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Topological recursive relations in $H^{2g}(\mathcal{M}_{g,n})$

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Abstract. We show that any degree at least g monomial in descendant or tautological classes vanishes on $\mathcal{M}_{g,n}$ when $g \geq 2$. This generalizes a result of Looijenga and proves a version of Getzler's conjecture. The method we use is the study of the relative Gromov-Witten invariants of $\mathbb{P}^1$ relative to two points combined with the degeneration formulas of [IP1].

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford compactification of the moduli space $\mathcal{M}_{g,n}$ of genus g smooth curves with n (distinct) marked points. (In this paper we work in the category of analytic orbifolds.) Let $L_i \to \overline{\mathcal{M}}_{g,n}$ be the relative cotangent bundle at the marked point x_i; the fiber of L_i over $(\Sigma, x_1, \ldots, x_n)$ is the cotangent space to Σ at x_i. The first Chern class of this bundle is denoted $\psi_i = c_1(L_i)$ and is sometimes called a (gravitational) descendant. If $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the map that forgets the last marked point then $\kappa_a = \pi_*(\psi_{n+1}^{a+1})$ is called a tautological class (or Mumford-Morita-Miller class); since $\kappa_a \in H^{2a}(\overline{\mathcal{M}}_{g,n})$ we define its degree to be a, while the degree of each ψ_i equals 1.

In [L2] Looijenga proved that in the Chow group $\mathcal{A}^*(\mathcal{C}_g^n)$ a product of descendant classes of degree at least $g+n-1$ vanishes, where $\mathcal{C}_g^n$ is the moduli space of smooth genus g curves with n (not necessarily distinct) points. In particular, in $\mathcal{M}_{g,0}$ any degree $g-1$ monomial in tautological classes vanishes. However, with the above definition of tautological classes, this not true anymore in $\mathcal{M}_{g,n}$, for $n \geq 1$ (for example in $\mathcal{M}_{2,1}$ $\kappa_1 = \psi_1 \neq 0$).

In this paper, we obtain the following generalization of Looijenga's result:

Theorem 0.1. *When $g \geq 2$, any product of degree at least g (or at least $g-1$ when $n = 0$) of descendant or tautological classes vanishes when restricted to $H^*(\mathcal{M}_{g,n}, \mathbb{Q})$.*

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Note that when $g \leq 1$, is has been known for a long time that ψ_j and κ_a with $a \geq 1$ vanish on $\mathcal{M}_{g,n}$.

The proof of Theorem 0.1 is a simple consequence of the degeneration formula for relative Gromov-Witten invariants (cf. [IP1]). The idea is to start with the moduli space $\mathcal{Y}_{d,g,n}$ of degree d holomorphic maps from a smooth genus g surface with n marked points into $\mathbb{P}^1$ which have a fixed ramification pattern over k marked points in the target $\mathbb{P}^1$. In Sect. 1 we describe the structure of $\mathcal{Y}_{d,g,n}$ and that of its compactification $\overline{\mathcal{Y}}_{d,g,n}$. The relatively stable map compactification $\overline{\mathcal{Y}}_{d,g,n}$ is closely related to both the space of admissible covers (introduced by Harris-Mumford in [HMu]) and the space of twisted covers (recently defined by Abramovich-Vistoli in [AV]). Moreover, it comes with two natural maps st and q that record respectively the domain and the target of the cover. One of the key ideas of the paper is then to pull back by q known relations in the cohomology of the target and then push them forward by st to get relations in the cohomology of the domain. So we need to know that the space $\overline{\mathcal{Y}}_{d,g,n}$ carries a fundamental class (over $\mathbb{Q}$). The discussion in Sect. 1 shows this assertion, so in particular $st_*\overline{\mathcal{Y}}_{d,g,n}$ defines a cycle in $\overline{\mathcal{M}}_{g,n}$; the codimension of this cycle is at most g when $\overline{\mathcal{Y}}_{d,g,n}$ is a 2-point ramification cycle (i.e. all but two of the branch points are simple).

We next choose the degree d and a 2-point ramification cycle so that the stabilization map $st : \overline{\mathcal{Y}}_{d,g,n} \to \overline{\mathcal{M}}_{g,n}$ has finite, nonzero degree. Theorem 2.2 then shows that any product of descendants on the domain is a linear combination of (generalized) 2-point ramification cycles on $\overline{\mathcal{M}}_{g,n}$. There are three main ingredients in its proof. We first relate the relative cotangent bundle of the domain to the pull back via q of the relative cotangent bundle of the target. Next, it is known that when genus is zero then (nontrivial) products of descendants are Poincare dual to boundary cycles D in $\overline{\mathcal{M}}_{0,k}$ (see for example [K]). This relates a product of descendants on the domain to cycles of type st_*q^*D, and the degeneration formula (1.23) completes the proof of Theorem 2.2.

Corollary 2.5 then implies that the Poincare dual of any degree m product of descendant and tautological classes can be written as a linear combination generalized 2-point ramification cycles of codimension m. But the codimension of a 2-point ramification cycle is at most g; Proposition 2.8 proves that the cycles of codimension exactly g vanish on $\mathcal{M}_{g,n}$, thus finishing the proof of Theorem 0.1. All degenerations used in this paper are in fact linear equivalences, so an algebraic-geometric proof of the degeneration formula (1.23) would in fact give not only the vanishing in cohomology, but also in the Chow ring, as in Looijenga's Theorem.

From Theorem 2.2 we see that the 2-point ramification cycles on $\overline{\mathcal{M}}_{g,n}$ generate a subring that contains the descendant and tautological classes. In fact, we believe that this subring is not larger then the one generated by descendant, tautological classes and their pullbacks by the attaching maps of the boundary strata of $\overline{\mathcal{M}}_{g,n}$. At least when restricted to $\mathcal{M}_{g,n}$, the arguments in Sect. 7 of [Mu] easily extend to show that any 2-point ramification constraint can be expressed as a polynomial in descendants and tautological classes. It is not clear at this moment how to generalize this argument to the compactification $\overline{\mathcal{M}}_{g,n}$.

On the other hand, when the genus is low ($g \leq 5$) one can prove that all 2-point ramification constraints appearing in Theorem 2.2 are in fact polynomials in only descendant and tautological classes supported on the boundary. Moreover, the coefficients of this polynomial can be determined by keeping track of the coefficients in (1.23). Relations expressing products of descendant classes as polynomials in descendant and tautological classes supported on the boundary are known as topological recursive relations (TRR). The $g = 0$ and $g = 1$ TRR's were known classically. In genus 2, Mumford ([Mu] §8) derived a formula for ψ_1^2 and Getzler ([G]) for $\psi_1\psi_2$. In the same recent paper [G], Getzler made the conjecture that for any genus g there are degree g TRR's.

When the genus is 3 for example, Theorem 0.1 implies the following new relations (modulo boundary terms): $\psi_1^2\psi_2 = \psi_1\psi_2\psi_3 = 0$ (as Getzler conjectured), plus the unexpected relation $\kappa_1\psi_1\psi_2 = 0$. Unfortunately, if we keep track of the boundary terms, the number of terms in the TRR increases very fast as the genus grows. The genus 0 and genus 1 TRR's have 1 and 2 terms respectively, but the genus 2 TRR in [G] has 18 boundary terms. We leave the actual TRR formulas in low genus ($3 \leq g \leq 5$) for another paper.

Note that the degree g is the lowest degree in which one could hope that some *monomial* in descendants would vanish on $\mathcal{M}_{g,n}$. The reason is that the class $\psi_2 \ldots \psi_n \lambda_g \lambda_{g-1}$ vanishes on $\partial\overline{\mathcal{M}}_{g,n}$ (cf [Fa], where $\lambda_i = c_i(E)$ are the Chern classes of the Hodge bundle), while Faber's conjecture ([Fa]), which also agrees with Virasoro predictions (see [GP]) gives

$$\psi_1^{a_1}\psi_2^{a_2+1}\ldots\psi_n^{a_n+1}\lambda_g\lambda_{g-1} = \frac{(2g-3+n)!}{(2a_1-1)!!(2a_2+1)!!\ldots(2a_n+1)!!} \cdot \frac{|B_{2g}|}{2^{2g}g(2g-1)!} \neq 0$$

when $\sum_{i=1}^{n} a_i = g-1$. On the other hand, for large genus, there are most likely lower degree (homogeneous) polynomials in descendants which vanish on $\mathcal{M}_{g,n}$.

While this paper was under revision, the author heard a conjecture made by Graber and Vakil [V]. They essentially conjectured that in the Chow group any degree m monomial in κ and ψ classes on $\overline{\mathcal{M}}_{g,n}$ is pulled back from the strata with at least $m+1-g$ genus 0 components. In the cohomology ring, this conjecture follows immediately from the results of this paper, and was added as the final Proposition 2.9. As mentioned above, an algebraic-geometrical proof of the degeneration formula (1.23) would also give the result in the Chow group.

1. The space of relatively stable covers

We start by defining a space of degree $d \geq 1$, Euler characteristic χ covers of $\mathbb{P}^1$ with prescribed ramification pattern over several points of $\mathbb{P}^1$. The ramification indices at each point $p \in \mathbb{P}^1$ will be encoded by an ordered sequence of positive multiplicities $I = (s_1, \dots, s_\ell)$. For any such I, we define

$$\ell(I) = \ell \qquad \deg I = \sum_{i=1}^{\ell} s_i \qquad |I| = \prod_{i=1}^{\ell} s_i.$$

We also allow some of the points in the inverse image of p to be marked points on the domain.

Definition 1.1. *Consider $I_1, \dots, I_k$ ordered sequences of multiplicities with* $\deg\,(I_j) = d \geq 1$ *for all j, and let $N_1, \dots, N_k$ be an ordered partition of the set $\{x_1, \dots, x_n\}$ (where some of the N_j's might be empty). For all $j = 1, \dots, k$ assume that $0 \leq \ell(N_j) \leq \ell(I_j)$, where $\ell(N_j)$ denotes the cardinality of N_j. We define*

$$\Xi_{d,\chi}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) \tag{1.1}$$

to be the infinite dimensional manifold consisting of data $(f, \Sigma, J, x_1, \dots, x_n; p_1, \dots, p_k)$ such that:

(i) J is a complex structure on Σ, a smooth two dimensional real manifold (not necessarily connected) with Euler characteristic χ;
(ii) $x_1, \dots x_n$ and $p_1, \dots, p_k$ are distinct points on Σ and respectively $\mathbb{P}^1$;
(iii) $f : (\Sigma, J) \to \mathbb{P}^1$ is a degree d holomorphic map, which has moreover positive degree on each component of Σ;
(iv) for each $j = 1, \dots, k$, there exist distinct points $(x_{n_{ij}})_{i=\ell(N_j)+1,\dots,\ell(I_j)}$ on Σ, distinct from $x_1, \dots, x_n$ such that

$$f^{-1}(p_j) = \sum_{i=1}^{\ell(I_j)} s_{ij}\, x_{n_{ij}}$$

(i.e. f is ramified at $x_{n_{ij}}$ of index s_{ij}), where $I_j = (s_{ij})_{i=1,\dots,\ell(I_j)}$ and $N_j = (x_{n_{ij}})_{i=1,\dots,\ell(N_j)}$.

By convention, the space (1.1) is empty when $\ell(N_j) > \ell(I_j)$ or $\deg I_j \neq d$.

We say that $b_{I_j}(N_j)$ describes the ramification pattern of f over the point $p_j \in \mathbb{P}^1$. Note that when $\deg I_j > \ell(I_j)$ the point p_j is a branch point of multiplicity $\deg I_j - \ell(I_j)$. For example $b_{2,1^{d-2}}(x_1)$ means that x_1 is a simple ramification point while $b_{1^d}(x_1, x_2)$ means that x_1 and x_2 are conjugate points of the cover.

In this context, we can think of $b_{I_j}(N_j)$ as imposing a $(\deg I_j - \ell(I_j) + \ell(N_j))$-dimensional condition on a generic degree d covering map $f : (\Sigma, x_1, \ldots, x_n) \to (\mathbb{P}^1, p_1, \ldots, p_k)$. In particular, we usually work with ramification patterns $b_{I_j}(N_j)$ that satisfy $\deg I_j - \ell(I_j) + \ell(N_j) \geq 1$.

The space (1.1) has several components, depending on the topological type of the domain Σ; the component corresponding to a fixed Σ will be denoted by

$$\Xi_{d,\Sigma}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$$

Definition 1.2. *The groups* $\mathrm{Diff}(\Sigma)$ *of diffeomorphisms of* Σ *and* $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2, \mathbb{C})$ *of automorphisms of* $\mathbb{P}^1$ *act on* $\Xi_{d,\Sigma}\left(\prod_{j=1}^{k} b_j(N_j)\right)$ *by*

$$(g, h) \cdot (f, \Sigma, J, x_1, \ldots, x_n, p_1, \ldots, p_k) = (h \circ f \circ g, \Sigma, g^*J, g^{-1}(x_1), \ldots, g^{-1}(x_n), h(p_1), \ldots, h(p_k))$$

where $g \in \mathrm{Diff}(\Sigma)$ *and* $h \in \mathrm{Aut}(\mathbb{P}^1)$. *Consider the two quotients*

$$\widehat{\mathcal{X}}_{d,\Sigma}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) = \Xi_{d,\Sigma}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) \Big/ \mathrm{Diff}(\Sigma)$$

and

$$\mathcal{X}_{d,\Sigma}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) = \Xi_{d,\Sigma}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) \Big/ \mathrm{Diff}(\Sigma) \times \mathrm{Aut}(\mathbb{P}^1). \quad (1.2)$$

The latter is called the moduli space of smooth *degree d covers of* $\mathbb{P}^1$ *by* Σ *with ramification pattern* $b_{I_j}(N_j)$ *at points* $p_j \in \mathbb{P}^1$ *for* $j = 1, \ldots, k$. *The corresponding union of spaces* $\mathcal{X}_{d,\Sigma}$ *over different topological types* Σ *with the same Euler characteristic* χ *is denoted by*

$$\mathcal{X}_{d,\chi}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right).$$

An element $f \in \mathcal{X}_{d,\chi}$ is an equivalence class of triples consisting of a smooth domain $C = (\Sigma, j, x_1, \ldots, x_n)$, the (marked) target $(\mathbb{P}^1, p_1, \ldots, p_k)$ and the covering map. The groups $\mathrm{Diff}(\Sigma)$ and $\mathrm{Aut}(\mathbb{P}^1)$ have induced actions on the domain and respectively the target. Therefore the space $\mathcal{X}_{d,\chi}$ comes with two natural projections

$$\overline{\mathcal{M}}_{0,k} \xleftarrow{\ q\ } \mathcal{X}_{d,\chi}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) \xrightarrow{\ st\ } \widetilde{\mathcal{M}}_{\chi,n} \tag{1.3}$$

defined by $q(f) = (\mathbb{P}^1, p_1, \ldots, p_k)$ and $st(f) = C$, where $\widetilde{\mathcal{M}}_{\chi,n}$ is the moduli space of complex structures with n marked points on a possibly disconnected curve with Euler characteristic χ. In fact, after choosing some ordering the m components of Σ we see that

$$\widetilde{\mathcal{M}}_{\chi,n} = \bigsqcup_{m=1}^{\infty} \left(\bigsqcup \overline{\mathcal{M}}_{g_1,n_1} \times \ldots \times \overline{\mathcal{M}}_{g_m,n_m} \right) / S_m$$

where the second union is over all g_i, n_i and distributions of the n marked points on the m components such that $\sum_{i=1}^{m}(2g_i - 2) = \chi$, $\sum_{i=1}^{m} n_i = n$; the symmetric group S_m acts by permuting the m components.

Restricting to a fiber of q in the fibration (1.3) gives us a corresponding moduli space of covers with prescribed ramification pattern at *k fixed points* in $\mathbb{P}^1$, denoted

$$\mathcal{X}_{d,\Sigma}\left(\prod_{j=1}^{k} B_{I_j}(N_j)\right).$$

The k points are suppressed in the notation for convenience.

Remark 1.3. Since the degree of the covering map f is required to be positive on each component of Σ and the group $\mathrm{Diff}(\Sigma)$ acts on $\Xi_{d,\Sigma}$ with finite stabilizers then $\widehat{\mathcal{X}}_{d,\Sigma}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$ has a natural orbifold structure of dimension

$$\begin{aligned}\dim \widehat{\mathcal{X}}_{d,\Sigma}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) &= 2d - \chi(\Sigma) + k + n \\ &\quad - \sum_{j=1}^{k}(\deg(I_j) - \ell(I_j) + \ell(N_j)) \\ &= 2d - \chi(\Sigma) - \sum_{j=1}^{k}(\deg(I_j) - \ell(I_j)) + k.\end{aligned}$$

When moreover $k \geq 3$ then $\mathrm{Aut}(\mathbb{P}^1)$ also acts with finite stabilizers, so in this case the quotient $\mathcal{X}_{d,\Sigma}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$ is naturally an orbifold of dimension

$$\dim \mathcal{X}_{d,\Sigma}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) = 2d - \chi(\Sigma) - \sum_{j=1}^{k}(\deg(I_j) - \ell(I_j)) + k - 3.$$

When $k \leq 2$, $\mathrm{Aut}(\mathbb{P}^1)$ has a $3-k$ dimensional subgroup which acts trivially and so $\mathcal{X}_{d,\Sigma}$ still has an orbifold structure, but of dimension $2d - \chi(\Sigma) - \sum_{j=1}^{k}(\deg(I_j) - \ell(I_j))$.

Similarly, when $2g-2+n \geq 1$, the moduli space $\overline{\mathcal{M}}_{g,n}$ has an orbifold structure of dimension $3g-3+n$ (obtained by adding Pyrm structures as described in [L1]), while when $2g-2+n \leq 0$ it has a (nonstandard) orbifold structure of dimension g. For this paper we take $\overline{\mathcal{M}}_{0,n} = \overline{\mathcal{M}}_{0,3} = pt$ when $n \leq 2$ and similarly $\overline{\mathcal{M}}_{1,0} = \overline{\mathcal{M}}_{1,1}$.

The space $\mathcal{X}_{d,\chi}$ also comes with a collection of intrinsic line bundles. Denote by $L_{x_i} \to \widetilde{\mathcal{M}}_{\chi,n}$ and $L_{p_j} \to \overline{\mathcal{M}}_{0,k+r}$ the relative cotangent bundles at the marked points x_i and p_j respectively. Next, let $\mathcal{L}_{x_i} \to \mathcal{X}_{d,g}$ be the relative cotangent bundle to the (unstabilized) domain C at the marked point x_i and $\mathcal{L}_{p_j} = q^*L_{p_i} \to \mathcal{X}_{d,g}$ be the relative cotangent bundle to the target $\mathbb{P}^1$ at p_j. The fiber at $f \in \mathcal{X}_{d,g}$ of $\mathcal{L}_{x_i}$ is $T^*_{x_i}C$ while that of $\mathcal{L}_{p_j}$ is $T^*_{p_j}\mathbb{P}^1$. To eliminate the possibility of confusion, throughout this paper x will denote a marked point of the domain and p will denote a marked point of the target.

We next want to compactify $\mathcal{X}_{d,\chi}$ so that the maps in the diagram (1.3) extend continuously and so that $st_*\overline{\mathcal{X}}_{d,\chi}$ defines a cycle in $\widetilde{\mathcal{M}}_{\chi,n}$. For that, we use the *relatively stable* maps compactification of the space of smooth holomorphic maps into $\mathbb{P}^1$ relative to the collection of marked points $\{p_1, \dots, p_k\}$ in the (target) $\mathbb{P}^1$ (cf. Sect. 6 of [IP2]). This compactification is similar in spirit to the usual 'stable maps into $\mathbb{P}^1$' compactification (as described for example in [P]) but it is much finer. The difference is that not only the domain can bubble (or equivalently gets rescaled) when for example two marked points start colliding, but also the target $\mathbb{P}^1$ gets rescaled around p_j when a ghost component (i.e. collapsed component) starts forming or the points p_j get too close to each other.

The strata in the usual stable map compactification that have ghost components not only have the wrong dimension, but more importantly, if the ghost component is sent to p, the ramification constraint above the point p becomes undefined. Making the target bubble yields in the limit a holomorphic map to a degenerate $\mathbb{P}^1$, but without any ghost components over p.

More precisely, consider a sequence (f_n) of smooth degree d stable holomorphic maps to $\mathbb{P}^1$ that have a fixed ramification pattern $b_I(N)$ above p. Suppose that their usual stable map limit f has some ghost components C_2 over p. Let $f_1 : C_1 \to \mathbb{P}^1$ be the restriction of f to the other components of C and let b_S be its ramification pattern above $p = p_0$ (in general $S \neq I$). After rescaling the target $\mathbb{P}^1$ around p (and passing to a subsequence) we obtain in the limit a second nontrivial cover $f_2 : C_2 \to \mathbb{P}^1$ that has the same ramification pattern b_S over p_∞, and fewer (if any) ghost components over p. If f_2 still has ghost components over p, we continue rescaling. Otherwise, f_2 has the ramification pattern $b_I(N)$ over p and all together the limit map is a degree d cover

$$f = f_1 \cup f_2 : C_1 \underset{\substack{y_i^1 = y_i^2 \\ i=1,\dots,\ell}}{\cup} C_2 \to \mathbb{P}^1 \underset{p_0 = p_\infty}{\cup} (\mathbb{P}^1, p) \tag{1.4}$$

of a degenerate $\mathbb{P}^1$ (with an ordinary double point). The cover f has no ghost components over p or the nodal point $p_0 = p_\infty$, and $f_1^{-1}(p_0) = \sum s_i y_i^1$, $f_2^{-1}(p_\infty) = \sum s_i y_i^2$ so f_1, f_2 have the same ramification pattern b_S over the node $p_0 = p_\infty$.

To have a good compactification of $\mathcal{X}_{d,\chi}\left(\prod_{j=1}^k b_{I_j}(N_j)\right)$ we must use the rescaling process around at least all the points p_j for $j = 1, \dots, k$, so that the limit map still satisfies the ramification constraints $b_{I_j}(N_j)$ at the points p_j. However, things become simpler to describe if there are no other branch points. For the rest of this paper we restrict our attention to the moduli space of stable maps where *all* the branch points are marked:

Definition 1.4. *Define a moduli space of possibly disconnected* smooth *covers*

$$\mathcal{Z}_{d,\chi}\left(\prod_{j=1}^k b_{I_j}(N_j)\right) \overset{def}{=} \mathcal{X}_{d,\chi}\left(\prod_{j=1}^k b_{I_j}(N_j)\ (b_{2,1^{d-2}})^r\right) \tag{1.5}$$

where the last r branch points are simple and ordered, with r given by

$$r = 2d + \chi - \sum_{j=1}^k (\deg I_j - \ell(I_j)). \tag{1.6}$$

When $\chi = 2 - 2g$ let

$$\mathcal{Y}_{d,g}\left(\prod_{j=1}^k b_{I_j}(N_j)\right) \subset \mathcal{Z}_{d,\chi}\left(\prod_{j=1}^k b_{I_j}(N_j)\right) \tag{1.7}$$

denote the subspace of connected covers.

Recall from Definition 1.2 that an element f of the space $\mathcal{X}_{d,\chi}$ is a triple consisting of a marked domain, marked target and a degree d covering map with a specified ramification pattern at marked points in the target. All the images of marked points in the domain are marked; some of the preimages of the marked points of the target might also marked. However, there are possibly many unmarked ramified points mapping to marked or unmarked points of the target. An element f of the space $\mathcal{Z}_{d,\chi}$ has the extra property that all its branch points are marked in the target, and in particular the ramification pattern of f is completely determined.

Moreover, when $k+r \geq 3$ the space $\mathcal{Z}_{d,\chi}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$ has a canonical orbifold structure of dimension

$$\begin{aligned}\dim \mathcal{Z}_{d,\chi}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) &= 2d-\chi-\sum_{j=1}^{k}(\deg(I_j)-\ell(I_j))+k-3 \\ &= r+k-3. \qquad (1.8)\end{aligned}$$

When $k+r \leq 2$ Lemma 1.5 below shows that the space $\mathcal{Z}_{d,\chi}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$ is 0-dimensional.

Lemma 1.5. *Consider the space* $\mathcal{Y} = \mathcal{Y}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$ *and let r be as in Definition 1.4 while* $n = \sum_{j=1}^{k} \ell(N_j)$. *If* $2g+n \geq 3$ *then* $k+r \geq 3$. *Moreover, if* $k+r \leq 2$, *then* $\mathcal{Y}$ *consists of only one element; the domain of this cover is an unstable* $g=0$ *curve and the covering is totally ramified at two points.*

Proof. When $k=2$ relation (1.6) becomes $r = 2g-2+\ell(I_1)+\ell(I_2)$. So $r > 0$ unless $g=0$ and $\ell(I_j)=1$. Similarly, when $k=1$ then $r = d+2g-2+\ell(I_1) > 1$ unless $g=0$ and $d+\ell(I_1) \leq 3$. Since $\ell(I_1) \leq d$ then $\ell(I_1)=1$ and $d \leq 2$. Finally, when $k=0$ then $r = d+2g-2$. Since there is no $d=1$ holomorphic cover of S^2 by a smooth T^2 then $r > 2$ unless $g=0$ and $d \leq 2$.

Note that since $\ell(N_j) \leq \ell(I_j)$ then $n \leq 2$ in all above cases. □

This lemma motivates the following:

Definition 1.6. *If* $k+r \leq 2$, *the unique element of the space* $\mathcal{Y}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$ *described in Lemma 1.5 will be called a* trivial cover.

The advantage of working with the space $\mathcal{Z}_{d,\chi}$ is that after 'marking' the location of all the branch points in the target (which in particular means rescaling any time two of them come close to each other) the limit map has no ghost components at all and the double points of the domain occur only above the double points of the target. This is because whenever we start with a sequence of *smooth* maps there cannot be any ghost components forming or double points appearing unless some branch points ran into each other in the target. Therefore in this case the limit can be thought as an *admissible cover* of an element of $\overline{\mathcal{M}}_{0,k+r}$ (as described for example on pp. 180–186 of [HMo]):

Definition 1.7. *Assume* $k+r \geq 3$. *The compactification* $\overline{\mathcal{Z}}_{d,\chi}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$ *of the space (1.5) consists of stable maps* $f : C \to A$ *such that:*

(i) *the domain* C *is a possibly disconnected curve with Euler characteristic* χ *and marked points* $x_1, \ldots, x_n$ *so* $st(C) \in \widetilde{\mathcal{M}}_{\chi,n}$*;*

(ii) *the target* $A \in \overline{\mathcal{M}}_{0,k+r}$ *is a stable genus* 0 *curve with marked points* $p_1, \ldots, p_{k+r}$*;*

(iii) *over the smooth part of* A *the curve* C *is smooth and* f *is a degree* d *cover which has ramification pattern* $b_{I_i}(N_i)$ *over* p_i *for* $1 \leq i \leq k$, *is simply branched over the rest of* p_i, $k+1 \leq i \leq k+r$ *and has no other branch points;*

(iv) *the inverse image of each node of* A *consists of nodes of* C *with matching ramification patterns. More precisely, if* A_1, A_2 *are the two components of* A *joined at the node* $q_1 = q_2$ *let* $C_i = f^{-1}(A_i)$ *and* $f^{-1}(A_1 \underset{q_1=q_2}{\cup} A_2) = C_1 \underset{\substack{y_i^1=y_i^2\\ i=1,\ldots,\ell}}{\cup} C_2$. *Then the multiplicity* s_i *of* $f_1 = f|_{C_1}$ *at* y_i^1 *equals that of* $f_2 = f|_{C_2}$ *at* y_i^2.

Let $\overline{\mathcal{Y}}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) \subset \overline{\mathcal{Z}}_{d,\chi}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$ *denote the corresponding compactification of the space of connected covers (1.7).*

An element f of $\overline{\mathcal{Z}}_{d,\chi}$ is an equivalence class of triples consisting of the (marked) domain and target plus the covering map. Thus (1.3) extends to

$$\begin{array}{ccccccc}
L_{p_j} & \quad \mathcal{L}_{p_j} & & & & \mathcal{L}_{x_i} \quad & L_{x_i} \\
\downarrow & & \searrow & & \swarrow & & \downarrow \\
\overline{\mathcal{M}}_{0,k+r} & \xleftarrow{\quad q \quad} & & \overline{\mathcal{Z}}_{d,\chi}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) & & \xrightarrow{\quad st \quad} & \widetilde{\mathcal{M}}_{\chi,n}
\end{array} \tag{1.9}$$

where $\mathcal{L}_{x_i} \to \overline{\mathcal{Z}}_{d,\chi}$ and $\mathcal{L}_{p_j} \to \overline{\mathcal{Z}}_{d,\chi}$ are the relative cotangent bundles to the (unstabilized) domain C at x_i and respectively to the target A at p_j. Note that in the setup above $q^*L_{p_j} = \mathcal{L}_{p_j}$ but in general $st^*L_{x_i} \neq \mathcal{L}_{x_i}$. This is because A is a stable curve, but C might have unstable components, which get collapsed under the stabilization map.

Moreover, the compactification $\overline{\mathcal{Y}}_{d,g}$ has a natural stratification which comes from the standard stratification of $\overline{\mathcal{M}}_{0,k+r}$ combined with data of the covering map which includes the ramification multiplicity at each node of C and the degree of f on each component of C. Each (open) stratum of the compactification is a smooth orbifold of (complex) dimension $\dim \mathcal{Y}_{d,g} - \#\{\text{double points of } A\}$. We will show below that the space $\overline{\mathcal{Y}}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$ (as well as its cousin $\overline{\mathcal{Z}}_{d,\chi}$) carries a fundamental class (over $\mathbb{Q}$) of dimension $\max(k+r-3, 0)$, which we will call a *ramification class*. In particular, the image under the stabilization map $st : \overline{\mathcal{Y}}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) \to \overline{\mathcal{M}}_{g,n}$ defines a cycle

$$st_*\overline{\mathcal{Y}}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$$

on $\overline{\mathcal{M}}_{g,n}$ called a *ramification cycle*. We can think of this cycle as a condition on a curve $C \in \overline{\mathcal{M}}_{g,n}$, in which case it will be called a *ramification constraint*. Note that if for some j we have $\deg(I_j) - \ell(I_j) + \ell(N_j) = 0$ then the corresponding ramification cycle vanishes in $\overline{\mathcal{M}}_{g,n}$ by dimensional reasons.

Moreover, suppose $M_j \subset N_j$ for all $j = 1, \ldots, k$, $M = \bigsqcup_{j=1}^{k} M_j$ and let ρ, π denote the projections that forget those marked points which are not in M:

$$\begin{array}{ccc} \overline{\mathcal{Y}}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) & \xrightarrow{\rho} & \overline{\mathcal{Y}}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(M_j)\right) \\ \Big\downarrow{\scriptstyle st_n} & & \Big\downarrow{\scriptstyle st_m} \\ \overline{\mathcal{M}}_{g,n} & \xrightarrow{\pi} & \overline{\mathcal{M}}_{g,m} \end{array} \tag{1.10}$$

where $m = \ell(M)$. Then ρ is a finite covering map so the image under π_* of a ramification cycle in $\overline{\mathcal{M}}_{g,n}$ is a multiple of a ramification cycle in $\overline{\mathcal{M}}_{g,m}$.

Given a space $\overline{\mathcal{Z}}_{d,\chi}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$ we can decompose each cover into connected components. In particular, for each connected component of the cover we can forget the marking of those points p_j, $j = k+1, \ldots, k+r$ of the target over which that component is unramified. This defines a map

u which fits in the diagram

$$\begin{array}{ccc} \overline{\mathcal{Z}}_{d,\chi}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) & \xrightarrow{u} & \bigsqcup_m \left(\bigsqcup \prod_{a=1}^{m} \overline{\mathcal{Y}}_{d_a,g_a}\left(\prod_{j=1}^{k} b_{I_{j,a}}(N_{j,a})\right)\right) \Big/ S_m \\ \downarrow st & & \downarrow \prod st \\ \widetilde{\mathcal{M}}_{\chi,n} & \xrightarrow{=} & \bigsqcup_m \left(\bigsqcup \prod_{a=1}^{m} \overline{\mathcal{M}}_{g_a,n_a} \right) \Big/ S_m \end{array} \tag{1.11}$$

where in the upper right hand side of the diagram the second union is over all (a) degrees $d_a \geq 1$ with $\sum_{a=1}^{m} d_a = d$; (b) genera g_a with $\sum_{a=1}^{m}(2-2g_a) = \chi$; (c) partitions $(I_{j,a})_{a=1}^{m}$ of I_j for each $j = 1, \ldots, k$; (d) partitions $(N_{j,a})_{a=1}^{m}$ of N_j for each $j = 1, \ldots, k$ and (e) all possible distribution of the r simple branch points on the connected components. As before, the symmetric group S_m acts by permuting the m domain components.

We will be mostly interested in those ramification cycles with complicated ramification patterns only over two points.

Definition 1.8. *When $k = 2$ the cycle*

$$st_* \overline{\mathcal{Y}}_{d,g}\left(b_{I_1}(N_1) b_{I_2}(N_2)\right)$$

on $\overline{\mathcal{M}}_{g,n}$ is called a 2-point ramification cycle.

For a 2-point ramification cycle $st_* \overline{\mathcal{Y}}_{d,g}\left(b_{I_1}(N_1) b_{I_2}(N_2)\right)$ relation (1.6) becomes

$$r = 2g - 2 + \ell(I_1) + \ell(I_2).$$

So $r = 0$ only for a trivial cover (see Definition 1.6). For a non-trivial cover, $r \geq 1$ and

$$\dim\ st_* \overline{\mathcal{Y}}_{d,g}\left(b_{I_1}(N_1) b_{I_2}(N_2)\right) = 2g - 3 + \ell(I_1) + \ell(I_2) = r - 1. \tag{1.12}$$

If moreover $2g + n \geq 3$ then the codimension of $st_* \overline{\mathcal{Y}}_{d,g}\left(b_{I_1}(N_1) b_{I_2}(N_2)\right)$ in $\overline{\mathcal{M}}_{g,n}$ (which equals the dimension of the constraint it imposes) is

$$\begin{aligned} \text{codim } st_* \overline{\mathcal{Y}}_{d,g}\left(b_{I_1}(N_1) b_{I_2}(N_2)\right) &= g + n - \ell(I_1) - \ell(I_2) \\ &= g - \sum_{j=1}^{2}(\ell(I_j) - \ell(N_j)). \end{aligned} \tag{1.13}$$

In particular, in genus 0

$$st_* \overline{\mathcal{Y}}_{d,0}\left(b_{I_1}(N_1) b_{I_2}(N_2)\right) = 0 \qquad \text{if} \qquad \ell(I_1) + \ell(I_2) > n \geq 3.$$

More generally, when $2g + n \geq 3$ relation (1.13) combined with the inequalities $\ell(I_j) \geq \ell(N_j)$ and $\ell(I_j) \geq 1$ implies that

$$\operatorname{codim} st_* \overline{\mathcal{Y}}_{d,g} \left(b_{I_1}(N_1) b_{I_2}(N_2)\right) \leq \min(g, g+n-2). \tag{1.14}$$

For a trivial cover

$$st_* \overline{\mathcal{Y}}_{d,0} \left(b_d(N_1) b_d(N_2)\right) = \frac{1}{d}[\overline{\mathcal{M}}_{0,n}] \in H_0(\overline{\mathcal{M}}_{0,n}) \cong \mathbb{Q}. \tag{1.15}$$

This follows from the diagram

$$\begin{array}{ccc} \overline{\mathcal{Y}}_{d,0}(b_d(x_1) b_d(x_2) b_{1^d}(x_3)) & \xrightarrow{\rho} & \overline{\mathcal{Y}}_{d,0}(b_d(N_1) b_d(N_2)) \\ \big\downarrow{\scriptstyle st_1} & & \big\downarrow{\scriptstyle st} \\ \overline{\mathcal{M}}_{0,3} & \xrightarrow{=} & \overline{\mathcal{M}}_{0,n} \end{array}$$

after noting that the maps ρ and st_1 have degrees d and 1 respectively.

Remark 1.9. Consider the diagram (1.11) when $k = 2$, and fix both a topological type for the domains of the covers in the moduli space $\overline{\mathcal{Z}}_{d,\chi} = \overline{\mathcal{Z}}_{d,\chi}\left(b_{I_1}(N_1) b_{I_2}(N_2)\right)$ as well as a particular distribution of the degree and of the branching constraints on each component of the domain. This data picks up a certain component $\mathcal{C}$ of the moduli space $\overline{\mathcal{Z}}_{d,\chi}$ which is mapped by u to a quotient by the symmetric group of one of the components $\prod_{a=1}^{m} \overline{\mathcal{Y}}_{d_a,g_a}$. As usual, let r be the number (1.6) of simple branch points for $\overline{\mathcal{Z}}_{d,\chi}$ and suppose that, on $\mathcal{C}$, r_a of them land on the component of the cover which lies in $\overline{\mathcal{Y}}_{d_a,g_a}$. In particular, $r = \sum_{a=1}^{m} r_a$. But $\overline{\mathcal{Z}}_{d,\chi}$ has dimension $\max(r-1, 0)$ while the dimension of $\prod_{a=1}^{m} \overline{\mathcal{Y}}_{d_a,g_a}$ is only $\sum_{a=1}^{m} \max(r_a - 1, 0)$. Diagram (1.11) then implies that $st_*(\mathcal{C}) = 0$ unless the covers in $\mathcal{C}$ are trivial on all but at most one of their connected components (see Definition 1.6). Moreover, if $\mathcal{C}$ is a component of $\overline{\mathcal{Z}}_{d,\chi}$ where all but at most one of the connected components of each cover are trivial, then the restriction of the map u to $\mathcal{C}$ is an isomorphism. Therefore, the cycle $st_* \overline{\mathcal{Z}}_{d,\chi}$ is a linear combination of products of 2-point ramification cycles; in each product, all but at most one of the factors comes from a trivial cover (see equation (1.15) for the contribution of a trivial cover).

Next we describe in more detail how the strata of $\overline{\mathcal{Z}}_{d,\chi}$ fit together. We start with the set-theoretical picture. First notice that there is another (coarser) stratification of $\overline{\mathcal{Z}}_{d,\chi}$ that records a stratification of $\overline{\mathcal{M}}_{0,k+r}$ together with the ramification pattern b_S over the nodes of A and the Euler characteristics of the preimages of the components of A. Take for example an (open) stratum where A has only 2 components A_1 and A_2, joined at the double point $q_1 = q_2$. Assume moreover that the first k_1 of the points p_i are on A_1, the next $k_2 = k - k_1$ on A_2, while the remaining r simple branch points are distributed in all possible ways on the two components. Denote the closure of this stratum in $\overline{\mathcal{M}}_{0,k+r}$ by D_Γ where Γ is the dual graph which has 2 vertices A_i joined by an edge corresponding to the node $q_1 = q_2$ and tails (half edges) $p_1, \ldots, p_{k_1}$ on A_1 and $p_{k_1+1}, \ldots, p_k$ on A_2; sometimes we denote this stratum by $(p_1, \ldots, p_{k_1} \mid p_{k_1+1}, \ldots, p_k)$.

Using the notation from Definition 1.7, given $f \in \overline{\mathcal{Z}}_{d,\chi}$ we start by choosing an ordering of the ℓ double points of C that lie above the node $q_1 = q_2$. We then get an ordered sequence S of multiplicities, two smooth curves C_1, C_2 and two stable maps $f_i = f|_{C_i}$, $f_i : C_i \to A_i$ such that

(a) the curve C_i is in $\widetilde{\mathcal{M}}_{\chi_i,n_i+\ell(S)}$, where its last $\ell(S)$ marked points are $y^i_1, \ldots, y^i_\ell$;

(b) $C = C_1 \bigcup_{\substack{y^1_i = y^2_i \\ i=1,\ldots,\ell}} C_2$ so in particular $\chi = \chi_1 + \chi_2 - 2\ell(S)$ and $n = n_1 + n_2$;

(c) $f_1 \in \mathcal{Z}_{d,\chi_1}\left(\prod_{j=1}^{k_1} b_{I_j}(N_j) b_S(M^1)\right)$ and $f_2 \in \mathcal{Z}_{d,\chi_2}\left(b_S(M^2)\prod_{j=k_1+1}^{k} b_{I_j}(N_j)\right)$ where $M^i = (y^i_1, \ldots, y^i_\ell)$.

Consider the attaching map that (pairwise) identifies the last $\ell(S)$ points of C_1 and C_2

$$\xi : \widetilde{\mathcal{M}}_{\chi_1,n_1+\ell(S)} \times \widetilde{\mathcal{M}}_{\chi_2,n_2+\ell(S)} \to \widetilde{\mathcal{M}}_{\chi,n}$$

given by $(C_1, C_2) \mapsto C_1 \bigcup_{\substack{y^1_i = y^2_i \\ i=1,\ldots,\ell}} C_2$. Then all together, the data above gives a parameterization F of a stratum of $\overline{\mathcal{Z}}_{d,\chi}$. More precisely, F fits in the diagram

$$\begin{array}{ccc}
\overline{\mathcal{Z}}_{d,\chi_1}\left(\prod_{j=1}^{k_1} b_{I_j}(N_j) b_S\right) \times \overline{\mathcal{Z}}_{d,\chi_2}\left(b_S \prod_{j=k_1+1}^{k} b_{I_j}(N_j)\right) & \xrightarrow{F} & \overline{\mathcal{Z}}_{d,\chi} \\
\Big\downarrow {\scriptstyle st\times st} & & \Big\downarrow {\scriptstyle st} \\
\widetilde{\mathcal{M}}_{\chi_1,n_1+\ell(S)} \times \widetilde{\mathcal{M}}_{\chi_2,n_2+\ell(S)} & \xrightarrow{\xi} & \widetilde{\mathcal{M}}_{\chi,n}
\end{array} \tag{1.16}$$

where to define F we used the attaching map ξ to identify the corresponding points in the inverse image of f_1 and f_2 over b_S (and thus also their images q_1 and q_2). As before, these points over b_S are considered marked and ordered, even though they do not appear in the notation. The parameterization F is a local embedding, but not necessarily injective, as the ordering of the

$\ell = \ell(S)$ double points of C is not part of the original data. To keep notation simple, we will denote by

$$\overline{\mathcal{Z}}_{d,\chi_1}\left(\prod_{j=1}^{k_1} b_{I_j}(N_j)\ b_S\right) \underset{\xi}{\times} \overline{\mathcal{Z}}_{d,\chi_2}\left(b_S \prod_{j=k_1+1}^{k} b_{I_j}(N_j)\right) \tag{1.17}$$

the pushforward by F of the fundamental class of the domain of the parameterization (1.16).

The inverse image of the stratum $D_\Gamma = (p_1, \ldots, p_{k_1} \mid p_{k_1+1}, \ldots, p_k)$ of $\overline{\mathcal{M}}_{0,k+r}$ under q can then be parameterized by

$$F : \bigsqcup_{\chi_i, S} \overline{\mathcal{Z}}_{\chi_1,d}\left(\prod_{j=1}^{k_1} b_{I_j}(N_j)\ b_S\right) \times \overline{\mathcal{Z}}_{\chi_2,d}\left(b_S \prod_{j=k_1+1}^{k} b_{I_j}(N_j)\right) \longrightarrow q^{-1}(D_\Gamma) \tag{1.18}$$

where the union is over all χ_1, χ_2, ordered sequences S of degree d with $\chi = \chi_1 + \chi_2 - 2\ell(S)$ and all possible distributions of the r simple branch points. As the target of a sequence of stable maps in $\mathcal{Z}_{d,\chi}$ degenerates into an element of D_Γ, the limit is an element of $q^{-1}(D_\Gamma)$. Going backwards, we next need to understand all possible smoothings of elements of $q^{-1}(D_\Gamma)$ into elements of $\mathcal{Z}_{d,\chi}$.

Recall that an element of $\overline{\mathcal{Z}}_{d,\chi}$ is a triple consisting of domain, target and a covering map. We start by looking at smoothings of the domain and of the target. In the setup above, the normal direction to D_Γ inside $\overline{\mathcal{M}}_{0,k+r}$ is parameterized by the line bundle $\mathcal{L}^*_{q_1} \otimes \mathcal{L}^*_{q_2}$ whose fiber at $A_1 \underset{q_1=q_2}{\cup} A_2$ is $T_{q_1}A_1 \otimes T_{q_2}A_2$. Similarly, the normal bundle of the ℓ-nodal stratum in $\widetilde{\mathcal{M}}_{\chi,n}$ is $\bigoplus_{i=1}^{\ell} \mathcal{L}^*_{y_i^1} \otimes \mathcal{L}^*_{y_i^2}$, whose fiber at $C = C_1 \underset{\substack{y_i^1=y_i^2\\ i=1,\ldots,\ell}}{\cup} C_2$ is $\bigoplus_{i=1}^{\ell} T_{y_i^1}C_1 \otimes T_{y_i^2}C_2$. However, for a fixed smoothing A_λ of A not all smoothings C_μ of C give rise to a stable map $f : C_\mu \to A_\lambda$; here $\lambda \in T_{q_1}A_1 \otimes T_{q_2}A_2$ and $\mu = (\mu_1, \ldots, \mu_\ell)$ with $\mu_i \in T_{y_i^1}C_1 \otimes T_{y_i^2}C_2$. This can be best seen in local coordinates $z_{m,i}$ at y_m^i and w_i at q_i. In these coordinates

$$w_i = f_i(z_{m,i}) = a_{m,i} \cdot (z_i)^{s_m} + \text{ higher order} \tag{1.19}$$

while the smoothings of C and A are given by $z_{m,1} \cdot z_{m,2} = \mu_m, m = 1, \ldots, \ell$ and $w_1 \cdot w_2 = \lambda$. Therefore $f : C \to A$ can be extended to a smooth cover $f_{\mu,\lambda} : C_\mu \to A_\lambda$ only when

$$\lambda = a_{m,1}\, a_{m,2}\, \mu_m^{s_m} \qquad \text{for all } m = 1, \ldots, \ell$$

to highest order. For example this fact is proven (in a more general setting) using PDE methods in [IP2]. It was also stated in the original Harris-

Mumford paper [HMu]. Moreover, in the algebraic-geometrical setting, the deformation argument of Caporaso and Harris [CH] could be extended to this case. After all, in [CH] they have studied stable maps into $\mathbb{P}^2$ with prescribed contact constraints along a line L, and the case above is simply a dimensional reduction where the pair $(\mathbb{P}^2, L)$ gets replaced by $(\mathbb{P}^1, p)$.

Summarizing, given a pair (f_1, f_2) in the domain of the parameterization (1.16), equation (1.19) defines a canonical section

$$\sigma_q : \mathcal{Z}_{d,\chi_1}(\ldots b_S) \times \mathcal{Z}_{d,\chi_2}(b_S \ldots) \longrightarrow \bigoplus_{i=1}^{\ell} \left(\mathcal{L}_{x_{n_i}} \otimes \mathcal{L}_{y_{n_i}}\right)^{s_i} \otimes \left(\mathcal{L}^*_{q_1} \otimes \mathcal{L}^*_{q_2}\right) \tag{1.20}$$

given by $\sigma_q = (a_1, \ldots, a_\ell)$ with $a_m = a_m^1 \cdot a_m^2$. For a fixed smoothing of the target $\lambda \in \mathcal{L}^*_{q_1} \otimes \mathcal{L}^*_{q_2}$ the possible smoothings of the domain $\mu = (\mu_1, \ldots, \mu_\ell)$ correspond to solutions of the equations

$$\lambda = a_1 \mu_1^{s_1} = \cdots = a_\ell \mu_\ell^{s_\ell}. \tag{1.21}$$

There are $|S| = \prod s_i$ many such solutions, differing by roots of unity. This describes the local model in the normal direction to a stratum parameterized by $\mathcal{Z}_{d,\chi_1}(\ldots b_S) \times \mathcal{Z}_{d,\chi_2}(b_S \ldots)$ inside the compactification $\overline{\mathcal{Z}}_{d,\chi}$. Moreover, this shows that as cycles, the pullback of D_Γ is

$$q^*(D_\Gamma) = \bigsqcup_{\chi_i, S} \frac{|S|}{\ell(S)!}\ \overline{\mathcal{Z}}_{\chi_1,d}\left(\prod_{j=1}^{k_1} b_{I_j}(N_j)\ b_S\right) \times_{\xi} \overline{\mathcal{Z}}_{\chi_2,d}\left(b_S \prod_{j=k_1+1}^{k} b_{I_j}(N_j)\right) \tag{1.22}$$

where the union is over all χ_1, χ_2, ordered sequences S of degree d with $\chi = \chi_1 + \chi_2 - 2\ell(S)$ and all possible distributions of the r simple branch points. The $\frac{1}{\ell(S)!}$ weight comes from the fact that the ordering of the $\ell(S)$ double points of C is not part of the original data of an element in $\overline{\mathcal{Z}}_{d,\chi}$.

Remark 1.10. Note that the solution space to the equations (1.21) has several branches intersecting at the origin (which corresponds to the boundary stratum) so the compactification $\overline{\mathcal{Z}}_{d,\chi}$ described in Definition 1.7 is not in general an orbifold. However, it can be desingularized by including as part of the data besides the triple $f : C \to A$ a choice of roots of unity for the leading term section (1.20). This desingularized compactification becomes then a version of the space of *twisted covers* defined in [AV]. In any event, we will only use the fact that (each component of) $\overline{\mathcal{Z}}_{d,\chi}$ carries a fundamental class (with rational coefficients) and so $st_*[\overline{\mathcal{Z}}_{d,\chi}]$ defines a class on $\widetilde{\mathcal{M}}_{\chi,n}$.

As a particular case of (1.22) we get the following

Theorem 1.11. *Let* $q : \overline{\mathcal{Z}}_{d,\chi}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) \to \overline{\mathcal{M}}_{0,k+r}$ *be as in (1.9) and let* D_Γ *be the codimension one stratum of* $\overline{\mathcal{M}}_{0,k+r}$ *where the first* k_1 *points are on a bubble, the next* $k_2 = k - k_1$ *points are on a different bubble and the remaining r points are distributed all possible ways. Then as cycles in* $\overline{\mathcal{M}}_{\chi,n}$

$$st_* q^*(D_\Gamma) = \sum \frac{|S|}{\ell(S)!}\ st_*\left(\overline{\mathcal{Z}}_{\chi_1,d}\left(\prod_{j=1}^{k_1} b_{I_j}(N_j)\ b_S\right)\right.$$
$$\left.\times_{\xi}\ \overline{\mathcal{Z}}_{\chi_2,d}\left(b_S \prod_{j=k_1+1}^{k} b_{I_j}(N_j)\right)\right) \tag{1.23}$$

where the sum is over all χ_1*,* χ_2*, ordered sequences S of degree d with* $\chi = \chi_1 + \chi_2 - 2\ell(S)$ *and all possible distributions of the r simple branch points.*

When q is restricted to the space $\overline{\mathcal{Y}}_{d,g}$ *of connected covers then we get cycles in* $\overline{\mathcal{M}}_{g,n}$ *and in the sum above we keep only those configurations of domains* C_1*,* C_2 *whose image under the attaching map* ξ *is connected.*

Note that the equal sign in (1.23) is only an equality in homology, because the proof in [IP2] (which is done in the symplectic category) only shows that the compactification $\overline{\mathcal{Z}}_{\chi,n}$ is diffeomorphic to the local model (1.21). However, an algebraic-geometrical proof of the local model (1.21) would give the equality in the Chow ring.

Example 1.12. Suppose $k = 2$, $k_1 = k_2 = 1$ and $2g + n \geq 3$. The right hand side of (1.23), when restricted to connected genus g covers, involves terms of type

$$\overline{\mathcal{Z}}_{\chi_1,d}\left(b_{I_1}(N_1)\ b_S\right)\ \times_{\xi}\ \overline{\mathcal{Z}}_{\chi_2,d}\left(b_S\ b_{I_2}(N_2)\right). \tag{1.24}$$

The pushforward by *st* of such term, using relation (1.17) and diagram (1.16), is equal to

$$\xi_*\left(st_*\overline{\mathcal{Z}}_{\chi_1,d}\left(b_{I_1}(N_1)\ b_S\right)\ \times\ st_*\overline{\mathcal{Z}}_{\chi_2,d}\left(b_S\ b_{I_2}(N_2)\right)\right).$$

By Remark 1.9, each component of $st_*\overline{\mathcal{Z}}_{\chi_i,d}$ is a multiple of a product of 2-point ramification cycles; the factors in the product correspond to (unstabilized) domain components. Moreover, the discussion following Definition 1.8 implies that on all genus 0 components the 2-point ramification cycles either vanish or else are multiples of the fundamental class. Suppose we fix a topological type of the (unstabilized) domain, and a fixed distribution of the ramification patterns on each component of the domain. Then the pushforward by *st* of the corresponding component of (1.24) equals a rational multiple of products of 2-point ramification cycles on the components of the stabilized domain. More precisely, suppose the stabilized domain consists of components of genus g_a with n_a special points labeled by M_a, all glued together according to the dual graph via the attaching map

$$\xi : \prod_{a=1}^{h} \overline{\mathcal{M}}_{g_a,n_a} \to \overline{\mathcal{M}}_{g,n}. \tag{1.25}$$

By convention, when the dual graph has no edges (i.e. stabilized domain is smooth), the attaching map is the identity. Then the component of the right hand side of (1.23) corresponding to the attaching map (1.25) is a linear combination (with rational coefficients) of terms of type

$$\xi_* \left(\prod_{a=1}^{h} st_* \overline{\mathcal{Y}}_{d_a,g_a} \left(b_{I_{a1}}(N_{a1})\, b_{I_{a2}}(N_{a2}) \right) \right) \in H_*(\overline{\mathcal{M}}_{g,n}) \tag{1.26}$$

where $N_{a1} \sqcup N_{a2} = M_a$. Note that the term (1.26) vanishes unless on all genus 0 components $\ell(N_{a1}) = \ell(I_{a1})$ and $\ell(N_{a2}) = \ell(I_{a2})$ (see relation (1.13)). Moreover, since all terms in (1.23) are codimension one, then only terms of type (1.26) for which the domain has at most one node can appear (with nonzero coefficient) in the right hand side of (1.23).

Definition 1.13. *Consider ramification cycles* $\mathcal{C}_a = st_* \overline{\mathcal{Y}}_{d_a,g_a} \left(\prod_{i=1}^{k_a} b_{I_{ai}}(N_{ai}) \right)$ *on* $\overline{\mathcal{M}}_{g_a,n_a}$ *where* $2g_a + n_a \geq 3$. *For each attaching map* ξ *as in (1.25) the cycle* $\xi_* \left(\prod_{a=1}^{h} \mathcal{C}_a \right)$ *is called a* generalized ramification cycle *on* $\overline{\mathcal{M}}_{g,n}$. *In particular, such a cycle for which* $k_a = 2$ *for all* $a = 1, \ldots, h$ *will be called a* generalized 2-point ramification cycle.

With this definition, Theorem 1.11 implies in particular that when D_Γ is a codimension one boundary stratum of $\overline{\mathcal{M}}_{0,r+k}$, then $st_* q^* D_\Gamma$ is a linear combination of codimension one generalized 2-point ramification cycles.

Definition 1.14. *Let* Θ *be a linear combination of generalized ramification cycles on* $\overline{\mathcal{M}}_{g,n}$. *Those terms of* Θ *which are constructed using the attaching map of a boundary stratum of* $\overline{\mathcal{M}}_{g,n}$ *will be called* lower order terms. *The sum of the other terms forms the* symbol of Θ. *By convention, if all the terms are lower order, we take the symbol to be 0.*

Remark 1.15. For $M \subset \{x_1, \ldots, x_n\}$ consider the map $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,m}$ that forgets the marked points which are not in M. Suppose $\mathcal{C}$ is a generalized ramification cycle on $\overline{\mathcal{M}}_{g,n}$. Since the attaching maps commute with the forgetful maps, diagram (1.10) implies that $\pi_* \mathcal{C}$ is a (rational) multiple of a generalized ramification cycle on $\overline{\mathcal{M}}_{g,m}$. Note that even if $\mathcal{C}$ is nonzero, $\pi_* \mathcal{C}$ might vanish (by dimensional reasons for example).

Remark 1.16. Theorem 1.23 generalizes to higher codimensional boundary strata in $\overline{\mathcal{M}}_{0,k+r}$. In particular, let D denote the codimension $m-1$ boundary strata of $\overline{\mathcal{M}}_{0,2+r}$ consisting of linear chains of m $\mathbb{P}^1$'s such that p_1 is on the first bubble, p_2 on the last bubble and the other r points p_j, $j = 3, \ldots, r+2$ are distributed in some fixed way on the m components. Then

$$q^{-1}(D) \subset \overline{\mathcal{Z}}_{d,\chi}(b_{I_1}(N_1) b_{I_2}(N_2))$$

is similarly parameterized by a disjoint union of spaces

$$\overline{\mathcal{Z}}_{d,\chi_1}(b_{I_1}(N_1) b_{S_1}) \underset{\xi_1}{\times} \overline{\mathcal{Z}}_{d,\chi_2}(b_{S_1} b_{S_2}) \underset{\xi_2}{\times} \ldots \underset{\xi_{m-1}}{\times} \overline{\mathcal{Z}}_{d,\chi_m}(b_{S_{m-1}} b_{I_2}(N_2)) \quad (1.27)$$

where each attaching map ξ_i identifies the corresponding points over b_{S_i} for $i = 1, \ldots, m-1$. So $st_* q^*(D)$ can also be written as a linear combination of generalized 2-point ramification cycles of codimension $m-1$.

Next, fix a moduli space $\overline{\mathcal{Y}}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$ such that $2g + n \geq 3$ (so in particular $k + r \geq 3$ by Lemma 1.5). Assume in what follows that the point x_i has prescribed ramification index s_i and image p_j. We can consider the 'universal family' $\overline{\mathcal{Y}}_{d,g}\left(b_{1^d}(x_0) \prod_{j=1}^{k} b_{I_j}(N_j)\right)$ obtained by adding extra marked points x_0 to the domain and p_0 to the target, together with the diagram

$$\begin{array}{ccccc} \overline{\mathcal{M}}_{g,n+1} & \xleftarrow{st_{n+1}} & \overline{\mathcal{Y}}_{d,g}\left(b_{1^d}(x_0) \prod_{j=1}^{k} b_{I_j}(N_j)\right) & \xrightarrow{q_{n+1}} & \overline{\mathcal{M}}_{0,k+r+1} \\ {\scriptstyle x_i}\uparrow\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} & & {\scriptstyle p_j}\uparrow\downarrow{\scriptstyle \pi_2} \\ \overline{\mathcal{M}}_{g,n} & \xleftarrow{st_n} & \overline{\mathcal{Y}}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) & \xrightarrow{q_n} & \overline{\mathcal{M}}_{0,k+r} \end{array} \quad (1.28)$$

where π_0 is the map that forgets both the marked point x_0 on the domain and its image p_0 on the target. The images of the canonical sections x_i, p_j are the strata $D_{0,i} \subset \overline{\mathcal{M}}_{g,n+1}$ and respectively $D_{0,j} \subset \overline{\mathcal{M}}_{0,r+1}$ where x_0 and x_i and respectively p_0 and p_j are the only marked points on a genus 0 bubble.

The covers in the preimage $q_{n+1}^{-1}(D_{0,j}) \subset \overline{\mathcal{Y}}_{d,g}\left(b_{1^d}(x_0) \prod_{j=1}^{k} b_{I_j}(N_j)\right)$ have a very special form. Because on the genus 0 bubble containing p_0 and p_j there are no other branch points, then over this component the cover consists of $\ell(I_j)$ spheres totally ramified above p_j and p_∞ (p_∞ is the double point of the target where the bubble is attached). Only the sphere that contains x_0 is nontrivial, the rest are trivial covers (see Definition 1.6).

When the point x_0 is on the same bubble as the point x_i we denote the corresponding canonical section by

$$\sigma_i : \overline{\mathcal{Y}}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) \to \overline{\mathcal{Y}}_{d,g}\left(b_{1^d}(x_0)\prod_{j=1}^{k} b_{I_j}(N_j)\right) \tag{1.29}$$

and let Σ_i denote its image. Then $st_{n+1}\circ\sigma_i = x_i\circ st_n$ and $q_{n+1}\circ\sigma_i = p_j\circ q_n$ with the notations of (1.28). In particular, this discussion shows that

$$\pi_1^* st_{n*}\overline{\mathcal{Y}}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right) = st_{n+1*}\overline{\mathcal{Y}}_{d,g}\left(b_{1^d}(x_0)\prod_{j=1}^{k} b_{I_j}(N_j)\right). \tag{1.30}$$

Moreover,

Lemma 1.17. *Consider the space* $\overline{\mathcal{Y}}_{d,g} = \overline{\mathcal{Y}}_{d,g}\left(\prod_{j=1}^{k} b_{I_j}(N_j)\right)$ *where all the preimages of all the marked points of the target (including all branch points) are marked. Suppose moreover that* x_i *is a marked point in the domain with image* p_j *and ramification index* s_i*. If* $L_{x_i}\to\overline{\mathcal{M}}_{g,n}$ *and* $L_{p_j}\to\overline{\mathcal{M}}_{0,r}$ *are the relative cotangent bundles to the domain and respectively the target then over* $\overline{\mathcal{Y}}_{d,g}$ *we have*

$$st^* L_{x_i}^{s_i} \;=\; q^* L_{p_j}. \tag{1.31}$$

Proof. Since $L_{x_i} = x_i^*\mathcal{O}(-D_{0,i})$ and $L_{p_j} = p_j^*\mathcal{O}(-D_{0,i})$ then

$$st_n^* L_{x_i} = st_n^* x_i^*\mathcal{O}(-D_{0,i}) = \sigma_i^* st_{n+1}^*\mathcal{O}(-D_{0,i})$$
$$q_n^* L_{p_j} = q_n^* p_j^*\mathcal{O}(-D_{0,j}) = \sigma_i^* q_{n+1}^*\mathcal{O}(-D_{0,j}).$$

But all the points over p_j are marked so all the covers in $q_{n+1}^{-1}(D_{0,j})$ have domains with x_0 and at least one of the other points over p_j on the same bubble. Moreover, the only instance where x_0 and x_i are the only two marked points on a genus 0 bubble are those covers in Σ_i. Then (1.23) implies that

$$q_{n+1}^*\mathcal{O}(-D_{0,j}) = \mathcal{O}(-s_i\Sigma_i) \qquad \text{along } \Sigma_i$$

where s_i is the ramification index of point x_i. The condition that all the preimages of all the marked points of the target (including all branch points) are marked implies in particular that all the domains of the covers are stable curves and therefore

$$st_{n+1}^*\mathcal{O}(-D_{0,i}) = \mathcal{O}(-\Sigma_i) \qquad \text{along } \Sigma_i.$$

Combining the last four displayed equations we then get (1.31). □

2. Polynomials in descendants

In this section we describe how to express a product of $\psi_i = c_1(L_{x_i})$ classes on $\overline{\mathcal{M}}_{g,n}$ (or more precisely the intersection product of their Poincare duals) as a linear combination of generalized ramification cycles.

The basic idea is simple: to begin with we choose a 2-point ramification cycle $\overline{\mathcal{Y}}_{d,g}$ so that the map $st : \overline{\mathcal{Y}}_{d,g} \to \overline{\mathcal{M}}_{g,n}$ is of finite (nonzero) degree. Then we use equation (1.31) to relate $st^* L_{x_i} \to \overline{\mathcal{Y}}_{d,g}$ to the pull back $q^* L_{p_j}$ of the relative cotangent bundle L_{p_j} to the target $\mathbb{P}^1$ at p_j, the image of x_i under the covering map. But we know that the Poincare dual of $c_1(L_{p_j})$ is a codimension 1 boundary cycle D_Γ in $\overline{\mathcal{M}}_{0,r}$. Then Theorem 1.11 implies that the Poincare dual of ψ_i is linear combination of generalized 2-point ramification cycles on $\overline{\mathcal{M}}_{g,n}$.

In what follows the descendant on the target $c_1(L_{p_j})$ will be denoted by $\widetilde{\psi}_j$ to avoid confusing it with the descendant on the domain $\psi_j = c_1(L_{x_j})$. Also, in the rest of the paper, we will often add or forget marked points. Note to begin with that if $\pi_0 : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the map that forgets the marked point x_0 then

$$\psi_i = \pi_0^* \psi_i + D_{i,0} \tag{2.32}$$

where $D_{i,0}$ is the boundary strata in $\overline{\mathcal{M}}_{g,n+1}$ consisting of domains where x_i and x_0 are the *only* points on a $g = 0$ bubble. Similarly, for tautological classes we have

$$\kappa_i = \pi_0^* \kappa_i + \psi_0^i. \tag{2.33}$$

Moreover, if ξ is the attaching map (1.25) of a boundary stratum of $\overline{\mathcal{M}}_{g,n}$ then the pullback by ξ of the relative cotangent bundle to x_i is the relative cotangent bundle to x_i, so

$$\xi^* \psi_1 = \psi_1. \tag{2.34}$$

Example 2.1. Let us illustrate the procedure described at the beginning of this section on the following example: when $g = 1$ it is known that $\psi_1 = \delta_0/12$ in $\overline{\mathcal{M}}_{1,1}$, where δ_0 is the boundary stratum which corresponds to a nodal sphere. To see this, we start by writing any element in $\overline{\mathcal{M}}_{1,1}$ as a degree 2 cover of $\mathbb{P}^1$ branched at 4 points such that the marked point x_1 is one of the branch points. As long as the branch points are not ordered, such a cover is unique. Fix now an ordering of the other 3 branch points and let $\overline{\mathcal{Y}}_{2,1}(b_2(x_1)b_2b_2b_2)$ denote the corresponding space of covers. Then the stabilization map $st : \overline{\mathcal{Y}}_{2,1}(b_2(x_1)b_2b_2b_2) \to \overline{\mathcal{M}}_{1,1}$ is a degree 3! cover so

$$st_* \left(st^* \psi_1 \cap [\overline{\mathcal{Y}}_{2,1}(b_2(x_1)b_2b_2b_2)\,] \right) = 6\psi_1 \cap [\overline{\mathcal{M}}_{1,1}]. \tag{2.35}$$

Next, relation (1.31) gives $2st^*\psi_1 = q^*\widetilde{\psi}_1$ on $\overline{\mathcal{Y}}_{2,1}$. On the other hand, on $\overline{\mathcal{M}}_{0,4}$ $\widetilde{\psi}_1$ is Poincare dual to the boundary stratum D_Γ which consists of $p_1,\ p_2$ on one bubble and $p_3,\ p_4$ on the other so

$$2\overline{\mathcal{Y}}_{2,1}(b_2(\psi_1)b_2b_2b_2) = q^*(D_\Gamma). \tag{2.36}$$

Now use the degeneration formula (1.23). Since the degree is 2 and total genus is 1, the only term that can appear is $S = (1, 1)$ with genus 0 on both sides, i.e.

$$st_*q^*(D_\Gamma) = \frac{1}{2}\,st_*\left(\overline{\mathcal{Y}}_{2,0}(b_2(x_1)b_2b_{1,1}) \underset{\xi}{\times} \overline{\mathcal{Y}}_{2,0}(b_{1,1}b_2b_2)\right).$$

But there is only one genus 0 degree two map, and under the stabilization map the component on the right gets collapsed to a point, while the one on the left is mapped to $2\delta_0 = [pt] \in H_0(\overline{\mathcal{M}}_{1,1})$. Therefore

$$st_*q^*(D_\Gamma) = \delta_0. \tag{2.37}$$

Combining (2.35), (2.36) and (2.37) gives the relation $\psi_1 = \delta_0/12$.

More generally,

Theorem 2.2. *Assume $g \geq 1$, $n \geq 1$ and $n+g \geq 3$. Then the Poincare dual of any degree m monomial in descendant classes on $\overline{\mathcal{M}}_{g,n}$ can be written as a linear combination of generalized 2-point ramification cycles on $\overline{\mathcal{M}}_{g,n}$, coming from a cover of degree at most $d = g + n - 1$. The nonzero terms appearing in the symbol are codimension m cycles of type*

$$st_*\overline{\mathcal{Y}}_{a,g}(b_{I_1}(N_1)b_{I_2}(N_2))$$

where $a \leq d$, $N_1 \sqcup N_2 = \{x_1, \ldots, x_n\}$ and $\ell(I_1) + \ell(I_2) = g + n - m$.

Note that $\ell(N_j) \leq \ell(I_j)$ so adding we get $n \leq g + n - m$. In particular, the Theorem implies that when $m \geq g + 1$ or $m \geq g + n - 1$ there are no nonzero terms in the symbol, and so the degree m monomial in descendant classes vanishes when restricted to $\mathcal{M}_{g,n}$.

Moreover, a closer analysis of the proof of Theorem 2.2 shows that the terms appearing in the symbol have either $\ell(I_1) = 1$ or $\ell(I_2) = 1$. But since this is irrelevant for this paper, we leave the details to the reader.

Proof of Theorem 2.2. Consider the ramification cycle (as defined in Sect. 1)

$$\overline{\mathcal{Y}}_{d,g,n} = \overline{\mathcal{Y}}_{d,g}(b_{1^d}(N)b_d)$$

where $N = (x_1, x_2, \ldots, x_n)$. Under the assumptions of the Theorem, when $d = g + n - 1$ Lemma 2.3 below shows that $st : \overline{\mathcal{Y}}_{d,g,n} \to \overline{\mathcal{M}}_{g,n}$ is map of finite, *nonzero* degree $\deg(st) \neq 0$.

Now let $\psi_1^{m_1} \dots \psi_n^{m_n}$ be a monomial on $\overline{\mathcal{M}}_{g,n}$ of degree $m = \sum m_j \geq 0$. The Poincare dual of $st^* \left(\psi_1^{m_1} \dots \psi_n^{m_n}\right)$ in $\overline{\mathcal{Y}}_{d,g}(b_{1^d}(N)\, b_d\,)$ is

$$st^* \left(\psi_1^{m_1} \dots \psi_n^{m_n}\right) \,\cap\, [\overline{\mathcal{Y}}_{d,g}(b_{1^d}(N)\, b_d\,)]$$

so the Poincare dual of $\psi_1^{m_1} \dots \psi_n^{m_n}$ is given by

$$PD\left(\psi_1^{m_1} \dots \psi_n^{m_n}\right) = (\deg(st))^{-1} \cdot st_* \left(st^* \left(\psi_1^{m_1} \dots \psi_n^{m_n}\right) \,\cap\, [\overline{\mathcal{Y}}_{d,g}(b_{1^d}(N)\, b_d\,)]\right). \tag{2.38}$$

The theorem then follows by induction on the degree m of the monomial $\psi_1^{m_1} \dots \psi_n^{m_n}$. The case $m = 0$ comes directly from relation (2.38). Now suppose the result is true for $m-1$, so we need to prove it for m. Consider a monomial $\psi_1^{m_1} \dots \psi_n^{m_n}$ of degree $m \geq 1$. Without loss of generality we may assume that $m_1 \geq 1$. Then relation (2.38) implies

$$PD\left(\psi_1^{m_1} \dots \psi_n^{m_n}\right) = (\deg(st))^{-1}\ \psi_1 \cap st_* \left(st^*\left(\psi_1^{m_1-1} \dots \psi_n^{m_n}\right) \cap [\overline{\mathcal{Y}}_{d,g}(b_{1^d}(N)\, b_d\,)]\right). \tag{2.39}$$

By induction, $st_* \left(st^*(\psi_1^{m_1-1} \dots \psi_n^{m_n}) \cap [\overline{\mathcal{Y}}_{d,g}(b_{1^d}(N)\, b_d\,)]\right) = (\deg st) \cdot \psi_1^{m_1-1} \dots \psi_n^{m_n}$ is a linear combination of generalized 2-point ramification cycles. Thus the cycle (2.39) is a linear combination of terms of type

$$\begin{aligned} &\psi_1 \cap \xi_* \left(\prod_{a=1}^{m} st_* \overline{\mathcal{Y}}_{d_a,g_a}\left(b_{I_{a1}}(N_{a1}) b_{I_{a2}}(N_{a2})\right)\right) \\ &\quad = \xi_* \left(\xi^* \psi_1 \cap \prod_{a=1}^{m} st_* \overline{\mathcal{Y}}_{d_a,g_a}\left(b_{I_{a1}}(N_{a1}) b_{I_{a2}}(N_{a2})\right)\right). \end{aligned}$$

Using relation (2.34) and applying Lemma 2.4 to the factor containing the marked point x_1 then completes the inductive step. □

Lemma 2.3. *Let $d = g + n - 1$. Then the degree of the map*

$$st : \overline{\mathcal{Y}}_{d,g}(b_{1^d}(N) b_d) \to \overline{\mathcal{M}}_{g,n}$$

is nonzero as long as $g \geq 1$, $n \geq 1$ and $g + n \geq 3$. Moreover, the degree of st vanishes when $g = 0$ or $n = 0$ or $g = n = 1$.

Proof. We begin by noting that when $d = g + n - 1$, dimension count shows that the domain and target of the map st have the same dimension. The vanishing part of the lemma follows immediately after noting that when $g = 0$ the domain of st is empty (since $\ell(N) > d$), while when $n = 0$ or $g = n = 1$ the fiber of st is one dimensional.

For $d = g \geq 2$ Mumford proved in §7 of [Mu] that the degree of the stabilization map $st : \overline{\mathcal{Y}}_{d,g}(b_d) \to \overline{\mathcal{M}}_{g,0}$ is nonzero. In particular, this implies that the degree of the map $st : \overline{\mathcal{Y}}_{d,g}(b_d b_{1^d}(x_1)) \to \overline{\mathcal{M}}_{g,1}$ is nonzero as well, because once we write a general Riemann surface as an element of $\overline{\mathcal{Y}}_{d,g}(b_d)$, adding a general marked point x_1 gives an element of $\overline{\mathcal{Y}}_{d,g}(b_d b_{1^d}(x_1))$. This proves the lemma in the case $n = 1$ and $g \geq 2$.

The case when $n \geq 2$ and $g \geq 1$ follows by methods similar to those of Sect. 5 of [HMu]. More precisely, fix a general (smooth) genus g Riemann surface C with n marked points x_i, $i = 1, \ldots, n$. It is enough to show that we can find g points $y_0, \ldots, y_{g-1}$ on C such that

$$\sum_{i=1}^{n} x_i + \sum_{i=1}^{g-1} y_i \sim d y_0.$$

Then as long as $g \geq 1$ and $n \geq 2$, a dimension count shows that the points $y_0, \ldots, y_{g-1}$ are distinct and distinct from the points $x_1, \ldots, x_n$, thus producing the required degree d cover. To show existence, let $J(C)$ be the Jacobian of C, $u : C \to J(C)$ be the Abel-Jacobi map, and $C_d = Sym^d(C)$. Consider the maps $v : C \to J(C)$ and $w : C_{g-1} \to J(C)$ given by $v(y) = u(dy) = du(y)$ and $w(D) = u(D) + u(\sum_{i=1}^{n} x_i)$. We need to show that the intersection between the image of v and that of w is nonempty. But the image of w is a translate of the Θ divisor and moreover $v^* w_*[C_{g-1}] = v^*([\Theta]) = dg \neq 0$. □

Lemma 2.4. *Fix a 2-point ramification cycle* $st_* \overline{\mathcal{Y}}_{d,g}(b_I(N) b_J(M))$ *on* $\overline{\mathcal{M}}_{g,n}$ *where* $N \sqcup M = \{x_1, \ldots, x_n\}$. *Then the cycle*

$$\psi_1 \cap st_* \overline{\mathcal{Y}}_{d,g}(b_I(N) b_J(M)) = st_* \left(st^* \psi_1 \cap \overline{\mathcal{Y}}_{d,g}(b_I(N) b_J(M)) \right)$$

can be written as a linear combination of generalized 2-point ramification cycles; its symbol consists of terms of type

$$st_* \overline{\mathcal{Y}}_{a,g}(b_{I_1}(N_1) b_{J_1}(M_1))$$

where $a \leq d$, $N_1 \sqcup M_1 = \{x_1, \ldots, x_n\}$ *and* $\ell(I_1) + \ell(J_1) = \ell(I) + \ell(J) - 1$.

Proof. The result is trivially true when $r = 0$, i.e. $\overline{\mathcal{Y}}_{d,g}(b_I(N) b_J(M))$ is zero dimensional (see Lemma 1.5). So we may assume $r > 0$.

The first step is to replace $st^* \psi_1$ by a multiple of $q^* \widetilde{\psi}_1$, where $\widetilde{\psi}_1 = c_1(L_{p_1})$ is the first Chern class of the relative cotangent bundle to the target $\mathbb{P}^1$ at p_1. For that, we temporarily mark the location of the other $\ell(I) - \ell(N)$ points in the preimage of p_1, $\ell(J) - \ell(M)$ points in the preimage of p_2 and each of the $d - 1$ points in the preimage of each of the other $r = 2g - 2 + n - \ell(I) - \ell(J)$ simple branch points. All together, we add $b = r(d - 2) + 2g - 2$ extra marked points, getting a corresponding 2-point cycle $\overline{\mathcal{Y}}_{d,g,n+b}$ in which all the preimages of all the branch points are marked. Consider the diagram

$$\begin{array}{ccccc} \overline{\mathcal{M}}_{g,n+b} & \xleftarrow{st_b} & \overline{\mathcal{Y}}_{d,g,n+b} & & \\ \downarrow{\pi_b} & & \downarrow{\rho_b} & \searrow^{q} & \\ \overline{\mathcal{M}}_{g,n} & \xleftarrow{st} & \overline{\mathcal{Y}}_{d,g,n} & \xrightarrow{q} & \overline{\mathcal{M}}_{0,2+r} \end{array} \tag{2.40}$$

where $\overline{\mathcal{Y}}_{d,g,n} = \overline{\mathcal{Y}}_{d,g}(b_I(N)\, b_J(M))$. Then $\rho_b : \overline{\mathcal{Y}}_{d,g,n+b} \to \overline{\mathcal{Y}}_{d,g,n}$ has finite *nonzero* degree deg ρ_b given by $(\ell(I)-\ell(N))!\cdot(\ell(J)-\ell(M))!\cdot(d-1)!^r \neq 0$, so

$$(\deg \rho_b) \cdot st^*\psi_1 = \rho_{b*}\, \rho_b^*\, st^*\psi_1 = \rho_{b*}\, st_b^*\, \pi_b^*\psi_1.$$

Moreover, the stabilization map $st_b : \overline{\mathcal{Y}}_{d,g,n+b} \to \overline{\mathcal{M}}_{g,n+b}$ does not collapse any components of the domain. Therefore, the relative cotangent bundle $\mathcal{L}_{x_1} \to \overline{\mathcal{Y}}_{d,g,n+b}$ to the domain is equal to the pullback by st_b of $L_{x_1} \to \overline{\mathcal{M}}_{g,n+b}$. Using formula (2.32) repeatedly and pulling back by st_b gives then the relation

$$st_b^*\, \pi_b^*\psi_1 = c_1(\mathcal{L}_{x_1}) - st_b^*\mathbf{D}_1$$

on $\overline{\mathcal{Y}}_{d,g,n+b}$, where $\mathbf{D}_1 = \sum_L D_{1,L}$ and $D_{1,L}$ is the boundary strata in $\overline{\mathcal{M}}_{g,n+b}$ where the marked point x_1 and a subset L of the b new marked points are the only points on a genus 0 bubble.

Now on $\overline{\mathcal{Y}}_{d,g,n+b}$ all the preimages of the marked points of the target are marked so the relation (1.31) implies that $\mathcal{L}_{x_1}^{s_1} = q^*L_{p_1}$ so

$$c_1(\mathcal{L}_{x_1}) = \frac{1}{s_1} \cdot q^*(c_1(L_{p_1})) = \frac{1}{s_1} \cdot q^*(\widetilde{\psi}_1).$$

Combining the last three displayed equations we get

$$\psi_1 \cap \overline{\mathcal{Y}}_{d,g,n}(b_I(x_1) b_J(M)) = \frac{1}{s_1 \deg(\rho_b)} \cdot st_* q^*(\widetilde{\psi}_1) - \frac{1}{\deg(\rho_b)} \cdot \rho_{b*} st_b^* \mathbf{D}_1. \tag{2.41}$$

Next, we use the fact that in $\overline{\mathcal{M}}_{0,2+r}$ we have $r \cdot \widetilde{\psi}_1 = D$ where $D = \sum_{j=3}^{r+2} D_{\Gamma_j}$ and D_{Γ_j} is the boundary strata that has the marked point p_1 on a bubble and p_2, p_j on a different bubble, while the remaining $r-1$ branch points are distributed all possible ways. Note that in D the strata which has a bubble containing p_1 and precisely r_1 of the points p_j with $j \geq 3$ appears with coefficient $r_2 = r - r_1$. Applying the degeneration formula (1.23) for each

j and summing then gives

$$\begin{aligned} st_* q^*(\widetilde{\psi}_1) &= \frac{1}{r}\, st_* q^*(D) \\ &= \sum \frac{|S|}{\ell(S)!} \cdot \frac{r_2}{r} \cdot\, st_* \left(\overline{\mathcal{Z}}_{d,\chi_1}(b_I(N) b_S) \underset{\xi}{\times} \overline{\mathcal{Z}}_{d,\chi_2}(b_S b_J(M)) \right) \end{aligned} \tag{2.42}$$

where the sum is over all χ_1, χ_2, r_1, r_2, ordered sequences S such that $\deg S = d$, $\chi_1 + \chi_2 - 2\ell(S) = 2g - 2$, $r_1 + r_2 = r$, over all possible identifications that lead to a connected domain and over all possible distributions of the r simple branch points such that r_1 are on the left component. In any case, this shows that the first term on the right hand side of equation (2.41) is a linear combination of generalized 2-point ramification cycles.

On the other hand $\rho_{b*} st_b^* \mathbf{D}_1$ is also equal to a linear combination of similar generalized 2-point ramification cycles. This is because st_b doesn't collapse any components, so $st_b^* \mathbf{D}_1$ consists of stable maps in $\overline{\mathcal{Y}}_{d,g,n+b}$ whose domain is an element of $\mathbf{D}_1$. In particular, the target of these maps must be a bubble tree with p_1 on one side and p_2 on the other.

Using (2.41) and (2.42) we then conclude that on $st_* \overline{\mathcal{Y}}_{d,g}(b_I(N) b_J(M))$, ψ_1 can be written as a linear combination of the generalized 2-point ramification classes. The statement about the structure of the symbol follows immediately by a dimension count. □

Because of Remark 1.15, an immediate consequence of Theorem 2.2 is the following:

Corollary 2.5. *Assume $g \geq 2$, $g + n \geq 3$ and let $\Pi_k : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-k}$ be a forgetful map. Then the Poincare dual of the class $\Pi_{k*}\left(\psi_1^{m_1} \dots \psi_n^{m_n}\right)$ on $\overline{\mathcal{M}}_{g,n-k}$ can be written as linear combination of generalized 2-point ramification cycles on $\overline{\mathcal{M}}_{g,n-k}$ whose symbol consists of codimension $\sum_{j=1}^{n} m_j - k$ terms of type*

$$st_* \overline{\mathcal{Y}}_{a,g}(b_{I_1}(N_1) b_{I_2}(N_2))$$

where $a \leq d$, $N_1 \sqcup N_2 = \{x_1, \dots, x_{n-k}\}$ and $\ell(I_1) + \ell(I_2) = g + n - \sum_{j=1}^{n} m_j$.

Note that since $\ell(N_j) \leq \ell(I_j)$ then in particular the symbol vanishes when $\sum_{j=1}^{n} m_j > g + k$.

Remark 2.6. If one is interested not only in the shape of the symbol, but in the actual formula then it is convenient to start with a cover of degree as small as possible, so there would be fewer terms to consider. In this context, one can use the fact that any complex structure can be written as a degree

$d = \left[\frac{g+1}{2}\right] + 1$ cover of $\mathbb{P}^1$ to adapt the proof of Theorem 2.2 to get the following:

Proposition 2.7. *Any polynomial in descendant classes on $\overline{\mathcal{M}}_{g,n}$ can be written as a linear combination of generalized ramification constraints coming from covers of degree at most* $\left[\frac{g+1}{2}\right] + 1$.

For example, when g is odd, one would start with the space $\overline{\mathcal{Y}}_{g,d}\left(\prod_{i=1}^{n} b_{1^d}(x_i)\right)$ for which the degree of the stabilization map is nonzero, while when g is even, one would use instead the space $\overline{\mathcal{Y}}_{g,d}\left(b_{2,1^{d-2}}(x_1)\prod_{i=2}^{n} b_{1^d}(x_i)\right)$. Then one uses the fact that in $\overline{\mathcal{M}}_{0,r+n}$ the Poincare dual of any monomial in descendant classes $\widetilde{\psi}_j$ can be expressed as a linear combination of boundary strata corresponding to linear chains of $\mathbb{P}^1$'s. In the end, after using Remark 1.16, one would get generalized ramification cycles with at most two complicated branch points on each component of the target (but not technically 2-point ramification cycles, because of the presence of constraints of type $b_{1^d}(x_i)$).

2.1. Proof of Theorem 0.1

Suppose we start with a degree m monomial in ψ and κ classes on $\overline{\mathcal{M}}_{g,n}$. Then using the formulas (2.32) and (2.33) we can express any such polynomial as a linear combination of terms of type

$$\Pi_{k*}\big(\psi_1^{m_1}\ldots\psi_{n+k}^{m_{n+k}}\big)$$

for some k's, where $\Pi_k : \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$ is the map that forgets the last k marked points, and $\sum_{j=1}^{n+k} m_j = m + k$. For example,

$$\kappa_a\kappa_b = \Pi_{2*}\big(\psi_1^{a+1}\psi_2^{b+1}\big) - \Pi_{1*}\big(\psi_1^{a+b+1}\big).$$

It is therefore enough to prove Theorem 0.1 for classes of type

$$\Pi_{k*}\left(\psi_1^{m_1}\ldots\psi_{n+k}^{m_{n+k}}\right) \in H^{m-k}(\overline{\mathcal{M}}_{g,n}) \tag{2.43}$$

where $m = \sum_{i=1}^{n+k} m_i \geq g + k$. We actually prove that the Poincare dual of such class can be written as a linear combination of generalized ramification cycles with vanishing symbol on $\overline{\mathcal{M}}_{g,n}$, i.e. all terms are coming from the boundary $\partial\overline{\mathcal{M}}_{g,n}$.

Corollary 2.5, with n replaced by $n+k$, implies that the Poincare dual of the class (2.43) can be written as a linear combination of generalized ramification cycles whose symbol consists of terms of type

$$\overline{\mathcal{Y}}_{a,g}(b_{I_1}(N_1)b_{I_2}(N_2))$$

with $a \leq d$, $N_1 \sqcup N_2 = \{x_1, \ldots, x_n\}$ and $\ell(I_1) + \ell(I_2) = g + n + k - m$. When $m \geq g + k$ we have

$$n = \ell(N_1) + \ell(N_2) \leq \ell(I_1) + \ell(I_2) = g + n + k - m \leq n,$$

so all terms in the symbol vanish unless $\ell(N_j) = \ell(I_j)$ for $j = 1, 2$ and $m = g + k$.

When $n \leq 1$ there are no such terms since $\ell(I_j) \geq 1$, so the symbol vanishes. Moreover, note that when $n = 0$ even for $m = g + k - 1$ a similar string of inequalities shows that the symbol also vanishes, implying Looijenga's result [L2] (in cohomology).

When $n \geq 2$, Proposition 2.8 below completes the proof of Theorem 0.1. □

Proposition 2.8. *Suppose* $g \geq 1$ *and* $\ell(I_i) = \ell(N_i)$ *for* $i = 1, 2$. *Then the codimension* g *cycle on* $\overline{\mathcal{M}}_{g,n}$

$$\mathcal{C} = st_* \overline{\mathcal{Y}}_{d,g}(b_{I_1}(N_1)b_{I_2}(N_2))$$

can be written as a linear combination of generalized ramification cycles of type $\xi_*(\prod_{a=1}^{h} \pi_a^* \mathcal{C}_a)$ *where* $\xi : \prod_{a=1}^{h} \overline{\mathcal{M}}_{g_a,n_a} \to \overline{\mathcal{M}}_{g,n}$ *is the attaching map of some* boundary strata *of* $\overline{\mathcal{M}}_{g,n}$, $\pi_a : \overline{\mathcal{M}}_{g_a,n_a} \to \overline{\mathcal{M}}_{g_a,m_a}$ *is a forgetful map (this includes the identity map in the case* $m_a = n_a$*) and* $\mathcal{C}_a$ *is a 2-point ramification cycle on* $\overline{\mathcal{M}}_{g_a,m_a}$ *coming from a degree* $d_a \leq d$ *cover. In particular, the symbol of this linear combination vanishes.*

Proof. We prove the statement by induction on both the degree d and the number of marked points n. It is enough to prove that the cycle $\mathcal{C}$ can be written as a linear combination of cycles of type $\xi_*(\prod_{a=1}^{h} \pi_a^* \mathcal{C}_a)$ which either come from the boundary or else have only one component (i.e. $h = 1$) and for this component either $d_1 < d$ or $m_1 < n_1 = n$.

Assume $x_1 \in N_1$ and let $N_1' = N_1 \setminus \{x_1\}$. Consider the cycle

$$\overline{\mathcal{Y}}_{d,g}(B_{I_1}(N_1')B_{1^d}(x_1)B_{2,1^{d-2}}B_{I_2}(N_2))$$

which corresponds to fixing the location of the marked points $p_1, \ldots, p_4$ on the target (while the remaining $r - 1$ simple branch points of the target are allowed to move). But on $\overline{\mathcal{M}}_{0,3+r}$ the divisor corresponding to fixing the location of $p_1, \ldots, p_4$ is linearly equivalent to the boundary stratum $D = (p_1 p_2 | p_3 p_4)$ where p_1, p_2 are on a bubble and p_3, p_4 are on a different bubble. For simplicity we denote $q^*(p_1 p_2 | p_3 p_4) = \overline{\mathcal{Y}}_{d,g}(b_{I_1}(N_1')b_{1^d}(x_1) \mid b_{2,1^{d-2}} b_{I_2}(N_2))$. Since the stratum $(p_1 p_2 | p_3 p_4)$ is linearly equivalent to the stratum $(p_1 p_3 | p_2 p_4)$ then

$$\begin{aligned} &st_* \overline{\mathcal{Y}}_{d,g}(b_{I_1}(N_1')b_{1^d}(x_1) \mid b_{2,1^{d-2}} b_{I_2}(N_2)) \\ &\qquad = st_* \overline{\mathcal{Y}}_{d,g}(b_{I_1}(N_1')b_{2,1^{d-2}} \mid b_{1^d}(x_1) b_{I_2}(N_2)) \end{aligned} \tag{2.44}$$

as codimension g cycles in $\overline{\mathcal{M}}_{g,n}$. But the degeneration formula (1.23) implies that both sides of (2.44) are linear combination of pushforwards by st of terms of type

$$\overline{\mathcal{Z}}_{d,\chi_1}(b_{I_1}(N_1')b_{1^d}(x_1)b_S) \underset{\xi}{\times} \overline{\mathcal{Z}}_{d,\chi_2}(b_S b_{2,1^{d-2}} b_{I_2}(N_2)) \text{ and} \tag{2.45}$$

$$\overline{\mathcal{Z}}_{d,\chi_1}(b_{I_1}(N_1')b_{2,1^{d-2}} b_S) \underset{\xi}{\times} \overline{\mathcal{Z}}_{d,\chi_2}(b_S b_{1^d}(x_1) b_{I_2}(N_2)) \tag{2.46}$$

respectively. We need to show that the only term not lying in the boundary of $\overline{\mathcal{M}}_{g,n}$ and with $d_1 = d$, $m_1 = n_1$ is the term $\mathcal{C}$; moreover $\mathcal{C}$ should appear in (2.44) with nonzero coefficient. Let $\mathcal{C}'$ be such a term appearing after stabilization in the symbol of (2.45) or (2.46). This means that before stabilization we have a degree d genus g component on one side and all the components on the other side are genus 0 totally ramified over the node of the target; otherwise collapsing them would produce a double point of the (stabilized) domain. Moreover, before stabilization we can have at most one marked point on each genus 0 component (since when $g \geq 1$ the strata of $\overline{\mathcal{M}}_{g,n}$ having stable $g = 0$ components are in the boundary).

Suppose first that $\mathcal{C}'$ appears in the symbol of (2.45). We have two cases to consider:

(a) the genus g component is on the left. But since $\ell(I_2) = \ell(N_2)$ the genus 0 component on the right which contains the simple ramification point cannot be totally ramified over p_4 so will have to contain two of the marked points in N_2, contradiction.

(b) the genus g component is on the right. Since $\ell(I_1) = \ell(N_1') + 1$ there can be at most one genus 0 component which is not totally ramified over p_1 (otherwise two of the points in N_1' would be on the same genus 0 component). But one of the genus 0 components must also contain x_1, so the only possibility is if all genus 0 components were totally ramified over p_1 and moreover x_1 would be on the only genus 0 component not containing a point from N_1'. After pushing forward by st_* this term contributes

$$s_1 st_*(b_{I_1}(N_1) b_{I_2}(N_2)) = s_1 \mathcal{C}$$

to the right hand side of (2.44), where s_1 is the multiplicity of x_1 in I_1.

Next suppose that $\mathcal{C}'$ appears in the symbol of (2.46). We also have two cases to consider:

(a) the genus g component is on the left. Since $\ell(I_2) = \ell(N_2)$ then each genus 0 component on the right has at least one of the marked points of N_2. But one of these genus 0 components must also have x_1, contradiction.

(b) the genus g component is on the right. Since $\ell(I_1) = \ell(N_1')+1$ there can be at most one genus 0 component which is not totally ramified over p_1, and this component can have at most 2 points over p_1 (otherwise two of the points in N_1' would land on the same genus 0 component). This genus 0 component must contain the simple ramification point and only one of the points $x_a \in N_1'$, the other point over p_1 being unmarked. The order of ramification over the node of the target of this component must then be equal to the sum of the multiplicities of the points over p_1. Denote by $\widehat{I}$ the sequence obtained from I_1 by erasing the multiplicity corresponding to x_1 and adding it to the multiplicity corresponding to x_a. After collapsing the genus 0 components this term is equal to a multiple of

$$st_*\overline{\mathcal{Y}}_{d,g}\left(b_{\widehat{I}}(N_1')b_{1^d}(x_1)b_{I_2}(N_2)\right)$$

where $\ell(\widehat{I}) = \ell(N_1')$. By relation (1.30) this term is equal to $\pi_1^* st_*\overline{\mathcal{Y}}_{d,g}\left(b_{\widehat{I}}(N_1')b_{I_2}(N_2)\right)$ where π_1 is the map that forgets the marked point x_1. Therefore it is pulled back from a moduli space with fewer marked points.

This concludes the inductive step and with it the proof of Proposition 2.8. □

We finish this paper by proving the following result, which was recently conjectured by Graber and Vakil [V] for the Chow group.

Proposition 2.9. *The Poincare dual of any degree m monomial in descendant or tautological classes on $\overline{\mathcal{M}}_{g,n}$ can be written as a linear combination of classes coming from the strata of $\overline{\mathcal{M}}_{g,n}$ which have at least $m+1-g$ genus 0 components.*

Proof. The result is already known in genus 0 or 1, so we prove it for $g \geq 2$. As in the proof of Theorem 0.1, Corollary 2.5 implies that the Poincare dual of any degree m monomial in ψ and κ classes can be written as a linear combination of codimension m generalized 2-point ramification cycles. Each such generalized 2-point ramification cycle is of type $\xi_*(\prod_{a=1}^{m} \mathcal{C}_a)$ where $\xi : \prod_{a=1}^{m} \overline{\mathcal{M}}_{g_a,m_a} \to \overline{\mathcal{M}}_{g,n}$ is the attaching map of some stratum of $\overline{\mathcal{M}}_{g,n}$ (including possibly the top stratum) and each $\mathcal{C}_a$ is a 2-point ramification cycle of type $\mathcal{C}_a = st_*\overline{\mathcal{Y}}_{d_a,g_a}(b_{I_{a1}}(N_{a1})b_{I_{a2}}(N_{a2}))$. The codimension of such

$\mathcal{C}_a$ is at most g_a by relation (1.14). But by induction (on the dimension of the moduli space $\overline{\mathcal{M}}_{g,n}$) we can prove that any 2-point ramification cycle $\mathcal{C} = st_* \overline{\mathcal{Y}}_{d,g}(b_{I_1}(N_1) b_{I_2}(N_2))$ can be written as a linear combination of generalized 2-point ramification cycles of type

$$\xi_* \left(\prod_{a=1}^{m} \mathcal{C}_a \right), \quad \text{where } \mathcal{C}_a = \pi_a^* st_* \overline{\mathcal{Y}}_{d_a, g_a}(b_{I_{a1}}(N_{a1}) b_{I_{a2}}(N_{a2})) \quad (2.47)$$

where moreover codim $\mathcal{C}_a \leq g_a - 1$ on all $g_a \geq 1$ components. This is because either $\mathcal{C}$ already has codimension less then g or else Proposition 2.8 shows that it can be written as a linear combination of generalized ramification cycles of type (2.47) coming from a boundary strata (in which case each $\mathcal{C}_a$ comes from a lower dimensional moduli space).

Therefore the Poincare dual of any degree m monomial in κ and ψ classes can be written as a linear combination of codimension m generalized ramification cycles of type (2.47) for which codim $\mathcal{C}_a \leq g_a - 1$ on all $g_a \geq 1$ components. Fix such a codimension m generalized ramification cycle. We only need to show that the domain of the corresponding attaching map ξ has at least $m + 1 - g$ genus 0 components. Let k be the number of double points and ℓ be the number of genus 0 components of the corresponding stratum of $\overline{\mathcal{M}}_{g,n}$. Then

$$\begin{aligned} m = k + \sum_{a=1}^{m} \text{codim } \mathcal{C}_a &\leq k + \sum_{g_a \geq 1} (g_a - 1) \\ &= k + \sum_{a=1}^{m} (g_a - 1) + \ell = g - 1 + \ell \end{aligned}$$

where the last equality follows from the Euler characteristic relation $2 - 2g = \sum_{a=1}^{m} (2 - 2g_a) - 2k$. Therefore $\ell \geq m + 1 - g$. □

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