second argument y.

b') **(Pointwise bound – simplified version of *b*))** *Let M be the constant of b). If f is a holomorphic section in L over $V_{\pm j}$, then $\sup_{U_j} |If|_A \leq M \sup_{V_j \cup V_{-j}} |f|_A$.*

c) **(L^2–bound.)** *There is a constant M, independent of j, such that the following holds: if f is a holomorphic section in L over $V_j \cup V_{-j}$, then*

$$\int_{U_j} |If|_A^2 \Omega \leq M \sum_{\pm} \int_{V_{\pm j}} |f|_A^2 \Omega .$$

In the case of degenerate handles we can simplify this by replacing U_j by D_j on the left hand side and $\sum_\pm \int_{V_{\pm j}}$ by $\int_{V_j}$ on the right hand side.

Proof:

a) We estimate the three terms in the definition of If:
For the first term: $\int\limits_{U_j} |f\chi_0|_A \Omega \le \int\limits_{U_j \cap U_0} |f|_A \Omega.$
The second term is bounded by

$$\int\limits_{x\in U_j} \left| \int\limits_{z\in V_{\pm j}} f(z) C_h(x,z) \wedge d\chi_j(z) \right|_A \Omega_x$$
$$\le \left(\sup_{z\in V_{\pm j}} \int\limits_{x\in U_j} |C_h(x,z)|_{AB} \Omega_x \right) \left(\int\limits_{z\in V_{\pm j}} |f(z)|_A \Omega_z \right) \sup_{z\in V_{\pm j}} |d\chi_j(z)|_2 \ .$$

Inserting Lemma 3.19 and Lemma 3.21 we get

$$\int\limits_{x\in U_j} \left| \int\limits_{z\in V_j\cup V_{-j}} f(z) C_h(x,z) \wedge d\chi_j(z) \right|_A \Omega_x \le \text{const} \int\limits_{z\in V_j\cup V_{-j}} |f(z)|_A \Omega_z \ .$$

Finally for the third term, using Lemma 3.21 again:

$$\int\limits_{x\in U_j} \left| \chi_j(x) \sum_z (\mathrm{res}_z f) C_h(x,z) \right|_A \Omega_x$$
$$\le \sum_z |\mathrm{res}_z f|_{B,\mathrm{dual}} \cdot \int\limits_{x\in U_j} |C_h(x,z)|_{AB} \Omega_x$$
$$\le \text{const} \cdot \alpha_j^{-1} \sum_z |\mathrm{res}_z f|_{B,\mathrm{dual}}.$$

Summing up the three terms, the result follows.

b) Similarly to a) we estimate the three terms in the definition of $I_1 f$:
First term:

$$\sup_{x\in U_j} \int\limits_{y\in U} |f(x,y)\chi_0(x)|_{AB} \tilde{\Omega}(y) \le \sup_{x\in V_j\cup V_{-j}} \int\limits_{y\in U} |f(x,y)|_{AB} \tilde{\Omega}(y) \ .$$

For the second summand, we get:

$$\sup_{x\in U_j}\int_{y\in U}\left|\int_{z\in V_{\pm j}} f(z,y)C_h(x,z)\wedge d\chi_j(z)\right|_{AB}\tilde{\Omega}(y)$$
$$\leq \left(\sup_{z\in V_{\pm j}}\int_{y\in U}|f(z,y)|_{AB}\tilde{\Omega}(y)\right)$$
$$\times\left(\sup_{x\in U_j}\int_{z\in V_{\pm j}}|C_h(x,y)|_{AB}\Omega_z\right)\sup_{z\in V_{\pm j}}|d\chi_j(z)|_2$$
$$\overset{3.21}{\leq} \text{const}\cdot\sup_{z\in V_{\pm j}}\int_{y\in U}|f(z,y)|_{AB}\tilde{\Omega}(y)\ .$$

The third term is missing since f has no poles by assumption.

Summing up, we get the result b).

b') This may be viewed as the special case of b) when $\tilde{\Omega}$ is supported in one point y; the one dimensional complex vector space $\check{L}^{1,0}_y$ may be identified with $\mathbb{C}$.

c) We omit the simpler case of degenerate handles.

For the first summand in the definition of If we again get:

$$\int_{U_j}|f\chi_0|^2_A\Omega\leq\sum_{\pm}\int_{V_{\pm j}}|f|^2_A\Omega\ .$$

The third summand in the definition of If vanishes again, and the second summand is an evaluation of the integral operator with integral kernel $F_j(x,z)=C_h(x,z)\wedge d\chi_j(z)$ to the section f. The two L^1–L^∞–norms of F_j are bounded by Lemma 3.19 and Lemma 3.21:

$$\sup_j\sup_{x\in U_j}\int_{y\in V_j\cup V_{-j}}|C_h(x,y)\wedge d\chi_j(y)|_{A\check{A}}\Omega_y<\infty\ ,$$
$$\sup_j\sup_{y\in U_j}\int_{x\in V_j\cup V_{-j}}|C_h(x,y)\wedge d\chi_j(y)|_{A\check{A}}\Omega_x<\infty\ .$$

Hence the result follows from Lemma 3.5 a). □

Next, we shall show that If is meromorphic – although it is defined by using a partition of unity:

Let $a_1, \dots, a_n$ be the poles of f. First assume that $x \in V_j \cup V_{-j}$. Cut small disks Δ_k, $(k = 1, \dots, n)$ and Δ_x out around a_k, x respectively, and call the resulting domain of integration $G = (V_j \cup V_{-j}) \setminus \Delta_x \setminus \bigcup_k \Delta_k$. By Stokes theorem,

$$\int_{z \in G} f(z) C_h(x,z) \wedge d\chi_j(z) = \tag{13}$$

$$\begin{aligned}
&= \quad -\int_{z\in G} d_z(f(z)C_h(x,z)\chi_j(z)) \\
&= \quad -\sum_{\pm} \int_{z\in\partial V_{\pm j}} f(z)C_h(x,z)\chi_j(z) + \int_{z\in\partial\Delta_x \cup \bigcup_k \partial\Delta_k} f(z)C_h(x,z)\chi_j(z) \\
&\overset{\text{radii}\to 0}{\longrightarrow} -\sum_{\pm} \int_{\substack{z\in \\ \text{inner boundary} \\ \text{of } V_{\pm j}}} f(z)C_h(x,z) \\
&\qquad\qquad +2\pi i \sum_k \chi_j(a_k)(\operatorname{res}_{a_k} f) C_h(x,a_k) - f(x)\chi_j(x) \ ,
\end{aligned}$$

where we used $\operatorname{res}_x C_h(x,\cdot) = -1/(2\pi i) \in \mathcal{E}_x \cong L_x \otimes \check{L}_x$. Of course the last term $-f(x)\chi_j(x)$ is missing in the remaining case that $x \in U_j \setminus (V_j \cup V_{-j})$.

This leads to:

Lemma 3.23 (Contour integral form of the interpolation operator)

a) The interpolation operator may be expressed by

$$If(x) = \sum_{\pm} \int_{\substack{z\in \\ \text{inner boundary} \\ \text{of } V_{\pm j}}} f(z)C_h(x,z) + \begin{cases} 0 & \text{if } x \notin V_{\pm j} \ , \\ f(x) & \text{if } x \in V_{\pm j} \ . \end{cases}$$

Consequently If is a meromorphic function and $If - f$ is holomorphic.

b) If f extends to a holomorphic section on the whole handle U_j, then $If = f$.

Proof:

a) This is an immediate consequence of the above calculation and of

$$\chi_0 = 0 \quad \text{over } U_j \setminus (V_j \cup V_{-j}) \ .$$

b) We use a) and Cauchy's formula: The integrals over the inner boundary of $V_{\pm j}$ in a) cancel. ◻

We are going to justify the name "interpolation of f through the handle" for If for sections f that extend to disks as stated by Ext-Disk(f/ψ_{0A}):

We replace all handles by degenerate handles with affine linear transition functions, just as in the definition of the codegree. Let $D^0_{\pm j}$ denote the two connected components of the degenerate handle replacing U_j. We glue the trivial line bundle over $D^0_{\pm j}$ to the line bundle L over U_0 by identifying the unit section $1 =: \psi^0_{jA}$ in the trivial line bundle over $D^0_{\pm j}$ with $g_{jA}\psi_{0A}$ over $V_{\pm j}$. Let C_0 denote the analogue to C_h over the modified surface, i. e.

$$C_0(x, y) = \frac{1}{2\pi i}\frac{\psi^0_{jA}(x)}{\psi^0_{jA}(y)}\frac{dz_0(y)}{z_0(x) - z_0(y)} \quad \text{if } x, y \in D^0_j, \tag{14}$$

$C_0(x, y) = 0$ for x, y in different connected components. The postulate (L2) guarantees that Ext-Disk$(g^{\pm 1}_{jA})$ holds for every handle. Let $I^0 f$ denote If for the modified surface. Here is Lemma 3.23b) for the modification:

Lemma 3.24 Ext-Disk$(f/\psi_{0A}) \quad \Longrightarrow \quad I^0 f = f.$ □

As a consequence we conclude:

Lemma 3.25 (Interpolation through handles – Bounds for the error term) Ext-Disk*(f/ψ_{0A}) implies*

$$\sup_{V_{\pm j}} |If - f|_A \le O(o_j)\alpha_j^2 \left[\int\limits_{V_j \cup V_{-j}} |f|_A \Omega + \alpha_j^{-1} \sum_z |\mathrm{res}_z f|_{B,\mathrm{dual}} \right].$$

A similar estimate holds for $\check{L}^{1,0}$–valued sections with reversed roles of $A \leftrightarrow B$.

Proof:

$$\begin{aligned} |If(x) - f(x)|_A &= |If(x) - I^0 f(x)|_A = \\ &= \Bigg| -\int\limits_z f(z)(C_h(x,z) - C_0(x,z)) \wedge d\chi_j(z) \\ &\quad + 2\pi i \sum_z \chi_j(z) \mathrm{res}_z f\, (C_h(x,z) - C_0(x,z)) \Bigg|_A \\ &\le \sup_z |C_h(x,z) - C_0(x,z)|_{AB} \\ &\quad \times \left[\sup_z |d\chi_j|_2 \int\limits_{V_j \cup V_{-j}} |f|_A \Omega + 2\pi \sum_z |\mathrm{res}_z f|_{B,\mathrm{dual}} \right] \end{aligned}$$

Using Lemma 3.19 the result follows from the following lemma. □

Lemma 3.26 (Coordinate change for the Cauchy kernel)

$$\sup_{(V_j\cup V_{-j})^2} |C_h - C_0|_{AB} = \alpha_j\, O(o_j)\ .$$

Proof: Let $x, y \in (V_j \cup V_{-j})^2$. We have two cases to distinguish.

Case 1: x, y belong to different components, say $x \in V_j$, $y \in V_{-j}$. Then

$$|C_h(x,y) - C_0(x,y)|_{AB} = |C_h(x,y)|_{AB} = \frac{|\psi_{jA}(x)|_A |\psi_{jB}(y)|_B}{|z_j(x) - z_j(y)|}$$
$$\leq \mathrm{const}_1 \alpha_j \frac{|z_j(y)|}{|Q_{-j}|} \leq \mathrm{const}_2 \alpha_j \frac{|t_j|}{|Q_{-j}|} \sim \alpha_j \sqrt{t_j} = \alpha_j\, O(o_j)\ .$$

We used (9), (10), (X2) and that $|z_j(x) - z_j(y)|$ can be bounded from below by a positive constant not depending on j. The boundedness of $|z_j(x) - z_j(y)|$ follows from the inequality $\epsilon_2^2 > \sup_j t_j$ in (X2).

Remark: This harmless looking estimate is the crucial point where the nontriviality of the transition functions for the basis sections enters: it is essential that $|\psi_{jB}|_B$ is very small over V_{-j}.

Case 2: x, y belong to the same component, say $x, y \in V_j$. Then writing $s := \psi^0_{0A}/\psi_{jA}$, we get

$$|C_h(x,y) - C_0(x,y)|_{AB} = \tag{15}$$
$$= \left| \frac{1}{z_j(x) - z_j(y)} - \frac{s(x)}{s(y)} \frac{\frac{dz_0}{dz_j}(y)}{z_0(x) - z_0(y)} \right| |\psi_{jA}(x)|_A |\psi_{jB}(y)|_B$$
$$\sim \alpha_j \left| \frac{1 - \frac{s(x)}{s(y)}}{z_j(x) - z_j(y)} + \frac{s(x)}{s(y)} \left(\frac{1}{z_j(x) - z_j(y)} - \frac{\frac{dz_0}{dz_j}(y)}{z_0(x) - z_0(y)} \right) \right|\ .$$

To estimate the first summand in the last sum, we note

$$\left|\frac{s(x)}{s(y)} - 1\right| = \left|\exp\left(\int_y^x \frac{ds}{s}\right) - 1\right| \overset{(L2)}{\leq} \exp\left(\int_y^x (1 + O(o_j)) \left|\frac{ds}{dz_j}\right| |dz_j|\right) - 1$$
$$\overset{(L2),(X2)}{=} \exp\left[O(o_j)|z_j(x) - z_j(y)|\right] - 1 = O(o_j)|z_j(x) - z_j(y)|\ .$$

Here we have used that $|z_j(x) - z_j(y)|$ is bounded, so the Taylor expansion of exp is justified; the boundedness is a consequence of (X2). This implies the desired estimate for the first summand in (15):

$$\left| \frac{1}{z_j(x) - z_j(y)} \cdot \left(1 - \frac{s(x)}{s(y)}\right) \right| = O(o_j)\ .$$

To estimate the second summand in (15), we note that Taylor expansion with Lagrange error terms yields:

$$\left|\frac{1}{z_j(x)-z_j(y)} - \frac{\frac{dz_0}{dz_j}(y)}{z_0(x)-z_0(y)}\right| = \frac{\left|\int_x^y (z_j(t)-z_j(x))\frac{d^2z_0}{dz_j^2}(t)\,dz_j(t)\right|}{|z_j(x)-z_j(y)||z_0(x)-z_0(y)|}$$

$$\leq \frac{O(|z_j(y)-z_j(x)|)\sup_{V_j}\left|\frac{d^2z_0}{dz_j^2}\right|}{|z_0(x)-z_0(y)|} \overset{(X4)}{\leq} \left(c_j \cdot \sup_{V_j}\left|\frac{d^2z_0}{dz_j^2}\right|\right) \overset{(X4)}{\leq} O(o_j)\ .$$

Together with $s(x)/s(y) = 1 + O(o_j)$ we get the desired bound $O(o_j)$ for the second summand in (15). □

Now we have prepared all the tools to define the missing pieces of the integral kernel K:

$$K_{h0} = I_1K_{00}, \quad K_{0h} = I_2K_{00}, \quad K_{hh} = I_1I_2(K_{00} - C_0) + C_h.$$

Here the notation I_1 and I_2 means that the interpolation operator I is applied to the first or second argument respectively.

Remark: The reason for not defining K_{hh} simply to be $I_1I_2K_{00}$ is that the interpolation operator cannot produce poles in the interior $U_j \setminus V_{\pm j}$ of every handle. Consequently the poles of the Cauchy kernel K_{hh} in $U_j \setminus V_{\pm j}$ have to be modelled separately: we therefore use C_h. Related to this observation is the following fact: although Ext-Disk($K_{00}(x,\cdot)/\psi_{0B}$) and Ext-Disk($K_{00}(\cdot,y)/\psi_{0A}$) are true for every $x, y \in V_{\pm j}$, Ext-Disk($I_2K_{00}(\cdot,y)/\psi_{0A}$) does not hold for $y \in U_j \setminus V_{\pm j}$ since $I_2K_{00}(\cdot,y)$ has a pole at y. The situation for $K_{00} - C_0$ is different, as Ext-Disk($I_2(K_{00} - C_0)(\cdot,y)/\psi_{0A}$) is valid, simply because $K_{00} - C_0$ extends to a *holomorphic* section over $(D^0_{\pm j})^2$.

3.3.3 Verification of the hypotheses K

We first show that the total volume of X is finite.

Lemma 3.27 (Volume of the handles; total volume)

a) $\int_{U_j} \Omega \sim \sum_{\pm} \int_{V_{\pm j}} \Omega$, *i. e. only a bounded fraction of the volume of any handle is not included in the regular sheet.*

b) $\int_{V_j} \Omega \sim \alpha_j^{-2}$

c) $\int_X \Omega < \infty$

Proof: We prove a) and b) simultaneously; the calculation is best done in polar coordinates for $z_{\pm j}$:

$$\int_{U_j} \Omega \sim 2\alpha_j^{-2} \int_{\sqrt{\tau_j}}^{\text{const}} \frac{2\pi r}{r}\, dr \sim \alpha_j^{-2}\ , \quad \int_{V_j} \Omega \sim \alpha_j^{-2} \int_{\text{const}_1}^{\text{const}_2} \frac{2\pi r}{r}\, dr \sim \alpha_j^{-2}\ .$$

To prove c), we use a) to get

$$\sum_j \int_{U_j} \Omega \le \text{const} \sum_j \int_{V_{\pm j}} \Omega \le \text{const} \int_{U_0} \Omega \le \text{const} \int_{z\in\mathbb{C}} \frac{d^2z}{1+|z|^4} < \infty\ ,$$

hence $\int_X \Omega \le \int_{U_0} \Omega + \sum_j \int_{U_j} \Omega < \infty$. Recall that we have transiently assumed the compact piece of the surface to be empty. □

We state the main result of this section.

Proposition 3.28 *Suppose that the compact piece of the surface is empty. Then the hypotheses (X) and (L) imply the statements (K1-4) on the integral kernel K.*

Proof:

(K1) By construction, all K_{ab} are sections in the external tensor product $L \otimes_{X\times X} \check{L}^{1,0}$. Since K_{00} only has a pole on the diagonal, $K_{h0} = I_1 K_{00}$ and $K_{0h} = I_2 K_{00}$ are holomorphic outside the diagonal, too, because of Lemma 3.23. As $K_{00} - C_0$ is holomorphic, $I_1 I_2 (K_{00} - C_0)$ is holomorphic on $U_h^2 = (\bigcup_{j\in J} U_j)^2$, that is why $K_{hh} = I_1 I_2 (K_{00} - C_0) + C_h$ has the same singular part as C_h. When we patch the four pieces together, (K1) follows.

(K2) It suffices to check that the K_{ab} are of the form

$$K_{ab} = (\mathcal{C}^\infty\text{–section}) + \frac{1}{2\pi i} \frac{1}{z(x) - z(y)} \frac{\omega_x}{\omega_y}\, dz(y) \qquad (16)$$

near the diagonal for some local coordinate z and some local basis section ω. This is obvious for K_{00}, hence the statement follows for K_{0h} and K_{h0} since $K_{0h} - K_{00}$ and $K_{h0} - K_{00}$ are holomorphic in $U_0 \times (U_0 \cap U_h)$ resp. $(U_0 \cap U_h) \times U_0$. (16) is obvious for K_{hh} too when $I_1 I_2 (K_{00} - C_0)$ plays the role of the first summand in (16) while C_h plays the role of the second one.

We prepare the proof of (K3) and (K4) with some lemmas:

Lemma 3.29 **(L^∞–L^1–norm for the Cauchy kernel on the regular sheet)**

a) $\sup_{x\in U_0} \int_{y\in U_0} |K_{00}(x,y)|_{AB}\Omega_y$ *and*

$\sup_{y\in U_0} \int_{x\in U_0} |K_{00}(x,y)|_{AB}\Omega_x$ *are finite.*

b) $\sup_j \alpha_j \sup_{y\in U_0} \int_{x\in V_j} |K_{00}(x,y)|_{AB}\Omega_x$ *and*

$\sup_j \alpha_j \sup_{x\in U_0} \int_{y\in V_j} |K_{00}(x,y)|_{AB}\Omega_x$ *are finite.*

Proof:

a) We interpret K_{00} geometrically over the regular sheet: Recall that we have chosen the order $\nu = -1$ at infinity for the weight functions in the definition of $|\cdot|_A$ and $|\cdot|_B$. During this proof, we identify the regular sheet U_0 with a subset of the Riemann sphere $\mathbb{P}_1$ by the identification $x \equiv (z_0(x) : 1)$. Then the volume form Ω is identified (up to a bounded factor) with the standard volume form on $\mathbb{P}_1$. Next we identify both $L|U_0$ and $\check{L}^{1,0}|U_0$ with the tautological bundle

$$\{((x_1 : x_2), (\lambda x_1, \lambda x_2)) \in \mathbb{P}_1 \times \mathbb{C}^2 \mid (x_1, x_2) \in \mathbb{C}^2 \setminus \{0\}, \lambda \in \mathbb{C}\}$$

over $\mathbb{P}_1$ restricted to U_0; the vector space operations in the fibres refer to the second component $(\lambda x_1, \lambda x_2)$. The tautological bundle may be viewed as the *disjoint* union of all one-dimensional subspaces of $\mathbb{C}^2$; these subspaces are indexed by the projective space $\mathbb{P}_1$. The identification is described by

$$\lambda\psi_{0A}(x) \equiv ((z_0(x) : 1), (\lambda z_0(x), \lambda)) \ ,$$
$$\lambda\psi_{0B}(x) \equiv ((z_0(x) : 1), (\lambda z_0(x), \lambda)) \ .$$

With this identification $|\cdot|_A$ and $|\cdot|_B$ have just become (modulo $\sim$) the standard Euclidean norm:

$$|\psi_{0A}|_A \sim |\psi_{0B}|_B \sim (1 + |z_0|^4)^{\frac{1}{4}} \sim |(z_0, 1)| \ ,$$

and K_{00} becomes

$$K_{00}((x_1 : x_2), (y_1 : y_2)) = \frac{1}{\frac{x_1}{x_2} - \frac{y_1}{y_2}} \begin{pmatrix} \frac{x_1}{x_2} \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \frac{y_1}{y_2} \\ 1 \end{pmatrix} = \frac{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}}$$

which is – after extension to $\mathbb{P}_1 \times \mathbb{P}_1$ – clearly invariant under the canonical operation of SU(2). The Euclidean norms in the fibre and the standard volume form are invariant under the action of SU(2) too. Consequently – since the action of SU(2) on $\mathbb{P}_1$ is transitive – the integral

$$\int_{x\in\mathbb{P}_1} |K_{00}(x,y)|_{AB}\Omega_x$$

does not depend on y (up to a bounded factor). The only singularity at $x = y$ is integrable, implying finiteness of the integral.

b) The calculation is best done using the geometric interpretation of a) again. We first note that V_j is contained in a disk $D_j \subseteq \mathbb{P}_1$ with total volume $\int_{D_j} \Omega \sim \alpha_j^{-2}$. We now apply an action of an element $g \in$ SU(2) to transform D_j into a disk D'_j centered at $(0:1) \in \mathbb{P}_1$ with a radius R_j:

$$D'_j = \{(z:1) \in \mathbb{P}_1 \mid |z| < R_j\} \ .$$

The invariance of the volume $\int_{D_j} \Omega \sim \int_{D'_j} \Omega$ shows $R_j^2 = O(\alpha_j^{-2})$. We remark that the sequence (R_j) is bounded. The transformed integral can now easily be estimated: let $z = z_0(gy)$ denote the transformed coordinate of y. Then

$$\begin{aligned}\int_{x\in D_j} |K_{00}(x,y)|_{AB}\Omega_x &\sim \int_{x\in D'_j} |K_{00}(x,y)|_{AB}\Omega_x \\ &\sim \int_{\substack{x\in\mathbb{C}\\ |x|<R_j}} \frac{1}{|x-z|}(1+|z|^2)^{\frac{1}{2}}(1+|x|^2)^{\frac{1}{2}}\frac{d^2x}{1+|x|^4} \\ &\overset{R_j \text{ bounded}}{\sim} \int_{\substack{x\in\mathbb{C}\\ |x|<R_j}} \frac{(1+|z|^2)^{\frac{1}{2}}}{|x-z|}\, d^2x \ . \end{aligned} \tag{17}$$

We distinguish two cases:

– If $|z| \le 2R_j$ then the last integral is bounded by

$$\sqrt{1+4R_j^2}\int_0^{3R_j} \frac{2\pi r}{r}\, dr = O(R_j) = O(\alpha_j^{-1}) \ .$$

Here we use polar coordinates with center z.

– In the case $|z| > 2R_j$, we use polar coordinates with center 0 to bound (17) by

$$\int_0^{R_j} \frac{\sqrt{1+|z|^2}}{|z|/2} 2\pi r\, dr \leq 4\pi \sqrt{\frac{1}{(2R_j)^2}+1} \int_0^{R_j} r\, dr$$
$$= O(R_j) = O(\alpha_j^{-1}) \; .$$

Hence the first statement of b) is proved, and the second one has the same proof when we use the symmetry $A \leftrightarrow B$. □

Corollary 3.30 *The following expressions are finite:*

$$\sup_j \alpha_j \sup_{x \in V_j} \int_{y \in V_j} |C_0(x,y)|_{AB} \Omega_y \quad \textit{and} \quad \sup_j \alpha_j \sup_{y \in V_j} \int_{x \in V_j} |C_0(x,y)|_{AB} \Omega_y \; .$$

Proof: This is an immediate consequence of the previous lemma and the fact that

$$\frac{|C_0(x,y)|_{AB}}{|K_{00}(x,y)|_{AB}} = \left| \frac{g_{jA}(x)}{g_{jA}(y)} \right|$$

is bounded as was stated in (L2). □

The next lemma provides bounds for the differences between different pieces K_{ab} on their common domain:

Lemma 3.31 (Bounds for the differences of the pieces for K)

a) $\sup_{V_j \times U_0} |K_{00} - K_{0h}|_{AB} = O(o_j)\alpha_j$,

b) $\sup_{U_0 \times V_j} |K_{00} - K_{h0}|_{AB} = O(o_j)\alpha_j$,

c) $\sup_{V_j \times U_h} |K_{hh} - K_{0h}|_{AB} = O(o_j)\alpha_j$,

d) $\sup_{U_h \times V_j} |K_{hh} - K_{h0}|_{AB} = O(o_j)\alpha_j$.

Proof: Because of the symmetry $A \leftrightarrow B$ it suffices to prove a) and d).

a) Using Ext-Disk($K_{00}(\cdot,y)/\psi_{0A}$) for $y \in U_0$ and the fact that $K_{00}(\cdot,y)$ has at most a pole of first order at y with residue $\frac{1}{2\pi i} \in (\check{L}_y^{1,0})^\vee \otimes \check{L}_y^{1,0}$, we get uniformly in y:

$$\sup_{V_j} |K_{00}(\cdot,y) - K_{h0}(\cdot,y)|_{AB} = \sup_{V_j} |K_{00}(\cdot,y) - I_1 K_{00}(\cdot,y)|_{AB}$$
$$\overset{3.25}{=} O(o_j)\alpha_j^2 \left[\int_{V_{\pm j}} |K_{00}(\cdot,y)|_{AB} \Omega + \alpha_j^{-1} \right] \overset{3.29}{=} O(o_j)\alpha_j \; .$$

d) We express

$$K_{hh} - K_{h0} = (I_2 I_1(K_{00} - C_0) - I_1(K_{00} - C_0)) - I_1(C_0 - C_h) \ . \tag{18}$$

Here we use that I_1 and I_2 commute and that $I_1 C_h = C_h$, which follows from Lemma 3.23 applied to $IC_h(\cdot, y)$, $y \in V_j$; the integral over the inner boundary of $V_{\pm j}$ vanishes there, since $C_h(\cdot, y)$ is holomorphic in U_h except of the pole in y.
We estimate the first term in (18): using Ext-Disk($I_1(K_{00} - C_0)(x, \cdot)/\psi_{0B}$), we get with the notation $V_h = \bigcup_{j \in J} V_j$:

$$\begin{aligned} &\sup_{U_h \times V_j} |I_2 I_1(K_{00} - C_0) - I_1(K_{00} - C_0)|_{AB} \\ &\overset{3.25}{\leq} O(o_j)\alpha_j^2 \sup_{x \in U_h} \int_{y \in V_{\pm j}} |I_1(K_{00} - C_0)(x, y)|_{AB} \Omega_y \\ &\overset{3.22b)}{\leq} O(o_j)\alpha_j^2 \sup_{x \in V_h} \int_{y \in V_{\pm j}} |(K_{00} - C_0)(x, y)|_{AB} \Omega_y \ . \end{aligned}$$

Lemma 3.29 b) and Corollary 3.30 show that $O(\alpha_j^{-1})$ is an upper bound for the last supremum, hence

$$\sup_{U_h \times V_j} |I_2 I_1(K_{00} - C_0) - I_1(K_{00} - C_0)|_{AB} \leq \alpha_j \, O(o_j) \ .$$

The second term in (18) remains to be bound:

$$\sup_{U_h \times V_j} |I_1(C_0 - C_h)|_{AB} \overset{3.22b')}{\leq} M \sup_{V_h \times V_j} |C_0 - C_h|_{AB} \overset{3.26}{\leq} \alpha_j \, O(o_j) \ .$$

The result is gained by summing up the two terms. □

We are now ready to prove (K3): We only estimate the first norm k_1, the estimate for k_2 is similar.

$$\sup_{x \in X} \int_{y \in X} |K(x, y)|_{AB} \Omega_y \leq \sum_{a, b \in \{0, h\}} \sup_{x \in U_a} \int_{y \in U_b} |K_{ab}(x, y)|_{AB} \Omega_y \ .$$

We shall show that every summand is finite:
For $a = b = 0$ this is the statement of Lemma 3.29 a).

For $a = h$, $b = 0$ we get (subtracting C_h if necessary to remove the pole)

$$\begin{aligned}
&\sup_{x\in U_h} \int_{y\in U_0} |K_{h0}(x,y)|_{AB}\Omega_y \\
&\quad \overset{(I_1 C_h = C_h)}{\leq} \sup_{x\in U_h} \int_{y\in V_h} \left[|(I_1(K_{00} - C_h))(x,y)|_{AB} + |C_h(x,y)|_{AB}\right]\Omega_y \\
&\qquad\qquad + \sup_{x\in U_h} \int_{y\in U_0\setminus V_h} |I_1 K_{00}(x,y)|_{AB}\Omega_y \\
&\quad \overset{3.22b)}{\leq} M \sup_{x\in V_h} \int_{y\in V_h} |(K_{00} - C_h)(x,y)|_{AB}\Omega_y \\
&\qquad\qquad + \sup_{x\in U_h} \int_{y\in V_h} |C_h(x,y)|_{AB}\Omega_y \\
&\qquad\qquad + M \sup_{x\in V_h} \int_{y\in U_0\setminus V_h} |K_{00}(x,y)|_{AB}\Omega_y \overset{3.29,3.21}{<} \infty\,.
\end{aligned}$$

Recall that $C_h(x, y)$ vanishes for x, y in different handles and that α_j^{-1} is bounded by (X6).

For $a = 0$, $b = h$:

$$\begin{aligned}
&\sup_{x\in U_0} \int_{y\in U_h} |I_2 K_{00}(x,y)|_{AB}\Omega_y \\
&\qquad \overset{3.22a)}{\leq} M \left[\sup_{x\in U_0} \int_{y\in V_h} |K_{00}(x,y)|_{AB}\Omega_y + O(1)\right] \overset{3.29}{<} \infty\,.
\end{aligned}$$

For $a = b = h$:

$$\begin{aligned}
&\sup_{x\in U_h} \int_{y\in U_h} |K_{hh}(x,y)|_{AB}\Omega_y \leq \\
&\leq \sup_{x\in U_h} \int_{y\in U_h} |I_1 I_2(K_{00} - C_0)(x,y)|_{AB}\Omega_y + \sup_{x\in U_h} \int_{y\in U_h} |C_h(x,y)|_{AB}\Omega_y < \infty\,.
\end{aligned}$$

We used Lemma 3.22 a) and b) to remove the interpolation operators in the last step, and then Lemma 3.29 a) and Lemma 3.21.

The proof of (K3) is finished. □

We finally are going to prove (K4): With the symmetry $A \leftrightarrow B$ in mind, we restrict ourselves to prove that the first Hilbert–Schmidt norm h_B is finite; the proof for h_A is similar. K_{ab} being holomorphic outside the diagonal, we get for $x \neq y$:

$$\begin{aligned}\overline{\partial_x} K(x, y) &= \sum_{a,b} (\overline{\partial_x} \chi_a(x)) \chi_b(y) K_{ab}(x, y) \\ &= \sum_b (\overline{\partial_x} \chi_0(x)) \chi_b(y) \, [K_{0b}(x, y) - K_{hb}(x, y)]\end{aligned}$$

which vanishes if $x \notin V_h$. For $x \in V_j$, $y \in U_b$ one estimates uniformly in x and y:

$$\begin{aligned}|\overline{\partial_x} K(x, y)|_{\check{B}B} &\leq \sum_b |\overline{\partial_x} \chi_0(x)|_2 \chi_b(y) |(K_{0b} - K_{hb})(x, y)|_{AB} \\ &\overset{3.31}{\leq} |\overline{\partial} \chi_0(x)|_2 O(o_j) \alpha_j \leq O(o_j) \alpha_j^2 \, .\end{aligned}$$

Hence, using that $\overline{\partial}_1 K$ is supported in $V_h \times X$,

$$\begin{aligned}h_b^2 &\leq \sum_j \left(\int_{x \in V_j} \Omega \right) \left(\int_{y \in X} \Omega \right) [O(o_j) \alpha_j^2]^2 \\ &\overset{3.27}{\leq} \sum_j \alpha_j^{-2} (O(o_j) \alpha_j^2)^2\end{aligned}$$

which is finite by (X6).

Now the proof of (K4) and also of Proposition 3.28 is finished. □

Remark: If a sufficiently large but finite number of handles is removed and replaced by two disks each (as described in the definition of Ext-Disk), then h_B^2 can be made arbitrarily small. This is necessary to make Corollary 3.16 applicable.

To summarize, we have proved so far that the Cauchy Riemann operator $\overline{\partial}_V$ is bounded with a bounded inverse when the compact piece and a sufficiently large but finite set of handles is replaced by disks.

3.4 Gluing in the compact piece: exchange lemma

We want to compare the indices of the Cauchy–Riemann operator on four surfaces X_{ac}, X_{ad}, X_{bc}, X_{bd} endowed with holomorphic line bundles L_{ac}, L_{ad}, L_{bc}, L_{bd}. The four surfaces are defined as follows:

- $X = X_{ac}$ is the surface we started with. We view it as being obtained by gluing two pieces X_a and X_c together:
  - The piece X_a just consists of the union of the regular sheet with the handles "near infinity"; only a compact subset is removed.
  - The piece X_c is the compact piece of the heat curve.
- When we fill the hole in the regular sheet where the compact piece can be glued in by a piece X_d of the complex plane – just by extending the coordinate z_0 – we call the resulting surface $X_{ad} = X_a \cup X_d$.
- We may also remove all the handles from X_a, fill the resulting holes with disks and insert one additional point ∞ at infinity, just as it was described in the definition of the codegree. The resulting surface, which may be viewed as a neighbourhood of $\infty = (1 : 0)$ in $\mathbb{P}_1$, is called X_b. When the compact piece X_c is glued to X_b, the compact Riemann surface X_{bc} comes out.
- Finally we may also glue X_b to X_d; the resulting surface is just the standard Riemann sphere $X_{bd} = \mathbb{P}_1$.

We identify all the intersections $X_a \cap X_c \equiv X_a \cap X_d \equiv X_b \cap X_c \equiv X_b \cap X_d =: U$. Over X_i, $i = a, b, c, d$ we prescribe holomorphic line bundles:

- $L_a = L|X_a$ over X_a,
- $L_c = L|X_c$ over X_c,
- trivial line bundles $L_b = X_b \times \mathbb{C}$, $L_d = X_d \times \mathbb{C}$ over X_b respectively X_d.

We identify $L_a|U \equiv L_b|U \equiv L_c|U \equiv L_d|U$ via $(L_a)_x = (L_c)_x \ni \psi_{0A}(x) \equiv (x, 1) \in (L_b)_x, (L_d)_x$. We now choose $\mathcal{C}^\infty$–partitions of unity over X_{ac}, X_{ad}, X_{bc}, X_{bd}, adapted to the covering $X_{kl} = X_k \cup X_l$ $(k = a, b; l = c, d)$. Let the partition of unity be given by χ_k over X_k, $k = a, b, c, d$, extended by 0 to X_{kl} $(k = a, b)$ respectively X_{lk} $(k = c, d)$. Just to simplify the notation below, we include an additional constant $\frac{\pi}{2}$: $\chi_k + \chi_l = \pi/2$ over X_{kl}, $\chi_k = \pi/2$ over $X_k \setminus U$.

We want to compare the Cauchy–Riemann–operators over $X_{ac} \dot{\cup} X_{bd}$ and $X_{ad} \dot{\cup} X_{bc}$, although these are different surfaces. The method works quite generally: we relate the four surfaces by a "twisting operator" J, defined by

$$J : \mathcal{C}_c^\infty(X_{ac}, L_{ac}) \oplus \mathcal{C}_c^\infty(X_{bd}, L_{bd}) \to \mathcal{C}_c^\infty(X_{ad}, L_{ad}) \oplus \mathcal{C}_c^\infty(X_{bc}, L_{bc})$$

$$J = \begin{pmatrix} \sin\chi_a & -\sin\chi_d \\ \sin\chi_c & \sin\chi_b \end{pmatrix} . \tag{19}$$

Here $\sin\chi_a$ is viewed as a multiplication operator

$$\mathcal{C}_c^\infty(X_{ac}, L_{ac}) \xrightarrow{\cdot \sin\chi_a} \mathcal{C}_c^\infty(X_a, L_a) \xrightarrow{\subseteq} \mathcal{C}_c^\infty(X_{ad}, L_{ad}) ,$$

analogously for the other matrix elements of J. J is invertible; its inverse is given by the matrix of multiplication operators

$$J^{-1} = \begin{pmatrix} \sin\chi_a & \sin\chi_c \\ -\sin\chi_d & \sin\chi_b \end{pmatrix} .$$

We keep in mind that the pointwise evaluation of the matrix J over U just leads to the rotation matrix

$$\begin{pmatrix} \cos\chi_c & -\sin\chi_c \\ \sin\chi_c & \cos\chi_c \end{pmatrix} ,$$

which gives us the intuitive picture behind J: over the bundle $L|U \oplus L|U$ of rank 2, J provides a “twisting by 90 degree”.

We observe that the following diagram[1] commutes:

$$\begin{array}{ccc} \mathcal{C}_c^\infty(X_{ac},L_{ac})\oplus\mathcal{C}_c^\infty(X_{bd},L_{bd}) & \xrightarrow{\begin{pmatrix}\bar\partial & 0\\ 0 & \bar\partial\end{pmatrix}} & \mathcal{C}_c^\infty\left(X_{ac},L_{ac}^{0,1}\right)\oplus\mathcal{C}_c^\infty\left(X_{bd},L_{bd}^{0,1}\right) \\ J\Big\downarrow & & J\Big\downarrow \\ \mathcal{C}_c^\infty(X_{ad},L_{ad})\oplus\mathcal{C}_c^\infty(X_{bc},L_{bc}) & \xrightarrow{\begin{pmatrix}\bar\partial & (\bar\partial\chi_c)\\ (-\bar\partial\chi_c) & \bar\partial\end{pmatrix}} & \mathcal{C}_c^\infty\left(X_{ad},L_{ad}^{0,1}\right)\oplus\mathcal{C}_c^\infty\left(X_{bc},L_{bc}^{0,1}\right) \end{array}$$

Here the matrix elements $\bar\partial\chi_c$ are supported in U. They are viewed as multiplication operators: Take a $\phi \in \mathcal{C}_c^\infty(U,\mathbb{R})$ which equals 1 in a neighbourhood of the support of $\bar\partial\chi_c$. Then we factor multiplication with $\bar\partial\chi_c$ as

$$\begin{aligned} \mathcal{C}_c^\infty(X_{ad},L_{ad}) &\xrightarrow{\cdot\phi} \mathcal{C}_c^\infty(U,L|U) \\ &\xrightarrow{\cdot(\bar\partial\chi_c)} \mathcal{C}_c^\infty(U,L^{0,1}|U) \hookrightarrow \mathcal{C}_c^\infty(X_{bc},L_{bc}^{0,1}) , \\ \mathcal{C}_c^\infty(X_{bc},L_{bc}) &\xrightarrow{\cdot\phi} \mathcal{C}_c^\infty(U,L|U) \\ &\xrightarrow{\cdot(\bar\partial\chi_c)} \mathcal{C}_c^\infty(U,L^{0,1}|U) \hookrightarrow \mathcal{C}_c^\infty(X_{ad},L_{ad}^{0,1}) . \end{aligned}$$

By completion, we now pass from the $\mathcal{C}_c^\infty$–theory to the Hilbert space / Fredholm theory: just as in Sect. 3.2.1 choose Hermitian metrics over L_a, L_b, L_c, L_d that coincide over L (all four metrics called $|\cdot|_A$). Similarly we choose metrics over $L_a^{0,1}$, $L_b^{0,1}$, $L_c^{0,1}$, $L_d^{0,1}$ (called $|\cdot|_{\check B}$), so that these metrics

[1] By a slight abuse of notation we call the vertical map at the right J again, since it is given by the same matrix (19).

coincide with the one chosen earlier over the surface X_{ac}. The volume forms compatible with these choices are again called Ω. Just as in Sect. 3.2.1, we define the Hilbert spaces A_{kl}, $\check{B}_{kl}$, V_{kl} to be the completion of $\mathcal{C}_c^\infty(X_{kl}, L_{kl})$, $\mathcal{C}_c^\infty(X_{kl}, L_{kl}^{0,1})$, $\mathcal{C}_c^\infty(X_{kl}, L_{kl})$ with respect to the norms $s \mapsto \left(\int_{X_{kl}} |s|_A^2 \Omega\right)^{\frac{1}{2}}$, $\omega \mapsto \left(\int_{X_{kl}} |\omega|_{\check{B}}^2 \Omega\right)^{\frac{1}{2}}$, $s \mapsto \left(\int_{X_{kl}} |s|_A^2 \Omega + \int_{X_{kl}} |\overline{\partial} s|_{\check{B}}^2 \Omega\right)^{\frac{1}{2}}$ respectively. We denote the completion of the Cauchy–Riemann operators by $\overline{\partial}_{kl} : V_{kl} \to \check{B}_{kl}$. Using Theorem 3.15 and the fact that X_{bc} is a *compact* Riemann surface without boundary we conclude that

$$\begin{pmatrix} \overline{\partial}_{ad} & 0 \\ 0 & \overline{\partial}_{bc} \end{pmatrix} : V_{ad} \oplus V_{bc} \to \check{B}_{ad} \oplus \check{B}_{bc}$$

is a Fredholm operator.

Next, we want to show that the nondiagonal terms $(\overline{\partial}\chi_c)$ form only a *compact* perturbation. The key to this fact is the following lemma.

Lemma 3.32 (Compactness of the inclusion map) *Let $V_U \subset V_{kl}$ be the closure of $\mathcal{C}_c^\infty(U, L) \subset \mathcal{C}_c^\infty(X_{kl}, L_{kl})$ in V_{kl}, $U \subset\subset X_{kl}$. Then the inclusion map $i_U : V_U \hookrightarrow A_{kl}$ is compact.*

Proof/Reference: We cover U with a finite number of coordinate domains U_n, $1 \le n \le N$. Cutting i_U into N pieces supported in U_n using a partition of unity, we see, that it suffices to show that every inclusion map $i_{U_n} : V_{U_n} \to A_{kl}$ is compact. Now the problem is local; but the weight functions for the Hermitian metrics are locally bounded from above and from below by positive constants. The assertion follows e. g. from Theorem 10.1.10 in [4]. □

As a consequence, we get the promised result:

Lemma 3.33 (Compactness of the perturbation) *The following multiplication operator is compact:*

$$\begin{pmatrix} 0 & (\overline{\partial}\chi_c) \\ (-\overline{\partial}\chi_c) & 0 \end{pmatrix} : V_{ad} \oplus V_{bc} \to \check{B}_{ad} \oplus \check{B}_{bc}$$

Proof: We factor $(\overline{\partial}\chi_c) : V_{k'l'} \to \check{B}_{kl}$ into $V_{k'l'} \xrightarrow{\cdot\phi} V_U \overset{i_U}{\hookrightarrow} A_{kl} \xrightarrow{\cdot(\overline{\partial}\chi_c)} \check{B}_{kl}$. The second map is compact, the first and the third map are bounded since ϕ and $\overline{\partial}\chi_c$ are compactly supported; hence the composition is compact. □

Let us harvest the result of this section:

Proposition 3.34 (Exchange lemma) *All operators $\overline{\partial}_{kl} : V_{kl} \to \check{B}_{kl}$ ($k = a, b$; $l = c, d$) are Fredholm operators. The following relation holds:*

$$\text{index}\,\overline{\partial}_{ac} + \text{index}\,\overline{\partial}_{bd} = \text{index}\,\overline{\partial}_{ad} + \text{index}\,\overline{\partial}_{bc}\ .$$

Proof: We have

$$\begin{pmatrix} \overline{\partial}_{ac} & 0 \\ 0 & \overline{\partial}_{bd} \end{pmatrix} = J^{-1} \circ \left[\begin{pmatrix} \overline{\partial}_{ad} & 0 \\ 0 & \overline{\partial}_{bc} \end{pmatrix} + \begin{pmatrix} 0 & (\overline{\partial}\chi_c) \\ (-\overline{\partial}\chi_c) & 0 \end{pmatrix} \right] \circ J \ ,$$

where we have called the completion of J again J. Here $J : V_{ac} \oplus V_{bd} \to V_{ad} \oplus V_{bc}$ and $J : \check{B}_{ac} \oplus \check{B}_{bd} \to \check{B}_{ad} \oplus \check{B}_{bc}$ are bounded and invertible,

$$\begin{pmatrix} \overline{\partial}_{ad} & 0 \\ 0 & \overline{\partial}_{bc} \end{pmatrix}$$

is a Fredholm operator and

$$\begin{pmatrix} 0 & (\overline{\partial}\chi_c) \\ (-\overline{\partial}\chi_c) & 0 \end{pmatrix}$$

is a compact perturbation. Since perturbations of Fredholm operators by compact operators are again Fredholm operators with unchanged index (see e. g. [12], Chapter VII, Cor. 1 to Thm. 2 and Cor. to Thm. 4), it follows that

$$\begin{pmatrix} \overline{\partial}_{ac} & 0 \\ 0 & \overline{\partial}_{bd} \end{pmatrix}$$

is a Fredholm operator and that

$$\operatorname{index} \begin{pmatrix} \overline{\partial}_{ad} & 0 \\ 0 & \overline{\partial}_{bc} \end{pmatrix} = \operatorname{index} \begin{pmatrix} \overline{\partial}_{ac} & 0 \\ 0 & \overline{\partial}_{bd} \end{pmatrix} .$$

□

Corollary 3.35 (Index version of the Riemann Roch Theorem) *If the compact piece X_c is chosen sufficiently large then*

$$\operatorname{index} \overline{\partial}_{ac} = \operatorname{index} \overline{\partial}_{bc} - 1 = -\operatorname{codeg}(L, \psi_{0A}) \ .$$

Proof: Remember that we still assume order $\nu = -1$ at infinity. We show that index $\overline{\partial}_{bd} = 1$, index $\overline{\partial}_{ad} = 0$ and index $\overline{\partial}_{bc} = 1 - \operatorname{codeg}(L, \psi_{0A})$.

- $X_{bd} = \mathbb{P}_1$, and L_{bd} is a trivial line bundle over $\mathbb{P}_1$, hence index $\overline{\partial}_{bd} = 1$ by the classical Riemann–Roch Theorem (or simply by elementary function theory on the Riemann sphere).
- For a compact piece chosen sufficiently large, we know by the previous sections that $\overline{\partial}_{ad}$ is invertible, hence index $\overline{\partial}_{ad} = 0$.
- For the line bundle L_{bc} over the *compact* Riemann surface X_{bd}, the classical Riemann–Roch Theorem tells us

$$\operatorname{index} \overline{\partial}_{bc} = 1 - \operatorname{genus}(X_{bc}) + \deg(L_{bc}) = 1 - \operatorname{codeg}(L, \psi_{0A}).$$

Using the previous proposition the result follows. □

3.5 Limit at infinity of square integrable sections

Technically, we have used so far the condition of having finite weighted L_2-norms for global holomorphic sections. But the Riemann Roch Theorem 2.1 for infinite genus Riemann surfaces imposed pointwise bounds in a neighbourhood of infinity for these holomorphic sections. The goal of the section is to show that these two concepts are equivalent. Using the symmetry $A \leftrightarrow B$, $L \leftrightarrow \check{L}^{1,0}$, we restrict our examinations to the case of $L, |\cdot|_A$.

The next lemma shows that square integrability of global holomorphic sections can be decided in the regular sheet:

Lemma 3.36 (Reduction to the regular sheet; $V \cap \mathcal{O} = A \cap \mathcal{O}$**)** *For every global holomorphic section f in L over X the following statements are equivalent:*

a) $\int_{U_0} |f|_A^2 \Omega < \infty$*; U_0 can be replaced by any neighbourhood of infinity,*

b) $\int_X |f|_A^2 \Omega < \infty$*, i. e.* $f \in A$*,*

c) $f \in V$*.*

Proof: b)⇒a) is trivial. a)⇒b): We do the proof for U_0 only; for an arbitrary neighbourhood of ∞ it is similar: We first estimate the total L^2–norm over the handles in terms of the total L^2–norm over the regular sheet: Since f is holomorphic in the handles, $If = f$ by Lemma 3.23. Hence we get with the help of Lemma 3.22 c):

$$\sum_{j \in J} \int_{U_j} |f|_A^2 \Omega \leq \text{const}_1 \sum_j \int_{V_{\pm j}} |f|_A^2 \Omega \leq \text{const}_1 \int_{U_0} |f|_A^2 \Omega \ .$$

It remains to estimate the total L^2–norm over the compact piece U_c in terms of the total L^2–norm over the regular sheet, i. e. we have to show:

$$\int_{U_c \setminus U_0} |f|_A^2 \Omega \leq \text{const}_2 \int_{U_0 \cap U_c} |f|_A^2 \Omega \leq \text{const}_2 \int_{U_0} |f|_A^2 \Omega \ .$$

Since X is noncompact we know that L is a trivial holomorphic line bundle (see e. g. [3], Satz 30.4). (Of course we expect any basis section to raise fast at infinity.) We are only interested in the relatively compact domain U_c, hence we may reduce our considerations to the case of $\mathbb{C}$–valued holomorphic *functions* f, $|\cdot|_A \sim |\cdot|$ over U_c. As a consequence of the maximum principle and of Cauchy's integral formula, we can estimate the sup–norm (and therefore also the L^2–norm) of $f|(U_c \setminus U_0)$ in terms of the L^2–norm of $f|(U_c \cap U_0)$ in this case, using that $U_c \setminus U_0$ is a relatively compact subset of U_c.

Summarising the considerations, we get an estimate

$$\int_X |f|_A^2 \Omega \leq \text{const} \int_{U_0} |f|_A^2 \Omega \ .$$

c)⇒b) is trivial again. b)⇒c): This is the most interesting statement of the lemma: it states that every square integrable global holomorphic section in L can be approximated in the $\|\cdot\|_V$–norm by smooth sections with compact support.

Instead of proving this directly, we note that it is an easy consequence of the parts of the Riemann–Roch theorem which are already proved:

We already know

$$\dim\ker(\overline{\partial} : V \to \check{B}) - \dim\operatorname{coker}(\overline{\partial} : V \to \check{B}) = -\operatorname{codeg}(L, \psi_{0A}) \ ,$$

writing $\overline{\partial}$ now for all Cauchy Riemann operators irrespective of their domain. Set $M := \{ f \in A \mid f \text{ is holomorphic}\}$, $N := \{\omega \in B \mid \omega \text{ is holomorphic}\}$. Then $\ker(\overline{\partial} : V \to \check{B}) = M \cap V$, and using Lemma 3.3 we know

$$\operatorname{coker}(\overline{\partial} : V \to \check{B})^\vee \cong \ker(\overline{\partial} : B \to \check{V}) = N \ .$$

Consequently

$$\dim(M \cap V) - \dim N = -\operatorname{codeg}(L, \psi_{0A}) \ . \tag{20}$$

We consider the dual situation, $A \leftrightarrow B$: similarly to the definition of V, let W be the completion of $\mathcal{C}^\infty_c(X, \check{L}^{1,0})$ with respect to the norm

$$\|\cdot\|_W : \omega \mapsto \left(\|\omega\|^2_B + \left\|\overline{\partial}\omega\right\|^2_{\check{A}}\right)^{\frac{1}{2}} \ .$$

Dual to (20), we get

$$\dim(N \cap W) - \dim M = -\operatorname{codeg}(\check{L}^{1,0}, \psi_{0B}) \ . \tag{21}$$

Adding (20) and (21) yields

$$\begin{aligned} &\operatorname{codim}(M \cap V, M) + \operatorname{codim}(N \cap W, N) \\ &\qquad = \operatorname{codeg}(L, \psi_{0A}) + \operatorname{codeg}(\check{L}^{1,0}, \psi_{0B}) \ . \end{aligned} \tag{22}$$

The right hand side vanishes: With the notations from Definition 2.3 and with $L_{bc}(\psi_{0A}) \otimes \check{L}^{1,0}_{bc}(\psi_{0A}^{-1}\, dz_0) \cong \mathcal{E}^{1,0}(dz_0)$ we obtain

$$\begin{aligned} &\operatorname{codeg}(L, \psi_{0A}) + \operatorname{codeg}(\check{L}^{1,0}, \psi_{0B}) = \\ &= 2\operatorname{genus}(X_{bc}) - \deg L_{bc}(\psi_{0A}) - \deg \check{L}^{1,0}_{bc}(\psi_{0A}^{-1}\, dz_0) \\ &= \deg \mathcal{E}^{1,0}(dz_\infty) + 2 - \deg L_{bc}(\psi_{0A}) - \deg \check{L}^{1,0}_{bc}(\psi_{0A}^{-1}\, dz_0) \\ &= \deg \mathcal{E}^{1,0}(dz_\infty) + 2 - \deg \mathcal{E}^{1,0}(dz_0) = 0 \ . \end{aligned}$$

We made use of the degree of any canonical divisor on the compact Riemann surface X_{bc} being $\deg \mathcal{E}^{1,0}(dz_\infty) = 2\operatorname{genus}(X_{bc}) - 2$ and of $dz_0/dz_\infty = -z_\infty^{-2}$ having order -2 at infinity.

Hence both codimensions on the left hand side of (22) vanish. This means $M \cap V = M$, $N \cap W = N$. This finishes the proof of b)⇒c). □

Finally we examine the pointwise behaviour at ∞ of square integrable holomorphic sections in L. We formulate the result for any order ν at ∞, for future references slightly more general than needed here:

Proposition 3.37 (limit at infinity) *Let the Hermitian metric $|\cdot|_A$ be defined using the order $\nu \in \mathbb{Z}$ at infinity, i. e. $|\psi_{0A}|_A \sim |z_0|^{-\nu}$ near infinity. Let f_∞ be a L–valued holomorphic section defined over an open neighbourhood U_∞ of ∞ and let f_j, $j \in J$, be L–valued holomorphic sections defined over the handles U_j. Assume that $\sup_{U_\infty \cap U_j} |f_\infty - f_j|_A \leq O(\alpha_j^{-1})$ and $\int_{U_\infty} |f_\infty|_A^2 \Omega < \infty$. Then $\lim_{P\to\infty} f_\infty/(z_0^\nu \psi_{0A})(P) \in \mathbb{C}$ exists.*

One case of special interest is $f_\infty = f_j$ on $U_\infty \cap U_j$:

Corollary 3.38 (L^2–bound $\iff$ pointwise bound near ∞) *A global holomorphic section f in L is square integrable (using $|\cdot|_A$ with the order ν at ∞) if and only if it is bounded near infinity in the regular sheet:*
$\limsup_{P\to\infty} \left| f/(z_0^\nu \psi_{0A}) \right| < \infty$.

Proof of the Corollary: We observe that $|f/(z_0^\nu \psi_{0A})| \sim |f|_A$, which holds for taking an order $\nu \in \mathbb{Z}$ at infinity in the definition of $|\cdot|_A$. Then "⇐" is a consequence of Lemma 3.36, while "⇒″ follows from Proposition 3.37. □

Proof of Proposition 3.37: Here is the strategy for the proof: first we reduce the problem to the Riemann sphere by removing once more all the handles U_j and filling the resulting holes with disks. The section f_∞ is interpolated "almost holomorphically" through the disks using the "disk–version" I^0 of the interpolation operator, which was introduced before Lemma 3.24. Finally the limit at ∞ is examined on the Riemann sphere.

By shrinking the regular piece if necessary we may assume without loss of generality that U_∞ contains the regular piece U_0. Note that this redefinition $U_{0,\text{new}} = U_0 \cap U_\infty$ of the regular piece depends on U_∞, hence all the cutoff functions χ_0, $(\chi_j)_{j\in J}$ depend on the choice of U_∞, too.

We view the regular sheet U_0 again both as a subset of the Riemann sphere $\mathbb{P}_1$ (just as in the proof of Lemma 3.29) and as a subset of the Riemann surface X. The disks that are glued to V_j are denoted by $D_j^0 \subseteq \mathbb{P}_1$ again.

We define the interpolated version f of f_∞ on $U_\infty \cup \bigcup_j D_j^0$ by

$$f = \chi_0 f_\infty + \sum_j \chi_j I^0 f_j \ . \tag{23}$$

Then f is holomorphic outside the intersections V_j. It is square integrable because

$$\int\limits_{U_\infty \cup \bigcup_j D_j^0} |f|_A^2 \,\Omega \quad \leq \quad 2\int\limits_{U_\infty} |f_\infty|_A^2 \,\Omega + 2\sum_j \int\limits_{D_j^0} \left|I^0 f_j\right|_A^2 \Omega \qquad (24)$$

$$\overset{\text{L.3.22}c)}{\leq} 2\int\limits_{U_\infty} |f_\infty|_A^2 \,\Omega + 2M\sum_j \int\limits_{V_j} \left|f_j\right|_A^2 \Omega$$

which is finite as

$$\int\limits_{V_{\pm j}} |f_j|_A^2 \Omega \leq 2\int\limits_{V_{\pm j}} (|f_\infty|_A^2 + |f_\infty - f_j|_A^2)\Omega \leq \int\limits_{V_{\pm j}} (2|f_\infty|_A^2 + O(\alpha_j^{-2}))\Omega$$

is summable over j. On V_j the deviation of f from holomorphy is bounded:

$$\sup_{V_j} |\overline{\partial} f|_{\check{B}} = \sup_{V_j} |(\overline{\partial}\chi_j)(f_\infty - I^0 f_j)|_{\check{B}}$$
$$\leq \sup_{V_j} |\overline{\partial}\chi_j|_2 (|f_\infty - f_j|_A + |f_j - I^0 f_j|_A)$$

For the first summand on the right hand side we have $\sup_j \sup_{V_j} |\overline{\partial}\chi_j|_2 |f_\infty - f_j|_A < \infty$ by Lemma 3.19 and the hypothesis on $|f_\infty - f_j|_A$. To estimate the second summand, we note that $If_j = f_j$ by Lemma 3.23 and that $I^0 f_j/(z_0^{\nu+1}\psi_{0A})$ satisfies the condition Ext-Disk in (L2). Consequently Lemma 3.25 and Lemma 3.27b) yield

$$\sup_{V_j} |I^0 f_j - f_j|_A \leq O(o_j)\alpha_j^2 \int\limits_{V_j \cup V_{-j}} |f_j|_A \Omega \leq O(o_j)\alpha_j \left(\int\limits_{V_j \cup V_{-j}} |f_j|_A^2 \Omega \right)^{\frac{1}{2}}$$
$$\leq O(o_j)\alpha_j \,.$$

We end up with the estimate

$$\sup_{V_j} |\overline{\partial} f|_{\check{B}} \leq O(1) + \alpha_j^2 O(o_j) \overset{(X6)}{\leq} O(1). \qquad (25)$$

In some punctured neighbourhood U^* of ∞ in $\mathbb{P}_1$ we write (24) and (25) in the coordinate $z_\infty = 1/z_0$ and use $F := f/(z_0^\nu \psi_{0A})$, $|f|_A \sim |F|$ and $|\overline{\partial} f|_{\check{B}} \sim |\partial F/\partial\overline{z_\infty}|\,|z_0^\nu \psi_{0A}|_A |dz_\infty|_2 \sim |\partial F/\partial\overline{z_\infty}|$. This yields $\int_{U^*} |F|^2\, d^2 z_\infty < \infty$ and $\sup_{U^*} |\partial F/\partial\overline{z_\infty}| < \infty$. The statement of Proposition 3.37 then is an immediate consequence of the following Lemma. □

Lemma 3.39 *Let $U \subseteq \mathbb{C}$ be an open neighbourhood of 0, $U^* = U \setminus \{0\}$. Assume that $F : U^* \to \mathbb{C}$ is $\mathcal{C}^1$ and that $\int_{U^*} |F(z)|^2 \, d^2z$ and $\sup_{z \in U^*} |\partial F(z)/\partial \overline{z}|$ are finite. Then $\lim_{z \to 0} F(z) \in \mathbb{C}$ exists.*

Proof: We may suppose that U is bounded. Define $G : U \to \mathbb{C}$ by convolution of $\frac{\partial F}{\partial \overline{z}}$ with the Cauchy kernel:

$$G(w) = \frac{1}{2\pi i} \int_{U^*} \frac{1}{z - w} \frac{\partial}{\partial \overline{z}} F(z) \, dz \wedge d\overline{z} \, .$$

Then $\partial G/\partial \overline{z} = \partial F/\partial \overline{z}$ on U^*. $\partial F/\partial \overline{z}$ being bounded and $\frac{1}{z-w}$ being locally integrable implies that G is a continuous function even at 0. $F - G$ is holomorphic on U^* and square integrable near 0, hence it extends to a holomorphic function on U. Consequently $\lim_{z \to 0} F(z) = \lim_{z \to 0}(F(z) - G(z)) + \lim_{z \to 0} G(z)$ exists. □

To summarize, we have shown that the pointwise asymptotic bound $f \in \mathcal{M}$ for holomorphic sections $f \in \mathcal{O}(X, L)$ used in the Riemann Roch Theorem is equivalent to the L^2-condition $f \in A$. By symmetry $A \leftrightarrow B$, $L \leftrightarrow \check{L}^{1,0}$ we conclude

$$\omega \in \mathcal{N} \quad \Leftrightarrow \quad \omega \in \mathcal{O}(X, \check{L}^{1,0}) \text{ and } \omega \in B.$$

Hence the index version 3.35 of the Riemann Roch Theorem is equivalent to the infinite genus Riemann Roch Theorem 2.1. This finishes the proof of the theorem.

We remark that as a byproduct of Proposition 3.37 we obtain that the limit at infinity of $f/(z_\infty^{-\nu} \psi_{0A})$ exists for $f \in \mathcal{M}$.

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