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Hurwitz numbers and intersections on moduli spaces of curves

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1. Introduction

1.1. Topological classification of ramified coverings of the sphere. For a compact connected genus g complex curve C let $f : C \to \mathbb{C}\mathbf{P}^1$ be a meromorphic function. We treat this function as a ramified covering of the sphere. Two ramified coverings $(C_1; f_1)$, $(C_2; f_2)$ are called *topologically equivalent* if there exists a homeomorphism $h : C_1 \to C_2$ making the following diagram commutative:

$$\begin{array}{ccc} C_1 & \xrightarrow{h} & C_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & \mathbb{C}\mathbf{P}^1 & \end{array}$$

The critical values of topologically equivalent functions, i.e., the ramification points of the coverings, coincide, as do the genera of the covering curves. In his famous paper [H] Hurwitz initiated the topological classification of such coverings in the case when exactly one of the ramification points is degenerate, and the remaining points are nondegenerate. Below we refer to the degenerate ramification point as “infinity”, and its preimages are called “poles”.

The space of all biholomorphic equivalence classes of generic (that is, with distinct critical values) meromorphic functions on genus g curves

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with the prescribed orders $k_1, \dots, k_n$ of poles carries a natural topology, a natural complex structure, and is connected (see [N]). The $\mathbb{C}$-dimension of this space is $\mu = K + n + 2g - 2$, where $K = k_1 + \cdots + k_n$ is the number of folds in the covering. The additive group $\mathbb{C}$ of complex numbers acts on this space by adding a constant $c \in \mathbb{C}$ to all meromorphic functions. The *Hurwitz space* is the space of orbits with respect to this action. There is a unique meromorphic function in each orbit such that the sum of its finite critical values is zero. Hence, the Hurwitz space is naturally identified with the space of meromorphic functions with zero sum of finite critical values. We denote the latter space by $H_{g;k_1,\dots,k_n}$.

For a given set of orders $k_1, \dots, k_n$, the number of the topological equivalence classes of ramified coverings with these orders of poles and prescribed nondegenerate ramification points is finite; it is called the *Hurwitz number* and denoted $h_{g;k_1,\dots,k_n}$. Hurwitz posed the problem of finding $h_{g;k_1,\dots,k_n}$ explicitly. Below we express the Hurwitz number in terms of intersection numbers for the Chern classes of certain bundles on the moduli space of complex curves with n marked points.

Let $\overline{\mathcal{M}}_{g;n}$ denote the Deligne–Mumford compactification of the moduli space of complex curves with n marked points, and let $\mathcal{L}_i$ be the line bundle on $\overline{\mathcal{M}}_{g;n}$ whose fiber at a point $(C; x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g;n}$ coincides with the cotangent space to C at x_i. The (first) Chern class of such a bundle is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

The main goal of the present paper is the proof of the following theorem.

Theorem 1.1. *For $g = 0$, $n \geqslant 3$, or for $g \geqslant 1$, the Hurwitz number $h_{g;k_1,\dots,k_n}$ equals*

(1)
$$h_{g;k_1,\dots,k_n} = \frac{(K+n+2g-2)!}{\#\mathrm{Aut}(k_1,\dots,k_n)} \prod_{i=1}^{n} \frac{k_i^{k_i}}{k_i!} \int_{\overline{\mathcal{M}}_{g;n}} \frac{c(\Lambda_{g;n}^{\vee})}{(1-k_1\psi_1)\dots(1-k_n\psi_n)}.$$

Here $\#\mathrm{Aut}(k_1, \dots, k_n)$ is the number of automorphisms of the n-tuple $(k_1, \dots, k_n)$, $\Lambda_{g;n}$ denotes the Hodge bundle of holomorphic 1-forms over $\overline{\mathcal{M}}_{g;n}$ (a precise definition of this bundle is given below), and $c(\Lambda_{g;n}^{\vee})$ is the total Chern class of the dual bundle. We admit as usual that the integral of a class with degree different from the dimension of the variety is zero.

Observe that the integral in (1) in case $g = 0$, $n \geqslant 3$ is equal to K^{n-3} (see Sect. 2.1). It follows that the two exceptional cases $g = 0$, $n = 1, 2$ are also covered by (1), provided the integral is understood formally as K^{n-3}.

Our main tool in the proof of the main theorem is the Lyashko–Looijenga mapping described below.

1.2. Lyashko–Looijenga mapping. The Lyashko–Looijenga mapping (the LL mapping in the text and $\mathcal{LL}$ in the formulas below) associates to a $\mathbb{C}$-valued holomorphic function the unordered set of its critical values (taking multiplicities into account). It is a classical tool in the study of the geometry of moduli spaces of meromorphic functions. Assume we have a family of functions such that for each member of the family the set of its critical values is finite and contains the same number of elements. Then the LL mapping can be considered as a mapping from the family to the space of monic (i.e. with leading coefficient 1) polynomials in one variable. To do this we associate to an unordered set of complex numbers the monic polynomial whose roots coincide with the elements of the set. The most interesting situation occurs when the LL mapping is of an algebraic nature, and one can compute its multiplicity. We are going to show that this is the case for the Hurwitz spaces.

The Riemann–Hurwitz formula implies that a function belonging to the Hurwitz space has $\mu = K + n + 2g - 2$ finite ramification points. Recall that the sum of the finite critical values equals zero. Therefore the LL mapping can be treated as the mapping to $\mathbb{C}^{\mu-1}$, where the target space is the space of polynomials of the form

$$t^{\mu} + d_2 t^{\mu-2} + \cdots + d_{\mu}, \tag{2}$$

and the coefficients d_i of these polynomials form a natural set of coordinates in this space. We denote the Lyashko–Looijenga mapping by $\mathcal{LL}$,

$$\mathcal{LL} : H_{g;k_1,\ldots,k_n} \to \mathbb{C}^{\mu-1}.$$

The multiplicity of the LL mapping is closely related to the enumeration problem for topological types of ramified coverings.

Lemma 1.2.

$$h_{g;k_1,\ldots,k_n} = \frac{\deg \mathcal{LL}}{\#\mathrm{Aut}(k_1, \ldots, k_n)}.$$

Proof. Indeed, consider a function $f : C \to \mathbb{C}\mathbf{P}^1$ with a prescribed set of μ distinct critical values. Such function determines a complex structure on C. A topological equivalence preserves the set of poles of the function, and it can permute only poles of the same order. If the poles are fixed, then, according to the Riemann theorem, the meromorphic function is unique. Thus, the multiplicity of the LL mapping is simply $\#\mathrm{Aut}(k_1, \ldots, k_n)$ times the number of topological types of meromorphic functions. □

1.3. Outline of the proof. In the present section we describe briefly the most important steps in the proof of the main theorem.

We start with the definition of the space $\mathcal{P}$ of generalized principal parts. This space is considered as a cone (in the sense of [Fu]) over the moduli space $\overline{\mathcal{M}}_{g;n}$ of stable curves with n marked points. The fiber of $\mathcal{P}$ at a point $(C; x_1, \ldots, x_n)$ consists essentially of n-tuples of principal parts of meromorphic germs with poles of order k_i at the marked points. This cone is simply a vector bundle if the orders k_i of all poles are equal to 1, but it loses the linearity in the case of larger orders.

Associating to a meromorphic function the n-tuple of its principal parts at the marked points we define an embedding of the Hurwitz space to $\mathcal{P}$. If $g = 0$, then the completed Hurwitz space simply coincides with the space of generalized principal parts. For $g > 0$, the Hurwitz space is a subcone in the cone $\mathcal{P}$. The principal parts $(p_1, \dots, p_n)$ of a meromorphic function $f : (C; x_1, \dots, x_n) \to (\mathbb{C}\mathbf{P}^1, \infty)$ at the marked points x_i must satisfy the requirement

$$\mathrm{Res}_{x_1} p_1\omega + \cdots + \mathrm{Res}_{x_n} p_n\omega = 0 \tag{3}$$

for any holomorphic 1-form ω on C. We *define* the completed Hurwitz space as the closure in $\mathcal{P}$ of the set of principal parts on *smooth curves* satisfying requirement (3). The completed Hurwitz space has a natural structure of a cone over $\overline{\mathcal{M}}_{g;n}$.

The LL mapping extends naturally to the completed Hurwitz space. Its multiplicity can be expressed in terms of the top Segre class of this cone. On the other hand, this top Segre class can be expressed in terms of the total Segre class of the cone of generalized principal parts and of the total Chern class of the Hodge bundle over $\overline{\mathcal{M}}_{g;n}$. The main difficulty here arises from the fact that relation (3) is satisfied not only by the points of the Hurwitz space, but as well by some other n-tuples of principal parts coming from meromorphic functions on singular curves. The *total* Segre class takes into account these additional components. Fortunately, however, their impact on the *top* Segre class, which is the one we are interested in, can be shown to be trivial, and we arrive at the integral (1).

1.4. Previous research. The study of topological types of ramified coverings of the sphere was initiated by A. Hurwitz in [H]. In particular, he suggested there, without a complete proof, an explicit formula for the number of genus 0 ramified coverings of the sphere. We reproduce this formula in Sect. 2. In 1995 it was partially (in the case of simple poles) rediscovered by physicists [CT]. Inspired by their result I. Goulden and D. Jackson [GJ1] came to a proof of the general Hurwitz formula (which they were not aware of). Their approach, mainly purely combinatorial, is presumably close to the original Hurwitz way of reasoning (see the reconstruction of Hurwitz's proof in [S]). The further development of the same approach resulted in obtaining explicit enumeration of "simple" toric coverings (these formulas are also reproduced in Sect. 2), and a number of other partial results and interesting conjectures in this direction [SSV,GJVn].

On the other hand, Mednykh [M1,M2] gave, in a sense, a complete answer to the enumeration problem under consideration. Unfortunately, the way of presentation of the results does not allow one to extract essential information about the behavior of the numbers and the underlying geometry remains covered.

The Lyashko–Looijenga mapping was involved in the subject by V.I. Arnold [A1]. The mapping itself was introduced by Lyashko (unpublished, see [A2]) and, independently, by Looijenga [L] as a main tool of investigating the topology of the complement to the discriminant in the space of versal deformations of simple singularities. Arnold reinterpreted the results of Lyashko and Looijenga for the singularities of the A_μ series as the topological classification of generic rational ramified coverings of the sphere with one pole and extended them to the case of two poles. Further exploitation of the same tools [GL] led to a new proof of Hurwitz's formula in all its generality. Our present approach is close to that of [GL], although we consider both a different compactification of the moduli space of curves, and a different fibration over this space. We do not know a direct generalization of the construction of [GL] to higher genera.[1]

On degenerate genus 0 coverings see [GJ2,Z,LZ].

The last years' flash of interest to the topology of Hurwitz spaces is due to the fact that it can be treated as one of the easiest examples of quantum cohomology calculations [KM,V1,GP]. It is also worth mentioning that the LL mapping is the main tool in Dubrovin's construction of various Frobenius structures in Hurwitz spaces [D].

The results of the present paper were announced in [ELSV] (with a number of mistakes that are corrected here). The main idea of the present proof follows that of [ELSV], although technical details are different. Since [ELSV] has been published, an independent proof of the formula (1) for the case of all simple poles, $k_i = 1$, appeared in [FnP]. More recently, a proof of (1) for the general case based on the same ideas of virtual localization as the proof presented in [FnP], appeared in [GV].

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2. Calculations

2.1. The genus zero case. In the genus zero case the integral in the right hand side of the main formula (1) has the form

$$\int_{\overline{\mathcal{M}}_{0;n}} \frac{1}{(1-k_1\psi_1)\dots(1-k_n\psi_n)}, \tag{4}$$

and in order to compute it it is sufficient to find the intersection numbers for the monomials

[1] See "Note added in proof" at the end of the article.

(5) $$\langle\tau_{m_1}\dots\tau_{m_n}\rangle_0 = \int_{\overline{\mathcal{M}}_{0;n}} \psi_1^{m_1}\dots\psi_n^{m_n},$$

which are special cases, for $g = 0$, of the monomials

$$\langle\tau_{m_1}\dots\tau_{m_n}\rangle_g = \int_{\overline{\mathcal{M}}_{g;n}} \psi_1^{m_1}\dots\psi_n^{m_n}.$$

The intersection numbers (5) are nonzero if and only if the degree of the integrand coincides with the dimension of the base,

$$m_1 + m_2 + \dots + m_n = n - 3.$$

They are totally determined by the initial condition $\langle\tau_0^3\rangle_0 = 1$ and the genus zero case of the *string equation*

(6) $$\langle\tau_0 \prod_{i=1}^{n} \tau_{m_i}\rangle_g = \sum_{j=1}^{n} \langle\tau_{m_j-1} \prod_{i\neq j} \tau_{m_i}\rangle_g$$

(Witten's theorem for genus 0, see [K,W]). An easy computation gives

$$\langle\tau_{m_1}\dots\tau_{m_n}\rangle_0 = \frac{(n-3)!}{m_1!\dots m_n!}.$$

This formula was known to physicists for about the last two decades.

Hence, the coefficient at $k_1^{m_1}\dots k_n^{m_n}$ in the expansion of (4) equals zero if $m_1 + \dots + m_n \neq n - 3$, and equals

$$\frac{(n-3)!}{m_1!\dots m_n!}$$

otherwise. Therefore, (4) is nothing but the expansion of $(k_1 + \dots + k_n)^{n-3}$. We thus arrive at the expression

(7) $$h_{0;k_1,\dots,k_n} = \frac{(K+n-2)!}{\#\mathrm{Aut}(k_1,\dots,k_n)} \prod_{i=1}^{n} \frac{k_i^{k_i}}{k_i!} \cdot K^{n-3},$$

where $K = k_1 + \dots + k_n$, which is known since the pioneering work of Hurwitz [H].

2.2. Calculation of some intersection numbers for higher genera. For $g \geqslant 1$ the factor $c(\Lambda_{g;n}^\vee)$ in the integrand in (1) is not 1 any more. It has the form

$$c\big(\Lambda_{g;n}^\vee\big) = 1 - \lambda_1 + \dots + (-1)^g\lambda_g,$$

where $\lambda_i \in H^{2i}(\overline{\mathcal{M}}_{g;n})$. Hence, the calculation of the integral is reduced to the calculation of the integrals of monomials of the form

$$\int_{\overline{\mathcal{M}}_{g;n}} \psi_1^{m_1} \dots \psi_n^{m_n} \lambda_i$$

for

$$m_1 + \dots + m_n = 3g + n - 3 - i$$

(for other monomials the integral vanishes). These integrals are called *Hodge integrals* (see [FP1,FP2]), and their values are known in some special cases.

In particular, it is known that

Theorem 2.1 [FP2].

$$\int_{\overline{\mathcal{M}}_{g;n}} \psi_1^{m_1} \dots \psi_n^{m_n} \lambda_g = \binom{2g+n-3}{m_1 \dots m_n} b_g,$$

where b_g is a constant independent of m_j and equal to

$$b_g = \begin{cases} 1, & g = 0, \\ \int_{\overline{\mathcal{M}}_{g;1}} \psi_1^{2g-2} \lambda_g = \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}, & g > 0, \end{cases}$$

B_{2g} being the Bernulli numbers.

From this result we immediately get the following corollary.

Corollary 2.2. *The term of the lowest degree in $k_1, \dots, k_n$ in the integral* (1) *equals, up to a nonzero constant factor,*

$$(k_1 + \dots + k_n)^{2g+n-3}.$$

The factor is b_g.

In particular, for $g = 1$ this statement entirely determines the “λ-part” of the integral in the right-hand side of (1). The “λ-free” part of the integral is, in principle, described by Witten’s conjecture (Kontsevich’s theorem, [W,K]) for all genera. In addition to the string equation (6), the description includes the *dilaton equation*

$$\langle \tau_1 \prod_{i=1}^{n} \tau_{m_i} \rangle_g = (2g - 2 + n) \langle \prod_{i=1}^{n} \tau_{m_i} \rangle_g. \tag{8}$$

Together with the initial value

$$\langle \tau_1 \rangle_1 = \int_{\overline{\mathcal{M}}_{1;1}} \lambda_1 = \frac{1}{24},$$

this gives a complete description of genus one Hurwitz numbers. In particular, it proves the following elegant closed formula conjectured in [GJVn] and first proved in [GJ3,V2].

Theorem 2.3. *For $g = 1$*

$$h_{1;k_1,\ldots,k_n} = \frac{(K+n)!}{24\#\mathrm{Aut}(k_1,\ldots,k_n)} \prod_{i=1}^{n} \frac{k_i^{k_i}}{k_i!} \left(K^n - \sum_{i=2}^{n} (i-2)! e_i K^{n-i} - K^{n-1} \right),$$

where $e_i = e_i(k_1,\ldots,k_n)$ is the ith elementary symmetric polynomial in $k_1,\ldots,k_n$.

Proof. It follows from the main theorem and Corollary 2.2 that it is enough to prove that $\langle \tau_{m_1} \ldots \tau_{m_n} \rangle_1$ equals the coefficient at the monomial $k^{m_1} \ldots k^{m_n}$ in the expression

$$\frac{1}{24} \left(K^n - \sum_{i=2}^{n} (i-2)! e_i K^{n-i} \right). \tag{9}$$

The latter coefficient is equal to

$$\alpha_{m_1,\ldots,m_n} = \frac{1}{24M} \left(n! - \sum_{i=2}^{n} (i-2)!(n-i)! e_i(m_1,\ldots,m_n) \right),$$

where $M = m_1! \ldots m_n!$.

First, assume that all m_j's are distinct from zero, and hence $m_1 = \cdots = m_n = 1$. Then, by the dilaton equation,

$$\langle \tau_1 \ldots \tau_1 \rangle_1 = \frac{(n-1)!}{24},$$

while the corresponding coefficient in (9) equals

$$\alpha_{1,\ldots,1} = \frac{1}{24} \left(n! - \sum_{i=2}^{n} (i-2)!(n-i)! \binom{n}{i} \right) = \frac{(n-1)!}{24}.$$

Now, assume that at least one among the m_j's is equal to zero, say, $m_1 = 0$. Define $J = \{ j : m_j \geqslant 1 \}$. It is easy to see that

$$\begin{aligned} &e_i(m_2,\ldots,m_j-1,\ldots,m_n) \\ &\qquad = e_i(m_2,\ldots,m_j,\ldots,m_n) - e_{i-1}(m_2,\ldots,0,\ldots,m_n) \end{aligned}$$

for $j \in J$ and

$$\sum_{j=2}^{n} m_j e_{i-1}(m_2,\ldots,0,\ldots,m_n) = i e_i(m_2,\ldots,m_j,\ldots,m_n).$$

Therefore we get

$$
\begin{aligned}
&\sum_{j\in J}\alpha_{m_2,\dots,m_j-1,\dots,m_n}\\
&=\frac{1}{24M}\sum_{j\in J}m_j\Bigg((n-1)!-\sum_{i=2}^{n-1}(i-2)!(n-1-i)!\\
&\qquad\qquad\times\big(e_i(m_2,\dots,m_j,\dots,m_n)-e_{i-1}(m_2,\dots,0,\dots,m_n)\big)\Bigg)\\
&=\frac{1}{24M}\left(n!-\sum_{i=2}^{n-1}(i-2)!(n-i)!e_i(m_2,\dots,m_j,\dots,m_n)\right)=\alpha_{0,m_2,\dots,m_n},
\end{aligned}
$$

since $e_i(m_2,\dots,m_j,\dots,m_n)=e_i(0,m_2,\dots,m_n)$ and $e_n(0,m_2,\dots,m_n)=0$.

So, the coefficients $\alpha_{m_1,\dots,m_n}$ obey the same string equation as the intersection numbers $\langle\tau_{m_1}\dots\tau_{m_n}\rangle_1$ and coincide with the latter for initial values of m_j's. Therefore, they coincide for arbitrary values of m_j's. □

Another application of the main theorem is the following generating function for Hodge integrals over moduli spaces with one marked point found first in [FP1].

Theorem 2.4.

$$
1+\sum_{g=1}^{\infty}t^{2g}\int_{\overline{\mathcal{M}}_{g;1}}\frac{k^g-k^{g-1}\lambda_1+\dots+(-1)^g\lambda_g}{1-k\psi}=\left(\frac{t/2}{\sinh\ t/2}\right)^{k+1}.
$$

Proof. Follows immediately from the main theorem and the generating function for the Hurwitz numbers $h_{g;k}$ obtained in [SSV]. □

Finally, as an immediate corollary of the main theorem we obtain the following statement, which generalizes a conjecture from [GJVn].

Theorem 2.5. *The factor in $h_{g;k_1,\dots,k_n}$ given by the integral is the sum of homogeneous symmetric polynomials in $k_1,\dots,k_n$ of degrees $n+3g-3$, $n+3g-4,\dots,n+2g-3$.*

For other applications of the main theorem see [GJVk].[2]

3. Completed Hurwitz spaces

3.1. Cones and projective cones. Let $\mathbb{S}^\bullet=\mathbb{S}^0\oplus\mathbb{S}^1\oplus\mathbb{S}^2\oplus\dots$ be a graded sheaf of $\mathcal{O}_X$-algebras over a scheme X. We suppose that $\mathbb{S}^\bullet$ is locally finitely generated.

[2] See "Note added in proof" at the end of the article.

Given $\mathbb{S}^\bullet$, we define two schemes over X, the *cone* $\mathcal{S} = \operatorname{Spec}(\mathbb{S}^\bullet)$ endowed with the action of the group $\mathbb{C}^*$ due to the grading in $\mathbb{S}^\bullet$, and the *weighted projective cone* $P\mathcal{S} = \operatorname{Proj}(\mathcal{S})$ (the *projectivization* of $\mathcal{S}$). Points of $P\mathcal{S}$ correspond to non-trivial orbits of the $\mathbb{C}^*$-action on $\mathcal{S}$. The projective cone $P\mathcal{S}$ is endowed with the canonical $\mathbb{Q}$-line bundle $\mathcal{O}(1)$ (which is an element of $\operatorname{Pic}(\operatorname{Proj}(S)) \otimes \mathbb{Q}$) of quasihomogeneous rational functions of weight 1.

Any vector bundle E has a natural structure of a cone determined by its structure sheaf $\operatorname{Sym}(E^\vee)$ endowed with the natural (integer-valued) grading by degrees.

For two cones $\mathcal{S}_1$, $\mathcal{S}_2$ over X defined by the sheaves $\mathbb{S}_1^\bullet$, $\mathbb{S}_2^\bullet$ their *direct sum* $\mathcal{S}_1 \oplus \mathcal{S}_2$ is defined by the graded sheaf $\mathbb{S}_1^\bullet \otimes \mathbb{S}_2^\bullet$.

3.2. Moduli space of stable curves. The base space of most of the bundles and cones considered in the present paper will be the *moduli space* $\overline{\mathcal{M}}_{g;n}$ of stable curves with n marked points. Its elements are the biholomorphic equivalence classes of connected compact curves of arithmetic genus g with n marked nonsingular points that are either smooth or have at most ordinary double points as singularities and that admit no continuous group of automorphisms. The last assumption (the *stability*) is equivalent to the requirement that each rational (of genus 0) irreducible component of the curve contains at least 3 special (marked or double) points, and each elliptic (of genus 1) irreducible component contains at least one special point.

The moduli space $\overline{\mathcal{M}}_{0;n}$, $n \geqslant 3$ of rational curves is a smooth and compact variety, while for $g \geqslant 1$ $\overline{\mathcal{M}}_{g;n}$ is a compact orbifold. By the Deligne–Mumford theorem, the space $\overline{\mathcal{M}}_{g;n}$ is an irreducible projective variety of pure dimension $\dim \overline{\mathcal{M}}_{g;n} = n + 3g - 3$. All the facts about moduli spaces we make use of can be found in [HMo].

A meromorphic function on a singular curve is a tuple of meromorphic functions, one for each irreducible component of the curve, such that the values of these functions on each two branches meeting at a double point coincide at this point.

3.3. The space of generalized principal parts at a point. Fix a positive integer k. Two germs of meromorphic functions $f_1, f_2 : (\mathbb{C}, 0) \to (\mathbb{C}\mathbf{P}^1, \infty)$ with poles of order k have the same principal part if their difference $f_1 - f_2$ has no pole at 0. A *principal part* is an equivalence class of germs with respect to this equivalence relation. The set of all principal parts with poles of order precisely k carries a natural complex structure. Below, we present a coordinate description of the space of principal parts.

Let x be the germ of a coordinate at the origin $0 \in \mathbb{C}$. Then a principal part can be written in the form

$$\left(\frac{u}{x}\right)^k + a_1\left(\frac{u}{x}\right)^{k-1} + \cdots + a_{k-1}\frac{u}{x}, \qquad u \neq 0. \tag{10}$$

This presentation is not unique: it depends on the choice of the parameter u, and there are k possibilities to make this choice. Hence, the space of expressions of the form (10) covers the space of principal parts with multiplicity k. The group of the covering is $\mathbb{Z}/k\mathbb{Z}$, and it acts on the space of expressions (10) according to the rule

$$(u, a_1, a_2, \dots, a_{k-1}) \mapsto \big(\zeta u, \zeta a_1, \zeta^2 a_2, \dots, \zeta^{k-1} a_{k-1}\big), \tag{11}$$

where ζ is a primitive root of unity of degree k generating $\mathbb{Z}/k\mathbb{Z}$ as a subgroup in $\mathbb{C}^*$. We call the parameters $u, a_1, \dots, a_{k-1}$ the *twisted Laurent coefficients of order k*.

The group $\mathbb{C}^*$ of nonzero complex numbers acts on the space of principal parts by multiplication. Up to the above action of $\mathbb{Z}/k\mathbb{Z}$, this action of $\mathbb{C}^*$ can be written in coordinates as

$$\begin{aligned} c : \left(\frac{u}{x}\right)^k + a_1\left(\frac{u}{x}\right)^{k-1} + \dots + a_{k-1}\frac{u}{x} \\ &\mapsto c\left(\left(\frac{u}{x}\right)^k + a_1\left(\frac{u}{x}\right)^{k-1} + \dots + a_{k-1}\frac{u}{x}\right) \\ &= \left(\frac{\eta u}{x}\right)^k + (\eta a_1)\left(\frac{\eta u}{x}\right)^{k-1} + \dots + (\eta^{k-1}a_{k-1})\frac{\eta u}{x}, \end{aligned}$$

where $\eta = c^{1/k}$. Therefore, it induces the following choice of *weights* of the parameters u, a_j:

$$\text{weight}(u) = \frac{1}{k}, \qquad \text{weight}(a_j) = \frac{j}{k}. \tag{12}$$

Consider the algebra of polynomials in twisted Laurent coefficients invariant with respect to the action (11) of $\mathbb{Z}/k\mathbb{Z}$. The weights of the coordinates endow this algebra with an integer-valued grading. Denote the spectrum of this algebra by P^k.

Lemma 3.1. 1) *The graded algebra of $\mathbb{Z}/k\mathbb{Z}$-invariant polynomials is independent of the choice of the local coordinate x.*

2) *The space of principal parts with poles of order k is naturally embedded in P^k as an open dense subset.*

3) *Its complement $A^{k-1} \subset P^k$ carries a natural structure of the quotient of the complex vector space $\widetilde{A}^{k-1} \cong \mathbb{C}^{k-1}$ modulo the $\mathbb{Z}/k\mathbb{Z}$-action.*

4) *The twisted Laurent coefficients $a_1, \dots, a_{k-1}$ are linear coordinates on $\widetilde{A}^{k-1}$. The action of $\mathbb{Z}/k\mathbb{Z}$ is diagonal in these coordinates.*

Proof. A direct calculation shows that a coordinate change $x = \alpha_1\tilde{x} + \alpha_2\tilde{x}^2 + \dots$ with $\alpha_1 \neq 0$ causes a polynomial change of the variables u, a_j of the form

$$\begin{aligned} \tilde{u} &= \frac{u}{\alpha_1^k}, \\ \tilde{a}_1 &= a_1 + \gamma_{11}u, \\ &\dots\dots\dots\dots \\ \tilde{a}_{k-1} &= a_{k-1} + \gamma_{k-1,1}a_{k-2}u + \dots + \gamma_{k-1,k-1}u^{k-1} \end{aligned}$$

for some constants γ_{ij}. Evidently, this action preserves the $\mathbb{Z}/k\mathbb{Z}$-invariance and the grading. Hence, the algebra of invariant polynomials is well-defined.

A principal part is given by an element in P^k with $u \neq 0$, and the set $u \neq 0$ is open and dense in P^k. Its complement $A^{k-1} \subset P^k$ is the spectrum of the quotient algebra of $\mathbb{Z}/k\mathbb{Z}$-invariant polynomials modulo the ideal of u-divisible polynomials.

Expression (10) is linear in the coefficients a_j. After setting $u = 0$ the vector space structure on the space $\widetilde{A}^{k-1}$ of these coefficients is introduced in the obvious way

$$c'\left(a'_1, \ldots, a'_{k-1}\right) + c''\left(a''_1, \ldots, a''_{k-1}\right) = \left(c'a'_1 + c''a''_1, \ldots, c'a'_{k-1} + c''a''_{k-1}\right).$$

Its independence of the choice of the coordinate x is the result of a direct computation. The space A^{k-1} is the quotient of $\widetilde{A}^{k-1}$ modulo the action (11) restricted to $\widetilde{A}^{k-1}$.

In order to prove that the coordinates in A^{k-1} can be chosen invariantly, fix a coordinate x and consider a smooth holomorphic 1-parameter family γ of principal parts through a, $\gamma : (\mathbb{C}^1, 0) \to (P^k, a)$, such that $\gamma(\tau) \notin A^{k-1}$ for $\tau \in \mathbb{C}^1$, $\tau \neq 0$. Then the elements of the family can be written in the form

$$\left(\frac{u(\tau)}{x}\right)^k + a_1(\tau)\left(\frac{u(\tau)}{x}\right)^{k-1} + \cdots + a_{k-1}(\tau)\frac{u(\tau)}{x}, \quad u(\tau) \to 0 \text{ as } \tau \to 0,$$

with $u(\tau)$, $a_j(\tau)$ chosen uniquely up to the action (11) of $\mathbb{Z}/k\mathbb{Z}$. One verifies immediately that the limit values $a_1(0), \ldots, a_{k-1}(0)$ depend neither on the choice of the coordinate x, nor on the choice of the 1-parameter family through a. □

Below, the term *generalized principal part of order k* means simply a point in P^k. Elements of A^{k-1} are just generalized principal parts *with zero leading coefficient.*

Now we are going to associate to a point in P^k the kth tensor power of a tangent vector at $0 \in \mathbb{C}^1$. Let L denote the cotangent line at 0. Writing a point $p \in P^k \setminus A^{k-1}$ in the form (10) we associate to this point the principal part u/x with pole of order 1. It is determined uniquely up to the action (11) of $\mathbb{Z}/k\mathbb{Z}$ on u. This principal part determines the tangent vector at 0 as the linear functional on L:

$$\omega \mapsto \operatorname{Res}_{x=0} \frac{u}{x}\omega,$$

ω being the germ of a 1-form representing a cotangent vector. The kth tensor power of the constructed tangent vector is an element in $(L^\vee)^{\otimes k}$, which depends neither on the choice of the coordinate x, nor on the choice of u. We denote this element by $\phi(p) \in (L^\vee)^{\otimes k}$. The mapping ϕ is extended continuously to the entire P^k by setting it identically 0 on $A^{k-1} \subset P^k$.

Lemma 3.2. *The dual mapping ϕ^* is a morphism of graded algebras. The zero locus of ϕ is A^{k-1}. The multiplicity of ϕ along A^{k-1} equals k.*

Proof. For a coordinate x fixed, the mapping ϕ is written as the polynomial u^k. By definition of P^k, it is a well-defined polynomial of weight 1 on P^k. This polynomial vanishes precisely on A^{k-1}. In order to compute the multiplicity of ϕ along A^{k-1} let us count the number of preimages of ϕ that glue together at a point $a = (a_1, \dots, a_{k-1}) \in A^{k-1}$ as their image u^k tends to zero. These are the principal parts with the coordinates $(u, a_1, \dots, a_{k-1})$, $(u, \zeta a_1, \dots, \zeta^{k-1} a_{k-1})$, $(u, \zeta^2 a_1, \dots, \zeta^{2(k-1)} a_{k-1})$, $\dots$, $(u, \zeta^{k-1} a_1, \dots, \zeta^{(k-1)(k-1)} a_{k-1})$. For a generic $a \in A^{k-1}$ all these k points are distinct, and the proof is completed. □

3.4. The cones of generalized principal parts. Denote by $\mathbb{P}_i$ the sheaf of graded algebras over $\overline{\mathcal{M}}_{g;n}$ whose stalk at a point $(C; x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g;n}$ is the algebra of $\mathbb{Z}/k\mathbb{Z}$-invariant polynomials in twisted Laurent coefficients of order k_i at x_i. Let $\mathcal{P}_i = \mathrm{Spec}(\mathbb{P}_i)$ be the corresponding cone over $\overline{\mathcal{M}}_{g;n}$. The sheaf $\mathbb{A}_i$ and the corresponding cone $\mathcal{A}_i = \mathrm{Spec}(\mathbb{A}_i)$ of generalized principal parts with zero leading coefficients are introduced similarly. Evidently, $\mathcal{A}_i$ is a subcone in $\mathcal{P}_i$.

The direct sum $\mathcal{P} = \mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_n$ of cones will be referred to as the *cone of generalized principal parts* over $\overline{\mathcal{M}}_{g;n}$. The cone $\mathcal{P}$ contains the subcone $\mathcal{A} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$; the defining sheaves are denoted by $\mathbb{P}$ and $\mathbb{A}$ respectively. Similarly, we consider the vector bundles $\widetilde{\mathcal{A}}_i$ and $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_1 \oplus \cdots \oplus \widetilde{\mathcal{A}}_n$ over $\overline{\mathcal{M}}_{g;n}$.

With each cone $\mathcal{P}_i$ we associate the cone morphism $\varphi_i : \mathcal{P}_i \to (\mathcal{L}_i^\vee)^{\otimes k_i}$, which takes a principal part of order k_i at the ith marked point to the k_ith power of a tangent vector at this point. The direct sum of these morphisms determines the morphism $\varphi : \mathcal{P} \to \mathcal{L}$, where $\mathcal{L} = (\mathcal{L}_1^\vee)^{\otimes k_1} \oplus \cdots \oplus (\mathcal{L}_n^\vee)^{\otimes k_n}$.

Lemma 3.3. 1) *The cone $\mathcal{A}_i$ is the quotient of the vector bundle $\widetilde{\mathcal{A}}_i$ modulo the fiberwise action of the group $\mathbb{Z}/k_i\mathbb{Z}$. The cone $\mathcal{A}$ is the quotient of the vector bundle $\widetilde{\mathcal{A}}$ modulo the fiberwise action of the group $\mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_n\mathbb{Z}$.*

2) *The sheaves $\mathbb{A}_i, \mathbb{A}$ are constant sheaves over $\overline{\mathcal{M}}_{g;n}$.*

3) *The zero locus of the morphism $\varphi_i : \mathcal{P}_i \to \mathcal{L}_i^{\vee \otimes k_i}$ is the cone $\mathcal{A}_i \subset \mathcal{P}_i$. The multiplicity of φ_i along $\mathcal{A}_i$ equals k_i. The zero locus of the morphism $\varphi : \mathcal{P} \to \mathcal{L}$ is the cone $\mathcal{A} \subset \mathcal{P}$. The multiplicity of φ along $\mathcal{A}$ equals $k_1 \dots k_n$.*

Proof. We need to prove only that $\mathbb{A}_i$ are constant sheaves. This follows from assertion 4) of Lemma 3.1: treating a_j, for $j = 1, \dots, k_i - 1$, as coordinates along fibers of $\mathcal{A}_i$ and setting $a_j = 1$, $a_l = 0$ for $l \neq j$ we obtain $k_i - 1$ sections of the cone $\mathcal{A}_i$ independent at each point of the base. Assertion 3) of the lemma follows immediately from Lemma 3.2. □

3.5. Hodge bundle. The *Hodge bundle* $\Lambda_{g;n} \to \overline{\mathcal{M}}_{g;n}$ is the rank g vector bundle over $\overline{\mathcal{M}}_{g;n}$ whose fiber at a point $(C; x_1, \dots, x_n)$ is the space of holomorphic sections of the dualizing sheaf over C. For a smooth C, this space simply coincides with the space of holomorphic 1-forms over C. For a singular curve C, an element of the fiber of $\Lambda_{g;n}$ is a meromorphic 1-form ω on C admitting poles of order at most 1 at the double points and no other poles and such that the sum of its residues along the two branches of the curve meeting at each double point is zero. Below, we use the term a *generalized holomorphic* 1*-form* as a synonym for an *element of the Hodge bundle*.

Let $(p_1, \dots, p_n)$ be an n-tuple of principal parts with poles of order k_i at the marked points x_i of a stable curve $(C; x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g;n}$. Each principal part p_i determines the linear functional

$$\omega \mapsto \mathrm{Res}_{x_i} p_i \omega$$

on the space of generalized holomorphic 1-forms, given by the residue. The residue mapping naturally extends to the entire fiber of $\mathcal{P}_i$ by setting the residue equal to 0 on the fiber of $\mathcal{A}_i$.

The following statement is classical, see e.g. [HMo].

Theorem 3.4. *An n-tuple $(p_1, \dots, p_n)$ is the n-tuple of generalized principal parts of a meromorphic function f over C with poles of order k_i at x_i and no other poles if and only if*

$$\mathrm{Res}_{x_1} p_1 \omega + \cdots + \mathrm{Res}_{x_n} p_n \omega = 0 \qquad (13)$$

for any generalized holomorphic 1-form ω on C. Two meromorphic functions with coinciding n-tuples of principal parts at the marked points differ by an additive constant.

3.6. Completed Hurwitz spaces as cones over moduli spaces of curves. Consider the mapping $R : \mathcal{P} \to \Lambda^\vee_{g;n}$ taking an n-tuple $(p_1, \dots, p_n)$ of principal parts to the linear functional $\omega \mapsto \mathrm{Res}_{x_1} p_1 \omega + \cdots + \mathrm{Res}_{x_n} p_n \omega$ on $\Lambda_{g;n}$. Let $\mathcal{Z}$ be the zero subscheme of R in $\mathcal{P}$, i.e., $\mathcal{Z}$ is given by equations (13).

Another way to understand $\mathcal{Z}$ is as follows. Consider a meromorphic function on a stable curve C with poles of order k_i at the marked points x_i and no other poles. Associating with f the n-tuple of its generalized principal parts at the marked points $x_1, \dots, x_n$ we obtain a point in $\mathcal{P}$. Now $\mathcal{Z}$ is the closure of the image of this mapping.

Definition 3.5. *The* completed Hurwitz space *$\mathcal{H}_{g;k_1,\dots,k_n}$ is the closure in $\mathcal{P}$ of the space of n-tuples of principal parts corresponding to meromorphic functions on smooth curves.*

The set of points of $\mathcal{H}_{g;k_1,\dots,k_n}$ that are not obtained from meromorphic functions on smooth curves will be called the *boundary* of $\mathcal{H}_{g;k_1,\dots,k_n}$, and denoted by $\partial\mathcal{H}_{g;k_1,\dots,k_n}$.

Lemma 3.6. *The completed Hurwitz space has a natural structure of a cone over* $\overline{\mathcal{M}}_{g;n}$.

Proof. Consider the sheaf of ideals defining $\mathcal{H}_{g;k_g,\dots,k_n}$ as a subscheme of $\mathcal{P}$. Over each smooth curve such an ideal is generated by the left hand sides of equations (13), which are homogeneous of weight 1, and hence the ideal is $\mathbb{C}^*$-invariant. Therefore, it is $\mathbb{C}^*$-invariant over each stable curve as well. Now, the sheaf of graded algebras defining $\mathcal{H}_{g;k_g,\dots,k_n}$ as a cone is just $\mathcal{P}$ modulo this sheaf of ideals. □

Remark 3.7. The completed Hurwitz space can be different from the entire subvariety $\mathcal{Z} \subset \mathcal{P}$ given by equations (13). In other words, not each meromorphic function on a singular curve is the limit of a family of meromorphic functions on smooth curves.

The simplest example arises for the case of two poles of order 1 on elliptic curves. Smooth elliptic curves degenerate into singular curves having a smooth elliptic component and a smooth rational component intersecting each other at a double point. The stability condition implies that both marked points belong to the rational component. For a smooth curve C, the space of pairs (p_1, p_2) of principal parts of meromorphic functions at the marked points is 1-dimensional because there is a restriction on the residues given by the holomorphic 1-form on C. For a singular curve C, there is no such restriction since each holomorphic 1-form is trivial on the rational component, and the space of principal parts of meromorphic functions is 2-dimensional. However, not each such function can be obtained as the limit of a family of meromorphic functions on smooth curves. The limit functions form a 1-dimensional subspace in the space of all meromorphic functions with given poles of order one on the rational component.

4. The Lyashko–Looijenga mapping and its extension

4.1. Functions with vanishing poles. In this section we describe a possible way of thinking about functions with vanishing principal parts. Let $(p_1, \dots, p_n) \in \mathcal{Z}$ be a tuple of generalized principal parts, and let $(C; x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g;n}$ be the underlying stable curve. If $(p_1, \dots, p_n)$ are nonvanishing, then there exists a meromorphic function f on C with given principal parts p_i at the marked points. This function is determined uniquely up to an additive constant. However, if there is a generalized principal part p_i with a vanishing leading coefficient, $p_i \in \mathcal{A}_i$, then, for $k_i > 1$, it cannot be reconstructed from the function f, since the principal part of f at x_i is 0. The following construction allows one to avoid this ambiguity.

We associate with the curve C and the given tuple of generalized principal parts another curve $\widetilde{C}$ with n marked points and a meromorphic function $\tilde{f}$ on $\widetilde{C}$ determined uniquely up to an additive constant. If neither of the generalized principal parts p_i has zero leading coefficient, then $\widetilde{C}$ simply coincides with C, and $\tilde{f}$ is the same as f. In the general case, $\widetilde{C}$ is obtained from C by adding rational (i.e., of genus 0) components, one for each generalized principal part with zero leading coefficient. The ith additional rational component intersects C transversally at the ith marked point. The new double point becomes unmarked, and a new marked point x_i distinct from the double point is added on the new rational component. To do this we introduce a new local coordinate $y = u/x$ and set x_i to be the point $y = \infty$.

Note that the curve $\widetilde{C}$ thus constructed is no longer stable since each new rational component contains only two special points: the double point and the marked point. However, recall that we are not adding this unstable curve to $\overline{\mathcal{M}}_{g;n}$, but rather provide a better understanding of generalized principal parts with zero leading coefficients over stable curves.

The function $\tilde{f}$ coincides with f on $C \subset \widetilde{C}$, and it is extended to the new rational components in the following way. Let us define the monic polynomial

$$a(y) = y^k + a_1 y^{k-1} + \cdots + a_{k-1} y.$$

According to assertion 4) of Lemma 3.1, the coefficients of the polynomial $a(y)$ are well-defined up to the action (11) of the group $\mathbb{Z}/k\mathbb{Z}$, or, what is the same, up to the change $y \mapsto \zeta y$ of the coordinate y. The extension of $\tilde{f}$ to the new rational component coincides, up to an additive constant, with this polynomial. The constant is chosen so as to make $\tilde{f}$ continuous, i.e., it coincides with the value $f(x_i)$ on C.

The following statement is now obvious.

Lemma 4.1. *There exists a one-to one correspondence between the set $\mathcal{Z}$ and the set of meromorphic functions on curves $\widetilde{C}$ with poles of order k_i at the ith marked point taken up to an additive constant.*

4.2. Extension of the LL mapping. Recall that the Lyashko–Looijenga mapping is defined on a dense subset $H_{g;k_1,\dots,k_n}$ in $\mathcal{H}_{g;k_1,\dots,k_n}$, and takes it to the space $\mathbb{C}^{\mu-1}$ of monic polynomials in one variable of the form

$$t^\mu + d_2 t^{\mu-2} + \cdots + d_\mu. \tag{14}$$

In this section we explain how to extend the LL mapping continuously to the completed Hurwitz space $\mathcal{H}_{g;k_1,\dots,k_n}$, and, moreover, to the whole $\mathcal{Z}$.

Theorem 4.2. *The Lyashko–Looijenga mapping is a finite covering outside the space of functions with coinciding critical values and it extends to a continuous mapping*

$$\mathcal{LL} : \mathcal{H}_{g;k_1,\dots,k_n} \to \mathbb{C}^{\mu-1}$$

equivariant with respect to the action of $\mathbb{C}^*$ on the left and on the right.

Proof. The assertion that the LL mapping is a covering over the subset of polynomials with distinct roots is a well-known corollary of the Riemann theorem (see, e.g., [A1]).

Let $(p_1, \dots, p_n) \in \partial\mathcal{H}_{g;k_1,\dots,k_n}$ be an n-tuple of generalized principal parts at the marked points of a curve $(C; x_1, \dots, x_n)$, which may be singular and have more than one irreducible component.

Let $\tilde{f} : \widetilde{C} \to \mathbb{C}\mathbf{P}^1$ be the function corresponding to $(p_1, \dots, p_n)$ with the constant chosen arbitrarily. It can happen that $\tilde{f}$ is constant on some irreducible components of $C \subset \widetilde{C}$. Denote by C' the union of all irreducible components of C, where $\tilde{f}$ is constant.

We *define* the set of critical values of $\tilde{f}$ as the union of the following sets:

(1) for each irreducible component of $\widetilde{C}$, where $\tilde{f}$ is not a constant, the critical values of the restriction of $\tilde{f}$ to this component;
(2) for each double point, the value of $\tilde{f}$ at this point taken with multiplicity 2;
(3) for each irreducible component of $\widetilde{C}$ of genus g' where $\tilde{f}$ is a constant, the value of $\tilde{f}$ on this component taken with multiplicity $2g' - 2$.

Now let us verify that the LL mapping thus extended is continuous.

Take a double point θ of the curve $\widetilde{C}$ and consider the preimage $\tilde{f}^{-1}(S^1)$ of a small circle S^1 in $\mathbb{C}$ centered at $\tilde{f}(\theta)$. We are interested in the intersection of this preimage with a small neighborhood of θ in $\widetilde{C}$. If $\tilde{f}$ is nonconstant on both branches meeting at θ, then this intersection consists of two circles, one on each branch. If $\tilde{f}$ is nonconstant only on one such branch, then there is only one circle, and it lies on this branch. Finally, if $\tilde{f}$ is constant on both branches, then the intersection in question is empty, and the corresponding double point is called an *inner* double point.

Each of the circles carries a natural number, the degree of its mapping to S^1 under $\tilde{f}$. This degree coincides with the order of the critical point of $\tilde{f}$ on the corresponding branch increased by 1.

Consider now a holomorphic deformation $F : \mathcal{C} \to \mathbb{C}\mathbf{P}^1$ of $\tilde{f}$, where $\mathcal{C} \to \mathbb{C}^1$ is a holomorphic family of marked genus g curves such that C_τ is smooth for $\tau \neq 0$ and $C_0 = \widetilde{C}$, and the restriction f_τ of F to C_τ is a meromorphic function, $f_0 = \tilde{f}$. We do not require that f_τ has zero sum of critical values.

For a fixed double point θ, consider the preimage $F^{-1}(S^1)$ of the same circle S^1 in $\mathbb{C}$ centered at $\tilde{f}(\theta)$. Take the connected component of this preimage containing one of the circles on $\widetilde{C}$ built above. For a sufficiently small $\tau \neq 0$, the intersection of this connected component with C_τ is a circle, and the degree of its mapping to S^1 under f_τ coincides with the number assigned to the circle on $\widetilde{C}$.

Applying the procedure described above for all double points of $\widetilde{C}$ we obtain a finite set of circles on each curve C_τ for τ small enough. These circles cut each curve C_τ into connected pieces of three different types:

(1) pieces, containing marked points. These pieces are in one-to-one correspondence with the irreducible components of $\widetilde{C}$, where $\tilde{f}$ is nonconstant;
(2) pieces, holomorphically equivalent to the annulus. These pieces are in one-to-one correspondence with those double points of $\widetilde{C}$, where $\tilde{f}$ is nonconstant on both branches meeting at this point;
(3) pieces of positive genus without marked points. These pieces are in one-to-one correspondence with the connected components of the curve $C' \subset C$, the constant locus of $\tilde{f}$.

Now let us follow the behavior of the critical points and critical values of f_τ on pieces of all three types. Let the index j run over the pieces.

(1) By the Riemann–Hurwitz formula, the number of critical points of f_τ on a piece of the first type is equal to $K_j + n_j + 2g_j - 2 - D_j + c_j$, where K_j is the total order of all poles on this piece, n_j is the number of marked points, g_j is the genus of the corresponding component of $\widetilde{C}$, D_j is the total degree of the circles bounding the piece, and c_j is the number of these circles. As τ tends to 0, these critical points tend to the critical points of $\tilde{f}$ on the corresponding component of $\widetilde{C}$ with the double points excluded, and the critical values tend to that of $\tilde{f}$.

(2) The mapping f_τ takes an annulus without marked points to the small disk bounded by S^1. The degree of the mapping is the sum of the integers assigned to the boundary circles. By the Riemann–Hurwitz formula, the number of critical points of f_τ on the annulus coincides with the degree. As τ tends to 0, these critical points tend to the double point, and the critical values tend to the value of $\tilde{f}$ at the double point.

(3) The mapping f_τ takes a piece of the third type to a disk. The degree of this mapping equals D_j, the sum of the numbers assigned to the boundary circles. The number of critical values on this piece is $2g_j - 2 + D_j + c_j$, where c_j is the number of boundary circles, and g_j is the (arithmetic) genus of the corresponding connected component $C' \subset \widetilde{C}$. Further, $2g_j - 2 = 2\sum(g_{ji} - 2) + 2l$, where l is the number of inner double points on C', and g_{ji} are the genera of irreducible components of C'. As τ tends to 0, the critical values tend to the constant value of $\tilde{f}$ on the limit curve.

Now, taking the union of all limit critical values over all pieces we conclude that the set of critical values of f_τ tends precisely to the set described above. □

It is now easy to get the following corollary.

Corollary 4.3. *The LL mapping extended on the whole $\mathcal{Z}$ as above is continuous.*

5. Top Segre classes

5.1. Segre classes of vector bundles and cones. This section follows (with slight modification) the approach of [Fu, Chaps. 3, 4, 8, 19].

Let X be a nonsingular variety or an orbifold of pure dimension d. Consider a cone $\mathcal{S}$ over X, and let $\pi : P\mathcal{S} \to X$ denote the projection of the corresponding weighted projective cone.

The *Segre class* of $\mathcal{S}$ is an element in the cohomology ring $H^*X = H^*(X, \mathbb{Q})$ of X with rational coefficients; it is defined as follows. Consider the value $c_1(\mathcal{O}(1))^i \cap [P\mathcal{S}]$ of the iterated first Chern class of the canonical line bundle on the fundamental cycle of $P\mathcal{S}$; here Chern classes of bundles over cones are considered in the operational sense (see [Fu, Chap. 3]). This value is an element in the group $A_*P\mathcal{S}$ of cycles modulo the rational equivalence on the projective cone $P\mathcal{S}$. The mapping $\pi_* : A_*P\mathcal{S} \to A_*X$ pushes this element to a cycle in X. The isomorphism $A_*X \cong A^{d-*}X$ takes this cycle to the Chow ring A^*X ([Fu, Sect. 8.3]). In its own turn, the Chow ring admits a homomorphic graded mapping $\mathrm{cl} : A^*X \to H^*X$, the class mapping ([Fu, Corollary 19.2]). Denote the composition of all these mappings by $h : A_*P\mathcal{S} \to H^*X$. The Segre class of $\mathcal{S}$ is defined by the expression

$$s(\mathcal{S}) = h\Bigg(\sum_{i\geqslant 0} c_1(\mathcal{O}(1))^i \cap [P\mathcal{S}]\Bigg).$$

It can be represented uniquely in the form

$$s(\mathcal{S}) = s_0(\mathcal{S}) + \cdots + s_d(\mathcal{S}), \qquad s_i(\mathcal{S}) \in H^{2i}X.$$

The class $s_d(\mathcal{S})$ is called the *top Segre class* of the cone $\mathcal{S}$ and denoted by $s_{top}(\mathcal{S})$; its *value* $\int_X s_{top}(\mathcal{S})$ on X is a rational number. Lemma 5.3 below expresses the multiplicity of the LL mapping in terms of the top Segre class of the cone $\mathcal{H}_{g;k_1,\ldots,k_n}$.

For a vector bundle E over X, the Segre class coincides with the inverse Chern class, $s(E) = c^{-1}(E)$; here the Chern class is understood in the usual sense, as an element in the cohomology ring H^*X.

5.2. Segre classes of cones of principal parts. Now we are going to compute the Segre classes we shall require below.

Lemma 5.1. *The Segre class of the cone* $\mathcal{A}$ *equals*

$$s(\mathcal{A}) = \prod_{i=1}^{n} \frac{1}{k_i} \frac{k_i^{k_i-1}}{(k_i-1)!}.$$

Proof. Since, by Lemma 3.3, the sheaf $\mathbb{A}_i$, which determines the cone $\mathcal{A}_i$, is a constant sheaf, the only nontrivial Segre class of $\mathcal{A}_i$ is its zero Segre class $s_0(\mathcal{A}_i)$. This class can be found locally, at a fiber of $P\mathcal{A}_i$ over a point in $\overline{\mathcal{M}}_{g;n}$. This fiber is the projectivized weighted vector space $\widetilde{A}^{k_i-1}$ modulo the action of the group $\mathbb{Z}/k_i\mathbb{Z}$. Its zero Segre class is the inverse product of the weights of the coordinates divided by the order of the acting group,

$$s_0(\mathcal{A}_i) = \frac{1}{k_i}\frac{k_i^{k_i-1}}{(k_i-1)!}.$$

In a similar way we get that $s_0(\mathcal{A})$ is the product of the Segre classes $s_0(\mathcal{A}_i)$. □

Lemma 5.2. *The Segre class of the cone $\mathcal{P}$ equals*

$$s(\mathcal{P}) = \prod_{i=1}^{n} \frac{k_i!}{k_i^{k_i}} \frac{1}{1-k_i\psi_i},$$

where $\psi_i = c_1(\mathcal{L}_i)$.

Proof. Consider the sequence of cone morphisms

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{P} \xrightarrow{\varphi} \mathcal{L} \longrightarrow 0,$$

where $\mathcal{A} \to \mathcal{P}$ is the embedding, and $\varphi : \mathcal{P} \to \mathcal{L}$ is as in Sect. 3.4.

Let $\pi : P\mathcal{P} \to \overline{\mathcal{M}}_{g;n}$ be the projection of the projective cone of principal parts to the base. This projection defines the bundle $\mathcal{O}(1) \otimes \pi^*(\mathcal{L}) \to P\mathcal{P}$. The mapping φ determines a section $\sigma_\varphi : P\mathcal{P} \to \mathcal{O}(1) \otimes \pi^*(\mathcal{L})$ of this bundle in the following way. The value of σ_φ at a point $\mathbb{C}^* p \in P\mathcal{P}$ is equal to $1_p \otimes \pi^*(\varphi(p))$; here 1_p is the element in the fiber of $\mathcal{O}(1)$ over $\mathbb{C}^* p$ that equals 1 on p.

The zero locus $Z(\sigma_\varphi) \subset P\mathcal{P}$ of the section σ_φ is the projectivization of the zero locus of φ. By Lemma 3.3, it is $P\mathcal{A} \subset P\mathcal{P}$ taken with multiplicity $k_1 \dots k_n$. It is of pure codimension n in $P\mathcal{P}$. Therefore, by [Fu, Proposition 14.1], this zero locus represents the top Chern class of the bundle $\mathcal{O}(1) \otimes \pi^*(\mathcal{L}) \to P\mathcal{P}$,

$$\begin{aligned}[Z(\sigma_\varphi)] &= c_n(\mathcal{O}(1) \otimes \pi^*(\mathcal{L})) \cap [P\mathcal{P}] \\ &= \sum_j c_{n-j}(\pi^*(\mathcal{L}))c_1(\mathcal{O}(1))^j \cap [P\mathcal{P}].\end{aligned}$$

Thus,

$$\begin{aligned}k_1 \dots k_n s(\mathcal{A}) &= \sum_l c_1(\mathcal{O}(1))^l \cap [Z(\sigma_\varphi)] \\ &= \sum_{l,j} c_{n-j}(\pi^*(\mathcal{L}))c_1(\mathcal{O}(1))^{l+j} \cap [P\mathcal{P}] \\ &= c(\mathcal{L})s(\mathcal{P}).\end{aligned}$$

Hence,

$$s(\mathcal{P}) = k_1 \dots k_n \frac{s(\mathcal{A})}{c(\mathcal{L})},$$

and the required assertion follows. □

5.3. Multiplicity of the LL mapping and Segre classes. Similarly to the proof of Lemma 5.2, one can associate to the mapping $R : \mathcal{P} \to \Lambda^{\vee}_{g;n}$ given by the sum of residues at the marked points the section

$$\sigma_R : P\mathcal{P} \to \mathcal{O}(1) \otimes \pi^*\big(\Lambda^{\vee}_{g;n}\big)$$

of the vector bundle over the projective cone $P\mathcal{P}$: σ_R associates to a point in $P\mathcal{P}$ the linear function on the fiber of $\pi^*(\Lambda_{g;n})$ determined by the residue with the element in the fiber of $\mathcal{O}(1)$. By Theorem 3.4, the zero locus of the section σ_R is the projectivization $P\mathcal{Z}$ of the space $\mathcal{Z}$ of meromorphic functions.

Recall that the completed Hurwitz space $\mathcal{H}_{g;k_1,\dots,k_n}$ is a subcone in the cone $\mathcal{P}$ of principal parts.

Lemma 5.3. *The multiplicity of the LL mapping $\mathcal{LL} : \mathcal{H}_{g;k_1,\dots,k_n} \to \mathbb{C}^{\mu-1}$ is equal to*

$$\mu! \int_{\overline{\mathcal{M}}_{g;n}} s(\mathcal{H}_{g;k_1,\dots,k_n}).$$

Proof. Denote by $\Sigma \subset \mathbb{C}^{\mu-1}$ the discriminant in the target space of the LL mapping, i.e., the subvariety of polynomials having at least two coinciding roots. On the Hurwitz space, the LL mapping is a local covering, and the image of the boundary $\partial\mathcal{H}_{g;k_1,\dots,k_n}$ is of codimension at least one in the target space. Indeed, for any element of $\partial\mathcal{H}_{g;k_1,\dots,k_n}$ the underlying curve is singular, and on a singular curve at least two critical values of a meromorphic function coincide, i.e., $\mathcal{LL}(\partial\mathcal{H}_{g;k_1,\dots,k_n}) \subset \Sigma$. Hence, in order to compute the multiplicity of LL it suffices to compute the number of preimages of a generic point in $\mathbb{C}^{\mu-1}$ or, what is the same, the number of $\mathbb{C}^*$-orbits in the preimage of a $\mathbb{C}^*$-orbit.

Define a *standard hypersurface* in the space $\mathbb{C}^{\mu-1}$ of polynomials as the image of a hyperplane in the space of roots with zero sum under the Vieta mapping $V : \mathbb{C}^{\mu-1} \to \mathbb{C}^{\mu-1}$ taking the set of roots to the set of coefficients. A generic orbit of the $\mathbb{C}^*$-action in the space of coefficients is an irreducible component in the intersection of $\mu - 2$ standard hypersurfaces in general position. Therefore, in order to compute the multiplicity of the LL mapping it is sufficient to compute the selfintersection index of the preimage of a standard hypersurface under LL.

The preimage of a standard hypersurface under the Vieta mapping consists of $\mu!$ hyperplanes, hence the preimage of a standard hypersurface under the LL mapping corresponds to $\mu! c_1(\mathcal{O}(1))$. The intersection index of $\mu-2$ such preimages is precisely the degree of

$$(\mu!)^{\mu-2}\big(c_1(\mathcal{O}(1))^{\mu-2} \cap [P\mathcal{H}_{g;k_1,\dots,k_n}]\big),$$

or, which is the same, the value of $s_{top}(\mathcal{H}_{g;k_1,\dots,k_n})$ on $\overline{\mathcal{M}}_{g;n}$ times $(\mu!)^{\mu-2}$.

Now consider the intersection of preimages of $\mu - 2$ standard hypersurfaces under the LL mapping. Evidently, the number of irreducible components in this intersection divided by the multiplicity of the LL mapping equals the number of intersections of preimages of the same standard hypersurfaces under the Vieta mapping divided by the multiplicity of the Vieta mapping. However, in the space of roots the preimages of standard hypersurfaces are just the sets of hyperplanes, so in this space the intersection contains $(\mu!)^{\mu-2}$ irreducible components. Recalling that the multiplicity of the Vieta mapping is $\mu!$ we obtain the required expression. □

By the definition of the LL mapping, $\mathcal{LL}^{-1}(\mathbb{C}^{\mu-1} \setminus \Sigma) \subset \mathcal{H}_{g;k_1,\dots,k_n}$. Denote by $\mathcal{Z}' \subset \mathcal{Z}$ the union of all irreducible components of $\mathcal{Z}$ that are not contained in $\mathcal{H}_{g;k_1,\dots,k_n}$. We have $\mathcal{Z} = \mathcal{H}_{g;k_1,\dots,k_n} \cup \mathcal{Z}'$ and $\mathcal{LL}(\mathcal{Z}') \subset \Sigma$.

Lemma 5.4. $c_1(\mathcal{O}(1))^{\mu-2} \cap [P\mathcal{Z}'] = 0$.

Proof. The inverse image $\mathcal{LL}^{-1}(0)$ is the zero section of $\mathcal{Z}$ (all twisted Laurent coefficients equal 0). Therefore, the projectivized LL mapping $P\mathcal{LL}$ is well-defined and it maps $P\mathcal{Z}$ to $P\mathbb{C}^{\mu-1}$ and $\mathcal{O}_{P\mathcal{Z}}(1) = P\mathcal{LL}^*\mathcal{O}_{P\mathbb{C}^{\mu-1}}(1)$. In particular, $\mathcal{O}_{P\mathcal{Z}'}(1) = (P\mathcal{LL}|_{P\mathcal{Z}'})^*\mathcal{O}_{P\Sigma}(1)$. But, since $\dim\ P\Sigma = \mu - 3$, the bundle $\mathcal{O}(1)$ over $P\mathcal{Z}'$ is induced from a line bundle on a variety of dimension $\mu - 3$, and the lemma is proved. □

Lemma 5.5.

$$\int_{\overline{\mathcal{M}}_{g;n}} s(\mathcal{H}_{g;k_1,\dots,k_n}) = \int_{\overline{\mathcal{M}}_{g;n}} c\big(\Lambda^\vee_{g;n}\big)s(\mathcal{P}).$$

Proof. Recall that $P\mathcal{Z} \subset P\mathcal{P}$ is the zero locus of the section $\sigma_R : P\mathcal{P} \to \mathcal{O}(1) \otimes \pi^*(\Lambda^\vee_{g;n})$ of the bundle $\mathcal{O}(1) \otimes \pi^*(\Lambda^\vee_{g;n})$ over $P\mathcal{P}$. We start with constructing a $(\mu - 2)$-cycle $D \subset P\mathcal{Z}$, $[D] \in A_{\mu-2}P\mathcal{Z}$ representing the localized top Chern class of the bundle $\mathcal{O}(1) \otimes \pi^*(\Lambda^\vee_{g;n})$ over $P\mathcal{P}$, $[D] = c_g(\mathcal{O}(1) \otimes \pi^*(\Lambda^\vee_{g;n})) \cap [P\mathcal{P}]$.

Such a construction works for an arbitrary vector bundle $E \to X$ over a pure dimensional scheme X and a section $\eta : X \to E$ of this bundle (see [Fu, Sects. 14.1, 6.1]). Let e denote the rank of E, and let $Z(\eta) \subset X$ be the zero locus of η. The normal cone N to $Z(\eta)$ in X is naturally embedded, as a subcone, in the total space of the vector bundle E restricted to $Z(\eta)$ in the following way. A tangent vector $\xi \in T_zX$ to X at a point $z \in Z(\eta)$ is taken to the tangent vector $d\eta(\xi) \in T_{\eta(z)}E$ treated as an element of the fiber E_z of E at z, which is identified naturally with the quotient space $E_z = T_{\eta(z)}E/T_zX$. The normal cone to each irreducible component of $Z(\eta)$ in X has the same dimension as X itself, and $[N]$ is a cycle of codimension e in $A_*E|_{Z(\eta)}$. Proposition 14.1 from [Fu] states that the image of N under the isomorphism $\tau : A_*E \cong A_{*-e}X$ is a codimension e cycle in $A_{*-e}X$ representing the localized top Chern class of the bundle E on X.

In particular, each irreducible component of $Z(\eta)$ of codimension e in X enters the class $[D]$ with some positive multiplicity, which is determined by the behavior of the section η along this component.

Applying the construction above to the vector bundle $\mathcal{O}(1) \otimes \pi^*(\Lambda_{g;n}^\vee)$ over $P\mathcal{P}$ and the section σ_R we obtain a cycle $[D] \in A_{\mu-2}P\mathcal{P}$ such that

$$\begin{aligned}[D] &= c_g\big(\mathcal{O}(1) \otimes \pi^*\big(\Lambda_{g;n}^\vee\big)\big) \cap [P\mathcal{P}] \\ &= \sum_i c_1(\mathcal{O}(1))^i c_{g-i}\big(\pi^*\big(\Lambda_{g;n}^\vee\big)\big) \cap [P\mathcal{P}].\end{aligned}$$

Therefore, one has

$$\begin{aligned}&\pi_*\Big(\sum_j c_1(\mathcal{O}(1))^j \cap [D]\Big) \\ &\quad = \pi_*\Big(\sum_j c_1(\mathcal{O}(1))^j \sum_i c_1(\mathcal{O}(1))^i c_{g-i}\big(\pi^*\big(\Lambda_{g;n}^\vee\big)\big) \cap [P\mathcal{P}]\Big) \\ &\quad = \pi_*\Big(\sum_{i,j} c_1(\mathcal{O}(1))^{i+j} c_{g-i}\big(\pi^*\big(\Lambda_{g;n}^\vee\big)\big) \cap [P\mathcal{P}]\Big) \\ &\quad = \pi_*\Big(\sum_i c_i\big(\pi^*\big(\Lambda_{g;n}^\vee\big)\big) \sum_{l=i+j} c_1(\mathcal{O}(1))^l \cap [P\mathcal{Z}]\Big).\end{aligned}$$

Hence $s(D) = c(\Lambda_{g;n}^\vee)s(\mathcal{P})$; in particular,

$$\int_{\overline{\mathcal{M}}_{g;n}} s(D) = \int_{\overline{\mathcal{M}}_{g;n}} c\big(\Lambda_{g;n}^\vee\big)s(\mathcal{P}).$$

The construction above provides a representative D of the class $[D]$ that is contained in $P\mathcal{Z}$. We set $[D_{\mathcal{H}}] = [D \cap P\mathcal{H}_{g;k_1,\dots,k_n}]$ and $[D'] = [D \cap P\mathcal{Z}']$ so that $[D] = [D_{\mathcal{H}}] \cup [D']$. Let us prove now that $P\mathcal{H}_{g;k_1,\dots,k_n}$ represents the class $[D_{\mathcal{H}}]$. As we have seen above, this is true up to multiplicity, since the codimension of $P\mathcal{H}_{g;k_1,\dots,k_n}$ in $P\mathcal{P}$ is g. It remains to prove that the section σ_R is transversal to the zero section of the bundle $\mathcal{O}(1) \otimes \pi^*(\Lambda_{g;n}^\vee)$ at a generic point of $P\mathcal{H}_{g;k_1,\dots,k_n}$, and hence the multiplicity in question is precisely 1.

In the Appendix we give a proof of this statement in a more general situation. Here we try to explain it without using sophisticated tools from algebraic geometry.

Fix a generic point in $P\mathcal{H}_{g;k_1,\dots,k_n}$. Such a point is a smooth stable curve $(C^0; x_1^0, \dots, x_n^0)$ endowed with an n-tuple $(p_1^0, \dots, p_n^0)$ of generalized principal parts at the marked points taken up to a common nonzero multiplier and such that:

1) the leading terms of the principal parts do not vanish;
2) the mapping $f^0 : C^0 \to \mathbb{C}\mathbf{P}^1$ with given principal parts has distinct critical values;
3) the principal parts p_i^0 satisfy equations (13) for any holomorphic 1-form ω on C^0.

For the sake of brevity, below we denote the chosen point simply by (C^0, f^0).

The tangent space to the total space of the bundle $\mathcal{O}(1) \otimes \pi^*(\Lambda^\vee_{g;n})$ over $P\mathcal{P}$ at (C^0, f^0) splits naturally into the direct sum of the horizontal tangent space along the base and the vertical tangent space along the fiber of the bundle. The vertical tangent space is naturally identified with the space $\Lambda^\vee(C^0)$ of linear functionals on the space of holomorphic 1-forms on C^0. The section σ_R is transversal to the zero section at the point (C^0, f^0) if and only if the projection of the image of $d\sigma_R$ at this point to the vertical tangent space is a surjection. We denote the composition of $d\sigma_R$ with the projection to the vertical tangent space by $d\tilde{\sigma}_R : T_{(C^0,f^0)}P\mathcal{P} \to \Lambda^\vee(C^0)$.

Below, we distinguish between two cases. In each case, we are going to construct a subspace L of the tangent space $T_{(C^0,f^0)}P\mathcal{P}$ such that the restriction of $d\tilde{\sigma}_R$ onto L is of rank g.

1. $k_1+\cdots+k_n > g$. We choose for L the tangent space to the fiber of π; that is, $L = T_{(C^0,f^0)}\pi^{-1}(C^0; x_1^0, \dots, x_n^0)$. Then the restriction of $d\tilde{\sigma}_R$ onto L is defined by the $g \times K$ Brill–Noether matrix (see [ACGH, Chap. 4]). Since in this case C^0 is a generic curve and f^0 is a generic function, the rank of the Brill–Noether matrix equals g.

2. $k_1 + \cdots + k_n \leqslant g$. This situation can happen only if $g \geqslant 2$ since there are no meromorphic functions on elliptic curves with a single pole of order 1.

We start with associating to each noncritical unmarked point $y \in C^0$ the germ of a holomorphic curve in $P\mathcal{P}$ passing through (C^0, f^0). For each such point $y \in C^0$, the function f^0 determines a coordinate $z_y = f^0 - f^0(y)$ in a neighborhood of y.

Fix the complex structure of C^0 outside a small disc $U_{2r} \subset C^0$ of radius $2r \leqslant 2$ (in the coordinate z_y) centered at y and not containing any of the critical points of f^0. Denote by $B \subset \mathbb{C}^1$ the standard unit disc with the coordinate w and let $W_\delta \subset B$ be the varying annulus $|w| > |\delta|^{1/2}$. Map W_δ to U_{2r} by setting $z_y = r\left(w + \frac{\delta}{w}\right)$. This amounts to identifying the annulus W_δ with its image obtaining thus a new complex curve, which we denote by C^δ. The tangent vector to the family C^δ of deformations of the complex structure thus constructed is called the *first-order Schiffer variation* centered at the point y, see [HMo, Chap. 3B]. It can be constructed starting from an arbitrary local coordinate z_y, and it does not depend, up to a constant factor, on the chosen coordinate.

Observe that the Schiffer variation centered at y preserves the curve C^0 outside U_{2r}. Let us fix the positions of the marked points, the corresponding local coordinates in their neighborhoods, and the Laurent polynomials defining principal parts with respect to these local coordinates. Then the first-order Schiffer variation lifts to a tangent vector $\tau_y \in T_{(C^0, f^0)} P\mathcal{P}$.

Let us prove that the mapping $d\tilde{\sigma}_R$ takes the vector τ_y to the following functional on the space of holomorphic 1-forms on C^0 closely related to the canonical mapping:

$$\omega \mapsto \mathrm{Res}_y \frac{\omega}{z_y}.$$

Indeed, let us fix an arbitrary holomorphic connection on the bundle $\Lambda_{g;n}$ restricted to C^δ, say the one preserving the integrals of holomorphic 1-forms over a-cycles. Then for each holomorphic 1-form ω^0 on C^0 we have a well-defined holomorphic family ω^δ of holomorphic 1-forms over C^δ.

The 1-form ω^δ written in the coordinate w looks like

$$\omega^\delta = \big(c_0 + c_1 w + c_2 w^2 + \dots\big) dw + \delta\eta + o(\delta),$$

where η is a holomorphic 1-form in the unit disc. Rewriting it in the coordinate z_y in the image of W_δ we get

$$\omega^\delta = \left(\frac{c_0}{r} + \frac{c_1}{r^2} z_y + \frac{c_2}{r^3} z_y^2 + \dots\right) dz_y + \frac{\delta c_0 r}{z_y^2} dz_y + \delta\bar{\eta} + o(\delta),$$

where $\bar{\eta}$ is a holomorphic 1-form in U_{2r}. Therefore,

$$\omega^0 = \left(\frac{c_0}{r} + \frac{c_1}{r^2} z_y + \frac{c_2}{r^3} z_y^2 + \dots\right) dz_y$$

and $c_0 = r\mathrm{Res}_y \frac{\omega^0}{z_y}$. Finally, the mapping $d\tilde{\sigma}_R$ takes τ_y to

$$\lim_{\delta\to 0} \frac{1}{\delta} \sum_i \mathrm{Res}_{x_i} p_i \omega^\delta = -\lim_{\delta\to 0} \frac{1}{\delta} \int_{\partial U_{2r}} f^0 \omega^\delta = -c_0 r,$$

and the required claim follows.

Choosing points $y_1, \dots, y_g \in C^0$ in general position we obtain g vectors $\tau_{y_1}, \dots, \tau_{y_g}$ whose images under the mapping $d\tilde{\sigma}_R$ are linearly independent, since the images of g points on C^0 in general position under the canonical mapping span a $(g-1)$-dimensional subspace.

Hence

$$c_1^{\mu-2}(\mathcal{O}(1)) \cap [D] = c_1^{\mu-2}(\mathcal{O}(1)) \cap [P\mathcal{H}_{g;k_1,\dots,k_n}] + c_1^{\mu-2}(\mathcal{O}(1)) \cap [D'].$$

By Lemma 5.4, $c_1(\mathcal{O}(1))^{\mu-2} \cap [D'] = 0$, and hence

$$\pi_*(c_1(\mathcal{O}(1))^{\mu-2} \cap [P\mathcal{H}_{g;k_1,\dots,k_n}]) = \pi_*\big(c_1(\mathcal{O}(1))^{\mu-2} \cap [D]\big),$$

and the result follows. □

Combining the assertions of Lemmas 5.2, 5.3, and 5.5, we obtain the proof of the main theorem.

Appendix: Local study of the LL map

Our setup will be the following. We have a smooth and proper map $f : C \to S$ of relative dimension 1. Even though many of the results will be true generally, we assume that $\mathbb{Q}$ is contained in $\Gamma(S, \mathcal{O}_S)$. We fix a genus g and a degree d and consider the moduli stack $\mathcal{M}_{g;d}(C/S)$ of families of semistable curves $D \xrightarrow{k} T$ over an S-scheme T of genus g together with an S-map $D \xrightarrow{h} C$ fiberwise finite and of degree d. Our aim is first to define a stratification by subschemes of $\mathcal{M}_{g;d}(C/S)$ on whose strata the local structure of the map $D \to S$ is fixed. For this we will need the following lemma.

Definition–Lemma A.1. *Let $\pi : X \to S$ be a finite flat map, where S is a scheme over* Spec $\mathbb{Q}$.

(i) *The relative trace map* $\mathrm{Tr} : \pi_*\mathcal{O}_X \to \mathcal{O}_S$ *induces a trace pairing* $\pi_*\mathcal{O}_X \bigotimes_{\mathcal{O}_S} \pi_*\mathcal{O}_X \to \mathcal{O}_S$ *by* $(f, g) \mapsto \mathrm{Tr}(fg)$. *If the map* $\pi_*\mathcal{O}_X \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\pi_*\mathcal{O}_X, \mathcal{O}_S)$ *induced by this trace paring has locally constant rank* (*in the schematic sense*), *then there exists a* (*unique*) *closed subscheme $X^{fred} \hookrightarrow X$ such that for any geometric point $\bar{s} \to S$, $X_{\bar{s}}^{fred}$ is the reduced subscheme of $X_{\bar{s}}$. Furthermore, X^{fred} is finite étale over S. We will call this subscheme the* fiberwise reduced subscheme.

(ii) *If S is reduced, then $\pi_*\mathcal{O}_X \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\pi_*\mathcal{O}_X, \mathcal{O}_S)$ is of locally constant rank if and only if the function that to a point of S associates the number of geometric points above it on X is locally constant.*

Proof. We begin by some (well-known) observations in the punctual case. Hence consider a finite dimensional commutative algebra R over an algebraically closed field k of characteristic zero. If m is the nilradical of R then as any $r \in m$ is nilpotent, the trace is zero on m and as m is an ideal, m will lie in the radical of the trace pairing. We now want to show that, as we are in characteristic zero, m equals the radical of the trace pairing. We can write R as a product of local rings $R = \prod_i R_i$. As the product of elements from different factors is zero, this decomposition is an orthogonal decomposition for the trace pairing and in particular the radical of the trace pairing is the direct sum of the radicals of the trace pairings on each factor. As the same is true of the nilradical we are reduced to the case when R is local. In that case m is of codimension 1 in R, so to prove that m and the radical are equal it is enough to show that the radical does not equal all of R, i.e., that the trace map is non-zero. However, $\mathrm{Tr}(1) = \dim_k R$ which is non-zero as an element in k.

Note further that, sticking to the punctual case, the radical of the trace pairing is precisely the kernel of the induced map $R \to \mathrm{Hom}_k(R, k)$. This means that our assumption is equivalent to $\pi_*\mathcal{O}_X \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\pi_*\mathcal{O}_X, \mathcal{O}_S)$ having constant rank and therefore its kernel $\mathcal{I}$ is a sub-vector bundle. This kernel is clearly also an ideal, and the closed subscheme defined by it is at each fibre the reduced subscheme of the fibre as the kernel is, as we have seen, the nilradical of the fibre. As $\mathcal{I}$ is a sub-vector bundle of $\pi_*\mathcal{O}_X$ this subscheme is flat and being fibrewise reduced it is finite étale.

Finally, if the base is reduced the kernel of $\pi_*\mathcal{O}_X \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\pi_*\mathcal{O}_X, \mathcal{O}_S)$ is a sub-vector bundle of constant rank precisely when this is true for each fibre. By what we have seen this is true precisely when the cardinality of the fibres is constant. This proves the second part. $\square$

We want to apply this lemma to the ramification and branch loci, but before we can do that we need to define these loci in the presence of singularities of the fibers of $D \to S$. To do this we first note that outside of the singular locus of $D \to S$ we have a map $h^*\Omega^1_{C/S} \to \Omega^1_{D/S}$, and by tensoring by the inverse of $h^*\Omega^1_{C/S}$ we get a global section of $\omega_{D/C}$. Now this sheaf is identified, by duality, with $\underline{\mathrm{Hom}}_{\mathcal{O}_C}(h_*\mathcal{O}_D, \mathcal{O}_C)$ and we claim that under this identification the section corresponds to the trace map $h_*\mathcal{O}_D \to \mathcal{O}_C$. Indeed, this can be verified on the open dense subset where h is étale and then, being local in the étale topology, we may assume that D is a disjoint union of copies of C in which case we reduce to the case of one copy by the fact that both sides are additive over disjoint unions. This latter case is obvious.

In our case we can now go backwards; as h is finite and f smooth, h is flat and it is assumed to be finite, so we may consider its trace map as a global section of $\omega_{D/C}$, the relative dualizing sheaf, and then tensoring with $h^*\omega_{C/S}$ we get a map $h^*\omega_{C/S} \to \omega_{D/S}$. Now these two sheaves are S-flat and the map, which commutes with base change, is fiberwise injective, and hence the map is injective with an S-flat cokernel. We tensor the map with the inverse of $\omega_{D/S}$ to define a finite and S-flat subscheme R of D by the exact sequence

$$0 \to \omega^{-1}_{D/C} \to \mathcal{O}_D \to \mathcal{O}_R \to 0.$$

This is by definition the *ramification locus* of h. We then define the *branch locus*, B, as $\mathrm{Div}(h_*\mathcal{O}_R)$ (cf. [Mu, Chap. 5, Sect. 3]). Then the branch locus also commutes with base change and is finite and flat over S.

Remark A.2. This definition is a special case of [FnP], the definition given here is somewhat more explicit. A consequence of that is that it is clear, by a local calculation, that it extends to the case when the base also is allowed to have nodes but the map then is required to have the same local degree on both branches when a node maps to a node (which is the condition of [HMu]).

A stratification by locally closed subschemes of $\mathcal{M}_{g;d}(C/S)$ will of course induce one on any family of maps. It will however be convenient to directly define this stratification for any family. Hence we consider a smooth proper curve $f : C \to S$ over a base S over $\mathrm{Spec}\,\mathbb{Q}$, a semi-stable curve $k : D \to S$ and a finite S-map $h : D \to C$.

We now stratify S by closed subschemes by the condition that the rank of the maps $k_*\mathcal{O}_R \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(k_*\mathcal{O}_R, \mathcal{O}_S)$ and $f_*\mathcal{O}_B \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(f_*\mathcal{O}_B, \mathcal{O}_S)$

induced by the trace maps is less than or equal to some integers. Looking at a locally closed stratum T that is the complement in one of the closed strata of the strata contained in it, we see that the restriction of R and B to T fulfill the conditions of Lemma A.1, and hence we may define R^{fred} and B^{fred}. Clearly the map $R \to B$ induces a map $R^{fred} \to B^{fred}$. We associate to each point of R^{fred} the length of R at that point. This is a locally constant function on R^{fred}, and as $R^{fred} \to T$ is finite étale, T can be written as a disjoint union of open subschemes over each of which this length is constant on each irreducible component of S^{fred}. We may similarly refine T so that the same is true for B^{fred} and the length of B. It is clear that over such a component the local degrees of h are the same. We call these components the *equisingular strata* of S. On each such stratum T we have associated a relative effective Cartier divisor B^{fred} of $C \to S$ of fixed degree. This map $T \to \mathrm{Div}(C/S)$ is the LL map.

Proposition A.3. *Let T be an equisingular stratum of $\mathcal{M}_{g;d}(C/S)$. Then the LL map $T \to \mathrm{Div}(C/S)$ is étale.*

Proof. To prove that it is enough to verify the infinitesimal lifting property, so we may assume that $S = \mathrm{Spec}\, A$ where A is a local artinian ring with algebraically closed residue field, A' is another artinian local ring, that we have a surjective map $A \to A'$, a relative effective Cartier divisor B of C/S, étale over S, and a lifting of it to a map $S' := \mathrm{Spec}\, A' \to T$ over A', and we want to show that there is a unique lifting over A.

Rearranging our data we have a semistable curve C over S, a relative effective Cartier divisor B of C/S, étale over S, a curve $D \to S'$ and an equisingular S'-map $f : D \to C_{|S'}$ whose branch locus is $B_{|S'}$. We then want to show that there is a unique equisingular extension of f to S whose branch locus is B. Outside of the support of B the map must be an étale cover, and the existence and uniqueness is clear. Hence we may work locally in the étale topology around the points of $\mathrm{Supp}\, B$. Hence we may assume that $C = \mathrm{Spec}\, A\{t\}$, where $A\{t\}$ is the strict henselisation of $A[t]$ at $t = 0$, and that B is given by $t = 0$. Then the fiberwise reduced subscheme of the ramification locus of f is a disjoint union of copies of $C_{|S'}$ and we may similarly restrict ourselves to one of these components, so that we may assume that also D is strict henselian and with an S'-section. Having arrived at this point we may also complete C along the branch locus and hence assume that $C = \mathrm{Spec}\, A[[t]]$ and then also D will be the spectrum of a complete local ring T.

Assume first that D is smooth over S'. We begin by showing that the lifting, if it exists, is unique. For this we assume that we can choose a generator s' for the defining ideal of the section of $D \to S'$ with the property that $s'^e = t$, where e is the (local) degree of f. We then make an arbitrary lift of s' to an element s of the defining ideal of the section of $D \to S$ (a lifting now supposed to exist). We then can write t as a power series $\sum_{i=0}^{\infty} a_i s^i$. As the deformation is supposed to be equisingular, we have that dt/ds is a unit times s^{e-1}. This forces $a_i = 0$ for $0 < i < e$, so that t is of the form $a_0 + s^e g$, where g is a unit, as it is so in $A'[[t]]$. Hence after possibly changing s (but keeping its image in $A'[[t]]$ the same) we may assume that $t = a_0 + s^e$. However, the fact that the reduced branch locus should be given by t forces $a_0 = 0$, which gives uniqueness. Existence is now trivial, as we may assume that $D_{|S'}$ has the standard form given by $s^e = t$.

The case when D is singular is very similar and left to the reader. □

For any finite set M of finite sets of integers $k_i \geqslant 2$, $M = \{\{k_1, k_2, \ldots, k_{n_1}\}, \ldots, \{\ldots, k_{n_m}\}\}$, such that $2g - 2 = (2g(C) - 2)d + \sum_i (k_i - 1)$ we define $\mathcal{M}_{g;M}(C/S)$ to be the equisingular stratum of $\mathcal{M}_{g;d}(C/S)$, with local degrees of ramification $\geqslant 2$ over the points of the branch locus given by M. Specializing even further, we let $\mathcal{M}'$ be the open subset of $\mathcal{M}_{g;M}(\mathbf{P}^1/\operatorname{Spec}\mathbb{Q})$ for which D is smooth, where M has the form $\{\{k_1, \ldots, k_n\}, \{2\}, \{2\}, \ldots, \{2\}\}$. By Proposition A.3, $\mathcal{M}'$ is smooth and so is then clearly also $\mathcal{M}$, the closed substack for which the point of the branch locus with ramification behavior $\{k_1, \ldots, k_n\}$ equals ∞. We now let $\mathcal{M}^f$ be the stack over $\mathcal{M}$ where the D has been provided with n sections such that the fiberwise reduced ramification locus over ∞ is the disjoint union of these sections together with a jet of a local coordinate up to order k_i at the ith section. The map $\mathcal{M}^f \to \mathcal{M}$ is evidently smooth and thus so is $\mathcal{M}$.

Suppose we have the following data: A family $k : D \to S$ of smooth proper connected curves, n disjoint sections s_i, a jet of order k_i of a local coordinate at the s_i and a polar part p_i with respect to the coordinate of order exactly k_i at the s_i. We then get a map $k_*\omega_{D/S} \to \mathcal{O}_D$ given by $\omega \mapsto \sum_i \operatorname{Res}_{s_i} p_i\omega$ and we assume that it is zero. This condition may also be interpreted as follows. Consider the line bundle $\mathcal{O}\big(\sum_i k_i s_i\big)$ on D. We have the natural section given by the fact that the divisor is effective, giving us a short exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}\bigg(\sum_i k_i s_i\bigg) \to \mathcal{D} \to 0,$$

where $\mathcal{D}$ is the sheaf of polar parts of the appropriate orders at the s_i. Pushing down by k we get an exact sequence

$$0 \to \mathcal{O}_S \to k_*\mathcal{O}\bigg(\sum_i k_i s_i\bigg) \to k_*\mathcal{D} \to R^1k_*\mathcal{O}_D.$$

The p_i give a section of $k_*\mathcal{D}$, and the map $k_*\omega_{D/S} \to \mathcal{O}_D$ is the image of this section in $R^1k_*\mathcal{O}_D$. The fact that it is zero thus gives a vector bundle $\mathcal{E}$ which is an extension of $\mathcal{O}_S$ by $\mathcal{O}_S$ mapping to $k_*\mathcal{O}\big(\sum_i k_i s_i\big)$. As a further datum we require a splitting of $\mathcal{E}$ making it a trivial bundle. The two sections of $k_*\mathcal{O}(\sum_i k_i s_i)$ we get in this way have no common base points, and thus give an S-map $D \to \mathbf{P}^1_S$. We further require that outside of ∞ this map only have simple singularities (in the sense of S being equal to the appropriate equisingular stratum). We will let $\mathcal{N}$ denote the moduli stack of such data.

Proposition A.4. *The two stacks $\mathcal{M}^f$ and $\mathcal{N}$ are naturally isomorphic. In particular, $\mathcal{N}$ is smooth.*

Proof. By looking at the polar parts of the rational function $1/x$ on $\mathbf{P}^1$ we get a map $\mathcal{M}^f \to \mathcal{N}$, and by looking at the map $D \to \mathbf{P}^1_{\mathcal{N}}$ given by the two sections of $k_*\mathcal{O}\left(\sum_i k_i s_i\right)$ we get a map for which all of $\mathcal{N}$ is the equisingular stratum given by $\{\{k_1, \ldots, k_n\}, \{2\}, \ldots, \{2\}\}$, as outside of ∞ this is built into the definition of $\mathcal{N}$, and at ∞ this follows from the specification of the polar parts of $1/x$. This gives a map $\mathcal{N} \to \mathcal{M}^f$, and it is clear that these two maps are each other's inverses. □

Observe that the closure of $\mathcal{N}$ represents the class $[D_{\mathcal{H}}]$ defined in Sect. 5.3, while $\mathcal{M}^f$ corresponds to $P\mathcal{H}_{g;k_1,\ldots,k_n}$. Therefore, Proposition A.4 implies that $[P\mathcal{H}_{g;k_1,\ldots,k_n}] = [D_{\mathcal{H}}]$, as required in the proof of Lemma 5.5.

References

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris, Geometry of Algebraic Curves, vol. I, Springer, 1984

[A1] V.I. Arnold, Topological classification of complex trigonometric polynomials and combinatorics of graphs with an identical number of vertices and edges, Funct. Anal. Appl. **30** (1996), 1–14

[A2] V.I. Arnold, Critical points of functions and the classification of caustics, Usp. Math. Nauk **29** (1974), 243–244

[CT] M. Crescimanno, W. Taylor, Large N phases of chiral QCD_2, Nuclear Phys. B **437** (1995), 3–24

[D] B.A. Dubrovin, Geometry of 2D topological field theories, Integrable Systems and Quantum Groups, Lecture Notes in Math. vol. 1620, 1996, pp. 120–348

[ELSV] T. Ekedahl, S.K. Lando, M. Shapiro, A. Vainshtein, On Hurwitz numbers and Hodge integrals, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 1175–1180, math.AG/9902104

[FP1] C. Faber, R. Pandharipande, Hodge integrals and Gromov–Witten theory, Invent. math. **139** (2000), 173–199, math.AG/9810173

[FP2] C. Faber, R. Pandharipande, Hodge integrals, partition matrices, and the λ_g-conjecture, math.AG/9908052

[FnP] B. Fantechi, R. Pandharipande, Stable maps and branch divisors, math.AG/9905104

[Fu] W. Fulton, Intersection Theory, Springer, 1998

[GL] V. Goryunov, S.K. Lando, On enumeration of meromorphic functions on the line, The Arnoldfest, Fields Inst. Commun., vol. 24, 1999, pp. 209–223, AMS, Providence, RI

[GJ1] I.P. Goulden, D.M. Jackson, Transitive factorization into transpositions and holomorphic mappings on the sphere, Proc. Amer. Math. Soc. **125** (1997), 51–60

[GJ2] I.P. Goulden, D.M. Jackson, The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group, Europ. J. Combin. **13** (1992), 357–365

[GJ3] I.P. Goulden, D.M. Jackson, A proof of a conjecture for the number of ramified coverings of the sphere by the torus, J. Combin. Theory Ser. A **88** (1999), 246–258, math.CO/9902009

[GJVn] I.P. Goulden, D.M. Jackson, A. Vainshtein, The number of ramified coverings of the sphere by the torus and surfaces of higher genera, Ann. Combin. **4** (2000), 27–46, math.CO/9902125

[GJVk] I.P. Goulden, D.M. Jackson, R. Vakil, The Gromow–Witten potential of a point, Hurwitz numbers, and Hodge integrals, Proc. London Math. Soc., to appear, math.AG/9910004
[GP] T. Graber, R. Pandharipande, Localization of virtual classes, Invent. math. **135** (1999), 487–518, math.AG/9708001
[GV] T. Graber, R. Vakil, Hodge integrals and Hurwitz numbers via virtual localization, math.AG/0003028
[HMo] J. Harris, I. Morrison, Moduli of Curves, Springer, 1998
[HMu] J. Harris, D. Mumford, On the Kodaira dimension of the moduli space of curves, Invent. math. **67** (1982), 23–88
[H] A. Hurwitz, Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten, Math. Ann. **39** (1891), 1–60
[K] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Comm. Math. Phys. **147** (1992), 1–23
[KM] M. Kontsevich, Yu.I. Manin, Gromov–Witten classes, quantum cohomology, and enumerative geometry, Comm. Math. Phys. **164** (1994), 525–562
[LZ] S.K. Lando, D. Zvonkine, On multiplicities of the Lyashko–Looijenga mapping on strata of the discriminant, Funct. Anal. Appl. **33** (1999), 178–188
[L] E. Looijenga, The complement of the bifurcation variety of a simple singularity, Invent. math. **23** (1974), 105–116
[M1] A.D. Mednykh, Nonequivalent coverings of Riemann surfaces with a prescribed ramification type, Sibirsk. Math. Zh. **25** (1984), 120–142
[M2] A.D. Mednykh, Branched coverings of Riemann surfaces whose branch orders coincide with the multiplicity, Comm. in Algebra **18** (1990), 1517–1533
[Mu] D. Mumford, Geometric Invariant Theory, Springer, 1965
[N] S. Natanzon, Topology of 2-dimensional coverings and meromorphic functions on real and complex algebraic curves, Selecta Math. Sovietica **12** (1993), 251–291
[S] V. Strehl, Minimal transitive products of transpositions – the reconstruction of a proof by A. Hurwitz, Sem. Lothar. Combinat. **37** (1996), Art. B37c, 12pp.
[SSV] B. Shapiro, M. Shapiro, A. Vainshtein, Ramified coverings of S^2 with one degenerate branching point and enumeration of edge-oriented graphs, AMS, Advances in Mathematical Sciences, vol. 34, 1997, pp. 219–228, Providence, RI
[V1] R. Vakil, Recursions for characteristic numbers of genus one plane curves. Ark. Math. **39** (2001), 157–180, math.AG/9803007
[V2] R. Vakil, Recursions, formulas, and graphic-theoretic interpretations of ramified coverings of the sphere by surfaces of genus 0 and 1, Trans. AMS, to appear, math.AG/9812105
[W] E. Witten, Two dimensional gravity and intersection theory on moduli space, Surveys in Differential Geometry, Lehigh Univ., 1991, pp. 243–310
[Z] D. Zvonkine, Multiplicities of the Lyashko–Looijenga map on its strata, C. R. Acad. Sci. Paris Ser. I Math. **324** (1997), 1349–1353

Notes added in proof:

[1] M. Kazaryan pointed out recently that the proof of Hurwitz's formula in [GL] contains a mistake. The approach used in the present paper provides a natural way to correct this mistake.

[2] Recently A. Okounov and R. Pandharipande used the main theorem of the present paper to obtain a new proof of Witten's conjecture.