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Equivariant first order differential operators on boundaries of symmetric spaces

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Introduction

Let G be a real semisimple Lie group and let, with the usual notations (described in detail in Sect. 2), MAN be a minimal parabolic subgroup. Any finite dimensional irreducible representation (ρ, V) of MAN is determined by an irreducible representation δ of M and a character ν of A; there is a corresponding homogeneous vector bundle $G \times_{MAN} V$ over G/MAN. The principal series of representations of G operates on the sections of such bundles. The G-equivariant linear operators from one such bundle to another are the intertwining operators for the principal series. In the present paper we determine all those intertwining operators which can be written as differential operators of order one.

The result is stated in Theorem 2.2; it amounts to the following. Take any irreducible representation δ of M and decompose its tensor product with the natural representation of M on a simple root space $\mathbf{g}_\lambda$ of the Lie algebra of G into irreducible representations (δ_j, V_j). Let $G \times_{MAN} V$ be the bundle determined by δ and the character ν of A, $G \times_{MAN} V_j$ the bundle determined by δ_j and $\nu + \lambda$. A necessary and sufficient condition for the existence of a G-equivariant first order differential operator from $G \times_{MAN} V$ to $G \times_{MAN} V_j$ is then written down as an explicit linear condition on ν. The operator itself is also explicitly written down, and this construction gives all G-equivariant first order operators.

For the case $G = SO_0(n+1, 1)$ this is an old result of Fegan [F]. In this case G/MAN is the unit sphere $S^n = \{x \in \mathbf{R}^{n+1} | \, |x| = 1\}$ and G is the group of conformal transformations of S^n. The exterior derivative d mapping k-forms to $(k+1)$-forms is an obvious example of equivariant differential operators; there are of course many others. There is one among them with great importance for the theory of quasiconformal mappings, the Ahlfors operator. In the above general construction it is obtained by taking for δ the (complexification of) the standard representation of $M = SO(n)$ on $\mathbf{R}^n$, and for δ_j the representation on traceless symmetric matrices. The study of this operator leads, in [R], to the result that quasiconformal deformations of S^n considered as the boundary of real hyperbolic space can be extended to quasiconformal deformations of real hyperbolic space itself. Extending Fegan's results step by step over the past several years we found, somewhat to our surprise, that they remain true for all semisimple groups. This is due to the fact that the analysis can be split into two parts. The first is a reduction to the real rank one case, which is based on Araki's work [A]. What remains for the second part are the simple groups of real rank one, which can be handled by a case-by-case analysis.

It should be mentioned that while our result is quite satisfying from the point of view of the theory of the principal series, it is less so from the point of view of geometry. In fact, many bundles of geometric interest (cf. e.g. [KR]) arise from non-irreducible and not fully reducible representations of MAN, and these are not included in our result. It seems that it would be quite difficult to make a meaningful general statement including all the non-irreducible cases; at any rate there is an open problem here.

We learned from B. Ørsted that he had discovered our Theorem 2.2 independently. His methods are rather different from ours. Furthermore, the referee informs us, that Zelobenko [Z] has constructed related operators. The construction is based on the theory of Verma modules and their duals.

The first section of the present paper contains generalities about equivariant differential operators. We need some statements that are more general than those in the literature (e.g. in [Wa]), therefore we give a concise self-contained treatment. The second section contains the main result, Theorem 2.2, and its proof except for the essential technical fact formulated as Theorem 2.1. The proof of this result, which uses the rank one reduction and a careful analysis of the group M is given in Sect. 3.

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1. Equivariant differential operators on homogeneous vector bundles

In this section, G denotes a Lie group with Lie algebra $\mathbf{g}$ and H a closed subgroup. The homogeneous vector bundles $G \times_H V$ over G/H arise from finite dimensional representations ρ of H on a vector space V. The space $G \times_H V$ is the quotient space of $G \times V$ under the equivalence relation $(g, \upsilon) \sim (gh, \rho(h^{-1})\upsilon)$ and the map $(g, \upsilon) \to gH$ induces a projection $p : G \times_H V \to G/H$. The group G acts on $G \times_H V$ by left translation. This action is denoted by τ.

Given local trivializations $U_o \times V$ around $o = eH$ and $U_g \times V$ around $go = gH$, τ_g induces a mapping again denoted by τ_g from a neighbourhood of $\{o\} \times V$ in $U_o \times V$ to $U_g \times V$.

If $s : G/H \to G \times_H V$ is a section, then its lift $f_s : G \to V$ is defined by

$$f_s(g) = \tau_g^{-1} s(go).$$

The lifts of the smooth sections make up the space $C^\infty(G, V)^H$ of C^∞-functions $f : G \to V$ satisfying

$$f(gh) = \rho(h^{-1}) f(g)$$

for all $g \in G,\ h \in H$. The action of G on $C^\infty(G, V)^H$ is by left translation.

Conversely, to any function $f \in C^\infty(G, V)^H$ there corresponds the section $s_f : G/H \to G \times_H V$ given by

$$s_f(gH) = (g,\ f(g)).$$

The left invariant first order differential operators on G are given by the Lie algebra vectors $X \in \mathbf{g}$

$$X f(g) = \frac{d}{dt}\Big|_{t=0} \quad f(g \exp tX)$$

and the left invariant differential operators of arbitrary order are obtained by extending this definition to the universal enveloping algebra $\mathcal{U}$ of $\mathbf{g}$. More generally, the left invariant differential operators $C^\infty(G, V_1) \to C^\infty(G, V_2)$ are given by $\mathcal{U} \otimes \mathrm{Hom}(V_1, V_2) \cong \mathcal{U} \otimes V_1^* \otimes V_2$; the element $U \otimes \upsilon_1^* \otimes \upsilon_2$ maps $f \in C^\infty(G, V_1)$ to $(U\langle f, \upsilon_1^*\rangle)\upsilon_2 \in C^\infty(G, V_2)$. In the following the equivariant differential operators from smooth sections of $G \times_H V_1$ to smooth sections of $G \times_H V_2$ will be described. They turn out to be the left invariant differential operators which map $C^\infty(G, V_1)^H \to C^\infty(G, V_2)^H$.

Let $\mathcal{U}(\mathbf{h})$ be the universal enveloping algebra of $\mathbf{h}$ and let $Y \to Y'$ be its principal anti-automorphism (i.e. the one determined by $Y \to -Y$ for elements $Y \in \mathbf{h}$). Given a representation ρ of H on a vector space V_1, it induces a representation ρ_* of $\mathbf{h}$ and of $\mathcal{U}(\mathbf{h})$ on V_1.

Let J denote the linear span of the elements

$$UY \otimes L - U \otimes (L \circ \rho_*(Y'))$$

where $U \in \mathcal{U}, Y \in \mathcal{U}(\mathbf{h})$ and $L \in \mathrm{Hom}(V_1, V_2)$ with V_2 a fixed vector space. We write

$$\mathcal{U} \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2) = (\mathcal{U} \otimes \mathrm{Hom}(V_1, V_2))/J.$$

(This is just the standard notion of tensor product of a left and a right $\mathcal{U}(\mathbf{h})$-module.)

Let $\mathbf{q}$ be a subspace of $\mathbf{g}$ complementary to $\mathbf{h}$. Let $Y_1, \ldots, Y_q$ be a basis of $\mathbf{q}$, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_q)$ of natural numbers we write Y^α for $Y_1^{\alpha_1} \ldots Y_q^{\alpha_q}$ [and $Y(\alpha)$ for the symmetrized product, i.e. for the coefficient of t^α in $(\sum t_j Y_j)^{\alpha_1 + \ldots + \alpha_q}$ where $t_1, \ldots, t_q$ stand for real numbers.]

Proposition 1.1. $\mathcal{U} \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2)$ *can be identified with the space of* G*-left invariant differential operators from the* C^∞*-sections of* $G \times_H V_1$ *to* $C^\infty(G, V_2)$*. Having chosen* $\mathbf{q}$ *and its basis* $\{Y_j\}$*, every element of* $\mathcal{U} \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2)$ *can uniquely be written in the "normal form"*

$$\sum_\alpha Y^\alpha \otimes L_\alpha$$

with $L_\alpha \in \mathrm{Hom}(V_1, V_2)$.

Proof. Let $X_1, \dots, X_h$ be a basis of $\mathbf{h}$. By the Poincaré–Birkhoff–Witt theorem, in a self-explanatory notation, all elements of $\mathcal{U} \otimes \mathrm{Hom}(V_1, V_2)$ can be written as finite sums $\sum_{\alpha,\beta} Y^\alpha X^\beta \otimes L_{\alpha\beta}$.

Using the definition of J, the factor X^β can be shifted to the other side of the tensor product, proving the existence of the normal form. [Since the Y^α can be written as linear combinations of the $Y(\alpha)$, this statement also applies to the other kind of normal form.]

Next we show that the action of $\mathcal{U} \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2)$ on $C^\infty(G, V_1)^H$ is well-defined: By linearity for this we must see only that the elements of J act trivially on $C^\infty(G, V_1)^H$. Since $\mathbf{h}$ generates $\mathcal{U}(\mathbf{h})$, it suffices to verify this for elements of the form $UY \otimes L$ with $Y \in \mathbf{h}$. Now

$$\begin{aligned}(UY \otimes L)\, f(g) &= L\, \tfrac{d}{dt}\Big|_o (Uf)(g \exp tY) = L\, \tfrac{d}{dt}\Big|_o \rho(\exp -tY)(Uf)(g) \\ &= L\rho_*(Y')(Uf)(g) = (U \otimes L \circ \rho_*(Y'))\, f(g)\end{aligned}$$

proving our statement.

The action of $\mathcal{U} \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2)$ on sections is defined by transfer to the lifted sections. This is indeed an action by differential operators. In fact, an operator on sections is by definition a differential operator if, in the neighbourhood of any point, in terms of some (hence any) local trivialization of the bundle, it is a (linear) differential operator. In terms of a local cross-section, e.g. $r : g \exp t_1 Y_1 \dots \exp t_q Y_q \cdot o \to g \exp t_1 Y_1 \dots \exp t_q Y_q$ of G/H in a neighbourhood of $g \cdot o$, there is a natural local trivialization in which a section s of the bundle becomes exactly $f_s \circ r$. From this our statement is easy to see.

We see that to each element of $\mathcal{U} \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2)$ there corresponds a G-left invariant differential operator. To see that the correspondence is one-to-one, suppose we have an element such that the corresponding operator is 0. We may assume that this element is given in normal form $\sum L_\alpha \otimes Y^\alpha$. Now the essential fact is that using a local trivialization, as above, every V_1-valued function on a neighbourhood of (say) o occurs as the (restriction of the) lift of some bundle section. Therefore if $L_\alpha \neq 0$ for some $\alpha = \alpha_o$, we can easily construct a section s such that $Y^\alpha f_s = 0$ for all $\alpha \neq \alpha$ but $L_{\alpha_o}(Y^{\alpha_o} f_s)(e) \neq 0$. With this argument we have also shown the uniqueness of the normal form.

Finally, to see that all equivariant differential operators arise from our construction, suppose that D is such an operator. Then writing D in terms of our local cross section, we have at e an expression of the form

$$\tilde{D}s(e) = \sum_{\alpha} L_\alpha \left.\frac{\partial^{|\alpha|}}{\partial t^\alpha}\right|_o f_s(\exp t_1 Y_1 \ldots \exp t_n Y_n) = \sum_{\alpha} L_\alpha (Y^\alpha f_s)(e).$$

This is then valid also for all G-translates of s, which shows that D arises from the element $\sum Y^\alpha \otimes L_\alpha$. ◻

Suppose we have two representations ρ_1 and ρ_2 of H on the vector spaces V_1, V_2. Then H acts naturally on $\mathrm{Hom}(V_1, V_2)$ by $h \in H$ sending any element L to $\rho_2(h) \circ L \circ \rho_1(h^{-1})$. H also acts on $\mathcal{U}$, by the adjoint representation. The tensor product action clearly leaves the subspace J invariant, hence we have an H-action on $\mathcal{U} \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2)$. We denote the subset of H-invariant elements by $(\mathcal{U} \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2))^H$.

Proposition 1.2. *Given two homogeneous vector bundles $G \times_H V_1$ and $G \times_H V_2$ over G/H, the space of G-equivariant differential from the first one into the second one is isomorphic to $(\mathcal{U} \otimes_{\mathcal{U}(\mathbf{h})} \mathrm{Hom}(V_1, V_2))^H$.*

Proof. From our earlier remarks it is clear that a G-equivariant differential operator from the sections of $G \times_H V_1$ to the sections of $G \times_H V_2$ is the same thing as an equivariant differential operator from $G \times_H V_1$ to $C^\infty(G, V_2)$ whose image is contained in $C^\infty(G, V_2)^H$. So, by Proposition 1.1, we will be finished if we can show that an element of $\mathcal{U} \otimes \mathrm{Hom}(V_1, V_2)$ maps $C^\infty(G, V_1)^H$ into $C^\infty(G, V_2)^H$ if and only if it is H-invariant modulo J.

Let $X \in \mathbf{g}$, $L \in \mathrm{Hom}(V_1, V_2)$, and suppose that f is in $C^\infty(G, V_1)^H$. Then, for $g \in G$, $h \in H$,

$$\begin{aligned}(X \otimes L) f(gh) &= L \left.\tfrac{d}{dt}\right|_o f(gh \exp tX) = L \left.\tfrac{d}{dt}\right|_o \rho_1(h^{-1}) f(g \exp t\mathrm{Ad}(h)X) \\ &= (\mathrm{Ad}(h)X \otimes L\rho_1(h^{-1}))\, f(g).\end{aligned}$$

This will be equal to $\rho_2(h^{-1})(X \otimes L) f(g)$ for all f and all g, h, if and only if $X \otimes L$ is invariant under the action of H modulo the kernel of the action of $\mathcal{U} \otimes \mathrm{Hom}(V_1, V_2)$, which we know to be equal to J. The same statement is then true for any element of $\mathcal{U}$ in place of X, and for any linear combination of terms of the form $X \otimes L$. ◻

Remarks. If H is compact the subspace J has an H-invariant complement, and therefore every equivariant operator can be representated by an H-invariant element of $\mathcal{U} \otimes \mathrm{Hom}(V_1, V_2)$.

In the general case, for an algebraically given element, even when it is in normal form, it may be difficult to determine whether it is H-invariant modulo J. Nevertheless, our results can be quite useful. For example, if we know that D is an equivariant operator of the type considered in Proposition 1.2., and we want to find its algebraic expression, it is enough to find an element $\hat{D}$ of $\mathcal{U} \otimes \mathrm{Hom}(V_1, V_2)$ such that $(\hat{D} f_s)(e) = f_{Ds}(e)$ for all sections s of $G \times_H V_1$. Then $\hat{D}$ is automatically $H-$invariant modulo J without any need of further proof.

For instance, the operator d from functions to 1-forms on G/H is equivariant by general principles. The cotangent bundle is naturally identified with $G \times_H (\mathbf{g}/\mathbf{h})^*$. Let $\{Y_j\}$ be a basis of $\mathbf{g}/\mathbf{h}$ and $\{Y_j^*\}$ the dual basis. It is clear that writing $\hat{d} = \sum_j Y_j \otimes Y_j^*$ the effect of d and of $\hat{d}$ at the base point is the same. Hence $\hat{d}$ is H-invariant modulo J (which is of course also rather easy to verify directly).

These remarks allow a slight simplification in the proofs in Sect. 4 of [KR]: When finding the "algebraic versions" of the operators discussed there, it is not necessary to give direct proofs of their H-invariance.

2. The first order differential operators on the maximal boundaries of symmetric spaces

In the following G will be a connected real simple Lie group with finite center and K a maximal compact subgroup of G. Our results are actually true more generally, with rather obvious modifications of the proofs: It would be enough to assume that G is semisimple, or even only that it is reductive and contained in the class considered in [Wo]. We make our more stringent hypothesis in order to be more concise and to concentrate on the essential points. Let $\mathbf{g} = \mathbf{k} + \mathbf{p}$ be a Cartan decomposition of the Lie algebra $\mathbf{g}$ of G, with Cartan involution ϑ and let us denote by B the Killing form. Choose a maximal abelian subalgebra $\mathbf{a} \subset \mathbf{p}$ and introduce an ordering in the dual $\mathbf{a}^*$ of $\mathbf{a}$. If $\sum$ is the set of non vanishing restricted roots of $\mathbf{g}$ with respect to $\mathbf{a}$ then the Lie algebra decomposes as

$$\mathbf{g} = \mathbf{g}_o + \sum_{\lambda \in \Sigma} \mathbf{g}_\lambda$$

with

$$\mathbf{g}_\lambda = \{X \in \mathbf{g} : [H, X] = \lambda(H)X \quad \text{for all } H \in \mathbf{a}\}.$$

The subspace

$$\mathbf{n} = \sum_{\lambda \in \Sigma^+} \mathbf{g}_\lambda$$

(with Σ^+ the positive roots) is a nilpotent subalgebra of $\mathbf{g}$, and the Lie algebra $\mathbf{g}$ has the Iwasawa decomposition

$$\mathbf{g} = \mathbf{k} + \mathbf{a} + \mathbf{n}.$$

If $\mathbf{m}$ denotes the centralizer of $\mathbf{a}$ in $\mathbf{k}$ and

$$\overline{\mathbf{n}} = \sum_{\lambda<0} \mathbf{g}_\lambda = \sum_{\lambda>0} \vartheta \mathbf{g}_\lambda$$

then the root decomposition can be written in the form

$$\mathbf{g} = \mathbf{m} + \mathbf{a} + \mathbf{n} + \overline{\mathbf{n}}.$$

The groups A and N are the analytic subgroups of G with Lie algebra $\mathbf{a}$ and $\mathbf{n}$ respectively. M is the centralizer of A in K, its Lie algebra is $\mathbf{m}$.

The subgroup MAN is a minimal parabolic subgroup and G/MAN is the maximal boundary of the symmetric space G/K. In this section we will describe the invariant first order differential operators between vector bundles $G \times_{MAN} V$ over the boundary under the assumption that the representations ρ of MAN on the finite dimensional complex vector spaces V are irreducible (hence M-irreducible, with scalar A- and trivial N-action).

Recall that a root $\lambda \in \Sigma$ is called simple, if it is positive and if it cannot be represented as a sum of two positive roots. The real rank one subalgebra generated by $\mathbf{g}_\lambda$ and ϑ will be denoted by $\mathbf{g}^\lambda$ and H_λ will be the element in $\mathbf{a}$ defined by

$$B(H, H_\lambda) = \lambda(H) \quad \text{for all } H \in \mathbf{a}.$$

Let (δ, V) be a complex irreducible unitary representation of M and denote by $(\mathrm{Ad}_{\mathbf{g}_\lambda, M}, \mathbf{g}_\lambda^{\mathbf{C}})$ the complexification of the adjoint representation restricted to M on the root space $\mathbf{g}_\lambda$. If no misunderstanding is possible we will suppress the index M and just write $\mathrm{Ad}_{\mathbf{g}_\lambda}$. The tensor product $(\mathrm{Ad}_{\mathbf{g}_\lambda, M} \otimes \delta, \mathbf{g}_\lambda^{\mathbf{C}} \otimes V)$ decomposes as a direct sum of irreducible representations

$$\mathrm{Ad}_{\mathbf{g}_\lambda, M} \otimes \delta = \oplus_l \, m_l \delta_l$$
$$\mathbf{g}_\lambda^{\mathbf{C}} \otimes V = \oplus \, V_l^{m_l}$$

with multiplicities m_l. The projection operator from $\mathbf{g}_\lambda^{\mathbf{C}} \otimes V$ onto $V_l^{m_l}$ will be denoted by pr_{δ_l}.

Theorem 2.1. *The multiplicities m_l which occur in the decomposition of*

$$\mathrm{Ad}_{\mathbf{g}_\lambda} \otimes \delta = \oplus_l m_l \delta_l$$

over M are one provided (V, δ) is an irreducible representation and λ is a simple restricted root.

The proof of this theorem will be given in Sect. 3.

The group MAN is the semidirect product of MA with the nilpotent group N and M commutes with A. If (ρ, V) is an irreducible complex representation of MAN, then its restriction to N is trivial and its restriction to M (denoted by δ) is still irreducible. On A the representation has to be scalar

$$\rho(a) = a^{\mu} I \qquad a \in A.$$

Here a^{μ} stands for $\exp\mu(H)$ when $a = \exp H$ and μ is a linear function on $\mathbf{a}$. The irreducible complex representation (ρ, V) of MAN is completely determined by the pair (δ, μ): a^{μ} is a character of A and (δ, V) is an irreducible complex representation of M.

Choose a basis $\{e_k\}$ of V and let $\{e_k^*\}$ be the dual basis. Also choose an orthonormal basis $\{Y_j\}$ of $\mathbf{g}_{-\lambda}$ with respect to the positive definite form $(.\,,\ .) = -B(.\,,\vartheta\,.)$. Then $\{\vartheta Y_j\}$ will be an orthonormal basis in $\mathbf{g}_{\lambda}$.

For an irreducible unitary M-representation the Casimir operator $C_{\delta} = -\sum_k (\delta_* Z_k)^2$ with $\{Z_k\}$ an orthonormal basis of $\mathbf{m}$ acts as a scalar

$$C_{\delta} = c(\delta) I_{\delta}.$$

The constant $c(\delta)$ depends on the choice of the scalar product in the Lie algebra $\mathbf{m}$ of the compact group M. The standard choice would be the Killing form of $\mathbf{m}$. However in the present context the scalar product is the restriction of $-B_{\mathbf{g}}(., \vartheta .)$ to $\mathbf{m}$ and as such C_{δ} is the “relative” Casimir operator.

Theorem 2.2. *Assume that λ is a simple restricted root, that $\rho = (\delta, \mu)$ is an irreducible complex MAN-representation on V and that (δ_l, V_l) is an irreducible component in the decomposition of $(\mathrm{Ad}_{\mathbf{g}_\lambda} \otimes \delta, \mathbf{g}_\lambda \otimes V)$ over M. If the representation of MAN on V_l is $(\delta_l, \mu + \lambda)$ and if the character μ of A satisfies*

$$2\mu(H_\lambda) = c(\delta) + c(\mathrm{Ad}_{\mathbf{g}_\lambda}) - c(\delta_l)$$

then

$$\nabla_{\mu,\delta_l} := \sum_{j,k} Y_j \otimes e_k^* \otimes pr_{\delta_l}(\vartheta Y_j \otimes e_k)$$

is a G-equivariant operator $C^\infty(G, V)^{MAN} \to C^\infty(G, V_l)^{MAN}$. Conversely, any first order equivariant differential operator $D : C^\infty(G, V)^{MAN} \to C^\infty(G, W)^{MAN}$ with irreducible actions of MAN on V and W is of the form $U \circ \nabla_{\mu,\delta_l}$ with U a MAN-equivariant mapping $V_l \to W$.

The proof of this theorem will occupy the remainder of this section. It is based on two auxiliary results.

Proposition 2.3. *If $\lambda \in \Sigma^+$ is a positive root which is not simple, then there exists a simple root $\mu > 0$ such that $[Z, Y] \neq 0$ whenever $0 \neq Z \in \mathbf{g}_\mu$ and $0 \neq Y \in \mathbf{g}_{-\lambda}$.*

Proof. Represent $\lambda > 0$ in the form $\lambda = \sum_{\mu\,\text{simple}} c_\mu \mu$ with $c_\mu \geq 0$. The Killing form B restricted to $\mathbf{a}$ is positive definite. Let $\langle .\,,\ .\rangle$ be the dual scalar product on $\mathbf{a}^*$ such that $\langle \mu, \lambda \rangle = B(H_\lambda, H_\mu)$. It is invariant under linear automorphisms preserving Σ.

From

$$0 < \langle \lambda, \lambda \rangle = \sum_{\mu \text{ simple}} c_\mu \langle \mu, \lambda \rangle$$

it follows that $\langle \mu, \lambda \rangle > 0$ for at least one simple root μ. Consequently $\lambda - \mu$ is a root (see [H] p. 457). Furthermore $\lambda + \mu$ is not a root except possibly if the root system is G_2. This is clear if λ is a multiple of μ. Otherwise, consider the μ-string of roots $\lambda + k\mu,\ \ p \le k \le q$. It satisfies ([H] p. 457).

$$p + q = -2\frac{\langle \lambda, \mu \rangle}{\langle \mu, \mu \rangle} := -a_{\lambda\mu} < 0$$

and hence $p + q$ can only be -3, -2 or -1.

If $\lambda + \mu$ were a root, then $q \ge 1$ and consequently

$$p = -q - a_{\lambda\mu} \le -2.$$

This would give a chain of length $q - p + 1 \ge 4$. The classification of root systems shows, that G_2 is the only root system with a chain of (non proportional) roots which is of length 4. Otherwise only chains of length ≤ 3 can occur. Hence $\lambda + \mu$ is not a root except possibly in the case G_2.

In all these cases, the Jacobi identity together with

$$[Z, \vartheta Z] = B(Z, \vartheta Z) H_\mu \quad Z \in \mathbf{g}_\mu$$

([H], p. 407) show that for $0 \ne Z \in \mathbf{g}_\mu,\ 0 \ne Y \in \mathbf{g}_{-\lambda}$

$$\begin{aligned}[][[Z, Y], \vartheta Z] &= [[Z, \vartheta Z], Y] + [Z, [Y, \vartheta Z]] \\ &= B(Z, \vartheta Z)\lambda(H_\mu) Y \\ &\ne 0\end{aligned}$$

since $\langle \lambda, \mu \rangle > 0$, and $[Y, \vartheta Z] = 0$.

The only symmetric space with restricted root system G_2 is G/K where G is the normal real form of the complex semisimple group of type G_2. Since it is a normal real form, the $\mathbf{R}$-subspace $\mathbf{a}$ is a real form of a complex Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$, and each restricted root space is a real form of an $\mathbf{h}$-root space, so has real dimension 1. If $\lambda - \mu$ is a root, then ([H], theorem 4.3 (iv), p. 168)

$$[\mathbf{g}_\mu, \mathbf{g}_{-\lambda}] = \mathbf{g}_{\mu - \lambda} \ne 0.$$

This proves the proposition. □

Proposition 2.4. *Let* $\{X_1, \ldots, X_n\}$ *be an orthonormal basis of the root space* $\mathbf{g}_\lambda$*, define the mappings* $E_{ij} \in \operatorname{Hom}(\mathbf{g}_\lambda, \mathbf{g}_\lambda)$ *by:*

$$\begin{aligned} X_i &\mapsto X_j \\ X_k &\mapsto 0 \qquad k \ne i. \end{aligned}$$

If $\{Z_1, \ldots, Z_m\}$ *is a basis for* $\mathbf{m}$*, which is orthonormal with respect to the positive definite form* $(.\,,\,.) = -B(.\,,\vartheta\,.)$*, then*

$$\sum_{i\neq j}[\vartheta X_i, X_j] \otimes E_{ij} = \sum_{k=1}^{m} Z_k \otimes \mathrm{ad}_{\mathbf{g}_\lambda} Z_k.$$

Proof. After identifying $\mathbf{m}$ with its dual under the positive definite form $(.\,,.) = -B(.\,,\vartheta.)$, the elements of $\mathbf{m} \otimes \mathrm{Hom}(\mathbf{g}_\lambda, \mathbf{g}_\lambda)$ can be regarded as elements of $\mathrm{Hom}(\mathbf{m}, \mathrm{Hom}(\mathbf{g}_\lambda, \mathbf{g}_\lambda))$. To show the equality it suffices to apply both sides to $Z \in \mathbf{m}$. The right hand side sends Z to $\mathrm{ad}_{\mathbf{g}_\lambda} Z$ (the mapping $\mathrm{ad}Z$ restricted to the subspace $\mathbf{g}_\lambda$):

$$\sum_k (Z, Z_k)\mathrm{ad}_{\mathbf{g}_\lambda} Z_k = \mathrm{ad}_{\mathbf{g}_\lambda}\left(\sum_k (Z, Z_k) Z_k\right) = \mathrm{ad}_{\mathbf{g}_\lambda} Z.$$

The left hand side sends Z to

$$\sum_{i\neq j} -B(Z, \vartheta[\vartheta X_i, X_j])E_{ij} = \sum_{i\neq j} -B([Z, X_i], \vartheta X_j)E_{ij}$$
$$= \sum_{i\neq j} (\mathrm{ad}_{\mathbf{g}_\lambda} Z(X_i), X_j)E_{ij}.$$

Now, $\mathrm{ad}_{\mathbf{g}_\lambda} Z$ is skew symmetric since M acts on $\mathbf{g}_\lambda$ by orthogonal transformations. Therefore the last sum is equal to

$$\sum_{i,j} (\mathrm{ad}_{\mathbf{g}_\lambda} Z(X_i), X_j)E_{ij} = \mathrm{ad}_{\mathbf{g}_\lambda} Z.$$

□

Proof of the theorem. Suppose that $D \neq 0$ is a first order equivariant differential operator $D : C^\infty(G, V)^{MAN} \longrightarrow C^\infty(G, W)^{MAN}$. It has a normal form

$$D = \sum_j Y_j \otimes L_j + L_o$$

with $\{Y_j\}$ an orthonormal basis of $\bar{\mathbf{n}}$, which is chosen so that each Y_j belongs to some $\mathbf{g}_{-\lambda}$. The corresponding $\lambda \in \Sigma^+$ will be denoted by $\lambda(j)$. Under the identification of $\mathrm{Hom}(V, W)$ with $V^* \otimes W$ the maps L_j take the form

$$L_j = \sum_k e_k^* \otimes f_{jk} \qquad f_{jk} \in W.$$

Introduce $U : \mathrm{Hom}(\bar{\mathbf{n}}, V) \longrightarrow W$ by

$$U : \vartheta Y_j \otimes e_k \longrightarrow f_{jk}$$

so that

$$D = U \circ \nabla + L_o$$

with $\nabla = \sum_{jk} Y_j \otimes e_k^* \otimes (\vartheta Y_j \otimes e_k)$. Thus the operator D can be factored through the gradient ∇.

Let us first consider the action of $a \in A$. On the vector spaces V and W, the action is scalar. It is determined by characters μ and ν and we will write a^μ for the action on V, a^ν on W. Similarly, the action on the root spaces $\mathbf{g}_{-\lambda}$ will be written as $a^{-\lambda}$.

$$\begin{aligned} a \cdot D &= a \cdot \left(\sum_{jk} Y_j \otimes e_k^* \otimes f_{jk} + \sum_k I \otimes e_k^* \otimes f_{ok} \right) \\ &= a^{-\mu} a^\nu \sum_{jk} a^{-\lambda(j)} Y_j \otimes e_k^* \otimes f_{jk} + a^{-\mu} a^\nu \sum_k I \otimes e_k^* \otimes f_{ok}. \end{aligned}$$

The operator $D - a \cdot D$ is in normal form. It vanishes modulo the kernel of the action of $\mathcal{U} \otimes V^* \otimes W$ if and only if it vanishes, and this happens in two cases

1) either $\nu = \mu$ and $f_{jk} = 0$ unless $j = 0$; in this case D is a zero order operator, $D = L_o$.
2) or $\nu = \mu + \lambda$ for some root λ and $f_{jk} = 0$ unless $\lambda(j) = \lambda$; in this case there is no zero order term; the differential operator is $D = U \circ \nabla_\lambda$ with ∇_λ the gradient restricted to the root space $\mathbf{g}_{-\lambda}$ (the expression for ∇_λ contains only a basis $\{Y_j\}$ of $\mathbf{g}_{-\lambda}$).

We discard the first case as trivial and next study the M-invariance of operators of the form $D = U \circ \nabla_\lambda$ with $\mathbf{g}_{-\lambda}$ a fixed root space. If δ is the representation of M on V, σ its representation on W, then

$$m \cdot U = \sigma(m) \circ U \circ \delta(m^{-1})$$

whereas clearly

$$m \cdot \nabla = \nabla \quad \text{and} \quad m \cdot \nabla_\lambda = \nabla_\lambda.$$

Altogether

$$mD = \sum_{jk} Y_j \otimes e_k^* \otimes \sigma(m) \circ U \circ \delta(m^{-1})(\vartheta Y_j \otimes e_k)$$

and as before $D - m \cdot D$ is in normal form. It vanishes if and only if

$$\sigma(m) \circ U \circ \delta(m^{-1})(\vartheta Y_j \otimes e_k) = \vartheta Y_j \otimes e_k \quad \text{for all } j, k.$$

This characterizes the M-intertwining operators $U : V \to W$.

Finally, consider the N-invariance for the operators $D = U \circ \nabla_\lambda$. The group N being connected it suffices to consider the infinitesimal action by $\mathbf{n}$. Since the simple root spaces generate all of $\mathbf{n}$, it remains to verify that

$$(\vartheta Y_i) \cdot D = \sum_{\lambda(j)=\lambda} \sum_k [\vartheta Y_i, Y_j] \otimes e_k^* \otimes U(\vartheta Y_j \otimes e_k) = 0$$

for all ϑY_i in simple root spaces.

We now apply Proposition 2.3. If λ is not simple, then there exists a simple root μ such that $[Z, Y_j] \neq 0$ for all $Z \in \mathbf{g}_\mu$, $Z \neq 0$. Setting $Z = \vartheta Y_i \in \mathbf{g}_\mu$, the $[\vartheta Y_i, Y_j]$ are linearly independent (for fixed i) and it is impossible that $(\vartheta Y_i) \cdot D = 0$ mod the kernel of the action of $\mathcal{U} \otimes V^* \otimes W$. The operator $U \circ \nabla_\lambda$ can therefore only be equivariant if λ is a simple root.

Assuming that λ is a simple root it follows that $[\vartheta Y_i, Y_j] \in \mathbf{a} + \mathbf{m}$ and that the bracket vanishes unless $\lambda(i) = \lambda$. Consequently

$$(\vartheta Y_i) \cdot D \equiv \sum_{\lambda(j)=\lambda} \sum_k I \otimes \rho_*^\vee([\vartheta Y_i, Y_j]) e_k^* \otimes U(\vartheta Y_i \otimes e_k)$$

modulo the kernel.

The condition for N-invariance becomes

$$\sum_{ijk} Y_i \otimes \rho_*^\vee([\vartheta Y_i, Y_j]) e_k^* \otimes U(\vartheta Y_j \otimes e_k) = 0$$

with summation over indices i, j such that $\lambda(i) = \lambda(j) = \lambda$. This tensor product can be interpreted as a linear transformation $A : \mathbf{g}_\lambda \otimes V \to \mathbf{g}_\lambda \otimes V$ followed by U. Setting $E_{ij} = Y_i \otimes \vartheta Y_j$ with the interpretation that E_{ij} is the linear transformation $\mathbf{g}_\lambda \to \mathbf{g}_\lambda$ mapping ϑY_i to ϑY_j and ϑY_k to 0 for $k \neq i$, we obtain A as a tensor product of mappings:

$$A = \sum_{ij} \rho_*[\vartheta Y_i, Y_j] \otimes E_{ij}.$$

Recall then ([H], p. 407) that

$$[\vartheta Y_i, Y_i] = B(Y_i, \vartheta Y_i) H_\lambda = -H_\lambda \in \mathbf{a}$$

and

$$[\vartheta Y_i, Y_j] \in \mathbf{m} \quad \text{for } i \neq j.$$

This leads to an expression

$$A = \sum_{i=j} \rho_*(-H_\lambda) \otimes E_{ij} + \sum_{i \neq j} \delta_*[\vartheta Y_i, Y_j] \otimes E_{ij}$$

to which Proposition 2.4 applies:

$$\begin{aligned} A &= \sum_i \mu(-H_\lambda) I_{\mathbf{g}_\lambda} \otimes E_{ii} + \sum_k \delta_* Z_k \otimes \operatorname{ad}_{\mathbf{g}_\lambda} Z_k \\ &= -\mu(H_\lambda) I_{\mathbf{g}_\lambda} \otimes I_V + \sum_k (\delta_* \otimes \operatorname{ad}_{\mathbf{g}_\lambda})(Z_k \otimes Z_k). \end{aligned}$$

As explained at the beginning of this section, for an irreducible M-representation the Casimir operator $C_\delta = -\sum_k (\delta_* Z_k^2)$ acts as a scalar:

$$C_\delta = c(\delta) I_\delta.$$

Upon decomposing $\delta \otimes \mathrm{Ad}_{\mathbf{g}_\lambda} = \oplus_l \delta_l, \;\; V \otimes \mathbf{g}_\lambda^C = \oplus V_l$, (the multiplicities are one according to Theorem 2.1) the Casimir operator $C_{\delta\otimes \mathrm{Ad}_{\mathbf{g}_\lambda}}$ is given by $\oplus_l c(\delta_l) I_{V_l}$. On the other hand

$$\begin{aligned} -\sum_k ((\delta \otimes \mathrm{Ad}_{\mathbf{g}_\lambda})_* Z_k)^2 &= -\sum_k (\delta_* Z_k \otimes I_\lambda + I \otimes \operatorname{ad}_{\mathbf{g}_\lambda} Z_k)^2 \\ &= -\sum_k \left\{ (\delta_* Z_k)^2 \otimes I + I \otimes (\operatorname{ad}_{\mathbf{g}_\lambda} Z_k)^2 + 2\delta_* Z_k \otimes \operatorname{ad}_{\mathbf{g}_\lambda} Z_k \right\} \\ &= C_\delta \otimes I + I \otimes C_{\mathrm{Ad}_{\mathbf{g}_\lambda}} - 2\sum_k (\delta_* \otimes \operatorname{ad}_{\mathbf{g}_\lambda})(Z_k \otimes Z_k) \\ &= (c(\delta) + c(\mathrm{Ad}_{\mathbf{g}_\lambda})) I_V \otimes I_{\mathbf{g}_\lambda} - 2\delta_* \otimes \operatorname{ad}_{\mathbf{g}_\lambda} \cdot \left(\sum_k Z_k \otimes Z_k \right). \end{aligned}$$

From this it follows that

$$-\sum_k (\delta_* \otimes \operatorname{ad}_{\mathbf{g}_\lambda})(Z_k \otimes Z_k) = \oplus_l \frac{1}{2}(c(\delta_l) - c(\delta) - c(\mathrm{Ad}_{\mathbf{g}_\lambda})) I_{V_l}$$

and finally the condition for N-invariance becomes

$$\begin{aligned} & U \circ \sum_{ij} \rho_*([\vartheta Y_i, Y_j]) \otimes E_{ij} \\ &\quad = U \circ \oplus_l (-\mu(H_\lambda) - \frac{1}{2}(c(\delta_l) - c(\delta) - c(\mathrm{Ad}_{\mathbf{g}_\lambda}))) I_{V_l} \\ &\quad = 0. \end{aligned}$$

If $\mu(H_\lambda) \neq \frac{1}{2}(c(\delta) + c(\mathrm{Ad}_{\mathbf{g}_\lambda}) - c(\delta_l))$ then U must vanish on V_l. Furthermore, U is an M-intertwining homomorphism onto the irreducible representation space W with M-representation σ. The only situation in which U does not vanish identically is that $\sigma \sim \delta_l$ and

$$2\mu(H_\lambda) = c(\delta) + c(\mathrm{Ad}_{\mathbf{g}_\lambda}) - c(\sigma).$$

In this case U factors over pr_{δ_l}. This proves the theorem. □

3. Real rank one reduction and the group M

As before, let

$$\mathbf{g} = \mathbf{m} + \mathbf{a} + \sum_{\lambda \in \Sigma} \mathbf{g}_\lambda$$

be the root space decomposition of the simple (real) Lie algebra $\mathbf{g}$.

Proposition 3.1. For $\lambda \in \Sigma^+$ an indivisible restricted root let $\mathbf{g}^\lambda$ denote the subalgebra of $\mathbf{g}$ generated by $\mathbf{g}_\lambda$ and $\mathbf{g}_{-\lambda}$. Then $\mathbf{g}^\lambda$ is a simple Lie algebra of real rank one. The Cartan decomposition of $\mathbf{g}^\lambda$ with respect to $\vartheta|_{\mathbf{g}^\lambda}$ is

$$\mathbf{g}^\lambda = (\mathbf{k} \cap \mathbf{g}^\lambda) + (\mathbf{p} \cap \mathbf{g}^\lambda) := \mathbf{k}^\lambda + \mathbf{p}^\lambda$$

and $\mathbf{a}^\lambda = \mathbf{R}H_\lambda$ with H_λ determined by $B(H_\lambda, H) = \lambda(H)$ for all $H \in \mathbf{a}$ is a maximal abelian subspace of $\mathbf{p}^\lambda$. The root space decomposition of $\mathbf{g}^\lambda$ is

$$\mathbf{g}^\lambda = \mathbf{g}_{-2\lambda} + \mathbf{g}_{-\lambda} + \mathbf{g}_o^\lambda + \mathbf{g}_\lambda + \mathbf{g}_{2\lambda}$$

with $\mathbf{g}_o^\lambda = \mathbf{a}^\lambda + \mathbf{m}^\lambda$, $\mathbf{m}^\lambda = [\mathbf{g}_\lambda, \mathbf{g}_{-\lambda}] \cap \mathbf{m}$ (where $\mathbf{g}_{-2\lambda}, \mathbf{g}_{2\lambda}$ have to be omitted if 2λ is not a restricted root).
(See e.g.[H], p. 407–409).

Proposition 3.2. [C]

$$\mathbf{m}^\lambda = span\{[X, \vartheta Y] : X, Y \in \mathbf{g}_\lambda, (X, Y) = 0\}.$$

Proof. Let $\mathbf{m}_o$ denote $span\{[X, \vartheta Y] : X, Y \in \mathbf{g}_\lambda, (X, Y) = 0\}$. Then $[\vartheta X, Y] = \vartheta[X, \vartheta Y] \in \mathbf{m}_o$ and $\mathbf{m}_o + \mathbf{a}^\lambda$ is both $span\{[X, \vartheta Y] : X, Y \in \mathbf{g}_\lambda\}$ and $span\{[\vartheta X, Y] : X, Y \in \mathbf{g}_\lambda\}$.

Now $\mathbf{m}_o + \mathbf{a}^\lambda$ is an ideal in $\mathbf{m}^\lambda + \mathbf{a}^\lambda$ since for any $M \in \mathbf{m}^\lambda$

$$[M, [X, \vartheta Y]] = [[M, X], \vartheta Y] + [X, \vartheta[M, Y]] \in \mathbf{m}_o + \mathbf{a}^\lambda.$$

Further, $[\mathbf{g}_{-2\lambda}, \mathbf{g}_{2\lambda}] \subset \mathbf{m}_o + \mathbf{a}^\lambda$ since, if $Z, Z' \in \mathbf{g}_{2\lambda}$, then there exist $X, Y \in \mathbf{g}_\lambda$ with $Z = [X, Y]$ and

$$[Z, \vartheta Z'] = [[X, Y], \vartheta Z'] = [[X, \vartheta Z'], Y] + [X, [Y, \vartheta Z']] \in \mathbf{m}_o + \mathbf{a}.$$

It follows that $\mathbf{g}_{-2\lambda} + \mathbf{g}_{-\lambda} + \mathbf{m}_o + \mathbf{a} + \mathbf{g}_\lambda + \mathbf{g}_{2\lambda}$ is an ideal in $\mathbf{g}^\lambda$, hence all of $\mathbf{g}^\lambda$ and consequently $\mathbf{m}^\lambda = \mathbf{m}_o$. □

Corollary. $\mathbf{m}^\lambda$ *is an ideal in* $\mathbf{m}$.

Proof. For $Z \in \mathbf{m}$ and $X, Y \in \mathbf{g}_\lambda$

$$[Z, [X, \vartheta Y]] = [[Z, X], \vartheta Y] + [X, [Z, \vartheta Y]] \subset \mathbf{m}^\lambda + \mathbf{a}^\lambda.$$

But if $X \perp Y$, then the left hand side is in $\mathbf{m}$ and hence

$$[Z, \mathbf{m}^\lambda] \subset \mathbf{m}^\lambda$$

for all $Z \in \mathbf{m}$. □

The group M is compact and hence reductive. Its Lie algebra decomposes into the direct sum

$$\mathbf{m} = \mathbf{z} + [\mathbf{m}, \mathbf{m}]$$

of its center $\mathbf{z}$ and the semisimple ideal $[\mathbf{m}, \mathbf{m}]$ ([Hu], p. 102). This decomposition is orthogonal with respect to the scalar product $(.\,,\,.) = -B(.\,, \vartheta .)$ defined via the Killing form B of $\mathbf{g}$. In fact, for $Z \in \mathbf{z}$ and $X, Y \in \mathbf{m}$

$$(Z, [X, Y]) = -B(Z, [X, Y]) = B([X, Z], Y) = B(0, Y) = 0.$$

The semisimple ideal $[\mathbf{m}, \mathbf{m}]$ decomposes into an orthogonal sum of simple ideals $\mathbf{m}_j$

$$[\mathbf{m}, \mathbf{m}] = \oplus_{j=1}^{n} \mathbf{m}_j$$

([Hu], p. 23) and the $\mathbf{m}_j$ are uniquely determined. Orthogonality follows from the relation

$$\begin{aligned} B(Z_i, Z_j) &= B([X_i, Y_i], Z_j) = -B(Y_i, [X_i, Z_j]) \\ &= -B(Y_i, 0) = 0 \end{aligned}$$

with $Z_j \in \mathbf{m}_j$ and $Z_i \in \mathbf{m}_i$ with Z_i of the form $Z_i = [X_i, Y_i]$ since $\mathbf{m}_i$ is simple.

The following proposition is well known.

Proposition 3.3. *Any ideal in* $\mathbf{m}$ *is of the form*

$$\mathbf{z}' + \oplus_{j \in J} \mathbf{m}_j$$

for some index set $J \subset \{1. \dots, n\}$ *and some subspace* $\mathbf{z}' \subset \mathbf{z}$.

Proposition 3.4. *Let* $\mathbf{m}^\alpha, \mathbf{m}^\lambda$ *be the subalgebras of the real rank one algebras* $\mathbf{g}^\alpha, \mathbf{g}^\lambda$ *which were defined above (with* α, λ *indivisible restricted roots). Then the* ad*-action of* $\mathbf{m}^\alpha$ *on* $\mathbf{g}^\lambda$ *is trivial if and only if* $\mathbf{m}^\alpha$ *is orthogonal to* $\mathbf{m}^\lambda$.

The ad-action of $\mathbf{m}^\alpha$ on $\mathbf{g}_\lambda$ is trivial if and only if $[U, W] = 0$ for all $W \in \mathbf{g}_\lambda$ and for all $U \in \mathbf{m}^\alpha$. The element $[U, W]$ lies in $\mathbf{g}_\lambda$, hence it vanishes if and only if

$$B([U, W], \vartheta Z) = B(U, [W, \vartheta Z]) = 0$$

for all $U \in \mathbf{m}^\alpha$ and $W, Z \in \mathbf{g}_\lambda$.

Since $\operatorname{span}\{[W, \vartheta Z] : W, Z \in \mathbf{g}_\lambda\} = \mathbf{m}^\lambda + \mathbf{a}^\lambda$, this condition is satisfied if and only if $\mathbf{m}^\alpha$ is orthogonal to $\mathbf{m}^\lambda$.

If the ad-action of $\mathbf{m}^\alpha$ is trivial on $\mathbf{g}^\lambda$, then *a fortiori* it is trivial on $\mathbf{g}_\lambda$. Conversely, the ad-action of $\mathbf{m}^\alpha$ will be trivial on $\mathbf{g}_{-\lambda}$ if it is trivial on $\mathbf{g}_\lambda$. The algebra $\mathbf{g}^\lambda$ being generated by $\mathbf{g}_\lambda$ and $\mathbf{g}_{-\lambda}$, the ad-action of $\mathbf{m}^\alpha$ will then also be trivial on $\mathbf{g}^\lambda$. □

The orthogonality relation $\mathbf{m}^\alpha \perp \mathbf{m}^\lambda$ implies that $\mathbf{m}^\alpha \cap \mathbf{m}^\lambda = 0$, because the Killing form B restricted to $\mathbf{m}$ is strictly negative definite, $\mathbf{m}$ being a compactly embedded subalgebra of $\mathbf{g}$ ([H], Prop. 6.8, p. 133).

Lemma 3.5. *Suppose $G = G_1 \cdot G_2$ is a commutative product of subgroups in the sense that each $g \in G$ can be written as $g = g_1 g_2$ with $g_i \in G_i$ $i = 1, 2$ and all elements of G_1 and G_2 mutually commute. Then every irreducible (complex) unitary representation (δ, V) of G is of the form $\delta(g_1 g_2) = \delta_1(g_1) \otimes \delta_2(g_2)$, $V = V_1 \otimes V_2$, with irreducible unitary representations (δ_i, V_i) of G_i $(i = 1, 2)$, such that for all $g \in G_1 \cap G_2$*

$$\delta_j(g) = \chi(g) I_{V_j} \quad j = 1, 2$$

with some (scalar) character χ of $G_1 \cap G_2$.

The lemma is a consequence of Schur's lemma (see e.g. [W], where essentially the same result is attributed to Burnside, or [B], p. 22). □

If G is a connected semisimple Lie group with finite center and Iwasawa decomposition $G = KAN$ then by a result of Satake [S] the centralizer M of A in K decomposes as a commutative product

$$M = Z_1 \, M_0$$

with M_0 the identity component of M and Z_1 finite. Let $\mathbf{g}^\lambda = \mathbf{g}_{-2\lambda} + \mathbf{g}_{-\lambda} + \mathbf{g}_0^\lambda + \mathbf{g}_\lambda + \mathbf{g}_{2\lambda}$ be a real rank one subalgebra of $\mathbf{g}$ as in Proposition 3.1 and let G^λ, K^λ and A^λ denote the analytic subgroups of G with Lie algebras $\mathbf{g}^\lambda, \mathbf{k}^\lambda$ and $\mathbf{a}^\lambda$. The centralizer of A^λ in K^λ is denoted by M^λ, its identity component by M_0^λ. Except in the case $\mathbf{g}^\lambda \cong sl(2, \mathbf{R})$, the quotient K^λ / M^λ is a sphere of dimension ≥ 2, so is simply connected, and therefore $M^\lambda = M_0^\lambda$.

Proposition 3.6. *There exists a subgroup M' of M_0 such that*

$$M_0 = M' M_0^\lambda \qquad \textit{(commutative product)}$$

and $Ad_{g_\lambda}(M') = \{I\}$.

The proof is based on the fact that $\mathrm{ad}_{g_\lambda}(\mathbf{m}^\lambda)$ consists of all derivations of $\mathbf{m}^\lambda = \mathbf{g}_\lambda + \mathbf{g}_{2\lambda}$, which are skew-symmetric with respect to the Killing form (see [CDKR2], Lemma 4.3).

Now $\mathrm{ad}_{g_\lambda}(\mathbf{m})$ also consists of skew-symmetric derivations of $\mathbf{m}^\lambda$. It follows that

$$\mathbf{m} = \mathbf{m}^\lambda \oplus \mathbf{m}'$$

with $\mathbf{m}'$ the kernel of the representation ad_{g_λ} of $\mathbf{m}$. M_0 is then the commutative product of M_0^λ with M', the analytic group with Lie algebra $\mathbf{m}'$. Furthermore $\mathrm{Ad}_{g_\lambda}(M') = \{I\}$. This proves the proposition. $\square$

As a consequence

$$\begin{aligned} M &= Z_1 M_0 = Z_1 M' M_0^\lambda \\ &= M'' M_0^\lambda = M_0^\lambda M'' \text{ with } M'' = Z_1 M'. \end{aligned}$$

It is proved in [S] that any $z_1 \in Z_1$ is of the form $z_1 = \exp iH$ with some $H \in \mathbf{a}$. In particular

$$\begin{aligned} \mathrm{Ad}(z_1) &= e^{ad(iH)} \\ \mathrm{Ad}_{g_\lambda}(z_1) &= e^{i\lambda(H)} I_{g_\lambda}. \end{aligned}$$

Since $\mathrm{Ad}(z_1)$ is a real linear transformation of $\mathbf{g}$ we must have $e^{i\lambda(H)} \in \mathbf{R}$, i.e. $= \pm I$. Together with the above proposition this shows that $\mathrm{Ad}_{\mathbf{g}_\lambda}(M'') = \{\pm I\}$.

For $m'' \in M''$ we write $\mathrm{Ad}_{\mathbf{g}_\lambda}(m'') = \chi(m'') I_{g_\lambda}$. Then χ is a character of M''.

Proposition 3.7. *Suppose G is a connected Lie group of real rank one. Let $\mathbf{g}, \mathbf{k}, \mathbf{a}, \mathbf{m}, K, A, M$ as above, and $\mathbf{g}_\lambda$ be the root space for the (unique!) indivisible positive root. Let δ be any complex irreducible representation of M and τ be the representation of M given by the adjoint action of M on $\mathbf{g}_\lambda^C$. Then the tensor product $\tau \otimes \delta$ decomposes into irreducible components with multiplicity one.*

Proof. From classification ([H], p. 513, or [CDKR1] for a more direct approach) we know that there are only the following four possibilities for $\mathbf{g}$:
(i) $\mathbf{so}(n+1, 1)$,
(ii) $\mathbf{su}(n+1, 1)$,
(iii) $\mathbf{sp}(n+1, 1)$,
(iv) $\mathbf{f}_{4(-20)}$
(in Helgason's notation). Assuming $n \geq 1$ in each case, this is a list without repetitions. The corresponding $\mathbf{m}'s$ are $\mathbf{so}(n)$, $\mathbf{su}(n) + \mathbf{R}$, $\mathbf{sp}(n) + \mathbf{sp}(1)$, $\mathbf{so}(7)$.

The adjoint action of M on $\mathbf{g}_\lambda$ is
(i) the natural action of $SO(n)$ on $\mathbf{R}^n$,
(ii) the natural action of $U(n)$ on $\mathbf{C}^n$,
(iii) the action of $Sp(n) \times Sp(1)$ on $\mathbf{H}^n$, one factor acting on the left, the other on the right,
(iv) the 8-dimensional spin-representation of *Spin*(7).
In case (ii), the complexification τ of this representation splits as $\tau = \tau_0 \oplus \overline{\tau}_0$, the natural representation of $U(n)$ on $\mathbf{C}^n$ and its complex conjugate. This leads to a generalization of the classical decomposition $d = \partial + \overline{\partial}$ of the outer derivative d. In the other cases τ remains irreducible.

As explained before Proposition 3.6, M is connected in all cases, except possibly when $\mathbf{g} = \mathbf{so}(2, 1)$. When $\mathbf{g} = \mathbf{so}(2, 1)$, τ is a scalar representation, and therefore our Proposition is trivial; we exclude this case in what follows. Then the representation δ is entirely determined by its highest weight. We may, and we will, regard δ as well as τ as a representation of the universal covering group $\tilde{M}$ of M.

For case (i) with n even and for case (iv) the Proposition can be proved as follows. In [Bou] Ch. VIII, §13 the fundamental representations of the classical groups are concretely described and their highest weights identified (cf. also [Si] sec. IX.8 for the same results). The list on p. 129 of [Bou] then shows that τ is a minuscule representation (note the misprint in [Bou]: the minuscule weights for B_l and C_l are interchanged). It is well known that for any ρ the tensor product with a minuscule representation decomposes with multiplicity one. The components will be exactly the representations whose highest weight are that of δ plus a weight of τ, whenever this sum is dominant. (For a proof in print cf. the argument in Lemma 11 of [Ma].)

A small modification of this argument works also for cases (ii) and (iii). In case (iii), $\tilde{M}$ is a product, so $\delta = (\delta', \delta'')$ with δ' and δ'' being irreducible representations of $Sp(n)$ resp. $Sp(1)$, and δ acts on the tensor product of the two representation spaces in the natural way.

Similarly $\tau = (\tau', \tau'')$ and [Bou] Ch. VIII, §13 shows that τ, τ' are minuscule weights for $Sp(n)$ resp. $Sp(1)$. So $\tau \otimes \delta = (\tau' \otimes \delta', \tau'' \otimes \delta'')$ with both terms between the parentheses decomposing with multiplicity one. Hence $\tau \otimes \delta$ also decomposes with multiplicity one.

In case (ii) we have $\tilde{M} = SU(n) \times \mathbf{R}$ and we can again write $\rho = (\rho', \rho'')$, $\tau_0 = (\tau_0', \tau_0'')$ where ρ'' and τ_0'' are now unitary characters of $\mathbf{R}$. We have

$$\tau \otimes \delta = (\tau_0 \oplus \overline{\tau}_0) \otimes \delta = (\tau_0' \otimes \delta', \tau_0'' \otimes \delta'') \oplus (\overline{\tau}_0' \otimes \delta', \overline{\tau}_0'' \otimes \delta'').$$

Again τ_0' and $\overline{\tau}_0'$ are minuscule, so the first components decompose with multiplicity one. Since τ_0'' is not trivial, the (one-dimensional) representations $\tau_0'' \otimes \delta''$ and $\overline{\tau}_0'' \otimes \delta''$ are different. Hence the final decomposition of $\tau \otimes \delta$ is still of multiplicity one.

There remains case (i) with n odd; here τ is not minuscule, so the above arguments don't work. This case is covered by Theorem 3.4 of [F] for which Fegan gives a sketchy proof. Another, perhaps simpler way to deal with it is to apply the result of Exercise 9 on p. 142 of [Hu]. We omit the details.

□

Remark. In the case $\mathbf{g} = so(n + 1, 1)$ Fegan [F] notes that for any δ, each irreducible component of $\tau \otimes \delta$ gives a different scalar value for the Casimir operator. This allows him to state his Theorem 1.1 in a little stronger form than our Theorem 2.2, namely he does not have to assume that the target bundle of D is associated to an irreducible representation. For general $\mathbf{g}$ of rank one such a sharpening is not possible as one can show with easy counterexamples when $\mathbf{g} = \mathbf{su}(n + 1, 1)$.

Proof of Theorem 2.1. If (δ, V) is an unitary (complex) representation of $M = M'' M_0^\lambda$ then by Lemma 3.7, $V = V'' \otimes V_0^\lambda$ and there are irreducible unitary representations δ'' and δ^λ of M'' and M_0^λ respectively such that

$$\delta(m'' m^\lambda) = \delta''(m'') \otimes \delta^\lambda(m^\lambda).$$

If $\mathrm{Ad}_{\mathbf{g}_\lambda}, \mathbf{g}_\lambda^C)$ is the complexification of the adjoint representation restricted to M on the root space $\mathbf{g}_\lambda$, then

$$\mathrm{Ad}_{\mathbf{g}_\lambda}(m^\lambda m'') \otimes \delta(m^\lambda m'') = \chi(m'')\mathrm{Ad}_{\mathbf{g}_\lambda}(m^\lambda) \otimes \delta^\lambda(m^\lambda) \otimes \delta''(m'').$$

The real rank-one result from Proposition 3.7 then gives

$$\mathrm{Ad}_{\mathbf{g}_\lambda} \otimes \delta(m^\lambda m'') = (\oplus_l \delta_l^\lambda(m^\lambda)) \otimes \chi(m'')\delta''(m'')$$

with multiplicities one. The representation $\chi(m'')\delta''(m'')$ of M'' is irreducible. It follows that for all l

$$\delta_l^\lambda(m^\lambda) \otimes \chi(m'')\delta''(m'')$$

is an irreducible representation of M. For different l, these representations are inequivalent. $\mathrm{Ad}_{\mathbf{g}_\lambda} \otimes \delta$ is their direct sum. □

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