Invent. math. 148, 117–141 (2002)
Digital Object Identifier (DOI) 10.1007/s002220100190

Inventiones mathematicae

Manifolds with positive sectional curvature almost everywhere

Burkhard Wilking★

University of Pennsylvania, 209 South Street, Philadelphia, PA 19104-6395, USA
(e-mail: `wilking@math.upenn.edu`)

Oblatum 1-VI-2001 & 19-IX-2001
Published online: 18 January 2002 –

1. Introduction

Synge's theorem is one of the few known obstructions for positive sectional curvature. It states that an orientable even dimensional compact Riemannian manifold of positive sectional curvature is simply connected, and that an odd dimensional compact Riemannian manifold of positive sectional curvature is orientable. As an immediate consequence of this theorem, $\mathbb{RP}^n \times \mathbb{RP}^m$ does not admit a metric with positive sectional curvature. This fact in turn is often quoted as an indication for the generalized

Hopf conjecture. $\mathbb{S}^n \times \mathbb{S}^m$ *does not admit a Riemannian metric with positive sectional curvature.*

Hopf originally only asked this in the case $n = m = 2$. This conjecture is one among very few that indicate what kind of general obstructions one should expect for positive sectional curvature. The reason for this dilemma is closely related to the lack of examples. In fact in dimensions above 24 the known examples are diffeomorphic to locally rank 1 symmetric spaces, i.e., quotients of $\mathbb{S}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathrm{Ca}\mathbb{P}^2$ by a finite group.

The first Riemannian manifolds that are not diffeomorphic to locally rank one symmetric spaces were found by Berger [1961]. He found two homogeneous examples; one in dimension 7 and one in dimension 13. Wallach [1972] and Bérard Bergery [1976] classified all simply connected homogeneous spaces with positive curvature in even and odd dimensions.

Eschenburg [1984] found the first examples of manifolds with positive sectional curvature that are not homotopically equivalent to homogeneous spaces. He found one such example in dimension six and an infinite family in dimension seven. Later on Bazaikin [1996] found a similar family in

★ The author was supported by a fellowship of the Deutsche Forschungsgemeinschaft

dimension 13. The examples of Bazaikin and Eschenburg are so called biquotients, i.e., they are given as orbit spaces $\mathsf{G}/\!/\mathsf{H}$ where G is a compact Lie group and H is a closed subgroup of $\mathsf{G}\times\mathsf{G}$ for which the natural action of H on G is free.

The first indication that one might find new examples among biquotients was given much earlier by Gromoll und Meyer [1974]. They found a seven dimensional biquotient (Σ, g) which is homeomorphic but not diffeomorphic to $\mathbb{S}^7$. The sectional curvature of (Σ, g) is only nonnegative, but there is a point $p \in \Sigma$ such that all planes at p have positive curvature. Gromoll und Meyer [1974] mention the following conjecture without saying that they support it.

Deformation conjecture. *Suppose (M, g) is a complete Riemannian manifold with nonnegative sectional curvature and suppose that there is one point $p \in M$ such that all planes based at p have positive curvature. Then M admits a metric with positive sectional curvature.*

There are several indications for this conjecture: Aubin [1970] and Ehrlich [1976] proved the analogous statements for scalar and Ricci curvature. Strake [1986] showed that the statement is true provided that the points at which zero curvature planes occur are contained in a convex set. According to Perelman's [1994] proof of the soul conjecture, a noncompact manifold M satisfying the hypothesis of the deformation conjecture is diffeomorphic to $\mathbb{R}^n$. In particular, M admits then a metric with positive sectional curvature.

However, the following theorem provides counterexamples.

Theorem 1. *Each of the following compact manifolds admits a Riemannian metric g with positive sectional curvature on an open and dense set of points.*

a) *The projective tangent bundles $P_{\mathbb{R}}T\mathbb{RP}^n$, $P_{\mathbb{C}}T\mathbb{CP}^n$ and $P_{\mathbb{H}}T\mathbb{HP}^n$ of $\mathbb{RP}^n$, $\mathbb{CP}^n$ and $\mathbb{HP}^n$, respectively. The corresponding dimensions are $2n-1$, $4n-2$ and $8n-4$.*
b) *The biquotient $\mathrm{SO}(2)\backslash\,\mathrm{SO}(2n+1)/\,\mathrm{SO}(2n-1)$ of dimension $4n-2$, $n \geq 2$.*
c) *The $(4n-1)$–dimensional manifold $M^{4n-1}_{kl} := \mathsf{U}(n+1)/\mathsf{H}_{kl}$ with*

$$\mathsf{H}_{kl} := \left\{ \begin{pmatrix} z^k & & \\ & z^l & \\ & & A \end{pmatrix} \;\middle|\; z \in \mathsf{S}^1,\, A \in \mathsf{U}(n-1) \right\},$$

provided that (k, l) is a pair of integers satisfying $k \cdot l < 0$, $n \geq 2$.
d) *The biquotient $\mathsf{Sp}(1)\backslash\mathsf{Sp}(n+1)/\mathsf{Sp}(1)\cdot\mathsf{Sp}(n-1)$ of dimension $8n-4$.*

Furthermore, $P_{\mathbb{K}}T\mathbb{KP}^2/\mathsf{S}_3$, i.e., a quotient of $P_{\mathbb{K}}T\mathbb{KP}^2$ by a free action of the symmetric group S_3, admits a metric with nonnegative sectional curvature and positive curvature at one point, $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathrm{Ca}\}$.

The odd dimensional manifold $P_{\mathbb{R}}T\mathbb{RP}^{2n+1}$ is not orientable. Thus, by Synge's theorem, it does not admit a metric with positive sectional curvature. Furthermore, the fundamental groups of the even dimensional manifolds $P_{\mathbb{C}}T\mathbb{CP}^2/\mathsf{S}_3$, $P_{\mathbb{H}}T\mathbb{HP}^2/\mathsf{S}_3$ and $P_{\mathrm{Ca}}T\mathrm{Ca}\mathbb{P}^2/\mathsf{S}_3$ are isomorphic to S_3. Again by Synge's theorem, these manifolds do not admit metrics of positive sectional curvature either. This proves

Corollary 2. *The deformation conjecture is wrong in dimensions* $4n+1$, $n \geq 1$, *as well as in dimensions* 6, 12 *and* 24.

Since the projective tangent bundles of $\mathbb{RP}^3$ and $\mathbb{RP}^7$ are trivial, we deduce

Corollary 3. *The manifolds* $\mathbb{RP}^3 \times \mathbb{RP}^2$ *and* $\mathbb{RP}^7 \times \mathbb{RP}^6$ *as well as their universal covers* $\mathbb{S}^3 \times \mathbb{S}^2$ *and* $\mathbb{S}^7 \times \mathbb{S}^6$ *admit metrics with positive sectional curvature on open dense sets.*

One could ask whether the deformation conjecture is true for simply connected manifolds. Although this problem remains unsettled, it is clear that a positive answer would provide a counterexample to the generalized Hopf conjecture.

The manifold M^{4n-1}_{kl} in the series c) is simply connected. Using the Gysin sequence of the fibration $\mathsf{S}^1 \to M^{4n-1}_{kl} \to P_{\mathbb{C}}T\mathbb{CP}^n$, it is easy to check that the order of $H^{2n}\big(M^{4n-1}_{kl}, \mathbb{Z}\big)$ is given by

$$q_{k,l} = k^n + k^{n-1}l + \cdots + kl^{n-1} + l^n = \tfrac{k^{n+1}-l^{n+1}}{k-l}$$

provided that k and l are relatively prime and $q_{k,l} \neq 0$; the case of $q_{k,l} = 0$ can only occur for $(n,k,l) = (2m+1, 1, -1)$ and $H^{4m+2}\big(M^{8m+3}_{1,-1}, \mathbb{Z}\big) \cong \mathbb{Z}$.

Corollary 4. *There are infinitely many different homotopy types of simply connected* $(4n-1)$*–dimensional manifolds with positive sectional curvature on open dense sets,* $n \geq 3$.

Remark 5. It is worth noticing that in low dimensions, i.e., for small n, the above examples a) – d) are already well known, although we endow these examples with different metrics.

1. Wallach [1972] showed that the flags, i.e., the projective tangent bundles of $\mathbb{CP}^2$, $\mathbb{HP}^2$ and $\mathrm{Ca}\mathbb{P}^2$, admit homogeneous metrics with positive sectional curvature.
2. For $n = 2$ the series in c) represents the well-known Aloff Wallach series. Aloff and Wallach [1975] showed that M^7_{kl} admits a homogeneous metric of positive sectional curvature unless $k \cdot l \cdot (k+l) = 0$. Notice that our construction provides a metric on $M^7_{1,-1} \cong M^7_{10}$ with positive sectional curvature on an open dense set. The integral cohomology ring of this manifold coincides with the integral cohomology ring of $\mathbb{S}^2 \times \mathbb{S}^5$.

3. Eschenburg [1984] showed that for $n = 2$ the biquotient in d) admits a metric with nonnegative sectional curvature and positive curvature at one point. This manifold has the same cohomology ring as $P_{\mathbb{H}}T\mathbb{HP}^2$ but a different homotopy type.
4. Petersen and Wilhelm [1999] constructed a metric with positive sectional curvature on an open dense set on the unit tangent bundle $T^1\mathbb{S}^4$ of $\mathbb{S}^4$ and on a six dimensional quotient of $T^1\mathbb{S}^4$. Notice that $T^1\mathbb{S}^4$ is the universal cover of $P_{\mathbb{R}}T\mathbb{RP}^4$. The six dimensional quotient corresponds to the example b) with $n = 2$.

For many of the examples in the theorem no obstruction to positive curvature applies and it would be interesting to know whether in some of the cases one can deform the metric into a metric with positive sectional curvature everywhere. Of course, it would also be interesting to find topological constraints explaining the existence of zero curvature planes. In either case a description of the points at which zero curvature planes occur might be a first step:

Proposition 6. *Consider the unit tangent bundle $T^1\mathbb{S}^n$ of $\mathbb{S}^n$ as a submanifold of $T\mathbb{R}^{n+1} = \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$, $n \geq 3$. The metric g of $P_{\mathbb{R}}T\mathbb{RP}^n$ of Theorem 1 induces naturally a metric $\hat{g}$ on its universal cover $T^1\mathbb{S}^n$. With respect to this metric the points in $T^1\mathbb{S}^n$ at which zero curvature planes occur are given by*

$$\left\{(p, v) \in T^1\mathbb{S}^n \;\middle|\; p \perp e_1\right\} \quad \cup \quad \left\{(p, v) \in T^1\mathbb{S}^n \;\middle|\; v \perp e_1\right\}$$

where e_1 is the first vector of the canonical basis of $\mathbb{R}^{n+1}$. Moreover, if $\mathrm{O}(n) \subset \mathrm{O}(n+1)$ denotes the subgroup that fixes e_1, then the natural action of $\mathrm{O}(n)$ on $T^1\mathbb{S}^n$ is isometric with respect to $\hat{g}$.

Both sets in the above proposition are diffeomorphic to $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ and they intersect in a copy of $T^1\mathbb{S}^{n-1}$ endowed with a homogeneous metric. For a description of the zero curvatures of $P_{\mathbb{C}}T\mathbb{CP}^n$ and $P_{\mathbb{H}}T\mathbb{HP}^n$, see Proposition 6.4.

Open problems. The class of examples of manifolds with nonnegative sectional curvature is huge compared to the known examples of manifolds with positive sectional curvature. It is the author's belief that it might be fruitful to have a class in between these two well studied classes, one that is more rigid than nonnegative sectional curvature but has a larger class of examples than the known examples of manifolds with positive sectional curvature.

By Theorem 1 the class of manifolds with positive sectional curvature is strictly smaller than the class of manifolds with positive sectional curvature on an open dense set. Since a manifold in the latter class has a finite fundamental group, this class in turn is strictly contained in the class of manifolds of nonnegative sectional curvature. But for simply connected manifolds the problem whether these inclusions are strict remains open. In the even dimensional case a good candidate for an obstruction might be the Euler-characteristic. Recall that one of the oldest conjectures in Riemannian geometry, the ’other’ Hopf conjecture, asks whether a compact even dimensional Riemannian manifold M with $\sec(M) > 0$ (resp. $\sec(M) \geq 0$) fulfills $\chi(M) > 0$ (resp. $\chi(M) \geq 0$). This imposes the following

Question 1. Does a compact even dimensional manifold M with positive sectional curvature on an open dense set satisfy $\chi(M) > 0$?

A positive answer would imply that an upper curvature bound of such a manifold yields via the Chern–Gauss–Bonnet formula a lower volume bound which in turn gives in presence of an upper diameter bound a lower bound for the injectivity radius. One could consider this as a substitute for Klingenberg’s injectivity radius estimate, and in particular it would follow that these manifolds are subject to the same type of finiteness results as manifolds of positive sectional curvature.

Notice that an affirmative answer would also imply that in contrast to $\mathbb{S}^3 \times \mathbb{S}^2$ the manifold $\mathbb{S}^3 \times \mathbb{S}^3$ does not admit a metric with positive sectional curvature on an open dense set.

One might hope that a study of the zero curvature planes in the examples yields some idea towards obstructions for positive sectional curvature, or possibly for constructing manifolds with positive sectional curvature.

Question 2. Suppose (M^n, g) is a compact Riemannian manifold with nonnegative sectional curvature and suppose there is an open set $U \subset M$ diffeomorphic to $\mathbb{R}^n$ such that $M^n \setminus U$ has positive sectional curvature. Does M admit a metric with positive sectional curvature?

In most examples of Theorem 1 the hypothesis of this question is not satisfied. However, the Gromoll Meyer sphere (Σ^7, g) satisfies the hypothesis.

According to Hamilton [1982], the deformation conjecture is true in dimension three and our counterexamples start in dimension 5. So it is natural to ask.

Question 3. Is the deformation conjecture true in dimension four?

Petersen and Wilhelm [1999] introduced the concept of quasi-positive sectional curvature. A complete Riemannian manifold is said to have quasi-positive sectional curvature if it has nonnegative sectional curvature and positive curvature at one point. They proposed that this class of manifolds deserves more attention. One of the problems they suggest to consider is

Question 4. Does a Riemannian manifold of quasi-positive sectional curvature admit a metric with positive sectional curvature on an open dense set?

Wilhelm [1999] showed that for the Gromoll Meyer sphere (Σ^7, g) the answer is yes. However, for the manifolds $P_{\mathbb{K}}T\mathbb{KP}^2/\mathsf{S}_3$ $\big(\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathrm{Ca}\}\big)$ the answer is not known.

It is my pleasure to thank Wolfgang Ziller for many useful discussions and comments. I am also indebted to the referee for suggesting several improvements.

2. Organization of the paper

The main idea for the proof of Theorem 1 relies on a basic observation which assigns to every biquotient a natural enlarged class of metrics. In this enlarged class of metrics one has a better chance to realize given curvature properties. We explain this basic observation in Sect. 3. In Sect. 4 we recall the concept of a normal biquotient and show that every zero curvature plane in such a manifold comes from a totally geodesic immersed flat.

Next we present the general idea for the proof of the theorem in the simplest and perhaps most interesting special case $P_{\mathbb{R}}T\mathbb{RP}^3 \cong \mathbb{RP}^3 \times \mathbb{RP}^2$, see Sect. 5. In particular, we will give a precise description of the zero curvature planes in this example.

In the proof of Theorem 1 a) we can make use of the large isometry groups and totally geodesic submanifolds to give a precise description of the points at which zero curvature planes occur, see Proposition 6.4. In fact by combing this rich structure with only one minor calculation we reduce the problem to the case of $n = 3$, i.e., it then remains to be seen that $P_{\mathbb{R}}T\mathbb{RP}^3$, $P_{\mathbb{C}}T\mathbb{CP}^3$ and $P_{\mathbb{H}}T\mathbb{HP}^3$ have positive curvature on open dense sets. This in turn also requires only a fairly small amount of calculation. Once part a) is established part b) and c) follow easily, see Sect. 7 and Sect. 8.

However, this sort of dimension reduction does not work for the series d) of the theorem. Here the isometry group is too small to be of any use. Instead we use a different technique, which might be useful in other contexts as well. We introduce a simple method which produces a polynomial equation for the points at which zero curvature planes occur. In general it is by no means clear that the equation is not satisfied at every point, but verifying for a point that the equation is not satisfied turns out to be a problem of linear algebra, see Sect. 9 for details.

Finally, we describe the construction for $P_{\mathbb{K}}T\mathbb{KP}^3/\mathsf{S}_3$ ($\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathrm{Ca}\}$) in Sect. 10. It is worth noticing that in these examples there are open subsets of points at which zero curvature planes occur.

3. Normalized description of biquotients

Let G be a compact Lie group and $\mathsf{H} \subset \mathsf{G} \times \mathsf{G}$ a compact subgroup such that the natural action of H on G given by

$$(h_1, h_2) \star g := h_1 g h_2^{-1} \quad \text{for all } (h_1, h_2) \in \mathsf{H},\ \ g \in \mathsf{G}$$

is effectively free, i.e., an element $h \in \mathsf{H}$ has a fix point if and only if h is in the kernel of the action. The orbit space of this action $\mathsf{G}/\!/\mathsf{H}$ has then the natural structure of a manifold. Any left invariant metric g on G for which the above action is isometric induces a metric on $\mathsf{G}/\!/\mathsf{H}$, i.e., the metric that turns the projection $\mathrm{pr}\colon \mathsf{G} \to \mathsf{G}/\!/\mathsf{H}$ into a Riemannian submersion. Of course, the same is valid for any right invariant metric on G invariant under the action of H.

So there are two natural families of metrics on the quotient. Of course one can consider instead the cone of metrics generated by these two families. However, it is more general and easier to make use of the following

Proposition 3.1. *Let* $\mathsf{H} \subset \mathsf{G} \times \mathsf{G}$ *be as above, and let* $\triangle\mathsf{G} \subset \mathsf{G} \times \mathsf{G}$ *denote the diagonal subgroup. Then the action of* $\triangle\mathsf{G} \times \mathsf{H}$ *on* $\mathsf{G} \times \mathsf{G}$ *given by*

$$(a, h) \star (c, d) = a \cdot (c, d) \cdot h^{-1} \text{ for } a \in \triangle\mathsf{G},\ \ h \in \mathsf{H}$$

is effectively free, the biquotient $\triangle\mathsf{G}\backslash\mathsf{G} \times \mathsf{G}/\mathsf{H}$ *is canonically diffeomorphic to* $\mathsf{G}/\!/\mathsf{H}$*, and the class of left invariant* Ad_{H}*–invariant metrics on* $\mathsf{G} \times \mathsf{G}$ *induces a cone of metrics on the quotient containing the two original families.*

At first sight one might think that one can iterate this process and get even a larger class of metrics on the biquotient. However, iterating does not increase the family of natural metrics any further.

Proof. The canonical diffeomorphism is induced by the map

$$\mathsf{G} \times \mathsf{G} \to \mathsf{G}, \quad (a, b) \mapsto a^{-1}b.$$

Consider all left invariant Ad_{H}–invariant product metrics $g_1 \times g_2$ on $\mathsf{G} \times \mathsf{G}$. Furthermore, we consider the subfamily of metrics for which g_i is a biinvariant metric on G, $i = 1, 2$. This first family of metrics induces a cone of metrics on $\mathsf{G}/\!/\mathsf{H}$ and it is straightforward to check that the two subcones corresponding to the two subfamilies coincide with the two orginal families of metrics on $\mathsf{G}/\!/\mathsf{H}$. □

4. Zero curvature planes in normal biquotients

A normal biquotient (M, g) is a Riemannian manifold that can be described as a biquotient $\mathsf{G}/\!/\mathsf{H}$ such that the metric g on M is induced by a biinvariant metric $\langle\cdot,\cdot\rangle$ on G. Since a Lie group endowed with a biinvariant metric has nonnegative curvature and since by O'Neill's formula Riemannian submersions are curvature nondecreasing, it follows that every normal biquotient has nonnegative sectional curvature. The main objective of this section is to prove

Proposition 4.1. *Let* M *be a normal biquotient. Suppose that* $\sigma \subset T_pM$ *is a plane satisfying* $\sec(\sigma) = 0$*. Then the map* $\exp\colon \sigma \to M$*,* $v \mapsto \exp(v)$ *is a totally geodesic isometric immersion.*

Proof. By the previous section we may restrict ourselves to biquotients that are given in the normalized description:

$$(B, g) := \triangle \mathsf{G}\big\backslash\big(\mathsf{G} \times \mathsf{G}, \langle\cdot,\cdot\rangle\big)\big/\mathsf{H}$$

where $\langle\cdot,\cdot\rangle$ denotes a biinvariant metric on $\mathsf{G} \times \mathsf{G}$. Recall that by definition of g the projection $\mathrm{pr}\colon \big(\mathsf{G} \times \mathsf{G}, \langle\cdot,\cdot\rangle\big) \to (B, g)$ is a Riemannian submersion. Let $\sigma \subset TB$ be a plane with curvature 0, and let $\hat{\sigma} \subset T_{(a,b)}\mathsf{G} \times \mathsf{G}$ be a horizontal lift of σ. Since Riemannian submersions are curvature nondecreasing and $\big(\mathsf{G} \times \mathsf{G}, \langle\cdot,\cdot\rangle\big)$ has nonnegative sectional curvature, it follows that $\sec(\hat{\sigma}) = 0$. Taking into account that $\big(\mathsf{G}\times\mathsf{G}, \langle\cdot,\cdot\rangle\big)$ is a symmetric space, we see that $\exp_{\hat{\sigma}}: \hat{\sigma} \to \mathsf{G} \times \mathsf{G}$ is a totally geodesic immersion. Furthermore we can find left invariant vectorfields X, Y such that for all $p \in \exp(\hat{\sigma})$ the vectors $X_{|p}, Y_{|p}$ form a basis of the tangent space of $\exp(\hat{\sigma})$. As $T_{(a,b)} \exp(\hat{\sigma})$ is perpendicular to the orbit of H, $T_p \exp(\hat{\sigma})$ is perpendicular to the orbit of H for all $p \in \exp(\hat{\sigma})$.

On the other hand we can also find right invariant vectorfields V, W such that $V_{|p}, W_{|p}$ form a basis of $T_p \exp(\hat{\sigma})$ for all $p \in \exp(\hat{\sigma})$. This implies similarly that $\exp(\hat{\sigma})$ intersects the orbits of $\triangle\mathsf{G}$ perpendicularly everywhere.

In summary we can say that the totally geodesic immersed flat $\exp(\hat{\sigma})$ is everywhere horizontal. Since horizontal geodesics in $\mathsf{G} \times \mathsf{G}$ project to geodesics in B, we deduce that $\exp_{|\sigma}: \sigma \to M$ is a totally geodesic immersion. □

Remark 4.2. The proof shows that every horizontal zero curvature plane in $\big(\mathsf{G}\times\mathsf{G}, \langle\cdot,\cdot\rangle\big)$ projects to a zero curvature plane of the quotient. A fact which also follows from the curvature formula for biquotients in [Eschenburg, 1984].

All the Riemannian manifolds of Theorem 1 are normal biquotients. This fact is not immediately clear from the construction used in the proof of Theorem 1, but it will follow from

Lemma 4.3. *Let* G *be a compact Lie group,* $\mathsf{K} \subset \mathsf{G}$ *a Lie subgroup and consider the corresponding Lie algebras* $\mathfrak{k} \subset \mathfrak{g}$*. Furthermore let* $P\colon \mathfrak{g} \to \mathfrak{k}$ *denote the orthogonal projection with respect to a biinvariant metric* $\langle\cdot,\cdot\rangle$*. We define a new left invariant metric by*

$$g(v, w) = \big\langle(1 - P)v, (1 - P)w\big\rangle + t\big\langle Pv, Pw\big\rangle \quad \textit{for all } v, w \in \mathfrak{g}$$

and for some $t \in (0, 1)$*. Suppose that for a compact subgroup* $\mathsf{H} \subset \mathsf{G}\times\mathsf{K} \subset \mathsf{G} \times \mathsf{G}$ *the natural action of* H *on* G *is effectively free. Then* $(\mathsf{G}, g)/\!/\mathsf{H}$ *is a normal biquotient.*

Proof of Lemma 4.3. Let $\langle\cdot,\cdot\rangle$ also denote the induced biinvariant metric on $\mathsf{K} \subset \big(\mathsf{G}, \langle\cdot,\cdot\rangle\big)$. Choose a $\lambda \in (0,\infty)$ such that $t = \frac{\lambda}{1+\lambda}$. It is straightforward to check that $(\mathsf{G}, g)/\!/\mathsf{H}$ is isometric to the normal biquotient

$$\mathsf{H}\big\backslash\big(\mathsf{G}, \langle\cdot,\cdot\rangle\big) \times \big(\mathsf{K}, \lambda\langle\cdot,\cdot\rangle\big)\big/\big\{(a, a) \mid a \in \mathsf{K}\big\},$$

compare [Wilking, 1999] for a similar argument. □

The left invariant metric considered in Lemma 4.3 has particularly good curvature properties if $\mathfrak{k} \subset \mathfrak{g}$ is a symmetric pair.

Lemma 4.4 (Eschenburg, 1984, Satz 231). *Let $\mathfrak{k} \subset \mathfrak{g}$ be a symmetric pair of compact type, and let $\mathsf{K} \subset \mathsf{G}$ denote the corresponding groups. We endow G with a left invariant metric g metric which is obtained from a biinvariant metric $\langle \cdot, \cdot \rangle$ by*

$$g(v, w) := \big\langle (1 - P)v, w \big\rangle + t\big\langle Pv, w \big\rangle \quad \textit{for all } v, w \in \mathfrak{g},$$

where $t \in (0, 1)$ and $P \colon \mathfrak{g} \to \mathfrak{k}$ is the orthogonal projection. Then two linear independent vectors $v, w \in \mathfrak{g}$ span a zero curvature plane with respect to g if and only if

$$[v, w] = \big[(1 - P)v, (1 - P)w\big] = \big[Pv, Pw\big] = 0.$$

In the special case of a rank 1 symmetric pair $\mathfrak{k} \subset \mathfrak{g}$ the vectors $(1 - P)v$ and $(1 - P)w$ are linear dependent.

5. The construction for $P_{\mathbb{R}}T\mathbb{RP}^3$

The main purpose of this section is to give a simple description of the construction of the metric g in the special case $P_{\mathbb{R}}T\mathbb{RP}^3 \cong \mathbb{RP}^3 \times \mathbb{RP}^2$. Moreover we will describe the zero curvature planes of $\big(P_{\mathbb{R}}T\mathbb{RP}^3, g\big)$. We will use this special case later in order to describe the points at which zero curvature planes exist on $\big(P_{\mathbb{R}}T\mathbb{RP}^n, g\big)$.

Consider, $\mathsf{Sp}(1) = \mathsf{S}^3 \subset \mathbb{H}$ with the biinvariant metric of constant curvature 1, and let $\langle \cdot, \cdot \rangle$ denote the induced product metric on $\mathsf{S}^3 \times \mathsf{S}^3$. We define a new left invariant metric g on $\mathsf{S}^3 \times \mathsf{S}^3$ by means of

$$g(x, y) = \big\langle x, y \big\rangle - \tfrac{1}{2}\big\langle Px, Py \big\rangle \quad \text{for all } x, y \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(1),$$

where $P \colon \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \to \triangle\mathfrak{sp}(1)$ denotes the orthogonal projection onto the diagonal subalgebra $\triangle\mathfrak{sp}(1) \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$.

It is an immediate consequence of Lemma 4.4 that a plane $\sigma \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ has curvature zero with respect to g if and only if σ is of the form

$$\sigma = \mathrm{span}_{\mathbb{R}}\big\{(v, 0), (0, v)\big\} \quad \text{for some } v \in \mathfrak{sp}(1) - \{0\}.$$

Clearly, right–translations by elements of the subgroup

$$\mathsf{H} := \Big\{(abe^{i\varphi}, ae^{i\varphi}) \mid \varphi \in \mathbb{R},\ b \in \{1, -1\},\ a \in \{1, j\}\Big\} \subset \mathsf{S}^3 \times \mathsf{S}^3$$

are isometries. The action of $\mathsf{S}^3 \times \mathsf{S}^3$ on S^3 induces a transitive action on the projective tangentbundle of $\mathbb{RP}^3 \cong \mathsf{S}^3/\pm 1$. Since H occurs as isotropy group of the latter action, the quotient $\mathsf{S}^3 \times \mathsf{S}^3/\mathsf{H}$ is diffeomorphic to $P_{\mathbb{R}}T\mathbb{RP}^3 \cong \mathbb{RP}^3 \times \mathbb{RP}^2$.

Following our normalized description we can define a metric g on $P_{\mathbb{R}}T\mathbb{RP}^3$ as the orbit metric on the space

$$\triangle(\mathsf{S}^3 \times \mathsf{S}^3)\Big\backslash\Big(\big(\mathsf{S}^3 \times \mathsf{S}^3, g\big) \times \big(\mathsf{S}^3 \times \mathsf{S}^3, g\big)\Big)\Big/\{1\} \times \mathsf{H}.$$

The group $\mathsf{D} := \big\{(a, a) \mid a \in \mathsf{Sp}(1)\big\}$ acts on the first factor of $\big(\mathsf{S}^3 \times \mathsf{S}^3, g\big)^2$ by right multiplication. This action is isometric and induces an isometric action on $\big(P_{\mathbb{R}}T\mathbb{RP}^3, g\big)$. The element $(-1, -1)$ is in the kernel of the latter action.

Notice that each orbit of the group $\triangle(\mathsf{S}^3 \times \mathsf{S}^3)$ intersects a point of the form $\big(\bar{a}, \bar{b}, 1, 1\big)$ in $\big(\mathsf{S}^3\big)^4$. In order to describe the horizontal vectors at $\big(\bar{a}, \bar{b}, 1, 1\big)$, it is useful to consider the self adjoint endomorphism G of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ that is characterized by

$$\big\langle x, Gy\big\rangle = g(x, y) \quad \text{for all } x, y \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(1).$$

The horizontal vectors at $\big(\bar{a}, \bar{b}, 1, 1\big)$ are then given by

$$\Big\{\Big(G^{-1}\big(-\operatorname{Ad}_a v, -\operatorname{Ad}_b w\big), G^{-1}\big(v, w\big)\Big) \;\Big|\; v, w \in \mathfrak{sp}(1), \langle v + w, i\rangle = 0\Big\}.$$

Suppose there exists a horizontal zero curvature plane σ at $\big(\bar{a}, \bar{b}, 1, 1\big)$. Using the characterization of zero curvature planes, we see that there is a basis

$$b_1 := \Big(G^{-1}\big(-\operatorname{Ad}_a v, 0\big), G^{-1}\big(v, 0\big)\Big),$$
$$b_2 := \Big(G^{-1}\big(0, -\operatorname{Ad}_b v\big), G^{-1}\big(0, v\big)\Big)$$

of σ such that $\operatorname{Ad}_a v$ and $\operatorname{Ad}_b v$ are linear dependent. Furthermore $v \perp i$. The fact that $\operatorname{Ad}_a v$ and $\operatorname{Ad}_b v$ are linear dependent leaves only two possibilities:

Case 1. $\operatorname{Ad}_a v = \operatorname{Ad}_b v$. This implies that $v = \operatorname{Ad}_{\bar{a}b} v$. Thus the imaginary part of $\bar{a}b$ and of v are linear dependent. Because of $v \perp i$ we deduce $\bar{a}b \perp i$.

Case 2. $\operatorname{Ad}_a v = -\operatorname{Ad}_b v$. This implies $v = -\operatorname{Ad}_{\bar{a}b} v$. In other words $\bar{a}b$ anti-commutes with v and hence $\bar{a}b \perp 1$.

In summary we can say that an orbit at which a zero curvature plane occurs can be represented by a point in the following set

$$\big\{(a, b, 1, 1) \mid a\bar{b} \perp i \;\text{ or }\; a\bar{b} \perp 1\big\}.$$

For a point in $P_{\mathbb{R}}T\mathbb{RP}^3$ represented by $(a, b, 1, 1)$ with $a\bar{b} \perp i$ and $a\bar{b} \neq \pm 1$ there is precisely one horizontal plane with zero curvature corresponding to Case 1. The analogous statement holds if $a\bar{b} \perp 1$ but $a\bar{b} \neq \pm i$. If $a\bar{b} \in \{\pm 1, \pm i\}$, then there is an one parameter family of zero curvature planes.

We define S_1 (respectively S_i) as the set of orbits in $P_{\mathbb{R}}T\mathbb{RP}^3$ represented by $(a, b, 1, 1)$ with $a\bar{b} \perp 1$ (respectively $a\bar{b} \perp i$). The submanifolds S_1 and S_i are both diffeomorphic to $\mathbb{RP}^2 \times \mathbb{RP}^2$.

The amount of the zero curvature planes at each point in $S_1 \cup S_i$ is enforced by the isometric action of $\mathsf{D}/\pm(1,1) \cong \mathrm{SO}(3)$ on $\big(P_{\mathbb{R}}T\mathbb{RP}^3, g\big)$. The induced actions of SO(3) on S_1 and S_i are of cohomogeneity one. They are equivalent to the diagonal action of SO(3) on $\mathbb{RP}^2 \times \mathbb{RP}^2$; in particular there is one two dimensional singular orbit and one exceptional three dimensional orbit. The set of points in S_1 (resp. S_i) at which an one parameter family of zero curvature planes occurs is given by the singular orbit of the SO(3)-action. The singular orbits in S_1 and S_i are diffeomorphic to $\mathbb{RP}^2$. The three dimensional exceptional orbit is given by the intersection $S_1 \cap S_i$ and it is diffeomorphic to $\mathrm{SO}(3)/(\mathbb{Z}/2\mathbb{Z})^2$.

Modulo isometries S_1 (resp. S_i) contains precisely one totally geodesic torus. The torus is vertizontal with respect to the projection $S_1 \to S_1/\,\mathrm{SO}(3)$.

The natural actions of SO(3) on the Grassmannians of S_1 and S_i induce cohomogeneity one actions on the two families of zero curvature planes. The principal isotropy groups of these actions are of order 2 whereas the isotropy groups of the exceptional orbits are isomorphic to $\big(\mathbb{Z}/2\mathbb{Z}\big)^2$.

6. The series $P_{\mathbb{R}}T\mathbb{RP}^n$, $P_{\mathbb{C}}T\mathbb{CP}^n$ and $P_{\mathbb{H}}T\mathbb{HP}^n$

In this section we construct the metrics on $P_{\mathbb{R}}T\mathbb{RP}^n$, $P_{\mathbb{C}}T\mathbb{CP}^n$ and $P_{\mathbb{H}}T\mathbb{HP}^n$. Before we characterize the points at which zero curvature planes occur we will show

Proposition 6.1. *The manifolds $P_{\mathbb{R}}T\mathbb{RP}^n$, $P_{\mathbb{C}}T\mathbb{CP}^n$ and $P_{\mathbb{H}}T\mathbb{HP}^n$ endowed with the metric of Theorem 1 admit an isometric cohomogeneity two action of* $\mathsf{O}(n)$, $\mathsf{U}(n)$ *and* $\mathsf{Sp}(1)\cdot\mathsf{Sp}(n)$ *respectively. The natural inclusions in the following diagram are totally geodesic embeddings.*

$$\begin{array}{ccccc}
P_{\mathbb{R}}T\mathbb{RP}^2 & \subset & P_{\mathbb{C}}T\mathbb{CP}^2 & \subset & P_{\mathbb{H}}T\mathbb{HP}^2 \\
\cap & & \cap & & \cap \\
\vdots & & \vdots & & \vdots \\
\cap & & \cap & & \cap \\
P_{\mathbb{R}}T\mathbb{RP}^n & \subset & P_{\mathbb{C}}T\mathbb{CP}^n & \subset & P_{\mathbb{H}}T\mathbb{HP}^n.
\end{array}$$

Proof of Proposition 6.1. Consider the symmetric pair $\mathsf{Sp}(1)\cdot\mathsf{Sp}(n) \subset \mathsf{Sp}(n+1)$ of Lie groups and the corresponding pair $\mathfrak{sp}(1)\oplus\mathfrak{sp}(n) \subset \mathfrak{sp}(n+1)$ of Lie algebras. Let $\langle\cdot,\cdot\rangle$ denote the biinvariant metric on $\mathsf{Sp}(n+1)$ given by

$$\langle X, Y\rangle = -\,\mathrm{Real}\big(\mathrm{trace}(XY)\big) \quad \text{for all } X, Y \in \mathfrak{sp}(n+1),$$

$P\colon \mathfrak{sp}(n+1) \to \mathfrak{sp}(1)\oplus\mathfrak{sp}(n)$ the orthogonal projection, and let

$$(1-P)\colon \mathfrak{sp}(n+1) \to \big(\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)\big)^{\perp}$$

be the projection onto the orthogonal complement of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$. We define a new left invariant metric g on $\mathsf{Sp}(n+1)$ by

$$g(X, Y) = \big\langle (1-P)X, (1-P)Y \big\rangle + \tfrac{1}{2}\big\langle PX, PY \big\rangle \quad \text{for all } X, Y \in \mathfrak{sp}(n+1).$$

Put

$$\mathsf{H}_{\mathbb{H}} := \left\{ \begin{pmatrix} a & & \\ & b & \\ & & C \end{pmatrix} \,\middle|\, a, b \in \mathsf{Sp}(1), C \in \mathsf{Sp}(n-1) \right\} \subset \mathsf{Sp}(1) \cdot \mathsf{Sp}(n).$$

Following our general description of Sect. 3 we define a metric on

$$P_{\mathbb{H}} T\mathbb{HP}^n = \mathsf{Sp}(n+1)/\mathsf{H}_{\mathbb{H}}$$

as the induced metric on the orbit space

$$\triangle \mathsf{Sp}(n+1) \big\backslash \big(\mathsf{Sp}(n+1), g\big) \times \big(\mathsf{Sp}(n+1), g\big) \big/ \{1\} \times \mathsf{H}_{\mathbb{H}}.$$

Notice that the action of $\mathsf{Sp}(1) \cdot \mathsf{Sp}(n)$ from the right on the first factor of $\mathsf{Sp}(n+1)^2$ is isometric and accordingly induces an isometric action on $\big(P_{\mathbb{H}} T\mathbb{HP}^n, g\big)$.

It is straightforward to check that a connected component of the fix point set of the isometry

$$\operatorname{diag}(1, \ldots, 1, -1) \in \mathsf{Sp}(1) \cdot \mathsf{Sp}(n) \subset \mathsf{Sp}(n+1)$$

is isometric to $\big(P_{\mathbb{H}} T\mathbb{HP}^{n-1}, g\big)$. As claimed in Proposition 6.1, we see that the natural inclusions $\big(P_{\mathbb{H}} T\mathbb{HP}^2, g\big) \subset \cdots \subset \big(P_{\mathbb{H}} T\mathbb{HP}^n, g\big)$ are totally geodesic embeddings. The fix point set of the isometry

$$\operatorname{diag}(i, \ldots, i) \in \mathsf{Sp}(1) \cdot \mathsf{Sp}(n) \subset \mathsf{Sp}(n+1)$$

is isometric to

$$\big(P_{\mathbb{C}} T\mathbb{CP}^n, g\big) := \triangle \mathsf{U}(n+1) \big\backslash \big(\mathsf{U}(n+1), g\big) \times \big(\mathsf{U}(n+1), g\big) \big/ \{1\} \times \mathsf{H}_{\mathbb{C}}$$

where g denotes the metric on $\mathsf{U}(n+1)$ that is induced by the inclusion $\mathsf{U}(n+1) \subset (\mathsf{Sp}(n+1), g)$ and $\mathsf{H}_{\mathbb{C}} := \mathsf{U}(1)^2 \cdot \mathsf{U}(n-1)$. This shows that the natural inclusion $\big(P_{\mathbb{C}} T\mathbb{CP}^n, g\big) \subset \big(P_{\mathbb{H}} T\mathbb{HP}^n, g\big)$ is a totally geodesic embedding.

Similarly, a connected component of the set fixed by each of the isometries

$$\operatorname{diag}(i, \ldots, i), \;\operatorname{diag}(j, \ldots, j) \in \mathsf{Sp}(1) \cdot \mathsf{Sp}(n) \subset \mathsf{Sp}(n+1)$$

is isometric to

$$\big(P_{\mathbb{R}} T\mathbb{RP}^n, g\big) := \triangle \mathsf{O}(n+1) \big\backslash \big(\mathsf{O}(n+1), g\big) \times \big(\mathsf{O}(n+1), g\big) \big/ \{1\} \times \mathsf{H}_{\mathbb{R}},$$

where $\mathsf{H}_{\mathbb{R}} = \mathrm{O}(1)^2 \cdot \mathrm{O}(n-1)$. Analogously it can be shown that the first two columns in Proposition 6.1 form chains of totally geodesic embeddings.

Notice that we can restrict the action of $\mathsf{Sp}(1)\cdot\mathsf{Sp}(n)$ on $\big(P_{\mathbb{H}}T\mathbb{HP}^n, g\big)$ to $\mathsf{U}(n)$ and $\mathrm{O}(n)$ in order to obtain isometric actions on $\big(P_{\mathbb{C}}T\mathbb{CP}^n, g\big)$ and on $\big(P_{\mathbb{R}}T\mathbb{RP}^n, g\big)$, respectively. Clearly the cohomogeneity of these actions is two. In fact each orbit of these actions intersects the totally geodesic three dimensional submanifold $P_{\mathbb{R}}T\mathbb{RP}^2$ perpendicularly in an orbit of the action of $\mathrm{O}(2)$ on $P_{\mathbb{R}}T\mathbb{RP}^2$. $\square$

Lemma 6.2. *Let* $\mathbb{K}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$.

a) *There is a Riemannian submersion*

$$s\colon \big(P_{\mathbb{K}}T\mathbb{KP}^n, g\big)\longrightarrow\big(\mathbb{KP}^n,\bar g\big)$$

for a suitable cohomogeneity one metric $\bar g$ *on* $\mathbb{KP}^n$.

b) *Every plane* σ *in the tangent space of* $\big(P_{\mathbb{K}}T\mathbb{KP}^n, g\big)$ *with* $\sec(\sigma)=0$ *is a vertizontal plane with respect to* s*, that is,* σ *is spanned by a horizontal and a vertical vector.*

c) *Let* σ *be as in b). There is an isometry* ι *of* $\big(P_{\mathbb{K}}T\mathbb{KP}^n, g\big)$ *such that* $\iota_*(\sigma)\subset T_pP_{\mathbb{K}}T\mathbb{KP}^4\subset T_pP_{\mathbb{K}}T\mathbb{KP}^n$ *with* $p\in P_{\mathbb{R}}T\mathbb{RP}^2\subset P_{\mathbb{K}}T\mathbb{KP}^2\subset P_{\mathbb{K}}T\mathbb{KP}^n$.

d) *Let* $p\in P_{\mathbb{R}}T\mathbb{RP}^2\subset P_{\mathbb{K}}T\mathbb{KP}^n$ *be point at which a zero curvature plane exists. Then there is a plane* $\sigma\subset T_pP_{\mathbb{K}}T\mathbb{KP}^3$ *with* $\sec(\sigma)=0$.

Proof. a). We consider first the case of $\mathbb{K}=\mathbb{H}$ and put

$$\big(\mathbb{HP}^n,\bar g\big):=\triangle\mathsf{Sp}(n+1)\Big\backslash\big(\mathsf{Sp}(n+1), g\big)^2/\{1\}\times\big(\mathsf{Sp}(1)\cdot\mathsf{Sp}(n)\big).$$

The action of $\mathsf{Sp}(1)\cdot\mathsf{Sp}(n)$ from the right on the first factor of $\mathsf{Sp}(n+1)^2$ is isometric and induces an isometric cohomogeneity one action on $\big(\mathbb{HP}^n,\bar g\big)$. Since each fiber of the projection $\mathsf{Sp}(n+1)^2\to P_{\mathbb{H}}T\mathbb{HP}^n$ is contained in a fiber of the projection $\mathsf{Sp}(n+1)^2\to\mathbb{HP}^n$, it follows that there is a Riemannian submersion

$$s\colon \big(P_{\mathbb{H}}T\mathbb{HP}^n, g\big)\longrightarrow\big(\mathbb{HP}^n,\bar g\big)$$

and s is equivariant with respect to the $\mathsf{Sp}(1)\cdot\mathsf{Sp}(n)$–actions on $\big(P_{\mathbb{H}}T\mathbb{HP}^n, g\big)$ and $\big(\mathbb{HP}^n,\bar g\big)$. Therefore the restriction of s to a fixed point set of an isometry in $\mathsf{Sp}(1)\cdot\mathsf{Sp}(n)$ is a Riemannian submersion onto its image. Hence $s_{|P_{\mathbb{C}}T\mathbb{CP}^n}$ and $s_{|P_{\mathbb{R}}T\mathbb{RP}^n}$ are Riemannian submersions onto $\mathbb{CP}^n$ and $\mathbb{RP}^n$, respectively.

b). It is sufficient to prove the statement for $\mathbb{K}=\mathbb{H}$. Suppose that (X_1, X_2) and (Y_1, Y_2) are linear independent left invariant vectorfields of $\big(\mathsf{Sp}(n+1), g\big)^2$ with $\sec\big(\mathrm{span}\big((X_1,X_2),(Y_1,Y_2)\big)\big)=0$. Suppose furthermore that both fields are horizontal at some point $(A,B)\in\mathsf{Sp}(n+1)^2$ with respect to the projection

$$\mathrm{pr}\colon\big(\mathsf{Sp}(n+1), g\big)^2\longrightarrow\big(P_{\mathbb{H}}T\mathbb{HP}^n, g\big).$$

By Lemma 4.4 this implies that $(1-P)X_2$ and $(1-P)Y_2$ are linear dependent. Thus we may assume $X_2 \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$. Since (X_1, X_2) and (Y_1, Y_2) are horizontal at some point, it follows that X_2 and Y_2 are orthogonal to the Lie algebra of $\mathsf{H}_{\mathbb{H}}$. Furthermore, we infer from Lemma 4.4 that $\big[PX_2, PY_2\big] = 0$. Combining these facts we deduce that PX_2 and PY_2 are linear dependent. Therefore without loss of generality $Y_2 \in \big(\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)\big)^{\perp}$.

But then $(Y_1, Y_2)_{(A,B)}$ is horizontal with respect to the projection $\mathsf{Sp}(n+1)^2 \to (\mathbb{HP}^n, \bar{g})$ and $(X_1, X_2)_{(A,B)}$ is vertical with respect to this projection. Evidently, the assertion follows.

c). This is an immediate consequence of the fact that every zero curvature plane is vertizontal.

d). Let $\sigma \subset T_p P_{\mathbb{K}} T\mathbb{KP}^4$ be a plane with $\sec(\sigma) = 0$. Let v be a vertical unit vector in σ with respect to s. There is an isometry ι of $P_{\mathbb{K}} T\mathbb{KP}^4$ such that $\iota_*(v) \in T_p P_{\mathbb{K}} T\mathbb{KP}^3$. Without loss of generality $v \in T_p P_{\mathbb{K}} T\mathbb{KP}^3$. Since $P_{\mathbb{K}} T\mathbb{KP}^3$ is a totally geodesic submanifold, it follows that the curvature endomorphism $R(\cdot, v)v$ of $T_p P_{\mathbb{K}} T\mathbb{KP}^4$ leaves the space $T_p P_{\mathbb{K}} T\mathbb{KP}^3$ invariant.

Let w be a horizontal unit vector in σ. Notice that the orthogonal projection w' of w to $T_p P_{\mathbb{K}} T\mathbb{KP}^3$ fulfills $R(w', v)v = 0$. If $w' \neq 0$ we are done. Thus we may assume $w' = 0$. There is an isometry $\bar{\iota}$ with $\bar{\iota}(p) = p$ and $\bar{\iota}\big(P_{\mathbb{K}} T\mathbb{KP}^3\big) \subset P_{\mathbb{K}} T\mathbb{KP}^4$ intersects $P_{\mathbb{K}} T\mathbb{KP}^3$ perpendicularly in $P_{\mathbb{K}} T\mathbb{KP}^2$. By assumption $w \in T\bar{\iota}\big(P_{\mathbb{K}} T\mathbb{KP}^3\big)$ and as above we can now argue that v is perpendicular to $\bar{\iota}\big(P_{\mathbb{K}} T\mathbb{KP}^3\big)$.

In particular, we may assume $v, w \perp P_{\mathbb{K}} T\mathbb{KP}^2$. Now we can find an isometry $\tilde{\iota}$ with $\tilde{\iota}_*(\sigma) \subset T_p P_{\mathbb{R}} T\mathbb{RP}^4$. In other words, without loss of generality $\mathbb{K} = \mathbb{R}$. Consider the projection $\mathrm{pr}\colon\ \mathrm{O}(5)^2 \to P_{\mathbb{R}} T\mathbb{RP}^4$. There are matrices

$$A = (a_{ij}) \in \mathrm{SO}(3) = \left\{ \begin{pmatrix} A & \\ & I \end{pmatrix} \,\middle|\, A \in \mathrm{SO}(3) \right\} \subset \mathrm{O}(5),$$

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{o}(5) \quad \text{with } x \in \mathbb{R}$$

and

$$Y = \begin{pmatrix} 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{o}(5) \quad \text{with } y \in \mathbb{R}$$

such that the tangent vectors

$$\hat{v} = \left(-G^{-1}\operatorname{Ad}_{A^{-1}} X, G^{-1}X\right)_{|(A,I)}, \quad \hat{w} = \left(-G^{-1}\operatorname{Ad}_{A^{-1}} Y, G^{-1}Y\right)_{|(A,I)}$$

in $T_{(A,I)}\,\mathrm{O}(5)^2$ are horizontal, $\mathrm{pr}_*(\hat{v}) = v$ and $\mathrm{pr}_*(\hat{w}) = w$; here G is the selfadjoint endomorphism of $\mathfrak{sp}(5)$ describing the change of the scalar product from $\langle\cdot,\cdot\rangle$ to g.

Since $\operatorname{Ad}_{A^{-1}} X$ and $\operatorname{Ad}_{A^{-1}} Y$ span a zero curvature plane, it follows that the orthogonal projections of these vectors to $\mathfrak{o}(4)^\perp$ are linear dependent. But this implies $a_{21}\cdot a_{11} = 0$, which in turn shows that we can find a plane $\sigma' \subset T_p P_{\mathbb{R}} T\mathbb{RP}^3$ with $\sec(\sigma') = 0$. ☐

Lemma 6.3. *Let $p \in P_{\mathbb{R}} T\mathbb{RP}^2 \subset P_{\mathbb{H}} T\mathbb{HP}^n$ be point at which a plane with zero curvature exists.*

a) *Then there is a plane $\sigma \subset T_p P_{\mathbb{C}} T\mathbb{CP}^3$ with $\sec(\sigma) = 0$.*
b) *Suppose that there is no zero curvature plane in $T_p P_{\mathbb{R}} T\mathbb{RP}^3$. Then there is a matrix $A = (a_{ij}) \in \mathrm{SO}(3)$ with $a_{31}^2 = \frac{1}{2}$ and the image of $(A, 1) \in \mathrm{O}(3)^2$ under the natural projection*

$$\mathrm{O}(3)^2 \to P_{\mathbb{R}} T\mathbb{RP}^2$$

is p. Conversely, at all these points there are zero curvature planes in $P_{\mathbb{C}} T\mathbb{CP}^3$.

Proof. a). By the previous lemma $n = 3$ without loss of generality. Consider a plane $\sigma \subset T_p P_{\mathbb{H}} T\mathbb{HP}^3$ with $\sec(\sigma) = 0$. By Lemma 6.2 b) there is a vertical unit vector $v \in \sigma$ and a horizontal unit vector $w \in \sigma$. There are matrices $A = (a_{ij}) \in \mathrm{SO}(3) \subset \mathsf{Sp}(4)$,

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{x}_1 & -\bar{x}_2 \\ 0 & x_1 & 0 & 0 \\ 0 & x_2 & 0 & 0 \end{pmatrix} \in \mathfrak{sp}(4) \quad \text{with } x_1, x_2 \in \mathbb{H}$$

and

$$Y = \begin{pmatrix} 0 & 0 & -\bar{y}_1 & -\bar{y}_2 \\ 0 & 0 & 0 & 0 \\ y_1 & 0 & 0 & 0 \\ y_2 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sp}(4) \quad \text{with } y_1, y_2 \in \mathbb{H}$$

such that the tangent vectors

$$\hat{v} = \left(-\mathsf{G}^{-1}\operatorname{Ad}_{A^{-1}} X, G^{-1}X\right)_{|(A,I)}, \quad \hat{w} = \left(-G^{-1}\operatorname{Ad}_{A^{-1}} G^{-1}Y, Y\right)_{|(A,I)}$$

in $T_{(A,I)}\mathsf{Sp}(4)^2$ are horizontal, $\mathrm{pr}_*(\hat{v}) = v$ and $\mathrm{pr}_*(\hat{w}) = w$. Notice that the 21 coefficient of Y is 0 as $[X, Y] = 0$.

It is sufficient to consider the case of $a_{21} \cdot a_{11} \neq 0$, because otherwise we can find a plane $\sigma' \subset T_p P_{\mathbb{R}} T\mathbb{RP}^3$ with $\sec(\sigma') = 0$. If x_2 and y_2 are linear independent, then the 41 coefficients of the matrices $\mathrm{Ad}_{A^{-1}} X$ and $\mathrm{Ad}_{A^{-1}} Y$ are linear independent, too. But that is not possible as these vectors span a zero curvature plane.

Thus x_2 and y_2 are linear dependent and because of $[X, Y] = 0$ the same is true for x_1 and y_1. Clearly, we can conjugate X, Y with diagonal matrices of the form $\mathrm{diag}\big(a, a, a, b\big)$, $a, b \in \mathsf{S}^3$. Hence we may assume that $x_2, y_2 \in \mathbb{R}$ and $x_1, y_1 \in \mathbb{C}$. In other words, there is an isometry ι of $P_{\mathbb{H}} T\mathbb{HP}^3$ such that $\iota_*(\sigma) \subset T_p P_{\mathbb{C}} T\mathbb{CP}^3$.

b). We keep the above notation and continue to assume $x_2, y_2 \in \mathbb{R}$ and $x_1, y_1 \in \mathbb{C}$. Since $\mathrm{Ad}_{A^{-1}} X$ and $\mathrm{Ad}_{A^{-1}} Y$ span a zero curvature plane, it follows that $(1 - P)\,\mathrm{Ad}_{A^{-1}} X$ and $(1 - P)\,\mathrm{Ad}_{A^{-1}} Y$ are linear dependent. Thus we can find $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{0\}$ with

$$(1 - P)\,\mathrm{Ad}_{A^{-1}}\big(\lambda X + \mu Y\big) = 0.$$

Let $\bar{X}$ and $\bar{Y}$ denote the conjugate matrices. By assumption $(\bar{X}, \bar{Y}) \neq (X, Y)$. Consider

$$Z := \lambda X + \mu Y - \lambda\bar{X} - \mu\bar{Y} = \begin{pmatrix} 0 & 0 & bi & 0 \\ 0 & 0 & ci & 0 \\ bi & ci & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{u}(4)$$

with $(b, c) \in \mathbb{R}^2 \setminus \{0\}$. Using $(1 - P)\,\mathrm{Ad}_{A^{-1}} Z = 0$ we find

$$\begin{pmatrix} a_{12}a_{31} + a_{32}a_{11} & a_{22}a_{31} + a_{32}a_{21} \\ a_{31}a_{13} + a_{33}a_{11} & a_{23}a_{31} + a_{33}a_{21} \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = 0.$$

Consequently the determinant of the above 2×2-matrix is 0. Because of $A \in \mathrm{SO}(3)$ the determinant is given by

$$a_{31}\big(2a_{31}^2 - 1\big) = 0.$$

It is easy to see that in the case of $a_{31} = 0$ zero curvature does not occur unless $a_{21} \cdot a_{11} = 0$. Therefore $a_{31}^2 = \frac{1}{2}$.

Suppose now conversely that $a_{31}^2 = \frac{1}{2}$. Because of the isometric action of $\mathrm{SO}(2)$ on $P_{\mathbb{R}} T\mathbb{RP}^2 \subset P_{\mathbb{C}} T\mathbb{CP}^3$ we may restrict ourselves to the case of $a_{32} = 0$. Then the matrices

$$X := \begin{pmatrix} 0 & 0 & -a_{22}i & \sqrt{2}a_{21} \\ 0 & 0 & 0 & 0 \\ -a_{22}i & 0 & 0 & 0 \\ -\sqrt{2}a_{21} & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{u}(4)$$

and

$$Y := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a_{12}i & -\sqrt{2}a_{11} \\ 0 & a_{12}i & 0 & 0 \\ 0 & \sqrt{2}a_{11} & 0 & 0 \end{pmatrix} \in \mathfrak{u}(4)$$

commute and the tangent vectors

$$\hat{v} = \left(-G^{-1}\operatorname{Ad}_{A^{-1}} X, G^{-1}X\right)_{|(A,I)}, \quad \hat{w} = \left(-G^{-1}\operatorname{Ad}_{A^{-1}} Y, G^{-1}Y\right)_{|(A,I)}$$

in $T_{(A,I)}\mathsf{U}(4)^2$ span a horizontal zero curvature plane. □

Proposition 6.4. *Let $\mathsf{K}_{\mathbb{K},n}$ denote the isometry group of $\left(P_{\mathbb{K}}T\mathbb{K}\mathbb{P}^n, g\right)$. The orbit spaces $\left(P_{\mathbb{K}}T\mathbb{K}\mathbb{P}^n, g\right)/\mathsf{K}_{\mathbb{K},n}$, $n \geq 2$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, are mutually isometric. The boundary of the topological 2-manifold $\left(P_{\mathbb{K}}T\mathbb{K}\mathbb{P}^n, g\right)/\mathsf{K}_{\mathbb{K},n}$ is given by a geodesic triangle with all angles being equal to $\pi/2$. Furthermore, for $n \geq 3$ the points in $\left(P_{\mathbb{K}}T\mathbb{K}\mathbb{P}^n, g\right)$, at which zero curvature planes occur, map in the orbit space onto a set, which can be described as follows:*

a) *For $\mathbb{K} = \mathbb{R}$ the set is given by the union of two sides of the boundary triangle.*
b) *For $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$ the set consists of two sides of the boundary triangle and of an open interval lying in the interior and joining the two sides.*

Proof of Proposition 6.4. It is straightforward to check that the quotients

$$P_{\mathbb{H}}T\mathbb{H}\mathbb{P}^n/\mathsf{Sp}(1)\cdot\mathsf{Sp}(n), \quad P_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n/\mathsf{U}(n) \text{ and } P_{\mathbb{R}}T\mathbb{R}\mathbb{P}^n/\mathsf{O}(n)$$

are mutually isometric and given by a topological 2-manifold whose boundary is a geodesic triangle with all angles being equal to $\pi/2$.

Because of Lemma 6.2 d) the set of points at which zero curvature planes occur maps in the quotient onto a set which does not depend on $n \geq 3$. In the case of $\mathbb{K} = \mathbb{R}$ the description of the latter set follows from Sect. 5. In the case of $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$ the description is a consequence of Lemma 6.3.

Using this description of the points at which zero curvature planes occur, it is not hard to see that the actions of the isometry groups on $\left(P_{\mathbb{R}}T\mathbb{R}\mathbb{P}^n, g\right)$, $\left(P_{\mathbb{C}}T\mathbb{C}\mathbb{P}^n, g\right)$ and $\left(P_{\mathbb{H}}T\mathbb{H}\mathbb{P}^n, g\right)$ are orbit equivalent to the actions of $\mathsf{O}(n)$, $\mathsf{U}(n)$ and $\mathsf{Sp}(1)\cdot\mathsf{Sp}(n)$, respectively. □

Proof of Proposition 6. In the case of $n = 3$ the description follows from Sect. 5. The general case follows from this special case and from Lemma 6.2 d). □

7. The series $\mathrm{SO}(2)\backslash \mathrm{SO}(2n+1)/\mathrm{SO}(2n-1)$

By Proposition 6 $\left(T^1\mathbb{S}^{2n}, \hat{g}\right)$ has positive sectional curvature on an open and dense set. Furthermore the natural action of $\mathrm{SO}(2n)$ on $T^1\mathbb{S}^{2n}$ is isometric with respect to $\hat{g}$. Choose an embedding $\mathrm{SO}(2) \subset \mathrm{SO}(2n)$ such that $\mathbb{R}^{2n}$ splits into two dimensional equivalent subrepresentations of $\mathrm{SO}(2)$. Clearly the action of $\mathrm{SO}(2)$ on $T^1\mathbb{S}^{2n}$ is isometric and free. Consequently we can define

$$(B, g) := \left(T^1\mathbb{S}^{2n}, \hat{g}\right)/\mathrm{SO}(2) = \mathrm{SO}(2)\backslash \mathrm{SO}(2n+1)/\mathrm{SO}(2n-1).$$

Since $\left(T^1\mathbb{S}^{2n}, \hat{g}\right)$ has positive sectional curvature on an open dense set, the same is valid for (B, g).

Remark 7.1. a) The cohomology ring of $\mathrm{SO}(2)\backslash \mathrm{SO}(2n+1)/\mathrm{SO}(2n-1)$ is the same as the cohomolgy ring of $\mathrm{SO}(2n+1)/\mathrm{SO}(2)\cdot\mathrm{SO}(2n-1)$. However the Pontrjagin classes are different.

b) If $n = 2m$ is even the isometric action of $\mathrm{SO}(2)$ on T^1S^{4m} can be extended to an isometric free action of $\mathrm{SU}(2)$. For that reason the $8m-4$-dimensional manifold $\mathrm{SU}(2)\backslash \mathrm{SO}(4m+1)/\mathrm{SO}(4m-1)$ has positive curvature on an open dense set as well.

8. The generalized Aloff Wallach examples

Let g denote the left invariant $\mathrm{Ad}_{\mathsf{U}(1)\cdot\mathsf{U}(n)}$–invariant metric on $\mathsf{U}(n+1)$ that we have defined in Sect. 6, and let $\mathsf{H}_{k,l}$ be as in the theorem. Clearly, we may replace (k, l) by $(-k, -l)$ or by (l, k) without changing the topology of the quotient. Thus we may assume $k \geq -l > 0$. Put

$$\left(W_{kl}^{4n-1}, g\right) := \triangle\mathsf{U}(n+1)\backslash\left(\mathsf{U}(n+1), g\right) \times \left(\mathsf{U}(n+1), g\right)/\{1\} \times \mathsf{H}_{k,l}.$$

Lemma 8.1. a) *There is a Riemannian submersion*

$$\phi\colon \left(W_{kl}^{4n-1}, g\right) \longrightarrow \left(P_{\mathbb{C}}T\mathbb{CP}^n, g\right).$$

b) *Suppose $k > -l > 0$. Then a plane $\sigma \subset TW_{kl}^{4n-1}$ has curvature 0 if and only if it is a horizontal plane such that $\phi_*(\sigma)$ is a zero curvature plane in $P_{\mathbb{C}}T\mathbb{CP}^n$.*

c) *If $k = -l$, then a plane with zero curvature is either horizontal or vertizontal with respect to ϕ.*

Proof. a). This is an immediate consequence of the fact that each fiber of the projection $\left(\mathsf{U}(n+1), g\right)^2 \to \left(W_{kl}^{4n-1}, g\right)$ is contained in a fiber of the projection $\left(\mathsf{U}(n+1), g\right)^2 \to \left(P_{\mathbb{C}}T\mathbb{CP}^n, g\right)$.

b). Let $\sigma \subset T_{1,A}\mathsf{U}(n+1)^2$ be a horizontal zero curvature plane with respect to the projection $\big(\mathsf{U}(n+1), g\big)^2 \to \big(W_{kl}^{4n-1}, g\big)$. Let σ_2 be the projection of σ on the second component of

$$T_{1,A}\mathsf{U}(n+1)^2 = T_1\mathsf{U}(n+1) \oplus T_A\mathsf{U}(n+1).$$

By assumption $\sec(\sigma_2) = 0$ with respect to g and σ_2 is perpendicular to the Lie algebra $\mathfrak{h}_{kl}$ of H_{kl}. We can find a basis v, w of σ_2 with $v \in \mathfrak{u}(1) \oplus \mathfrak{u}(n)$ and $w \in \big(\mathfrak{u}(1) \oplus \mathfrak{u}(n)\big)^{\perp}$. In order to show that σ is horizontal with respect to the projection $\big(\mathsf{U}(n+1), g\big)^2 \to \big(P_{\mathbb{C}}T\mathbb{CP}^n, g\big)$, it is sufficient to show that the 11-coefficient of v is zero.

Let $\lambda \cdot i$ denote the 11-coefficient of v and assume, on the contrary that $\lambda > 0$ (if $\lambda < 0$ we can pass from v to $-v$). The 22-coefficient of v is then given by $\mu \cdot i$ for some $\mu > \lambda$. Because of $v \in \mathfrak{h}_{kl}^{\perp} \cap \big(\mathfrak{u}(1) \oplus \mathfrak{u}(n)\big)$, the matrix v has at most three non-vanishing eigenvalues. Using $\mu > \lambda$ we see that $\lambda \cdot i$ is an eigenvalue with multiplicity one. On the other hand v commutes with $w \neq 0$ and hence $\lambda \cdot i$ is an eigenvalue of v with multiplicity at least two – a contradiction.

By Remark 4.2 σ projects to a zero curvature plane in $(P_{\mathbb{C}}T\mathbb{CP}^n, g)$.

c.) We can carry out the same argument as in b). The difference is that in the present situation we can only conclude $\lambda = \mu$ and not $\mu > \lambda$. We may assume $\lambda \neq 0$. Using that $\lambda \cdot i$ is an eigenvalue of v with multiplicity at least two, it follows that all coefficients of v other than the 11 and the 22-coefficients of v are zero.

Consequently, $v \in \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(n-1)$. This is equivalent to saying that σ is vertizontal with respect to the projection $\big(\mathsf{U}(n+1), g\big)^2 \to \big(P_{\mathbb{C}}T\mathbb{CP}^n, g\big)$. $\square$

Proof of Theorem 1 c). If $k > -l > 0$, then Lemma 8.1 implies that a point $p \in W_{kl}^{4n-1}$ at which a zero curvature plane occurs, projects to a point $\bar{p} \in P_{\mathbb{C}}T\mathbb{CP}^n$ at which a zero curvature plane occurs. Taking into account that $P_{\mathbb{C}}T\mathbb{CP}^n$ has positive curvature on an open dense set, it follows that W_{kl}^{4n-1} has positive curvature on an open dense set as well.

If $(k, l) = (1, -1)$, then by Lemma 8.1 c) it is sufficient to rule out the possibility that vertizontal zero curvature planes in $W_{1,-1}^{4n-1}$ occur on an open set of points. Given a vertizontal zero curvature plane $\sigma \subset TW_{1,-1}^{4n-1}$ one can find an isometry such that $\iota_*(\sigma) \subset TW_{1,-1}^{7} \subset TW_{1,-1}^{4n-1}$. Since the natural inclusion $W_{1,-1}^{7} \subset W_{1,-1}^{4n-1}$ is a totally geodesic embedding, it suffices to prove the statement for $n = 2$. This in turn is a straightforward computation. $\square$

9. The series $\mathsf{Sp}(1)\backslash\mathsf{Sp}(n+1)/\mathsf{Sp}(1)\cdot\mathsf{Sp}(n-1)$

Consider the biinvariant metric $\langle\cdot,\cdot\rangle$ and the left invariant metric g on $\mathsf{Sp}(n+1)$ defined in Sect. 6. Put

$$\mathsf{K}_i := \left\{ \begin{pmatrix} a_1 & & & \\ & \ddots & & \\ & & a_{n+1-i} & \\ & & & C \end{pmatrix} \,\middle|\, a_1, \cdots, a_{n+1-i} \in \mathsf{Sp}(1),\ C \in \mathsf{Sp}(i) \right\},$$

and let $P_i \colon \mathfrak{sp}(n+1) \to \mathfrak{k}_i$ denote the orthogonal projection onto the Lie algebra of K_i. We set

$$g_n(v, w) := \langle v, w\rangle - \sum_{i=1}^{n} \frac{1}{2^i} \langle P_i v, P_i w\rangle.$$

Geometrically the metric g_n can be described as follows: Put $g_0 = \langle \cdot, \cdot \rangle$ and define g_{i+1} as the metric that is obtained from g_i by scaling down the fibers of the Riemannian submersion $(\mathsf{Sp}(n+1), g_i) \to (\mathsf{Sp}(n+1), g_i)/\mathsf{K}_{n-i}$ by a factor $\sqrt{2}$. Or equivalently put $(\mathsf{Sp}(n+1), g_{i+1}) := (\mathsf{Sp}(n+1), g_i) \times (\mathsf{K}_{n-i}, g_i)/\triangle\mathsf{K}_{n-i}$.

Let $Q_i \colon \mathfrak{sp}(n+1) \to \mathfrak{k}_i^{\perp} \cap \mathfrak{k}_{i+1}$ denote the orthogonal projection. Using the latter description of the metric g_n, it is straightforward to check that if $v, w \in \mathfrak{sp}(n+1)$ span a zero curvature plane with respect to g_n, then $Q_i v$ and $Q_i w$ are linear dependent, $i = 1, \ldots, n$. Put

$$\mathsf{D} := \{\mathrm{diag}(a, \cdots, a) \mid a \in \mathsf{Sp}(1)\},$$

$$\mathsf{H} := \left\{ \begin{pmatrix} a & & \\ & 1 & \\ & & A \end{pmatrix} \,\middle|\, a \in \mathsf{Sp}(1),\quad A \in \mathsf{Sp}(n-1) \right\},$$

and let $\mathfrak{d}$ and $\mathfrak{h}$ denote the Lie algebras of D and H, respectively. By Proposition 3.1 we can define a metric on the biquotient $B := \mathsf{D}\backslash\mathsf{Sp}(n+1)/\mathsf{H}$ by

$$(B, g) := \triangle\mathsf{Sp}(n+1)\Big\backslash(\mathsf{Sp}(n+1), g_n) \times (\mathsf{Sp}(n+1), g)\Big/\mathsf{D} \times \mathsf{H}.$$

Using the above description of the metric g_n and iterating the argument in the proof of Lemma 4.3 we see that (B, g) is a normal biquotient.

The isometry group of (B, g) is fairly small. Therefore a precise description of the points at which zero curvature planes occur seems to be hard. In order to show that (B, g) has positive sectional curvature on an open and dense set of points we apply a different strategy which can be briefly outlined as follows: Let V denote the subspace of $\Lambda_2(\mathfrak{h}^{\perp})$ generated by all oriented g-zero curvature planes in $\mathfrak{h}^{\perp}$, and let W be the subspace of $\Lambda_2(\mathfrak{d}^{\perp})$ generated by all oriented g_n-zero curvature planes in $\mathfrak{d}^{\perp}$. It turns out that a necessary condition for a horizontal zero curvature plane at $(I, A) \in \mathsf{Sp}(n+1)^2$ is $\widehat{\mathrm{Ad}}_A(V) \cap W \neq 0$, where $\widehat{\mathrm{Ad}}_A$ denotes the endomorphism of $\Lambda_2\mathfrak{sp}(n+1)$ induced by Ad_A. Furthermore it is easy to see that the condition $\widehat{\mathrm{Ad}}_A(V) \cap W \neq 0$ is a polynomial equation in the coefficients of A. Consequently, if one is able to verify for one particular matrix A that $\widehat{\mathrm{Ad}}_A(V) \cap W = 0$, then it follows that the biquotient has positive curvature on an open and dense set of points.

We observe that if a plane $\sigma \subset \mathfrak{h}^\perp$ satisfies $\sec(\sigma) = 0$ then σ has a basis v, w with $v \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$, $w \in \big(\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)\big)^\perp$ such that $[v, w] = 0$.

Put

$$U_1 := \left\{ \begin{pmatrix} 0 & 0 & -v^* \\ 0 & 0 & 0 \\ v & 0 & 0 \end{pmatrix} \;\middle|\; v \in \mathbb{H}^{n-1} \right\},$$

$$U_2 := \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -v^* \\ 0 & v & 0 \end{pmatrix} \;\middle|\; v \in \mathbb{H}^{n-1} \right\},$$

and let W_{ij} denote the subspace of matrices in $\mathfrak{sp}(n+1)$ for which all coefficients other that the (i, j) and the (j, i)-coefficients are zero. Clearly

$$\mathfrak{p} := \mathfrak{h}^\perp = U_1 \oplus U_2 \oplus W_{22} \oplus W_{12}.$$

By the above characterization of zero curvature planes, an oriented plane $\sigma \in \mathrm{Gr}_+\big(\mathfrak{h}^\perp\big) \subset \Lambda_2\big(\mathfrak{h}^\perp\big)$ with $\sec(\sigma) = 0$ fulfills

$$\sigma \in U_1 \otimes (W_{22} \oplus U_2) \subset \Lambda_2\big(\mathfrak{h}^\perp\big).$$

The map $Q_i : \mathfrak{sp}(n+1) \to \mathfrak{q}_i := \mathfrak{k}_i^\perp \cap \mathfrak{k}_{i+1}$ induces a map $\hat{Q}_i : \Lambda_2\mathfrak{sp}(n+1) \to \Lambda_2\mathfrak{q}_i$. Notice that all oriented g_n-zero curvature planes are contained in the kernel of $\hat{Q}_i$. Finally we put $\mathfrak{q} := \mathfrak{d}^\perp$ and let

$$\hat{S} \colon \Lambda_2\mathfrak{sp}(n+1) \longrightarrow \Lambda_2(\mathfrak{d}) \oplus \mathfrak{d} \otimes \mathfrak{q}$$

denote the orthogonal projection. The kernel of $\hat{S}$ is $\Lambda_2\mathfrak{q}$. Furthermore $\hat{S}$ is also the orthogonal projection with respect to $\langle \cdot, \cdot \rangle$.

Lemma 9.1. a) *Let $A \in \mathsf{Sp}(n+1)$. If the linear map*

$$\Phi_A \colon U_1 \otimes (W_{22} \oplus U_2) \longrightarrow \Lambda_2(\mathfrak{d}) \oplus (\mathfrak{d} \otimes \mathfrak{q}) \oplus \bigoplus_{i=1}^{n} \Lambda_2\mathfrak{q}_i$$

given by

$$\Phi_A := \Big(\hat{S} \oplus \hat{Q}_1 \oplus \cdots \oplus \hat{Q}_n\Big) \circ \widehat{\mathrm{Ad}}_{A|U_1 \otimes (W_{22} \oplus U_2)}$$

is injective, then there are no zero curvature planes at $(I, A) \in \mathsf{Sp}(n+1)^2$ which are horizontal with respect to the projection $\mathrm{pr}\colon \mathsf{Sp}(n+1)^2 \to (B, g)$.

b) *There is one matrix $A \in \mathsf{Sp}(n+1)$ such that the linear map Φ_A is injective.*

Notice that Φ_A is injective if and only if $\det\big(\Phi_A^*\Phi_A\big) \neq 0$. Of course, the quantity $\det\big(\Phi_A^*\Phi_A\big)$ is a polynomial in the coefficients of A. By part b) of the lemma the polynomial equation $\det\big(\Phi_A^*\Phi_A\big) = 0$ defines a proper subvariety of B. And by part a) every point at which a zero curvature plane occurs is contained in this subvariety.

Proof of Lemma 9.1. a). We let G_n and G denote the endomorphisms of $\mathfrak{sp}(n+1)$ describing the change of the scalar product from $\langle\cdot,\cdot\rangle$ to g_n and g, respectively. Suppose that the vectors

$$\big(G_n^{-1}u_1, -G^{-1}u_2\big),\ \big(G_n^{-1}v_1, -G^{-1}v_2\big) \in \mathfrak{sp}(n+1)^2$$

span a zero curvature plane that is horizontal at (I, A). Then $u_1 = \mathrm{Ad}_A\, u_2$ and $v_1 = \mathrm{Ad}_A\, v_2$. Moreover $\sigma = \mathrm{span}_{\mathbb{R}}(u_2, v_2) \subset \mathfrak{h}^\perp$ is a zero curvature plane and thus

$$\sigma \in U_1 \otimes \big(W_{22} \oplus U_2\big).$$

Finally

$$G_n^{-1}\,\mathrm{Ad}_A(\sigma) = G_n^{-1}\,\mathrm{span}_{\mathbb{R}}(u_1, v_1) \in \Lambda_2(\mathfrak{q}) = \mathrm{Ker}(\hat{S})$$

is a zero curvature plane as well. Therefore $\mathrm{Ad}_A(\sigma)$ is in the kernel of $\hat{Q}_i$ for all $i \in \{1, \ldots, n\}$. Consequently σ is in the kernel of Φ_A.

b). Put

$$B := \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and}$$

$$A := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & B \end{pmatrix} \cdots \begin{pmatrix} B & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \mathsf{Sp}(n+1),$$

where the right hand side in the definition of A consists of the product of $(n-1)$ matrices. By abuse of notation we call the last of these factors B as well.

Since A is in the centralizer of the group D, it follows that

$$\mathrm{Ker}\big(\hat{S} \circ \widehat{\mathrm{Ad}}_{A|U_1\otimes(U_2\oplus W_{22})}\big) = \mathrm{Ker}\big(\hat{S}_{|U_1\otimes(U_2\oplus W_{22})}\big) = U_1 \otimes U_2.$$

Put

$$U_1' := W_{31}^\perp \cap U_1 \quad \text{and} \quad U_2' := W_{32}^\perp \cap U_2.$$

Using the identity

$$Q_n \circ \mathrm{Ad}_A = \mathrm{Ad}_{AB^{-1}} \circ Q_n \circ \mathrm{Ad}_B$$

we see that $Q_n \circ \mathrm{Ad}_{A|U_i}$, is injective, $i = 1, 2$,

$$V := Q_n\big(\mathrm{Ad}_A(U_1')\big) = Q_n\big(\mathrm{Ad}_A(U_2')\big)$$

and

$$\mathfrak{q}_n = V \oplus Q_n\big(\mathrm{Ad}_A(W_{31})\big) \oplus Q_n\big(\mathrm{Ad}_A(W_{32})\big).$$

Hence

$$Z := \mathrm{Ker}\big(\hat{Q}_n \circ \widehat{\mathrm{Ad}}_{A|U_1 \otimes U_2}\big) \subset U_1' \otimes U_2'.$$

Let $\hat{L} \colon \Lambda_2\mathfrak{sp}(n+1) \subset \Lambda_2\mathfrak{sp}(n)$ be the orthogonal projection. Taking into account that Z is also the kernel of $\hat{Q}_n \circ \widehat{\mathrm{Ad}}_{B|U_1' \otimes U_2'}$ we deduce that $\hat{L} \circ \widehat{\mathrm{Ad}}_{B|Z}$ is injective and $\hat{L}\big(\widehat{\mathrm{Ad}}_B(Z)\big) \subset U_2' \otimes U_3 \subset \Lambda_2\mathfrak{sp}(n)$ where

$$U_3 := \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -v^* \\ 0 & 0 & v & 0 \end{pmatrix} \;\middle|\; v \in \mathbb{H}^{n-2} \right\}.$$

By induction we may assume that the map

$$\big(\hat{Q}_{n-1} \oplus \cdots \oplus \hat{Q}_1\big) \circ \widehat{\mathrm{Ad}}_{AB^{-1}|U_2' \otimes U_3}$$

is injective. Thus the result follows. □

10. The metrics on $P_{\mathbb{C}}T\mathbb{CP}^2/\mathsf{S}_3 \subset P_{\mathbb{H}}T\mathbb{HP}^2/\mathsf{S}_3 \subset P_{\mathrm{Ca}}T\mathrm{Ca}\mathbb{P}^2/\mathsf{S}_3$

Let $\mathsf{Spin}(9) \subset \mathsf{F}_4$ be the symmetric pair corresponding to the Cayley plane, and let $\langle \cdot, \cdot \rangle$ denote a biinvariant metric on F_4. We also consider the left invariant metric g defined by

$$g(v, w) = \big\langle w, v \big\rangle - \tfrac{1}{2}\big\langle Pv, Pw \big\rangle \quad \text{for all } v, w \in \mathfrak{f}_4$$

where $P \colon \mathfrak{f}_4 \to \mathfrak{spin}(9)$ denotes the orthogonal projection.

Recall that $\mathsf{Spin}(9)$ is the isotropy group of some point $p \in \mathrm{Ca}\mathbb{P}^2$. We choose a line $L \subset \mathrm{Ca}\mathbb{P}^2$ with $p \in L$. The subgroup of all isometries $\iota \in \mathsf{Spin}(9)$ satisfying $\iota(L) = L$ is isomorphic to $\mathsf{Spin}(8) \subset \mathsf{Spin}(9)$. Clearly, $\mathsf{F}_4/\mathsf{Spin}(8)$ corresponds to the natural description of $P_{\mathrm{Ca}}T\mathrm{Ca}\mathbb{P}^2$ as a homogeneous space. Wallach [1972] showed that $\big(\mathsf{F}_4, g\big)/\mathsf{Spin}(8)$ has positive sectional curvature.

Let N_{Ca} be the normalizer of $\mathsf{Spin}(8)$ in F_4. It is well known that $\mathsf{N}_{\mathrm{Ca}}/\mathsf{Spin}(8) \cong \mathsf{S}_3$ and that the isotropy representation of N_{Ca} corresponding to the homogeneous space $\big(\mathsf{F}_4, \langle \cdot, \cdot \rangle\big)/\mathsf{N}_{\mathrm{Ca}}$ is irreducible.

In particular, the metric g is not $\mathrm{Ad}_{\mathsf{N}_{\mathrm{Ca}}}$ invariant. However, we can define a metric on $P_{\mathrm{Ca}}T\mathrm{Ca}\mathbb{P}^2/\mathsf{S}_3$ as the orbit metric of $\mathsf{N}_{\mathrm{Ca}}\backslash(\mathsf{F}_4, g)$. Since the horizontal distribution at the identity element coincides with the horizontal distribution with respect to the projection $(\mathsf{F}_4, g) \to (\mathsf{F}_4, g)/\mathsf{Spin}(8)$, it follows that all sectional curvatures are positive at the orbit $\mathsf{N}_{\mathrm{Ca}} \in \mathsf{N}_{\mathrm{Ca}}\backslash(\mathsf{F}_4, g)$. Furthermore the sectional curvature is everywhere nonnegative as $\mathsf{N}_{\mathrm{Ca}}\backslash(\mathsf{F}_4, g)$ is a normal biquotient.

The group $\mathsf{Spin}(9)$ acts by isometries on $\mathsf{N}_{\mathrm{Ca}}\backslash(\mathsf{F}_4, g)$. There are subgroups $\mathrm{SU}(2) \subset \mathrm{SU}(3) \subset \mathsf{G}_2 \subset \mathsf{Spin}(9)$ such that the fix point set of $\mathrm{SU}(2)$ in $P_{\mathrm{Ca}}T\mathrm{Ca}\mathbb{P}^2/\mathsf{S}_3$ is diffeomorphic to $P_{\mathbb{H}}T\mathbb{H}\mathbb{P}^2/\mathsf{S}_3$ and the fix point set of $\mathrm{SU}(3)$ is diffeomorphic to $P_{\mathbb{C}}T\mathbb{C}\mathbb{P}^2/\mathsf{S}_3$. Equipped with the induced metrics these manifolds have nonnegative sectional curvature and positive curvature at one point, as well.

Remark 10.1. For the metric we constructed on $P_{\mathbb{K}}T\mathbb{K}\mathbb{P}^2/\mathsf{S}_3$, the set of points at which zero curvature planes occur contains an open subset, $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathrm{Ca}\}$. On the universal cover the metric can be deformed into a metric of positive sectional curvature, in fact there is a family $(g_\lambda)_{\lambda\in[0,1]}$ of metrics on $P_{\mathbb{K}}T\mathbb{K}\mathbb{P}^2$ such that the g_λ has positive sectional curvature for $\lambda \in (0, 1]$ and g_0 is the pull back metric from $P_{\mathbb{K}}T\mathbb{K}\mathbb{P}^2/\mathsf{S}_3$. It would be interesting to understand by more direct means why such a deformation can not carry over to the quotient $P_{\mathbb{K}}T\mathbb{K}\mathbb{P}^2/\mathsf{S}_3$.

References

S. Aloff, N. Wallach, An infinite family of 7–manifolds admitting positively curved Riemannian structures, Bull. Amer. Math. Soc. **81** (1975), 93–97

T. Aubin, Metriques riemanniennes et courbure, J. Differential Geometry **4** (1970), 383–424

Y.V. Bazaikin, On a certain family of closed 13–dimensional Riemannian manifolds of positive curvature, Sib. Math. J. **37**, No. 6 (1996), 1219–1237

L. Bérard Bergery, Sur certaines fibrations d'espaces homogènes riemanniens, Compositio Mathematica **30** (1975), 43–61

L. Bérard Bergery, Les variétés riemanniennes homogènes simplement connexes de dimension impaire à courbure strictement positive, J. Math. pure et appl. **55** (1976), 47–68

M. Berger, Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive, Ann. Scuola Norm. Sup. Pisa **15** (1961), 191–240

P. Ehrlich, Local convex deformations of Ricci and sectional curvature on compact manifolds, Proc. of Symposia in Pur Math. **27** (1975), 69–71

P. Ehrlich, Metric deformations of curvature. I, Geometriae Dedicata **5** (1976), 1–23

J.H. Eschenburg, New examples of manifolds with strictly positive curvature, Invent. math. **66** (1982), 469–480

J.H. Eschenburg, Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen, Schriftenr. Math. Inst. Univ. Münster **32** (1984)

J.H. Eschenburg, Inhomogeneous spaces of positive curvature, Diff. Geom. and its App. **2** (1992), 123–132

D. Gromoll, W. Meyer, An exotic sphere with non–negative sectional curvature, Ann. of Math. **100** (1974), 401–408

R. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geom. **17** (1982), 255–306

G. Perelman, Proof of the soul conjecture of Cheeger and Gromoll, J. Differential Geometry **40** (1994), 209–212

F. Wilhelm, P. Petersen Examples of Riemannian manifolds with positive curvature almost everywhere, Geom. Topol. **3** (1999), 331–367

M. Strake, Variationen von Metriken nichtnegativer Schnittkrümmung, Schriftenr. Math. Inst. Univ. Muenster, 2. Ser. **41**, 133 S. (1986)

N. Wallach, Compact homogeneous Riemannian manifolds with strictly positive curvature, Ann. of Math. **96** (1972), 277–295

F. Wilhelm, An exotic sphere with positive sectional curvature almost everywhere, J. Geom. Anal. (2001), No. 3

B. Wilking, The normal homogeneous space $SU(3) \times SO(3)/U^\bullet(2)$ has positive sectional curvature, Proceedings of the AMS **127** (1999), 1191–1994