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Valuation-like maps and the congruence subgroup property

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Abstract. Let D be a finite dimensional division algebra and N a subgroup of finite index in $D^\times$. A valuation-like map on N is a homomorphism $\varphi\colon N \to \Gamma$ from N to a (not necessarily abelian) linearly ordered group Γ satisfying $N_{<-\alpha} + 1 \subseteq N_{<-\alpha}$ for some nonnegative $\alpha \in \Gamma$ such that $N_{<-\alpha} \neq \emptyset$, where $N_{<-\alpha} = \{x \in N \mid \varphi(x) < -\alpha\}$. We show that this implies the existence of a nontrivial valuation v of D with respect to which N is (v-adically) open. We then show that if N is normal in $D^\times$ and the diameter of the commuting graph of $D^\times/N$ is ≥ 4, then N admits a valuation-like map. This has various implication; in particular it restricts the structure of finite quotients of $D^\times$. The notion of a valuation-like map is inspired by [27], and in fact is closely related to part (U3) of the U-Hypothesis in [27].

1. Introduction

This paper grew out of the preprint [20] which is entirely due to Rapinchuk and comprises of Sects. 2–5 of this paper and the preprint [29] which is entirely due to Segev and comprises of Sects. 6–7 of this paper. In Sect. 8 we prove "Nonexistence Theorem at Diameter ≥ 4" which is a combination of the results in previous sections. The idea in Sect. 8 to use induction on the

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transcendence degree of the center of the division algebra (over its prime subfield) is due to Rapinchuk.

The referee of [20] suggested that we put the preprints [20] and [29] together to form a joint paper. We decided to accept this suggestion as the combined result yields Theorem 3 (below) which is the best possible result in a sense which will be made precise shortly.

Throughtout this paper D is an infinite division algebra having center K. With the exception of Sects. 6–7, D is assumed to be finite dimensional. We let N be a subgroup of $D^\times$ of finite index. We will consider homomorphisms

$$\text{(i)} \qquad \varphi\colon N \to \Gamma$$

from N to a totally ordered group Γ. The homomorphism φ need not be surjective and Γ need not be commutative. Given $\gamma \in \Gamma$, we let

$$N_{<\gamma} := \{n \in N \mid \varphi(n) < \gamma\} \qquad \gamma \in \Gamma$$

(where the homomorphisn φ is always understood from the context); $N_{>\gamma}$, $N_{\leq\gamma}$, etc. are defined similarly.

Definition. Let D be a finite dimensional division algebra and let $N \subseteq D^\times$ be a subgroup of finite index. A *valuation-like map* on N is a nontrivial homomorphism $\varphi\colon N \to \Gamma$ from N to a totally ordered group Γ such that there exists a nonnegative $\alpha \in \Gamma$ for which

$$\text{(VL)} \qquad N_{<-\alpha} \neq \emptyset \qquad \text{and} \qquad N_{<-\alpha} + 1 \subseteq N_{<-\alpha}$$

where, of course, $N_{<-\alpha} + 1 = \{n + 1 \mid n \in N_{<-\alpha}\}$. We say that α is *a level* of φ.

The notion of a valuation-like map is inspired by the paper [27] where in Sects. 5–6 of that paper, Segev considers the “U-Hypothesis” which implies the existence of a homomorphism φ as in (i) such that

$$\text{(ii)} \qquad N_{>0} + 1 \subseteq N_{\leq 0}.$$

Using the equality $x+1 = x(x^{-1}+1)$, it is immediate that a homomorphism φ satisfying (ii) is a valuation-like map having any nonnegative $\alpha \in \Gamma$ as its level.

It is worth noticing that even in the simplest case $N = D^\times$, $\alpha = 0$, the conditions imposed on a valuation-like map are weaker than those in the definition of a valuation (no information about $\varphi(a+b)$ when $\varphi(a) = \varphi(b)$); general valuation-like maps resemble valuations even less. So, one of the intriguing results of this paper is that *any* valuation-like map on a finite index subgroup gives rise to a valuation, and the subgroup turns out to be open with respect to the topology defined by this valuation.

Theorem 1. *Let D be a finite dimensional division algebra over a finitely generated infinite field K, and $-1 \in N \subseteq D^\times$ be a subgroup of finite index. Then N is open with respect to a nontrivial valuation v of D if and only if N admits a valuation-like map.*

To keep the introduction less technical, we refer the reader to the beginning of Sect. 2 for the definition of a valuation and the notion of openness. We mention that in fact we prove a more precise statement than Theorem 1 (see Theorem 5.1), but Theorem 1 conveys the essence of the result.

Thus, once we obtain a valuation-like map on N, we get a nontrivial valuation of D with respect to which N is open. Questions dealing with openness of subgroups have been considered for quite some time for algebraic groups over global fields in the form of the congruence subgroup problem (cf. [21] for a recent survey), but its analysis for division algebras over arbitrary fields became possible only after the paper [27] and this subsequent paper which sharpens the results in this direction.

As we will see momentarily, having N open with respect to a valuation not only has high esthetic value, but is also practical and useful. We need to recall (from [27]) the following definition. The *commuting graph* $\Delta(H)$ of a finite group H, is the graph whose vertex set is $H \setminus \{1\}$ and whose edges are pairs of commuting elements. The surprising connection between the notion of the commuting graph and the notion of a valuation-like map comes from [27] and is revealed in the following theorem.

Theorem 2. *Let D be an infinite division algebra (not necessarily finite dimensional) over an arbitrary field and $N \subseteq D^\times$ be a normal subgroup of finite index. If the diameter of the commuting graph of $D^\times/N$ is ≥ 4, then N admits a valuation-like map.*

In fact, instead of Theorem 2 we will prove a more precise result (Theorem 6.1) which gives a constructive way of building a valuation-like map using a pair of elements in $D^\times$ whose images are at distance ≥ 4 in the commuting graph of $D^\times/N$. Note that if the diameter of the commuting graph of $D^\times/N$ is ≥ 3, then automatically $-1 \in N$, hence combining Theorem 1 and Theorem 2 we get

Theorem 3. *Let D be a finite dimensional division algebra over a finitely generated field and $N \subseteq D^\times$ be a normal subgroup of finite index. If the commuting graph of the quotient $D^\times/N$ has diameter ≥ 4, then N is open in D with respect to a nontrivial (height one) valuation of D.*

Before discussing some applications of Theorem 3, let us note two things. First, Theorem 3 is the best possible result in the following sense. In Sect. 8 we give an example (see Example 8.4) of a finite dimensional division algebra D and a normal subgroup N of $D^\times$ such that $D^\times/N \cong \Sigma_3 \times \Sigma_3$ (here Σ_3 is the symmetric group on 3 letters), but N is *not* open with respect to any single valuation of D. Since the diameter of the commuting graph of $\Sigma_3 \times \Sigma_3$ is 3, we see that the bound 4 can not be improved in Theorem 3.

Secondly, Theorem 3 gives the existence of nontrivial valuations of D in a variety of situations which is an interesting and important information per se. For example, if $D^\times/N \cong \Sigma_3$, then already D must admit a nontrivial valuation (!), because the commuting graph of Σ_3 is disconnected.

Our next (and last) result which restricts the structure of finite quotients of $D^\times$ is perhaps the most practical one.

Nonexistence Theorem at Diameter ≥ 4. *Let $\mathcal{G}$ be a class of finite groups. Call a member $G \in \mathcal{G}$* minimal *if no proper quotient of G belongs to $\mathcal{G}$ (i.e. if $1 \neq M \lhd G$, then $G/M \notin \mathcal{G}$). Assume that*

(1) *The members of $\mathcal{G}$ are not solvable.*
(2) *If $G \in \mathcal{G}$ and $N \lhd G$ with G/N solvable, then $N \in \mathcal{G}$.*
(3) *If $G \in \mathcal{G}$ and $N \lhd G$ is a solvable normal subgroup of G, then $G/N \in \mathcal{G}$.*
(4) *The commuting graph of minimal members of $\mathcal{G}$ has diameter ≥ 4.*

Then no member of $\mathcal{G}$ is a quotient of the multiplicative group of a finite dimensional division algebra.

Along the proof of "Nonexistence Theorem at Diameter ≥ 4", we use structural information on $D^\times/N$, when N is open with respect to a nontrivial valuation v of D, we pass to the residue division algebra of v and induct on the transcendence degree of the center K of D over its prime subfield (showing that there is no loss of generality in assuming it is finitely generated). Without getting into technical details, we mention that the idea of passing to the residue division algebra comes from [27], where the local ring R constructed in Sects. 7–10 of [27] is in fact a valuation ring of a valuation extending a homomorphism φ as in (i) (v in the notation of 6.9 in [27], see Appendix B) and satisfying (ii). This homomorphism is constructed in Sects. 2–6 of [27]. In Sect. 10 of [27] Segev passes to the residue division algebra of R and obtains a contradiction. As we noted, the idea of using induction on the transcendence degree of K is due to Rapinchuk.

Obviously, "Nonexistence Theorem at Diameter ≥ 4" can be used to restrict the structure of finite quotients of $D^\times$. For example, it was shown in [31] that the commuting graph of a nonabelian finite simple group either has diameter ≥ 5 or has diameter ≥ 4 and is balanced (see Appendix B for the definition of balance) which in view of [27] implies that no finite quotient of the multiplicative group of a finite dimensional division algebra is nonabelian simple. Now we can derive this fact from "Nonexistence Theorem at Diameter ≥ 4" (taking $\mathcal{G}$ to be the class of all nonabelian finite simple groups) and the information needed for this is just that the diameter of the commuting graph of a nonabelian finite simple group is ≥ 4 which is a much easier fact to establish then showing balance.

The fact that no finite quotient of the multiplicative group of a finite dimensional division algebra is nonabelian simple ([31]) completed the proof of the Margulis-Platonov conjecture (MP) for the groups of the form $SL(1, D)$. We refer the reader to Appendix A for a more detailed discussion of (MP).

We now turn to the following conjecture of Segev [28].

Conjecture F.So.Q. (Finite Solvable Quotients). *Finite quotients of the multiplicative group of a finite dimensional division algebra are solvable.*

This conjecture holds when the center of D is a global field (see [31] and Appendix A) and when the degree of D is 3 or 5 (see [24] and [25]). Now, the techniques in this paper bring us close to the resolution of this conjecture. Indeed, observe that "Nonexistence Theorem at Diameter $\geq$ 4" is only "distance one" away from the proof of Conjecture F.So.Q. Namely, the class $\mathcal{G} = \mathcal{NS}$ of all nonsolvable finite groups obviously satisfies conditions (1)–(3) of "Nonexistence Theorem at Diameter $\geq$ 4". As for condition (4), Segev [30] showed that the diameter of any minimal nonsolvable group is $\geq$ 3, and there are many examples where the diameter is precisely 3. Thus, Conjecture F.So.Q. would follow if one could replace the requirement "diameter $\geq$ 4" by "diameter $\geq$ 3" in condition (4). However, the proof of "Nonexistence Theorem at Diameter $\geq$ 4" relies on Theorem 3 which is simply not true under "diameter $\geq$ 3" hypothesis.

This was the status of Conjecture F.So.Q. at the time the manuscript of the present paper was submitted (July, 2000). Over the past several months, however, new progress was achieved in a joint work of the authors with Gary Seitz. Namely, we have been able to show that Theorem 3 can still be proven in the diameter $\geq$ 3 situation if one imposes one additional technical condition on $\Delta(D^\times/N)$. Futhermore, this condition can apparently be verified for all minimal nonsolvable groups, which allows one to extend the "Nonexistence Theorem" to the class $\mathcal{NS}$, proving thereby Conjecture F.So.Q. These results will be presented in our forthcoming paper [22] which we hope to complete soon. On the other hand, it would be interesting to find out what can be proved in the direction of Theorem 3 without any additional assumptions on $\Delta(D^\times/N)$. In this regard, we expect the answer to the following question to be affirmative.

Question: Let D be a finite dimensional division algebra over a finitely generated field and $N \subseteq D^\times$ be a normal subgroup of finite index. Does the fact that the commuting graph of the quotient $D^\times/N$ has diameter ≥ 3 imply that N is open in $D^\times$ with respect to a finite set T of nontrivial valuations of D?

We observe that the affirmative answer to this question would allow one to give an alternative proof of Conjecture F.So.Q. It could also be used to restrict the structure of (solvable) finite quotients of $D^\times$, which may eventually lead to a description of finite groups that can appear as quotients of $D^\times$ (we recall that for finite subgroups of $D^\times$ this was accomplished by Amitsur [1]).

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2. On subgroups open with respect to a valuation

Let D be a finite dimensional division algebra over its center K, $N \subseteq D^\times$ be a finite index subgroup of its multiplicative group. We always assume that K is infinite. For technical reasons, it is convenient to assume that $-1 \in N$. *This assumption* doesn't really result in a loss of generality and *will be kept throughout the paper.* Recall that a valuation of D is a group homomorphism $v : D^\times \to \tilde{\Gamma}$, from $D^\times$ onto a linearly ordered group $\tilde{\Gamma}$ satisfying $v(x + y) \geq \min\{v(x), v(y)\}$, whenever $x + y \neq 0$. The goal of Sects. 2–5 is to establish a criterion for N to be open in $D^\times$ with respect to a nontrivial valuation v of D.

Notation 2.1. Let Γ be a nontrivial linearly ordered group (not necessarily abelian but written additively!), and let $\varphi\colon N \to \Gamma$ be a homomorphism (which will always be clear from the context).

(1) For $\beta \in \Gamma$, we let $\Gamma_{<\beta}$ (resp., $\Gamma_{\leq\beta}$, $\Gamma_{>\beta}$, etc.) denote the set of $\gamma \in \Gamma$ satisfying $\gamma < \beta$ (resp., $\gamma \leq \beta$, $\gamma > \beta$, etc.).
(2) For a subset $M \subseteq N$, $M_{<\beta}$ (resp., $M_{\leq\beta}$, $M_{>\beta}$, etc.) denote the set of $m \in M$ satisfying $\varphi(m) < \beta$ (resp., $\varphi(m) \leq \beta$, $\varphi(m) > \beta$, etc.).
(3) For a subfield L of D, write $N_L := N \cap L$, and $\varphi_L := \varphi\,|_{N_L}\colon N_L \to \Gamma$.

Remark 2.2. We recall the well known fact that given a (surjective) valuation $v\colon D^\times \to \tilde{\Gamma}$, the ordered group $\tilde{\Gamma}$ must be abelian. To see this one shows easily that given $x_1, \ldots, x_k \in D^\times$ such that $v(x_1), \ldots, v(x_k)$ are in distinct cosets of $v(K^\times)$ in $\tilde{\Gamma}$, $x_1, \ldots, x_k$ are linearly independent over K. It follows that $|\tilde{\Gamma} : v(K^\times)| \leq \dim_K(D) < \infty$. Since $v(K^\times)$ is contained in the center of $\tilde{\Gamma}$, by [2], 33.9, p. 169, the commutator group of $\tilde{\Gamma}$ is finite. However $\tilde{\Gamma}$ is torsion free, so $\tilde{\Gamma}$ is abelian.

Suppose now that v is a valuation of D with the value group $\tilde{\Gamma} = v(D^\times)$ and the valuation ring $O_{D,v}$. Associated with any $\delta \in \tilde{\Gamma}$, $\delta \geq 0$, one has a two-sided ideal

$$\mathfrak{m}_{D,v}(\delta) = \{x \in D^\times \mid v(x) > \delta\} \cup \{0\}$$

of $O_{D,v}$. The ideals $\{\mathfrak{m}_{D,v}(\delta)\}$ for all nonnegative $\delta \in \tilde{\Gamma}$ form a fundamental system of open neighborhoods of zero for the natural topology on D associated with the valuation v (sometimes referred to as the v-adic topology), cf. details in [5], Sect. 5, n° 1. This topology makes $D^\times$ into a topological group, and the openness of a subgroup $N \subseteq D^\times$ is equivalent to the fact that there exists a $\delta \in \tilde{\Gamma}$ such that $1 + \mathfrak{m}_{D,v}(\delta) \subseteq N$. Of course, we will write O, $\mathfrak{m}$, etc. instead of $O_{D,v}$, $\mathfrak{m}_{D,v} \ldots$ if this may not lead to a confusion.

Lemma 2.3. *Let $v : D^\times \to \tilde{\Gamma}$ be a valuation of D, and let $N \subseteq D^\times$ be a v-adically open subgroup having finite index m. If $\varphi\colon N \to \Gamma := v(N)$ is the restriction of v, then φ is a valuation-like map.*

Proof. By our assumption there exists a $\delta \in \tilde{\Gamma}$, $\delta \geq 0$, such that $1+\mathfrak{m}(\delta) \subseteq N$. There exists an $\alpha \in \Gamma$ with the property $\alpha \geq \delta$ (one can take, for example, $\alpha = \delta^{m!}$ where $m = |D^\times : N|$); we claim that for any such α, $N_{<-\alpha} + 1 \subseteq N_{<-\alpha}$. Indeed, for any $x \in N_{<-\alpha}$ we have

$$1 + x = (1 + x^{-1})x \in N,$$

as $1+x^{-1} \in 1+\mathfrak{m}(\alpha) \subseteq 1+\mathfrak{m}(\delta) \subseteq N$. Furthermore, since v is a valuation, we have

$$\varphi(1 + x) = v(1 + x) = v(x) < -\alpha$$

as $v(x) < 0 = v(1)$. So, $1 + x \in N_{<-\alpha}$, as required. □

From now on our efforts will be focused on reversing this lemma, viz. on proving that if N admits a valuation-like map, then there exists a nontrivial valuation v of D with respect to which N is open. To describe the relationship between φ and v in precise terms, we need the following.

Definition 2.4. Let $\varphi\colon N \to \Gamma$ be a nontrivial homomorphism to a totally ordered group (not necessarily surjective). We say that a valuation $v\colon D^\times \to \tilde{\Gamma}$ is *associated* with φ if there exists a nontrivial homomorphism $\theta\colon \varphi(N) \to \tilde{\Gamma}$ of ordered groups such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & \varphi(N) \\ \iota \downarrow & & \downarrow \theta \\ D^\times & \xrightarrow{v} & \tilde{\Gamma} \end{array}$$

in which ι is the inclusion map, commutes. Furthermore, we say that v *extends* φ if θ is injective.

Remarks 2.5. (1) Note that given a nontrivial homomorphism $\varphi\colon N \to \Gamma$, the nontrivial valuation $v\colon D^\times \to \tilde{\Gamma}$ is associated with φ if and only if $\varphi(n) \geq 0$ implies $v(n) \geq 0$, for all $n \in N$.

(2) A special role in this paper is played by valuations of height one, for which the value group is isomorphic (as an ordered group) to a subgroup of the additive group $(\mathbb{R}, +)$ of the reals, and which therefore admit the associated absolute value. In this regard we observe that if $v\colon D^\times \to \tilde{\Gamma}$ is a valuation and $\mu\colon \tilde{\Gamma} \to \bar{\Gamma}$ is a homomorphism of ordered groups onto a (nontrivial) totally ordered group of height one, then $\bar{v} = \mu \circ v$ is a height one valuation. Moreover, if v is associated with $\varphi\colon N \to \Gamma$, then so is $\bar{v}$. Finally, if $\tilde{\Gamma}$ is commutative and has finite height, then a homomorphism μ as above exists (cf. [5], Ch. 6, Sect. 4, no. 4 for a discussion of the height of a totally ordered group\valuation).

After some preparations in Sect. 2, we will examine the existence of a valuation v associated with φ in the commutative (Sect. 4) and noncommutative (Sect. 5) cases, respectively. As the following proposition shows, once the existence of v has been established, the openness of N with respect to v follows automatically.

Proposition 2.6. *Let $-1 \in N \subseteq D^\times$ be a finite index subgroup which admits a valuation-like map $\varphi\colon N \to \Gamma$. Assume further that φ admits an associated valuation $v\colon D^\times \to \tilde{\Gamma}$. Then N is open in $D^\times$ in the v-adic topology.*

Proof. We need to show that there exists a $0 \leq \delta \in \tilde{\Gamma}$ such that

$$1 + \mathfrak{m}(\delta) \subseteq N. \tag{i}$$

For that we will show that for each coset Na of N in $D^\times$, there exists $0 \leq \gamma(Na) = \gamma \in \tilde{\Gamma}$, such that

$$1 + (Na \cap \mathfrak{m}(\gamma)) \subseteq N. \tag{ii}$$

Then, since N has a finite index in $D^\times$, the maximum, $\delta = \max \gamma(Na)$, taken over all cosets of N in $D^\times$, exists and obviously satisfies (i).

Let α be a level of φ. Pick $\alpha_0 \in \varphi(N)$ with the property $\alpha_0 > \alpha$. To establish the existence of $\gamma(Na)$, we need the following.

Lemma 2.7. (1) *For $m, n \in N$ such that $v(m) < v(n) - \theta(\alpha_0)$, the element $c = m + n$ belongs to N.*

(2) *For any $a \in D^\times$, there exists $\beta(a) \in \tilde{\Gamma}$ such that*

$$a + \{n \in N \mid v(n) < \beta(a)\} \subseteq N.$$

Proof. (1): We have $v(n^{-1}m) = -v(n) + v(m) < -\theta(\alpha_0)$. Since the group $\varphi(N)$ is totally ordered and θ preserves the order relation, this implies that $\varphi(n^{-1}m) < -\alpha_0 < -\alpha$, i.e. $n^{-1}m \in N_{<-\alpha}$. Then

$$n^{-1}m + 1 \in 1 + N_{<-\alpha} \subseteq N_{<-\alpha} \subseteq N$$

and therefore $c = n(n^{-1}m + 1) \in N$.

(2): Since D is infinite, $D = N - N$ (cf. [3], [33]), so there exists an $s \in N$ such that $a + s \in N$. Set

$$\beta(a) = \theta(\min(\varphi(s), \varphi(a + s)) - \alpha_0)$$

(where $\theta\colon \varphi(N) \to \tilde{\Gamma}$ is as in Definition 2.4).

Suppose now that $t \in N$ satisfies $v(t) < \beta(a)$. Then, in particular, $v(t) < v(s) - \theta(\alpha_0)$, so it follows from (1) that $t - s \in N$; moreover, $v(t - s) = v(t)$ as v is a valuation and $v(t) < v(s)$. Thus, $v(t - s) < v(a + s) - \theta(\alpha_0)$, and

$$a + t = (a + s) + (t - s) \in N$$

again according to (1). The proof of the lemma is complete. □

Now, fix a representative a of a given coset Na and let

$$\gamma = \gamma(Na) := |v(a)| + |\beta(a)|,$$

where $\beta(a)$ is as in Lemma 2.7.2 (here, as usual, for $\gamma \in \tilde{\Gamma}$, we denote $|\gamma| = \max\{\gamma, -\gamma\}$). Suppose $na \in Na \cap \mathfrak{m}(\gamma)$. Then

$$v(n) = v(na) - v(a) > (|v(a)| + |\beta(a)|) - v(a) \geq |\beta(a)|,$$

implying that

$$1 + na = n(n^{-1} + a) \in N$$

as $v(n^{-1}) < -|\beta(a)| \leq \beta(a)$, and therefore by Lemma 2.7.2, $n^{-1} + a \in N$. This shows (ii) and completes the proof of Proposition 2.6. ◻

3. Towards proving the existence of an associated valuation

Throughout this section we assume that $-1 \in N \subseteq D^\times$ is a finite index subgroup admitting a valuation-like map $\varphi\colon N \to \Gamma$ having some level $\alpha \in \Gamma$, $\alpha \geq 0$. We continue with the notation in 2.1. We will be exploiting the well-known connection between valuations and valuation rings. We let $\mathcal{A}$ denote the subring of D generated by $N_{>\alpha}$, observing that in fact it coincides with the set of all sums $b_1 + \cdots + b_l$ where all $b_i \in N_{>\alpha}$. Furthermore, the set $\mathcal{O} = \{x \in D \mid x\mathcal{A} \subseteq \mathcal{A}\}$, is obviously a subring of D and we let $\tilde{\mathcal{O}}$ denote the set of elements $x \in D$ that are integral over $\mathcal{O}$, i.e. satisfy an equation

$$x^d + a_1 x^{d-1} + \cdots + a_d = 0$$

for some $d \geq 1$ with $a_i \in \mathcal{O}$. (If D is commutative, then $\tilde{\mathcal{O}}$ is obviously the integral closure of $\mathcal{O}$ in D, however $\tilde{\mathcal{O}}$ may not be a subring for D noncommutative). Given a subfield L of D, we write $\tilde{\mathcal{O}}_L$ for the integral closure of $L \cap \mathcal{O}$ in L (which, of course, is a subring of L).

Theorem 3.1. (1) *For all $x \in D^\times$, either x or x^{-1} is in $\tilde{\mathcal{O}}$.*
(2) *Let L be a subfield of D. Then $\tilde{\mathcal{O}}_L$ is a valuation ring in L (i.e, for any $x \in L^\times$, either x or x^{-1} belongs to $\tilde{\mathcal{O}}_L$).*
(3) $\tilde{\mathcal{O}} \cap N_{<-\alpha} = \emptyset$.

Proof. Since $N_{\geq 0} N_{>\alpha} \subseteq N_{>\alpha}$, we get from the definition of $\mathcal{O}$ that $N_{\geq 0} \subseteq \mathcal{O}$. Let $m = |D^\times : N|$. Then for any $x \in D^\times$ one has $x^{m!} \in N$. Moreover, replacing x with x^{-1} if necessary, we may assume that $x^{m!} \in N_{\geq 0}$. Hence $x^{m!} \in \mathcal{O}$, implying that $x \in \tilde{\mathcal{O}}$. This shows (1), and (2) is proved in exactly the same way.

To prove (3) we first show that

(i) $$1 \notin \mathcal{A}.$$

Assume that (i) is false, and pick a presentation

$$1 = b_1 + \cdots + b_l, \quad b_i \in N_{>\alpha}$$

with minimal possible l (obviously, $l > 1$ as $1 \notin N_{>\alpha}$). We may assume that $\varphi(b_l) = \min_{i=\overline{1,l}} \varphi(b_i)$. Now $b_l^{-1} \in N_{<-\alpha}$ so, $b_l^{-1} - 1 \in N_{<-\alpha}$. It follows that $(b_l^{-1} - 1)^{-1} \in N_{>\alpha}$ and we conclude that

$$b_i b_l^{-1} \left(b_l^{-1} - 1\right)^{-1} \in N_{>\alpha} \qquad \text{for all } 1 \leq i \leq (l-1).$$

However,

$$1 = b_1 b_l^{-1}\left(b_l^{-1} - 1\right)^{-1} + \cdots + b_{l-1} b_l^{-1}\left(b_l^{-1} - 1\right)^{-1}$$

is a shorter presentation of 1, a contradiction, proving (i).

Now, suppose $z \in N_{<-\alpha} \cap \tilde{\mathcal{O}}$. Then z satisfies an equation

$$z^d + a_1 z^{d-1} + \cdots + a_d = 0$$

with $a_i \in \mathcal{O}$. It follows that

$$z = -\left(a_1 + \cdots + a_d z^{-(d-1)}\right) \in \mathcal{O}, \quad \text{i.e. } z\mathcal{A} \subseteq \mathcal{A}.$$

Since $z^{-1} \in N_{>\alpha}$, the inclusion $z\mathcal{A} \subseteq \mathcal{A}$ entails $1 \in \mathcal{A}$, contradicting (i). The proof of the theorem is complete. □

Lemma 3.2. *Suppose N admits a valuation-like map $\varphi\colon N \to \Gamma$ having a level α. Let v be a nontrivial valuation associated with φ. If Γ has height one, then v extends φ.*

Proof. By hypothesis there exists a commutative diagram

$$\begin{array}{ccc} N & \overset{\varphi}{\to} & \varphi(N) \\ \iota \downarrow & & \downarrow \theta \\ D^\times & \overset{v}{\to} & \tilde{\Gamma} \end{array}$$

as in Definition 2.4. Recall that

Definition 3.3. A subgroup Λ of a totally ordered group is called *isolated* if $0 \leq \gamma \leq \delta \in \Lambda$ implies $\gamma \in \Lambda$.

Since Γ has height one, so does $\varphi(N) \subseteq \Gamma$ (because having height one is equivalent to being a subgroup of $(\mathbb{R}, +)$). Since $\operatorname{Ker}\theta$ is a proper isolated subgroup of $\varphi(N)$ it follows that $\operatorname{Ker}\theta$ is trivial since $\varphi(N)$ has no nontrivial proper isolated subgroups (recall that one of the equivalent definitions of the *height* of a totally ordered group is the maximum length of a chain of isolated subgroups). □

4. The commutative case

The following theorem gives a criterion for a subgroup $N \subseteq K^\times$ of finite index, K is an arbitrary field, to be open with respect to a single valuation.

Theorem 4.1. *Let K be an infinite field, $-1 \in N \subseteq K^\times$ be a subgroup of finite index m. Then N is open with respect to some nontrivial valuation v of K if and only if N admits a valuation-like map.*

Suppose that N admits a valuation-like map $\varphi\colon N \to \Gamma$. Then

(1) *The valuation v above can be chosen to be associated with φ, and moreover to extend φ if either $\alpha = 0$ or Γ has height one.*

(2) *If K is finitely generated, then there exists a unique (up to equivalence) valuation of height one associated with φ, and N is open with respect to this valuation.*

Proof. If N is open with respect to some nontrivial valuation v, then by Lemma 2.3, N admits a valuation-like map. So, suppose N admits a valuation-like map φ. Consider $\mathcal{A}$, $\mathcal{O}$, and $\tilde{\mathcal{O}}$ introduced in the beginning of Sect. 3. Since K is commutative, $\tilde{\mathcal{O}}$ is a subring of K. Moreover, it follows from Theorem 3.1 that $\tilde{\mathcal{O}}$ is a valuation ring and $\tilde{\mathcal{O}} \neq K$. By ([5], Sect. 3, n° 2–3) or ([26], Ch.1, Thm.3), there exists a nontrivial valuation $v\colon K^\times \to \tilde{\Gamma} := K^\times/\tilde{\mathcal{O}}^\times$ whose valuation ring coincides with $\tilde{\mathcal{O}}$. It follows from our construction that $N_{\geq 0}$ is contained in $\tilde{\mathcal{O}}$, implying that $v(N_{\geq 0})$ is contained in $\tilde{\Gamma}_{\geq 0}$; by Remark 2.5.1, v is associated with φ. Then, by Lemma 3.2, v extends φ if Γ has height one. Suppose that $\alpha = 0$. If θ is not injective, then $\mathrm{Ker}\,\theta \cap \Gamma_{<0} \neq \emptyset$, implying $N_{<0} \cap \tilde{\mathcal{O}} \neq \emptyset$ which immediately contradicts assertion (3) of Theorem 3.1.

Once the existence of a valuation v associated with φ has been established, the openness of N in $K^\times$ with respect to the corresponding v-adic topology follows from Proposition 2.6.

Suppose now that K is finitely generated. Then it follows from Cor. 1 in n° 3 and Prop. 3 in n° 2 of [5], Sect. 10, that $\tilde{\Gamma} = v(K^\times)$ has finite height, so by Remark 2.5.2, there exists a valuation associated with φ having height one. To establish the uniqueness, suppose that v_1 and v_2 are two valuations with this property, i.e. for each $i = 1, 2$ there exists a commutative diagram of the form

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & \varphi(N) \\ \iota \downarrow & & \downarrow \theta_i \\ K^\times & \xrightarrow{v_i} & \tilde{\Gamma}_i \end{array} .$$

It is easy to see that for two isolated subgroups Λ_1 and Λ_2 (see Definition 3.3) of a totally ordered group one has one of the inclusions $\Lambda_1 \subseteq \Lambda_2$ or $\Lambda_2 \subseteq \Lambda_1$. By this fact, $\mathrm{Ker}\,\theta_1$ and $\mathrm{Ker}\,\theta_2$, being proper isolated subgroups of $\varphi(N)$ with quotients having height one, must coincide. This implies that the valuation rings $\mathcal{O}_{v_1}$ and $\mathcal{O}_{v_2}$ also coincide. Indeed, for an $x \in \mathcal{O}_{v_1}$, $x \neq 0$, one has $x^m \in N$, and there are two possibilities: 1) $\varphi(x^m) \geq 0$, and 2) $\varphi(x^m) < 0$. In the first case, $v_2(x^m) = \theta_2(\varphi(x^m)) \geq 0$, and eventually $v_2(x) \geq 0$. In the second case we have $v_1(x^m) = \theta_1(\varphi(x^m)) \leq 0$ which in conjunction with $v_1(x) \geq 0$ (as $x \in \mathcal{O}_{v_1}$) implies that actually

$$\varphi(x^m) \in \mathrm{Ker}\,\theta_1 = \mathrm{Ker}\,\theta_2,$$

so $v_2(x^m) = \theta_2(\varphi(x^m)) = 0$, and $v_2(x) = 0$. In either case, $v_2(x) \geq 0$, i.e. $x \in \mathcal{O}_{v_2}$, proving the inclusion $\mathcal{O}_{v_1} \subseteq \mathcal{O}_{v_2}$; by symmetry we obtain the opposite inclusion. Thus, $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$ which implies that v_1 and v_2 are equivalent (cf. [5], Sect. 3, Prop. 3). □

In connection with Theorem 4.1 we observe that in [6], Chevalley established the congruence subgroup property for an arbitrary finitely generated subgroup $E \subseteq K^\times$ where K is a number field. More precisely, he proved that given a subgroup $E' \subseteq E$ of finite index and any finite set S of valuations of K for which E is contained in the group $K_S^\times$ of S-units, one can pick finitely many (nonarchimedean) valuations $v_1, \dots, v_r \notin S$ and an integer n such that the congruence subgroup

$$\{x \in E \mid v_i(x-1) \geq n,\ i = 1, \dots r\}$$

is contained in E' (in other words, E' is open in E for the topology defined by a certain finite collection of valuations). However, this fact is no longer true if one considers infinitely generated subgroups, in particular, the multiplicative group $K^\times$ itself. For example, if $K = \mathbb{Q}$, then the group $\mathbb{Q}_+^\times$ of positive rationals is not open in $\mathbb{Q}^\times$ with respect to the topology defined by any finite set of nonarchimedean valuations (though, of course, it is open with respect to the archimedean one). As a more sophisticated example (in which even the archimedean valuation doesn't help) one can consider the subgroup

$$M = \left\{x \in \mathbb{Q}^\times \middle| x = \pm p_1^{\alpha_1} \cdots p_r^{\alpha_r} \text{ and } \alpha_1 + \cdots + \alpha_r \equiv 0 (\mathrm{mod}\ 2)\right\}.$$

To see that M is not open in $\mathbb{Q}$ with respect to any finite set of valuations, one needs to observe that any open subgroup must contain a certain arithmetic progression $1 + d\mathbb{Z}$ which always contains a prime while M does not. This example shows that to be open with respect to a nontrivial valuation (or valuations) a finite index subgroup $N \subseteq K^\times$ must satisfy some additional conditions (for example, like those described in Theorem 4.1).

5. The noncommutative case

We continue with the notation in 2.1. The purpose of this section is to prove Theorem 1 of the introduction. More precisely, we prove

Theorem 5.1. *Let D be a finite dimensional central division algebra over a finitely generated infinite field K, $-1 \in N \subseteq D^\times$ be a subgroup of finite index. Then N is open with respect to a nontrivial valuation v of D if and only if N admits a valuation-like map.*

Suppose that N admits a valuation-like map $\varphi\colon N \to \Gamma$. Then the valuation v above can be chosen to be associated with φ, and to have height one (it will extend φ if Γ has height one).

Throughout we let n be the degree of D (i.e. $\dim_K D = n^2$), m be the index of N in $D^\times$.

If N is open with respect to some nontrivial valuation v, then by Lemma 2.3, N admits a valuation-like map. So, suppose that N admits a valuation-like map $\varphi\colon N \to \Gamma$ having a level α. According to Proposition 2.6, N will be open with respect to *any* valuation of D associated with φ.

Also, if v is a valuation of D which is associated with φ, then by Lemma 3.2 v extends φ if Γ has height one. Thus, it remains to show (and this is the goal of this section) that φ admits an associated valuation of height one. Our strategy will be to use Theorem 4.1 to establish the existence of a valuation v_0 of K having height one and associated with the restriction $\varphi \mid N_K$ and then to extend v_0 to a height one valuation of D. We then show that v is forced to be associated with φ.

We need a general result which gives a sufficient condition for extending a height one valuation v_0 of an arbitrary field K to a central division algebra D over K (this result will be a basis for the extension step in our proof of Theorem 5.1). Pick a basis $a_1, \ldots a_{n^2}$ of D over K and define a norm $|| \; ||_{v_0}$ on D by

$$\text{(i)} \qquad || \alpha_1 a_1 + \cdots \alpha_{n^2} a_{n^2} ||_{v_0} = \max_{i=\overline{1,n^2}} | \alpha_i |_{v_0}$$

where $| \; |_{v_0}$ is the absolute value associated with v_0. (One easily shows that the topology notion of boundedness associated with two norms of the form (i) constructed using different bases, coincide.)

Theorem 5.2. *Let D be a central division algebra of degree n over an arbitrary field K, v_0 be a valuation of K having height one. Assume there exists a proper subring $\mathcal{B}$ of D such that*

(*a*) *$\mathcal{B}$ is open in D with respect to the topology defined by the norm $|| \; ||_{v_0}$.*
(*b*) *There exists a positive integer k and a subset $\Theta \subseteq D^\times$ such that $(D^\times)^k \subseteq \Theta \cup \Theta^{-1}$ and $\Theta\mathcal{B} \subseteq \mathcal{B}$, where $(D^\times)^k := \{x^k \mid x \in D^\times\}$.*

Then v_0 extends to a height one valuation v of D.

Proof. Let K_{v_0} be the completion of K with respect to v_0. It is well-known that v_0 extends to D if and only if the extended algebra $A := D \otimes_K K_{v_0}$ remains a division algebra. Now, $A \simeq M_d(\mathcal{D})$ for some integer $d \geq 1$ and some central division algebra $\mathcal{D}$ over K_{v_0}, and we need to show that under our assumptions one necessarily has $d = 1$. The valuation v_0 extends from K_{v_0} to a valuation u on $\mathcal{D}$ by the formula:

$$\text{(ii)} \qquad u(x) = \frac{1}{l} v_0(\mathrm{Nrd}_{\mathcal{D}/K_{v_0}}(x)) \quad \text{for any} \quad x \in \mathcal{D}^\times$$

where l is the degree of $\mathcal{D}$. Note that since v_0 has height one, the value group $v_0(K^\times)$ can be identified with a subgroup of the additive group $(\mathbb{R}, +)$ of the reals which gives meaning to the right-hand side of (ii). In particular, this implies that u has height one, and therefore admits the associated absolute value $| \; |_u$. Then one can introduce the following norm on $A \simeq M_d(\mathcal{D})$:

$$|| (a_{ij}) ||_u = \max_{i,j=\overline{1,d}} | a_{ij} |_u \, .$$

On the other hand, the equation (i) allows one to extend the norm $|| \; ||_{v_0}$ from D to A (just take the coefficients $\alpha_1, \ldots \alpha_{n^2}$ in K_{v_0} and think of v_0 as a valuation of K_{v_0}); we will denote this extension also by $|| \ ||_{v_0}$. Since both norms $|| \ ||_{v_0}$ and $|| \ ||_u$ are norms on A as a vector space over K_{v_0}, they are equivalent because $\dim_{K_{v_0}} A < \infty$ and K_{v_0} is complete (cf. [8]). It follows that the two norms give rise to the same topology and notion of boundedness on A. We also recall that for $a, b \in A$ the coordinates of ab are given by bilinear functions in terms of the coordinates of a and b, and for $a \in A^\times$ the coordinates of a^{-1} are given by polynomials in terms of the coordinates of a and $(\mathrm{Nrd}_{A/K_{v_0}}(a))^{-1}$, which makes the operations of taking the product and taking the inverse continuous on A and $A^\times$, respectively; in particular, $A^\times$ is a topological group.

We need the following lemma.

Lemma 5.3. *If $B \subseteq A$ is an open unbounded subring, then $B = A$.*

Proof. The openness of B means that there exists a nonnegative $\delta \in \bar{\Gamma} := u(\mathcal{D}^\times)$ such that $M_d(\mathfrak{d}(\delta)) \subseteq B$ where

$$\mathfrak{d}(\delta) = \{x \in \mathcal{D}^\times \mid u(x) > \delta\} \cup \{0\}.$$

(observe that $\mathfrak{d}(\delta) \neq 0$ as v_0, and hence u, are nontrivial). On the other hand, since B is unbounded, one can pick a pair of indices $s, t \in \{1, \dots, d\}$ satisfying the following property:

(*) for any $\mu \in \bar{\Gamma}$, $\mu > 0$, there exists a matrix $a(\mu) = (a_{ij}(\mu)) \in B$ with $u(a_{st}(\mu)) < -\mu$.

For $i, j \in \{1, \dots, d\}$ we let $e_{ij}(a)$ denote the matrix in which the (ij)-entry is equal to a, and all other entries are equal to zero, and for a subset $S \subseteq \mathcal{D}$ denote

$$E_{ij}(S) = \{e_{ij}(a) \mid a \in S\}.$$

We will show that $E_{st}(\mathcal{D}) \subseteq B$. Then picking a nonzero $a \in \mathfrak{d}(\delta)$, we will have

$$e_{is}(a) E_{st}(\mathcal{D}) e_{tj}(a) = E_{ij}(\mathcal{D}) \subseteq B$$

for all i, j, hence $B = A$.

Now,

$$E_{st}(\mathfrak{d}(\delta)) a(\mu) E_{st}(\mathfrak{d}(\delta)) = E_{st}(\mathfrak{d}(\delta) a_{st}(\mu) \mathfrak{d}(\delta)) \subseteq B.$$

But $\mathfrak{d}(\delta) a_{st}(\mu) \supseteq \mathfrak{d}(\delta - \mu)$ and $\cup_{\mu > 0} \mathfrak{d}(\delta - \mu) = \mathcal{D}$, implying that $\cup_{\mu > 0} \mathfrak{d}(\delta) a_{st}(\mu) \mathfrak{d}(\delta) = \mathcal{D}$, and therefore $E_{st}(\mathcal{D}) \subseteq B$, completing the proof of Lemma 5.3. □

The fact that $\mathcal{B}$ is open in D means that there exists a nonnegative $\gamma \in v_0(K^\times)$ such that

$$\sum_{i=1}^{n^2} \mathfrak{m}_{K,v_0}(\gamma) a_i \subseteq \mathcal{B}.$$

Since the closure of $\mathfrak{m}_{K,v_0}(\gamma)$ in K_{v_0} contains $\mathfrak{m}_{K_{v_0},v_0}(\gamma)$, we obtain that the closure $\bar{\mathcal{B}}$ of $\mathcal{B}$ in A contains

$$\sum_{i=1}^{n^2} \mathfrak{m}_{K_{v_0},v_0}(\gamma)a_i,$$

hence is open. Being open in D, the subring $\mathcal{B}$ is at the same time closed, implying that $\bar{\mathcal{B}} \cap D = \mathcal{B}$; in particular, $\bar{\mathcal{B}} \neq A$. It follows now from Lemma 5.3 that $\bar{\mathcal{B}}$ is bounded. Let us show that this is impossible under the assumptions made in (b) if $d > 1$.

The set

$$\mathcal{C} = \{x \in A \mid x\bar{\mathcal{B}} \subseteq \bar{\mathcal{B}}\}$$

is a subring of A containing $\bar{\mathcal{B}}$, hence open. Since $\bar{\mathcal{B}}$ is a proper subring, so is $\mathcal{C}$, and therefore it follows from Lemma 5.3 that $\mathcal{C}$ is bounded. But $\Theta\mathcal{B} \subseteq \mathcal{B}$ implies $\Theta\bar{\mathcal{B}} \subseteq \bar{\mathcal{B}}$, i.e. $\Theta \subseteq \mathcal{C}$, so Θ is bounded. It follows that the function $\psi\colon a \mapsto \min(\| a \|_{v_0}, \| a^{-1} \|_{v_0})$ is bounded on $\Theta_1 := \Theta \cup \Theta^{-1}$. Then the function ψ, which is defined on $A^\times$, will be bounded also on the closure of Θ_1 in $A^\times$. Since D is dense in A, the assumption made in (b) implies that this closure contains $(A^\times)^k$. Taking into account that the norms $\| \; \|_{v_0}$ and $\| \; \|_u$ are equivalent, we eventually obtain that the function $\eta(a) := \min(\| a \|_u, \| a^{-1} \|_u)$ is bounded on $(A^\times)^k$. Let us show that this is impossible if $d > 1$. Pick an element $s \in \mathcal{D}$ such that $| s |_u > 1$, and, assuming that $d > 1$, let $t = \mathrm{diag}(s, s^{-1}, 1, \ldots, 1) \in A^\times$. Then for any integer $l > 1$, we have $t^{kl} \in (A^\times)^k$; on the other hand,

$$\eta(t^{kl}) = | s^{kl} |_u = (| s |_u)^{kl} \longrightarrow \infty \quad \text{as} \quad l \to \infty,$$

a contradiction. Hence v_0 extends to a valuation v of D, and it is well known that v is given by the formula $v(x) = \frac{1}{n} v_0(\mathrm{Nrd}_{D/K}(x))$, for all $x \in D^\times$. Since v_0 has height one, v has height one as well. The proof of Theorem 5.2 is complete. □

Proof of Theorem 5.1. Pick any $a \in N_{<-\alpha}$ and consider the field $P = K(a)$. Then $(N_P)_{<-\alpha} \neq \emptyset$, so by Theorem 4.1 there exists a height one valuation v_P of P associated with φ_P, i.e. there exists a commutative diagram of the form:

$$\begin{array}{ccc} N_P & \xrightarrow{\varphi_P} & \varphi(N_P) \\ \iota \downarrow & & \downarrow \theta_P \\ P^\times & \xrightarrow{v_P} & \tilde{\Gamma}_P \end{array}$$

in which θ_P is a homomorphism of ordered groups. Let $v_0 = v_P \mid K$. We point out one fact to be used later. We have $r := |\tilde{\Gamma}_P : v_P(K^\times)| \leq [P : K] < \infty$ (cf. [26], Lemma 1.15). Then $v_P(a^r) \in v_P(K^\times)$, so since $v_P \mid K$ is nontrivial, there exists $b \in K^\times$ such that $v_P(b) < v_P(a^r)$. Then $b^m \in N_K$ and

$$v_P(b^m) \leq v_P(b) < v_P(a^r) \leq v_P(a)$$

as $v_P(a) \leq 0$, implying that

(iii) $$(N_K)_{<-\alpha} \neq \emptyset.$$

Furthermore, it follows from Theorem 4.1 that there exists a nonnegative $\beta \in v_0(K^\times)$ such that

$$1 + \mathfrak{m}_{K,v_0}(\beta) \subseteq N_K.$$

To prove that v_0 extends to a valuation v of D we are going to use Theorem 5.2. Let $\mathcal{A}$ be the subring of D generated by $N_{>\alpha}$. Then for

$$\Theta := N_{\geq 0}$$

one obviously has $\Theta N_{>\alpha} \subseteq N_{>\alpha}$, implying that $\Theta\mathcal{A} \subseteq \mathcal{A}$. Let $k = m!$, where, recall, $m = |D^\times : N|$. Then

$$(D^\times)^k \subseteq N = \Theta \cup \Theta^{-1}.$$

So, to use Theorem 5.2 to extend v_0 it remains to check that $\mathcal{A}$ is open in D with respect to the topology defined by v_0. Let $\mathbb{G}$ be the algebraic K-group associated with $D^\times$, i.e. $\mathbb{G}(K) = D^\times$ (we observe that over an algebraically closed field Ω containing K we have $D \otimes_K \Omega \simeq M_n(\Omega)$, and under this identification $\mathbb{G}$ corresponds to $GL_n(\Omega)$). Since $\mathbb{G}$ is connected and $D^\times$ is Zariski dense in $\mathbb{G}$, we conclude that so is N. Then

$$\mathbb{G} = \check{\Theta} \cup \check{\Theta}^{-1},$$

where $\check{}$ denotes the Zariski closure, and the connectedness of of $\mathbb{G}$ again implies that $\check{\Theta} = \mathbb{G}$. Finally, $\Theta N_{>\alpha} \subseteq N_{>\alpha}$ implies that $N_{>\alpha}$ is Zariski dense in $\mathbb{G}$, and therefore the direct product $N_{>\alpha} \times \cdots \times N_{>\alpha}$ is Zariski dense in $\mathbb{G} \times \cdots \times \mathbb{G}$ (n^2 factors). Since the set of linearly independent n^2-tuples in $\mathbb{G}^{n^2}$ is Zariski open, we conclude that there are elements $a_1, \ldots, a_{n^2} \in N_{>\alpha}$ forming a basis of D over K.

Using (iii), pick $b \in (N_K)_{>\alpha}$ and choose $c \in N_K$ such that $\gamma := v_0(c) > v_0(b)$. Since $v_0(1+\mathfrak{m}_{K,v_0}(\beta)) = \{0\}$ and v_0 is associated with φ_K, we obtain that

$$c(1 + \mathfrak{m}_{K,v_0}(\beta)) \subseteq N_{>\alpha} \subseteq \mathcal{A},$$

so for $\delta = \gamma + \beta$, one has

$$\mathfrak{m}_{K,v_0}(\delta) = c\mathfrak{m}_{K,v_0}(\beta) \subseteq \mathcal{A}.$$

Then

$$\sum_{i=1}^{n^2} \mathfrak{m}_{K,v_0}(\delta) a_i \subseteq \mathcal{A}.$$

This yields the openness of $\mathcal{A}$, and therefore, in view of Theorem 5.2, the existence of a height one valuation v of D extending v_0.

Finally, let us show that v is associated with φ, i.e. for $x \in N$, $\varphi(x) \geq 0$ implies $v(x) \geq 0$ (cf. Remark 2.5.1). So fix an $x \in N$ such that $\varphi(x) \geq 0$, and let $M \subseteq D$ be a maximal subfield containing x. In view of (iii), it follows from Theorem 4.1 that there exists a valuation v_M of M having height one and associated with φ_M. Then the restriction $v_M \mid_K$ is a height one valuation of K associated with φ_K, so $v_M \mid_K = v_0$ due to the uniqueness statement in Theorem 4.1, i.e. v_M is an extension of v_0. On the other hand, $v \mid_M$ is also an extension of v_0. Since $D \otimes_K K_{v_0}$ is a division algebra, $M \otimes_K K_{v_0}$ is a field implying that the extension of v_0 to M is unique (cf. [5], Sect. 8, Prop. 2), hence $v_M = v \mid_M$. Since v_M is associated with φ_M, we have $v(x) = v_M(x) \geq 0$, as required. □

Remark 5.4. Theorem 5.2 was implicit in [20] and was made explicit by Segev in the process of writing this paper.

The assumption in Theorem 5.1 that K be finitely generated is important for the existence of a *height one* valuation associated with φ, a fact to be used in the proof of the "Nonexistence Theorem at Diameter ≥ 4". However the existence of *some* nontrivial valuation associated with φ can probably be established over general fields. In the rest of this section we prove this under the additional assumption that φ is *conjugation invariant*, i.e. that N is normal in $D^\times$ and $\varphi(g^{-1}ng) = \varphi(x)$ for any $n \in N$ and $g \in D^\times$. We observe that the map considered in Sect. 6 of [27] is a conjugation invariant valuation-like map by construction. We also note that for conjugation invariant valuation-like maps one can give an alternative proof to Theorem 5.1 (see [19] or Appendix C). First, we need a lemma showing that in various situations one can pass to a divison algebra over a finitely generated field.

Lemma 5.5. *Let D be a finite dimensional division algebra (not necessarily central) over a field K. Then given a finite subset $S \subseteq D$ (resp. a surjective homomorphism $\psi\colon D^\times \to H$ onto a finite group H) there exists a finitely generated subfield $k \subseteq K$ and a division k-subalgebra $\tilde{D} \subseteq D$ with $D = \tilde{D} \otimes_k K$ such that $S \subseteq \tilde{D}$ (resp. the restriction $\psi \mid_{\tilde{D}^\times}\colon \tilde{D}^\times \to H$ is surjective).*

Proof. First, let $S \subseteq D$ be a finite subset. Without loss of generality we may assume that S contains a basis of D over K. Let k denote the subfield of K obtained by adjoining to the prime subfield $K_0 \subseteq K$ the coordinates (with respect to the fixed basis contained in S) of all elements of the form xy, where $x, y \in S$, and let $\tilde{D}$ be the k-linear span of S. By our construction $\tilde{D}$ satisfies $D = \tilde{D} \otimes_k K$ and is closed under multiplication, hence a k-algebra. Being a finite dimensional algebra without zero divisors, $\tilde{D}$ is a division algebra. This proves our assertion given a subset. To prove the assertion given a surjective homomorphism $\psi\colon D^\times \to H$, we pick a finite set S such that $\psi(S) = H$ and apply the first part. □

Theorem 5.6. *Let D be a finite dimensional central division algebra over an infinite field K, $-1 \in N \subseteq D^\times$ be a normal subgroup of finite index. Given a conjugation invariant valuation-like map $\varphi\colon N \to \Gamma$, there exists a nontrivial valuation v of D associated with φ and N is open with respect to this valuation.*

Proof. As in the proof of Theorem 5.1, we only need to prove the existence of v. Let $\Theta = N_{\geq 0}$ and $\tilde{\Theta} = \{x \in D^\times \mid x^m \in N_{\geq 0}\}$ where $m = |D^\times : N|$. Let $\mathcal{R}$ (resp., $\tilde{\mathcal{R}}$) be the subring of D generated by Θ (resp., $\tilde{\Theta}$). Since $(D^\times)^m \subseteq N$ and Γ is totally ordered, we have $D^\times = \tilde{\Theta} \cup \tilde{\Theta}^{-1}$, which implies that $\tilde{\mathcal{R}}$ is a valuation ring in D; besides, since φ is conjugation invariant, the ring $\tilde{\mathcal{R}}$ is also conjugation invariant[1]. Then associated with the valuation ring $\tilde{\mathcal{R}}$ one has a valuation v of D. Note that by Remark 2.5.1, since $N_{\geq 0} \subseteq \mathcal{R} \subseteq \tilde{\mathcal{R}}$, the valuation v is associated with φ, and we only need to prove that this valuation is nontrivial, i.e. $\tilde{\mathcal{R}} \neq D$. We observe that $\mathcal{R} \subseteq \mathcal{O}$, where $\mathcal{O}$ is the subring defined in Sect. 3, in particular, it follows from Theorem 3.1 that

(iv) $$N_{<-\alpha} \cap \mathcal{R} = \emptyset.$$

We will show that the assumption $\tilde{\mathcal{R}} = D$ eventually contradicts (iv). We need the following.

Lemma 5.7. *There exists $\mu \in (N_K)_{\geq 0}$ such that $\tilde{\Theta} \subseteq \mu^{-1}\mathcal{R}$.*

Proof. First, we note the following property

(v) For any $s \in N$ there exists $t \in N_K$ such that $\varphi(t) \geq \varphi(s)$.

Indeed, we may assume that $\varphi(s) \geq 0$. By Wedderburn's factorization theorem (see [23], p. 253), $t = \mathrm{Nrd}_{D/K}(s)$ is a product of n conjugates of s, implying that $t \in N_K$ and $\varphi(t) = n\varphi(s) \geq \varphi(s)$.

Let $a_1, \ldots, a_m$ be a transversal for N in $D^\times$, and suppose $x = a_i s$ with $s \in N$. Then

$$\varphi(x^m) = \varphi\big((a_i s a_i^{-1})(a_i^2 s a_i^{-2}) \cdots (a_i^m s a_i^{-m}) a_i^m\big) = m\varphi(s) + \varphi\big(a_i^m\big).$$

It follows that if $x \in \tilde{\Theta}$, then $m\varphi(s) \geq -\varphi(a_i^m)$, and therefore

(vi) $$m\varphi(s) \geq -\big|\varphi\big(a_i^m\big)\big|.$$

Using (v), pick $\mu_1 \in N_K$ such that

(vii) $$\varphi(\mu_1) \geq \max_{i=\overline{1,m}} \big|\varphi\big(a_i^m\big)\big|.$$

[1] Since D is finite dimensional, valuation subrings of D are in fact automatically conjugation invariant (a theorem due to P. Cohn, cf. [7])

Since $D = N - N$, it follows from (v) above that for any $a \in D$ there exists $\lambda \in (N_K)_{\geq 0}$ such that

$$\lambda a \in N_{\geq 0} - N_{\geq 0} \subseteq \mathcal{R}.$$

Therefore one can pick $\mu_2 \in (N_K)_{\geq 0}$ such that

$$\text{(viii)} \qquad \mu_2 a_i \in \mathcal{R} \quad \text{for all} \quad i = 1, \ldots, m.$$

We claim that $\mu = \mu_1 \mu_2$ will work. Indeed, if $x = a_i s \in \tilde{\Theta}$, then s satisfies (vi), and it follows from (vii) that $\mu_1 s \in N_{\geq 0}$. On the other hand, $\mu_2 a_i \in \mathcal{R}$ according to (viii). Thus, $\mu x = (\mu_2 a_i)(\mu_1 s) \in \mathcal{R}$, proving the lemma. □

Now, suppose $\tilde{\mathcal{R}} = D$, then in particular $\mathcal{R}[\mu^{-1}] = D$. By Lemma 5.5 there exists a finitely generated subfield $k \subseteq K$ and a division k-subalgebra $D_0 \subseteq D$ such that $D = D_0 \otimes_k K$, $(N \cap D_0)_{<-\alpha} \neq \emptyset$, and μ^{-1} belongs to the subring generated by $\tilde{\Theta} \cap D_0$ (take S in Lemma 5.5 to be a set consisting of an element in $N_{<-\alpha}$ and the elements in $\tilde{\Theta}$ required to write μ^{-1} as a polynomial in elements of $\tilde{\Theta}$). By Theorem 5.1 there exist a nontrivial valuation w_0 of D_0 associated with $\varphi\ |_{(N \cap D_0)}$; let $\mathcal{O}_{w_0}$ be the corresponding valuation ring. Given $x \in \tilde{\Theta} \cap D_0$ one has $\varphi(x^m) \geq 0$, implying $w_0(x^m) \geq 0$, and eventually $w_0(x) \geq 0$. This means that $\tilde{\Theta} \cap D_0 \subseteq \mathcal{O}_{w_0}$, and consequently $\mu^{-1} \in \mathcal{O}_{w_0}$. Since originally $\varphi(\mu) \geq 0$, we obtain that $w_0(\mu) = 0$. Let $\theta\colon \varphi(N \cap D_0) \to \tilde{\Gamma}_0 = w_0(D_0^{\times})$ be the connecting homomorphism of ordered groups (whose existence is guaranteed by the fact that w_0 is associated with $\varphi\ |_{N \cap D_0}$). Take some $s \in (N \cap D_0)_{<-\alpha}$ and using the fact that w_0 is nontrivial, pick $t \in N \cap D_0$ such that $w_0(t) < w_0(s)$. Since $\mathcal{R}[\mu^{-1}] = D$, there exists l such that $\mu^l t \in \mathcal{R} \cap D_0$. But $w_0(\mu^l t) = w_0(t) < w_0(s)$, so $\varphi(\mu^l t) < \varphi(s) < -\alpha$, i.e.

$$\mu^l t \in N_{<-\alpha} \cap \mathcal{R},$$

contradicting (iv) and proving Theorem 5.6. □

6. The order relation $\leq_{y^*}$ and its linearity

We continue with the notation of Sect. 2, however, here D is an arbitrary infinite division algebra over K (not necessarily finite dimensional) and N is a normal subgroup of $D^{\times}$ of finite index such that $-1 \in N$. Our goal is to prove Theorem 2 (of the introduction), i.e., to show that if the diameter of the commuting graph of $D^{\times}/N$ is ≥ 4, then N admits a valuation-like map $\varphi\colon N \to \Gamma$. As in [27], we let $G := D^{\times}$, $G^* := G/N$, and for $a \in G$ denote by a^* the image of a in G^* under the canonical homomorphism.

We start by defining the ordered group Γ and the map φ. A crucial role is played by the sets $N(y)$ introduced in [27]: for $y \in D^{\times}$ we let

$$N(y) := \{n \in N \mid y + n \in N\}.$$

Lemma 6.3 below gives some properties of the sets $N(y)$. We define the preorder relation $\mathfrak{P}_{y^*}$ on N by

$$m\,\mathfrak{P}_{y^*}\,n \quad\Longleftrightarrow\quad N(my) \subseteq N(ny).$$

As we will see, $\mathfrak{P}_{y^*}$ depends only on the coset $y^* = Ny$ and not on the coset representative y. Furthermore it does not depend on the "side", i.e. the relation defined by $N(ym) \subseteq N(yn)$ coincides with $\mathfrak{P}_{y^*}$ (see below). We then show that $\mathfrak{P}_{y^*}$ is compatible with the group structure, implying that

$$U_{y^*} := \{n \in N \mid N(ny) = N(y)\}$$

is a normal subgroup of N. This yields the partially ordered group

$$\Gamma_{y^*} := N/U_{y^*}$$

with the order relation induced by $\mathfrak{P}_{y^*}$, and the homomorphism (the canonical homomorphism)

$$\varphi_{y^*} : N \to \Gamma_{y^*}$$

Instead of Theorem 2 we will prove in Sects. 6–7 the following more precise result.

Theorem 6.1. *Let D be an infinite division algebra over an arbitrary field and $N \subseteq D^\times$ be a normal subgroup of finite index. If* $\mathrm{diam}(\Delta(D^\times/N)) \geq 4$*, then for some $y \in D^\times$, Γ_{y^*} is a totally ordered group and φ_{y^*} is a valuation-like map.*

As in [27], we denote by Δ the commuting graph of G^*, i. e., the graph whose vertex set is $G^* \smallsetminus \{1^*\}$, and whose edges are $\{a^*, b^*\}$ such that $a^* \neq b^*$ and $[a^*, b^*] = 1$ (i. e, a^* and b^* commute). We let $d : \Delta \times \Delta \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ be the usual distance function of Δ. Recall that the *diameter* of Δ is the largest distance between two vertices of Δ (it is ∞ if Δ is disconnected). We denote the diameter of Δ by $\mathrm{diam}(\Delta)$.

Once we have two elements in Δ at distance at least 4 from each other, Sects. 6–7 show that for y^* appropriately chosen amongst these two elements and their inverses, Γ_{y^*} and φ_{y^*} satisfy the assertions of Theorem 6.1. Sect. 6 concentrates on showing how to choose y^* so that Γ_{y^*} is totally ordered, and Sect. 7 concentrates on showing that y^* can be further specified so that φ_{y^*} has a level.

Remark 6.2. As in the previous sections, Γ_{y^*} is *written additively*. In particular the zero element of Γ_{y^*} is $0 = U_{y^*}$.

Let us start by recalling some properties of the sets $N(y)$.

Lemma 6.3. *Let $y \in G \smallsetminus N$ and $n \in N$. Then*

(1) *$N(ny) = nN(y)$ and $N(yn) = N(y)n$.*
(2) *For all $x \in G$, $N(y^x) = x^{-1}N(y)x$.*

(3) $N(y) \neq \emptyset$.
(4) *If* $n \in N(y^{-1})$, *then* $y + n^{-1} \in Ny$. *Consequently,* $n^{-1} \notin N(y)$. *In particular* $\emptyset \subsetneq N(y) \subsetneq N$.

Proof. This is 1.8 in [27]; for completeness we indicate that (1) and (2) follow directly from the definition of $N(y)$, (3) is an immediate consequence of the fact that $D = N - N$ (cf. [3], [33]). The first part of (4) follows from the definition of $N(y)$ and the rest of (4) is a consequence of the first part. □

We can now show that for any $y \in D^\times \setminus N$, the relation $\mathfrak{P}_{y^*}$ is a group preorder relation.

Lemma 6.4. *For any* $y \in D^\times \setminus N$, *the relation* $\mathfrak{P} := \mathfrak{P}_{y^*}$ *has the following properties*

(1) $\mathfrak{P}$ *is reflexive and transitive.*
(2) $m\mathfrak{P}n$ *implies* $(sm)\mathfrak{P}(sn)$ *and* $(ms)\mathfrak{P}(ns)$, *for any* $s \in N$.
(3) $m\mathfrak{P}n$ *and* $s\mathfrak{P}t$ *implies* $(ms)\mathfrak{P}(nt)$.
(4) $m\mathfrak{P}n$ *implies* $n^{-1}\mathfrak{P}m^{-1}$.

Proof. (1): This is immediate from the definition of $\mathfrak{P}$.

(2): That $(sm)\mathfrak{P}(sn)$ is immediate from the definition of $\mathfrak{P}$ and Lemma 6.3.1. For the second part of (2), we have $N(my) \subseteq N(ny)$ so multiplying on the right by elements of N and using Lemma 6.3.1, we see that

(i) $$N(my') \subseteq N(ny') \text{ for all } y' \in yN = Ny.$$

Taking in (i) $y' = sy$ we see that $N(msy) \subseteq N(nsy)$, i.e. $(ms)\mathfrak{P}(ns)$.

(3): Using (2) in the setup of (3) we obtain $(ms)\mathfrak{P}(mt)\mathfrak{P}(nt)$.

(4): This follows from (2) by multiplying on the left by m^{-1} and on the right by n^{-1}. □

We note that it is immediate from (i) that $\mathfrak{P}_{y^*}$ is independent of the coset representative y; furthermore, Lemma 6.3.2 implies that $N(my) \subseteq N(ny)$ iff $N(ym) \subseteq N(yn)$ (just conjugate both sides by y^{-1}). Since $U_{y^*} = \{n \in N \mid n\mathfrak{P}_{y^*}1 \text{ and } 1\mathfrak{P}_{y^*}n\}$, it follows from Lemma 6.4 that U_{y^*} is a normal subgroup of N, and the preorder relation $\mathfrak{P}_{y^*}$ defines the order relation $\leq_{y^*}$ on the quotient $\Gamma_{y^*} = N/U_{y^*}$, namely

$$mU_{y^*} \leq_{y^*} nU_{y^*} \iff N(my) \subseteq N(ny).$$

To continue the discussion we recall from Notation 2.1 that given $y \in D^\times \setminus N$,

$$\text{For } nU_{y^*} = \gamma \in \Gamma_{y^*}, \text{ we let } N_{<_{y^*}\gamma} = \{m \in N \mid mU_{y^*} <_{y^*} \gamma\},$$

and the sets $N_{\leq_{y^*}\gamma}$, $N_{>_{y^*}\gamma}$ etc. are defined similarly. Next, we let

$$\mathbb{P}_{y^*} = \{b \in Ny \mid 1 \in N(b)\}.$$

Note that it follows from Lemma 6.3.3 and 6.3.1 that $\mathbb{P}_{y^*} \neq \emptyset$. Here are some additional properties of the order relation $\leq_{y^*}$.

Lemma 6.5. *Let $y \in G \smallsetminus N$. Then*

(1) $N_{\leq_{y^*} 0} = \{n \in N \mid n \in N(b),$ *for all* $b \in \mathbb{P}_{y^*}\}$.

(2) $nU_{y^*} \leq_{y^*} 0$ *iff* $n^{-1}\mathbb{P}_{y^*} \subseteq \mathbb{P}_{y^*}$.

(3) *For all* $y' \in Ny$, *if* $n \in N(y')$ *and* $mU_{y^*} \leq_{y^*} nU_{y^*}$, *then* $m \in N(y')$.

Proof. (1): Suppose $N(ny) \subseteq N(y)$. Then, multiplying on the right with elements of N and using 6.3.1, we see that $N(ny') \subseteq N(y')$, for all $y' \in Ny$. In particular $n \in N(nb) \subseteq N(b)$, for all $b \in \mathbb{P}_{y^*}$.

Conversely, set $\mathbb{N} := \{n \in N \mid n \in N(b),$ for all $b \in \mathbb{P}_{y^*}\}$ and let $m \in N(y)$ and $n \in \mathbb{N}$. Then $ym^{-1} \in \mathbb{P}_{y^*}$, so $n \in N(ym^{-1})$, or $nm \in N(y)$. As this holds for all $m \in N(y)$, we see that $nN(y) \subseteq N(y)$.

(2): By definition, $nU_{y^*} \leq_{y^*} 0$ iff $N(nb) \subseteq N(b)$, for all $b \in \mathbb{P}_{y^*}$, iff $N(n^{-1}b) \supseteq N(b) \ni 1$, for all $b \in \mathbb{P}_{y^*}$ iff $n^{-1}\mathbb{P}_{y^*} \subseteq \mathbb{P}_{y^*}$.

(3): We have $mn^{-1}U_{y^*} \leq_{y^*} 0$, so by definition (and since $\leq_{y^*}$ is independent of the coset representative), $N(mn^{-1}y') \subseteq N(y')$. It follows from Lemma 6.3.1 that $m \in N(y')$. □

The following basic result appears in the proof of Proposition 7.1 in [18] (see also 2.1 in [27]).

Lemma 6.6. *Let $y \in G$ and $n \in N$ such that $y, y + n \in G \smallsetminus N$. Then* $d(y^*, (y + n)^*) \leq 1$.

Proof. We have $y + n = n(n^{-1}y + 1)$, so $(y + n)^* = (n^{-1}y + 1)^*$. But $(n^{-1}y + 1)^*$ commutes with $(n^{-1}y)^* = y^*$. □

Remark 6.7. As in [27], we remark that for all $a \in G$, $(-a)^* = a^*$. Also, note that by Lemma 6.6, given $x, y \in G \smallsetminus N$ and $n \in N$, if $x + y \in N$, or $x - y \in N$, then $d(x^*, y^*) \leq 1$. Also, if $n \notin N(y)$, then $d(y^*, (y+n)^*) \leq 1$. We use these facts without further reference.

Here is a slight improvement on 2.9 of [27].

Lemma 6.8. (1) *Let $x, y \in G \smallsetminus N$ and suppose that $d(x^*, y^*) > 2$. Then $x + y \notin N$ and $N(x + y) = N(x) \cap N(y)$.*

(2) *If $d(x^*, y^*) > 2 < d((x + y)^*, x^*)$, then $N(x + y) = N(y) \subseteq N(x) \cap N(-x)$.*

Proof. (1): By Remark 6.7, $x + y \notin N$. By 2.9.1 in [27], $N(x + y) \subseteq N(x) \cap N(y)$. Assume $N(x+y) \subsetneqq N(x) \cap N(y)$ and let $n \in (N(x) \cap N(y)) \smallsetminus N(x+y)$. Then, using Remark 6.7, we see that $x^*(x + y + n)^*y^*$ is a path in Δ, a contradiction.

(2): Follows from (1), since by (1), also $N(y) = N(x + y - x) = N(x + y) \cap N(-x)$. □

The following lemma is the main ingredient in the proof of the linearity of $\leq_{y^*}$, for an appropriately chosen y^*.

Lemma 6.9. *Let $x, y \in G \setminus N$. Let $a \in Nx$, $b \in Ny$ and $\epsilon \in \{1, -1\}$. Then*

(1) If $d(x^*, y^*) \geq 4$ and $\epsilon \notin N(b)$, then $N(ab) \cup N(ba) \subseteq N(a) \cap N(-a)$.
(2) If $d(x^*, y^*) \geq 3$ and $\epsilon \in N(b^{-1})$, then $N(ab) \cup N(ba) \subseteq N(a) \cap N(-a)$.
(3) If $d(x^*, y^*) \geq 3$ and $\epsilon \in N(a)$, then $N(b) \subseteq N(ab) \cap N(ba)$.

Proof. (1) & (2): We have $ab = ab + \epsilon a - \epsilon a$. Since $d(a^*, (ab)^*) > 2$, $N(ab + \epsilon a) \subseteq N(\epsilon a)$, by Lemma 6.8. Notice that $d((ab + \epsilon a)^*, a^*) \leq 2$ implies

$$(*) \qquad d(a^*, (b + \epsilon)^*) \leq 2.$$

Hence if $\epsilon \in N(b^{-1})$, then by Lemma 6.3.4 $(b + \epsilon)^* = b^*$ and then (*) implies that $d(a^*, b^*) \leq 2$, a contradiction; while if $d(x^*, y^*) \geq 4$, we get a contradiction since (*) implies that $d(a^*, b^*) \leq 3$. Hence $d((ab + \epsilon a)^*, a^*) > 2$, so by Lemma 6.8.2, $N(ab) \subseteq N(\epsilon a) \cap N(-\epsilon a)$. This shows that in (1) and (2), $N(ab) \subseteq N(a) \cap N(-a)$, and for the other inclusion conjugate by a using Lemma 6.3.2.

(3): Since $d(x^*, y^*) \geq 3$, we have $d((x^{-1})^*, (xy)^*) \geq 3$, and it follows from (2) (taking a^{-1} in place of b and ab in place of a), that $N(a^{-1}ab) \subseteq N(ab)$, that is, $N(b) \subseteq N(ab)$. The other inclusion is obtained by conjugating by b. □

We recall two binary relations which were crucial in [27] and are crucial for the proof of Theorem 2.

Notation 6.10. Let $x, y \in G \setminus N$.

(1) $In(x^*, y^*)$ is equivalent to: For any $(a, b) \in Nx \times Ny$, either $N(a) \subseteq N(b)$ or $N(b) \subseteq N(a)$. Note that $In(y^*, y^*)$ make sense.
(2) $Inc(y^*, x^*)$ is equivalent to: $In(x^*, y^*)$ and for any $b \in \mathbb{P}_{y^*}$ there exists $a \in \mathbb{P}_{x^*}$ such that $N(b) \supseteq N(a)$.

Proposition 6.11. *Let $x, y \in G \setminus N$, with $d(x^*, y^*) \geq 4$, then $In(x^*, y^*)$.*

Proof. Let $a \in Nx$ and $b \in Ny$. We must show that either $N(a) \subseteq N(b)$ or $N(b) \subseteq N(a)$. So assume $N(b) \nsubseteq N(a)$, and let $n \in N(b) \setminus N(a)$. Let $c := n^{-1}a$ and $d := n^{-1}b$. Of course, it suffices to show that $N(c) \subseteq N(d)$, but now, $N(d) \ni 1 \notin N(c)$. By Lemma 6.9.1 and 6.9.3 $N(c) \subseteq N(cd) \subseteq N(d)$, and we are done. □

The next lemma lists some properties of the relations In and Inc.

Lemma 6.12. *Let $x, y \in G \setminus N$. Then*

(1) *If $In(x^*, y^*)$, then either $Inc(x^*, y^*)$ or $Inc(y^*, x^*)$.*
Suppose $Inc(y^*, x^*)$, then
(2) *If $z \in G \setminus N$ is such that $In(x^*, z^*)$, then $In(y^*, z^*)$; in particular, $In(y^*, y^*)$.*
(3) *$\leq_{y^*}$ is a linear order relation.*
(4) *If $z \in G \setminus N$ is such that $Inc(x^*, z^*)$, then $Inc(y^*, z^*)$.*
(5) *$N_{\leq_{y^*} 0} \supseteq N_{\leq_{x^*} 0}$, and hence $U_{y^*} \supseteq U_{x^*}$.*

Proof. (1): This is 3.10 in [27]: Suppose $Inc(x^*, y^*)$ does not hold. Then there is $a \in \mathbb{P}_{x^*}$ such that $N(a) \subseteq N(b)$, for all $b \in \mathbb{P}_{y^*}$, so $Inc(y^*, x^*)$ holds.

(2): Let $b \in Ny$ and $c \in Nz$. Suppose that $N(b) \nsubseteq N(c)$. We must show that $N(b) \supseteq N(c)$. Let $N(b) \ni n \notin N(c)$, then replacing b by $n^{-1}b$ and c by $n^{-1}c$, we may assume that $N(b) \ni 1 \notin N(c)$. By $Inc(y^*, x^*)$, we can pick $a \in \mathbb{P}_{x^*}$, with $N(b) \supseteq N(a)$. As $N(a) \ni 1 \notin N(c)$, $In(x^*, z^*)$ implies that $N(a) \supseteq N(c)$, so $N(b) \supseteq N(a) \supseteq N(c)$, as asserted. Since we have $Inc(y^*, x^*)$ and $In(x^*, y^*)$, the second part of (2) follows from the first.

(3): By (2) we have $In(y^*, y^*)$. Hence, by definition, for $m, n \in N$ we either have $N(my) \subseteq N(ny)$ or $N(ny) \subseteq N(my)$, i.e., either $mU_{y^*} \leq_{y^*} nU_{y^*}$ or $nU_{y^*} \leq_{y^*} mU_{y^*}$.

(4): First, by (2) we have $In(y^*, z^*)$. Let $b \in \mathbb{P}_{y^*}$. By $Inc(y^*, x^*)$, there is $a \in \mathbb{P}_{x^*}$, with $N(b) \supseteq N(a)$. By $Inc(x^*, z^*)$, there is $c \in \mathbb{P}_{z^*}$, with $N(a) \supseteq N(c)$. Thus $N(b) \supseteq N(a) \supseteq N(c)$, and we get $Inc(y^*, z^*)$.

(5): This follows from Lemma 6.5.1. Indeed, let $b \in \mathbb{P}_{y^*}$, and (using $Inc(y^*, x^*)$) pick $a \in \mathbb{P}_{x^*}$ with $N(b) \supseteq N(a)$. Then $N(b) \supseteq N(a) \supseteq N_{\leq_{x^*} 0}$. As this holds for all $b \in \mathbb{P}_{y^*}$, $N_{\leq_{x^*} 0} \subseteq N_{\leq_{y^*} 0}$. The rest of (5) follows from the definitions. □

Corollary 6.13. *Let $x, y \in D^\times \smallsetminus N$, with $d(x^*, y^*) \geq 4$. Then, after perhaps interchanging x and y, we have that $\leq_{y^*}$ is a linear order relation.*

Proof. By Proposition 6.11, $In(x^*, y^*)$, and by Lemma 6.12.1, after perhaps interchanging x and y, we may assume that $Inc(y^*, x^*)$. Then by Lemma 6.12.3, $\leq_{y^*}$ is linear. □

Remark 6.14. We note that the preorder relation $\mathfrak{P}_{y^*}$ is precisely the preorder relation $\mathcal{R}_y$ of [20], where it was defined differently. The relation $\leq_{y^*}$ is a minor modification of the relation given is Sect. 6 of [27] (the relation in [27] uses the conjugacy class of y^*, while the relation $\leq_{y^*}$ uses just y^*).

7. The proof of Theorem 2

In this section we complete the proof of Theorem 6.1 and hence also of Theorem 2 (of the introduction). We let $x, y \in D^\times \smallsetminus N$ such that

$$d(y^*, x^*) \geq 4.$$

We start with

Lemma 7.1. *We may (and we do) assume that*

(1) *$Inc(y^*, x^*)$ and $Inc((y^{-1})^*, x^*)$.*

(2) *$Inc((y^{-1})^*, y^*)$.*

Proof. (1): By Proposition 6.11, for any $\epsilon, \nu \in \{1, -1\}$ we have $In((x^\epsilon)^*, (y^\nu)^*)$ and hence either $Inc((x^\epsilon)^*, (y^\nu)^*)$ or $Inc((y^\nu)^*, (x^\epsilon)^*)$ (Lemma 6.12.1). We may also assume that $Inc(y^*, x^*)$.

Assume that $Inc(x^*, (y^{-1})^*)$. Suppose also $Inc((x^{-1})^*, (y^{-1})^*)$. Then replacing y by x and x by y^{-1}, we are done.

Hence we may assume that $Inc((y^{-1})^*, (x^{-1})^*)$. But now we have $Inc(y^*, x^*)$, $Inc(x^*, (y^{-1})^*)$, $Inc((y^{-1})^*, (x^{-1})^*)$. So, Lemma 6.12.4 implies that $Inc(y^*, (x^{-1})^*)$. Replacing x by x^{-1}, we are done.

(2): First, since $Inc(y^*, x^*)$ and $In((y^{-1})^*, x^*)$ holds, Lemma 6.12.2 implies that $In((y^{-1})^*, y^*)$ holds. By Lemma 6.12.1, replacing y by y^{-1} if necessary, we may assume that $Inc((y^{-1})^*, y^*)$. □

We will show that for y^* as in Lemma 7.1, φ_{y^*} has all the properties asserted in Theorem 6.1. We simplify notation to

$$U = U_{y^*}, \quad \Gamma = \Gamma_{y^*} \quad \leq = \leq_{y^*} \quad \text{and} \quad \mathbb{P} = \mathbb{P}_{y^*}.$$

Note that by 6.12.5

$$U \supseteq U_{x^*} \subseteq U_{(y^{-1})^*}.$$

Since $Inc(y^*, x^*)$ holds, Corollary 6.13 implies that $\leq$ is a linear order relation. It remains to show that φ has a level.

Lemma 7.2. *Let $b \in \mathbb{P}$. Then $b + u \in U_{x^*}$, for all $u \in U_{x^*}$.*

Proof. The only hypotheseses required to prove this lemma is that $d(x^*, y^*) \geq 3$ and that $u \in U_{y^*}$ (which hold here). Since $b \in \mathbb{P}$ and $u \in U_{x^*} \subseteq U$, we have $b + u \in N$. By Lemma 6.8.2, $N(xb + xu) = N(xu) = N(x)$, because $(xb+xu)^* = x^*$, and therefore all distances are as required in Lemma 6.8.2. By definition, $b + u \in U_{x^*}$. □

The following proposition is the main step towards showing the existence of a level for φ.

Proposition 7.3. *Let $b \in \mathbb{P}$, $n \in N(b^{-1})$, and set $n^{-1}U =: \alpha \in \Gamma$. Then*
(1) $\alpha > 0$.
(2) $n \in N(b^{-1} + u)$*, for all* $u \in U$.
(3) $u + n \in N$ *and furhermore* $u + n \in N(b^{-1})$*, for all* $u \in U$.
(4) $n^{-1} + 1 \in N_{\leq 0}$.
(5) *We have* $N_{>\alpha} + 1 \subseteq N_{\leq 0}$.

Proof. (1): Let $a \in \mathbb{P}_{x^*}$. By Lemma 6.9.3 and 6.9.2 respectively, $N(b^{-1}) \subseteq N(ab^{-1}) \subseteq N(a)$. As this holds for all $a \in \mathbb{P}_{x^*}$, Lemma 6.5.1 says that $N(b^{-1}) \subseteq N_{\leq_{x^*} 0}$. By Lemma 6.12.5, $N(b^{-1}) \subseteq N_{\leq_{y^*} 0}$. Since $U \subseteq N(b)$, $U \cap N(b^{-1}) = \emptyset$, by Lemma 6.3.4, hence $n \notin U$, so $nU < 0$.

(2): We have $b^{-1} + u = b^{-1}(bu + 1)$. Now since $u \in U$, $bu \in \mathbb{P}$ (see 6.5.2), and it follows from Lemma 7.2 that $bu + 1 \in U_{x^*} \subseteq U_{(y^{-1})^*}$. Hence $n \in N(b^{-1}) = N(b^{-1}(bu + 1)) = N(b^{-1} + u)$.

(3): First we show that there exists $x' \in Nx$, with

(i) $$u \notin N(x') \ni n.$$

Pick $a \in \mathbb{P}_{x^*}$, with $N(b) \supseteq N(a)$ (using $Inc(y^*, x^*)$) and set $x' := na$. By Lemma 6.3.4 $n^{-1} \notin N(b)$, hence $1 \notin N(nb)$. Since $u \in U$, we get that $u \notin N(nbu) = N(nb) \supseteq N(na) = N(x')$. This shows (i).

Consider the element $b^{-1} + x' + u$. Notice that since $d(x^*, y^*) \geq 4$, we must have $d((b^{-1})^*, (x' + u)^*) \geq 3 \leq d((b^{-1} + u)^*, (x')^*)$. It follows from 6.8, that $N(b^{-1} + x' + u) = N(b^{-1} + u) \cap N(x') = N(b^{-1}) \cap N(x' + u)$. Note that $n \in N(b^{-1} + u) \cap N(x') \cap N(b^{-1})$ and it follows that $n \in N(x' + u)$.

We thus have $x' + (u + n) \in N \ni b^{-1} + (u + n)$ so if $u + n \notin N$, then $d((x')^*, (u + n)^*) \leq 1 \geq d((b^{-1})^*, (u + n)^*)$, contradicting $d(x^*, y^*) > 2$. Since $b^{-1} + u + n \in N$, we see that $u + n \in N(b^{-1})$.

(4): First note that $n^{-1} + 1 = n^{-1}(n + 1) \in N$, by (3). Let $c \in \mathbb{P}$ and set $u := c + 1 \in U_{x^*} \subseteq U$ (see 7.2). Then $c + 1 + n^{-1} = u + n^{-1} = u(n + u^{-1})n^{-1} \in N$, by (3). So, by Lemma 6.5.1, we have $1 + n^{-1} \in N_{\leq 0}$.

(5): Let $m \in N_{>\alpha}$, then $m^{-1}U < nU$. By Lemma 7.1, $Inc((y^{-1})^*, y^*)$ holds, so by Lemma 6.12.5, $N_{\leq_{(y^{-1})^*} 0} \supseteq N_{\leq 0}$. It follows that $m^{-1}U_{(y^{-1})^*} \leq nU_{(y^{-1})^*}$ and hence by Lemma 6.5.3 that $m^{-1} \in N(b^{-1})$. By (4), $m + 1 \in N_{\leq 0}$. □

Proof of Theorem 6.1. We already have our map $\varphi = \varphi_{y^*}$ and we already saw that $\Gamma = \Gamma_{y^*}$ is a totally ordered group. Take now $b \in \mathbb{P}$, pick $n \in N(b^{-1})$ and set $\alpha := n^{-1}U$. Then Proposition 7.3.5 together with the equality $x + 1 = x(x^{-1} + 1)$ immediatly implies that any $\beta \geq \alpha$ is a level of φ completing the proof of the theorem. □

Remark 7.4. One can show that if $d(x^*, y^*) \geq 5$, then φ has 0 as a level, see Sect. 6 in [27] (where it is shown for the valuation-like map v) or [20].

8. The proof of "Nonexistence Theorem at Diameter ≥ 4"

Since the assumption that $\text{diam}(\Delta(D^\times/N)) \geq 4$ automatically implies that $-1 \in N$, Theorem 3 (of the introduction) immediately follows from Theorems 1 and 2 (and moreover, we may take the resulting valuation v to have height one, cf. Theorem 5.1). Example 8.4 at the end of the section shows that Theorem 3 is the best possible.

In this section we will use Theorem 3 to eliminating some finite groups as possible quotients of $D^\times$. The idea, described already in [19], is based on replacing the division algebra D with the residue algebra $\bar{D}$ relative to the valuation constructed in Theorem 3 while keeping track of what part of the original finite quotient of $D^\times$ can still be obtained as a quotient of $\bar{D}^\times$. As noted in the introduction, the use of $\bar{D}$ was suggested by the use of the residue algebra R/I in [27] (see Appendix B).

We will continue to use the notation introduced in Sect. 2. In particular, given a valuation $v\colon D^\times \to \tilde{\Gamma}$, we let O (resp., $\mathfrak{m}$) denote the valuation ring (resp., the valuation ideal) in D, so that $\bar{D} = O/\mathfrak{m}$ is the corresponding residue (division) algebra. In addition, we let U be the group of v-adic units, i.e., $U = \{x \in D^\times \mid v(x) = 0\}$ and for $0 \leq \alpha \in \tilde{\Gamma}$, $\mathfrak{n}(\alpha) = \{x \in O \mid v(x) \geq \alpha\}$, observing that $U \lhd D^\times$, the quotient $D^\times/U \simeq \tilde{\Gamma}$ is abelian, and $\mathfrak{n}(\alpha)$ is a two-sided ideal of O.

To compare the quotients of $D^\times$ and $\bar{D}^\times$ we need the following.

Lemma 8.1. *Let v be a height one valuation of D and $N \subseteq D^\times$ be a v-adically open finite index normal subgroup. Then there exists a normal subgroup $\bar{N} \subseteq \bar{D}^\times$ in the multiplicative group of the residue algebra $\bar{D}$ such that the quotients $H := D^\times/N$ and $\bar{H} := \bar{D}^\times/\bar{N}$ are related as follows:*

$$\bar{H} \simeq H_1/M_1 \quad \textit{where} \quad M_1 \lhd H_1 \lhd H \quad \textit{and} \quad H/H_1,\ M_1 \quad \textit{are solvable}.$$

Proof. The reduction homomorphism $O \to \bar{D}$ induces a surjective group homomorphism $\rho\colon U \to \bar{D}^\times$ with the kernel $\operatorname{Ker}\rho = 1 + \mathfrak{m}$. Let $\bar{N} = \rho(N \cap U)$, $H_1 = UN/N$, and $M_1 = (1 + \mathfrak{m})N/N$ (observe that $1 + \mathfrak{m}$ is normal not only in U, but also in $D^\times$). Then $M_1 \lhd H_1 \lhd H$ and

$$\bar{H} = \rho(U)/\rho(N \cap U) \simeq U/(1 + \mathfrak{m})(N \cap U) \simeq H_1/M_1.$$

Furthermore, $H/H_1 \simeq D^\times/UN$ is a quotient of $\tilde{\Gamma}$, hence abelian. So, it remains to prove that M_1 is solvable. For this we need to recall a well-known commutator relation for the congruence subgroups.

Lemma 8.2. *Let $\alpha, \beta \in \tilde{\Gamma}$ be positive elements. Then*

(1) $1 + \mathfrak{n}(\alpha)$ *is a normal subgroup of* U;
(2) *the commutator subgroup* $[1+\mathfrak{n}(\alpha), 1+\mathfrak{n}(\beta)]$ *is contained in* $1+\mathfrak{n}(\alpha+\beta)$.

Proof. Indeed, (1) is an easy consequence of the fact that $\mathfrak{n}(\alpha)$ is a two-sided ideal of O contained in $\mathfrak{m}$. Now, for any $x \in \mathfrak{n}(\alpha)$, $y \in \mathfrak{n}(\beta)$ we obviously have

$$(1 + x)(1 + y) \equiv (1 + y)(1 + x) (\operatorname{mod} \mathfrak{n}(\alpha + \beta))$$

as $xy - yx \in \mathfrak{n}(\alpha + \beta)$, implying that the (multiplicative) commutator $[1 + x, 1 + y] \in 1 + \mathfrak{n}(\alpha + \beta)$, and (2) follows. □

Now, let $1+x_1, 1+x_2, \ldots, 1+x_r$ be a transversal for N in $(1+\mathfrak{m})N$. We let $\gamma_i := v(x_i)$ and $\gamma = \min_i \gamma_i, \gamma > 0$. Then $M = (1+\mathfrak{m})N$ coincides with $(1 + \mathfrak{n}(\gamma))N$. Consider the descending central series of M : $\mathfrak{C}^0(M) = M$, $\mathfrak{C}^{i+1}(M) = [M, \mathfrak{C}^i(M)]$ for $i \geq 0$. It easily follows from Lemma 8.2.2 that for any $i > 0$

(i) $$\mathfrak{C}^i(M) \subseteq (1 + \mathfrak{n}(i\gamma))N.$$

On the other hand, since N is open, there exists a nonnegative $\alpha \in \tilde{\Gamma}$ such that $1+\mathfrak{m}(\alpha) \subseteq N$. Pick a natural number l sufficiently large, so that $l\gamma > \alpha$ (which is possible since $\tilde{\Gamma}$ has height one). Then $\mathfrak{n}(l\gamma) \subseteq \mathfrak{m}(\alpha)$, which in view of (i) implies that $\mathfrak{C}^l(M) \subseteq N$, i.e. M_1 is nilpotent, completing the proof. □

Proof of Nonexistence Theorem at Diameter ≥ 4. Suppose some member of $\mathcal{G}$ is a quotient of the multiplicative group of some finite dimensional central division algebra D over K and choose such $H \in \mathcal{G}$ of minimal size. By Lemma 5.5 we may assume that K is finitely generated. Let $t = \text{tr.deg}_{K_0} K$ be the transcendence degree of K over the prime subfield $K_0 \subseteq K$. If there are presentations of H as a quotient of $D^\times$ with K having positive characteristic, pick such a presentation (with finitely generated K) for which t is minimal. Otherwise, all presentations have characteristic zero, and again we pick amongst these a presentation with minimal t. Let $N := \text{Ker}(D^\times \to H)$. The fact that H was chosen to have minimal size implies that H is minimal in $\mathcal{G}$ in the sense specified in the "Nonexistence Theorem at Diameter ≥ 4", so in view of hypothesis (4) therein and Theorem 3, N is v-adically open in $D^\times$ for some nontrivial valuation v of D having height one. It follows from Lemma 8.1 that the multiplicative group $\bar{D}^\times$ of the residue algebra $\bar{D}$ has a quotient $\bar{H}$ with the following structure: $\bar{H} = H_1/M_1$ where $M_1 \lhd H_1 \lhd H$ and the groups H/H_1 and M_1 are solvable. By hypotheseses (2) and (3) of "Nonexistence Theorem at Diameter ≥ 4", we obtain that $\bar{H} \in \mathcal{G}$. Since $\dim_{\bar{K}} \bar{D} \leq \dim_K D$ where $\bar{K}$ is the residue field for the restriction of v to K (cf. Cor. 1 on p. 20 in [26]), $\bar{D}$ is finite dimensional, so the minimality of size in the choice of H implies that $|\bar{H}| = |H|$ and $\bar{H} \simeq H$, i.e. H is a quotient also of $\bar{D}^\times$. It follows from our choice of D that $\bar{D}$ cannot have positive characteristic if D has characteristic zero; in other words,

$$\text{(ii)} \qquad \text{char } \bar{D} = \text{char } D.$$

Now, the restriction of v to K is nontrivial ([26], Lemma 1.13), however it follows from (ii) that the restriction of v to the prime subfield K_0 is trivial. Then the residue field $\bar{K}$ has the same prime subfield K_0 and according to [5], Sect. 10, n° 3, Cor. 4,

$$\text{(iii)} \qquad \text{tr.deg}_{K_0} \bar{K} < \text{tr.deg}_{K_0} K.$$

Applying Lemma 5.5 we obtain the existence of a finitely generated subfield $K_1 \subseteq \bar{K}$ and a division K_1-subalgebra $D_1 \subseteq \bar{D}$ such that $\bar{D} = D_1 \otimes_{K_1} \bar{K}$ and H is a quotient of $D_1^\times$. Then in view of (iii), $\text{tr.deg}_{K_0} K_1 < t$ - a contradiction, proving the theorem. □

Remark 8.3. The proof of "Nonexistence Theorem at Diameter ≥ 4" works in fact under more general assumptions. For example, instead of hypothesis (4) it is sufficient to require that the kernel of a surjective homomorphism $D^\times \to H$ onto a minimal member $H \in \mathcal{G}$ be open with repect to a finite set $T = \{v_1, \ldots, v_d\}$ of nontrivial valuations of D (then O needs to be replaced with the intersection of the valuation rings of the v_i's, $\bar{D}$ - with the direct product $\bar{D}_{v_1} \times \cdots \times \bar{D}_{v_d}$ of the corresponding residue algebras, etc). This explains why an affirmative answer to the question posed at the end of the introduction would allow one to give an alternative (i.e. different from the one obtained in [22]) proof of Conjecture F.So.Q.

We conclude Sect. 8 with an example showing that Theorem 3 is the best possible (cf., however, [22]).

Example 8.4. We construct here an example of a division algebra D and a finite index normal subgroup $N \subseteq D^{\times}$ such that $\mathrm{diam}(\Delta(D^{\times}/N)) = 3$, but N is not open with respect to *any* nontrivial valuation of D. The construction below is of general nature and allows one to produce such examples involving algebras of various degrees, however to keep the example simple we work with quaternion algebras.

Let K be a an algebraic number field that admits two inequivalent valuations v_1 and v_2 with the completion $K_{v_i} = \mathbb{Q}_2$ for $i = 1, 2$ (one can take, for example, $K = \mathbb{Q}(\sqrt{17})$). It follows from the description of the Brauer group of K (cf. [11], Sect. 18.7) that there exists a quaternion central division algebra D over K such that the algebras $D_{v_i} = D \otimes_K K_{v_i}$ for $i = 1, 2$, are division algebras (over $K_{v_i} = \mathbb{Q}_2$). To construct N we need one additional notation: given a finite set S of inequivalent (nontivial) valuations of K, we denote $D_S^{\times} = \prod_{v \in S} D_v^{\times}$ with the topology of the direct product (where, of course, $D_v = D \otimes_K K_v$ is endowed with the topology of a vector space over K_v), and let $\iota_S\colon D^{\times} \to D_S^{\times}$ denote the diagonal embedding. Next we observe that if $\mathcal{D}$ is a quarternion division algebra over $\mathcal{K} = \mathbb{Q}_2$, then $\mathcal{D}^{\times}$ possess a normal subgroup $\mathcal{N}_0$ such that $\mathcal{D}^{\times}/\mathcal{N}_0 \simeq \Sigma_3$, the symmetric group on three letters: indeed let $\mathcal{N}_0 = \mathcal{K}^{\times}(1+\mathfrak{m})$, where $\mathfrak{m}$ is the valuation ideal in $\mathcal{D}$; then one easily checks that $\mathcal{D}^{\times}/\mathcal{N}_0$ is a nonabelian group of order 6, hence isomorphic to Σ_3.

Now, take such a subgroup $\mathcal{N}_{v_i} \subseteq D_{v_i}^{\times}$ for $i = 1, 2$, and let $\mathcal{N} = \mathcal{N}_{v_1} \times \mathcal{N}_{v_2} \subseteq D_{\{v_1, v_2\}}^{\times}$ and $N = \iota_{\{v_1, v_2\}}^{-1}(\mathcal{N})$. We claim that $D^{\times}/N \simeq \Sigma_3 \times \Sigma_3$ and that N is not open with respect to any valuation of D. Since the diameter of the commuting graph of $\Sigma_3 \times \Sigma_3$ is 3, we will be done. To see the properties required from N we need the following.

Lemma 8.5. *Let S be a finite set of inequivalent valuations of K, $\mathcal{N} \subseteq D_S^{\times}$ be an open normal subgroup and $N = \iota_S^{-1}(\mathcal{N})$. Then*

(1) $D^{\times}/N \simeq D_S^{\times}/\mathcal{N}$.

(2) *For any nontrivial valuation $w \notin S$ of K, $\iota_w(N)$ is dense in $D_w^{\times}$.*

Proof. Easily follows from the weak approximation for $D^{\times}$ and will be omitted. □

It follows from Lemma 8.5.1 (applied to $S = \{v_1, v_2\}$) that $D^{\times}/N \simeq \Sigma_3 \times \Sigma_3$. Suppose N is open in $D^{\times}$ with respect to a nontrivial valuation $\tilde{v}$ of D, and let v denote the restriction of $\tilde{v}$ to K (it is a nontrivial valuation of K). Pick $w \in \{v_1, v_2\}$ distinct from v. Then by Lemma 8.5.2 (applied to $S = \{v\}$), $\iota_w(N)$ is dense in $D_w^{\times}$ which contradicts the fact that $\iota_w(N) \subseteq \mathcal{N}_w$ and the latter is a proper closed subgroup.

Appendices

Appendix A. A survey of the Margulis-Platonov conjecture

Although questions dealing with the openness of normal subgroups were not previously examined (or even raised) in the context of arbitrary fields, their analysis over special fields (primarily, local and global) has been an area of active research for quite some time. The main efforts (over global fields) were focused on proving the Margulis-Platonov conjecture which is briefly surveyed in this appendix (for a more detailed discussion see Ch. IX in [14]). We hope that this survey will provide the reader with an adequate perspective on the results of the current paper.

First, we recall the following general local result which follows from the classical theory of Lie groups over the field $\mathcal{K} = \mathbb{R}$ and was established by C. Riehm if $\mathcal{K}$ is a field complete with respect to a discrete valuation: if $\mathcal{G}$ is a simple algebraic group over such $\mathcal{K}$, then any noncentral normal subgroup of the group of rational points $\mathcal{G}(\mathcal{K})$ is open in the topology induced by that of $\mathcal{K}$. Moreover, it is known that if in addition $\mathcal{G}$ is simply connected, then $\mathcal{G}(\mathcal{K})$ in fact does not have any proper noncentral normal subgroups (i.e. is *projectively simple*) in the following cases: 1) $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$ (E. Cartan), and 2) $\mathcal{K}$ is a nonarchimedean locally compact field and $\mathcal{G}$ is $\mathcal{K}$-isotropic (a consequence of Platonov's proof of the Kneser-Tits conjecture). On the contrary, if $\mathcal{K}$ is a nonarchimedean locally compact field and $\mathcal{G}$ is $\mathcal{K}$-anisotropic, then the group $\mathcal{G}$ is compact and totally disconnected, hence profinite, and therefore is approximated by (open) normal subgroups of finite index. In fact, a more precise information is available in the latter case: $\mathcal{G} \simeq \mathbf{SL}_{1,\mathcal{D}}$, the algebraic group associated with the norm 1 group in a finite dimensional central division algebra $\mathcal{D}$ over $\mathcal{K}$; the valuation extends from $\mathcal{K}$ to $\mathcal{D}$, and then a simple analysis of the filtration given by the congruence subgroups modulo the powers of the valuation ideal in $\mathcal{D}$ shows that $\mathcal{G}(\mathcal{K})$ is an extension of a pro-p group by a finite cyclic group where p is the characteristic of the residue field for $\mathcal{K}$; in particular, for any noncentral normal subgroup $\mathcal{N} \subseteq \mathcal{G}(\mathcal{K})$, the quotient $\mathcal{G}(\mathcal{K})/\mathcal{N}$ is solvable.

Let now G be an absolutely simple simply connected algebraic group over a global field K. The Margulis-Platonov conjecture relates the normal subgroup structure of the group of K-rational points $G(K)$ to that of the "local" groups $G(K_v)$ over the completions K_v of K where v runs through all valuations of K. It follows from the discussion above that the group $G(K_v)$ is projectively simple unless v is a nonarchimedean valuation such that G is K_v-anisotropic, and in any case the noncentral normal subgroups of $G(K_v)$ are open in the v-adic topology. This motivates the following conjecture, known as the Margulis-Platonov conjecture:

(MP): *Let T be the set of all nonarchimedean valuations v of K for which the group G is K_v-anisotropic. Then for any noncentral normal subgroup $N \subseteq G(K)$ there should exist an open normal subgroup*

$W \subseteq G_T = \prod_{v \in T} G(K_v)$ such that $N = \delta^{-1}(W)$ where $\delta\colon G(K) \to G_T$ is the diagonal map; in particular, if $T = \emptyset$, then $G(K)$ should be projectively simple (no proper noncentral normal subgroups).

(Speaking about (MP), one should bear in mind two finiteness results: any noncentral normal $N \subseteq G(K)$ has finite index (cf. [9], [15]), and T is always finite which follows, for example, from Theorem 6.7 in [14]. Also, the (product) topology on the group G_T is sometimes called T-adic, so (MP) claims that all noncentral normal subgroups of $G(K)$ should be T-adically open in $G(K)$.)

Historically, the first question of this nature was raised by M. Kneser. In 1956 he established the projective simplicity of $G(K)$ for $G = \mathrm{Spin}_n(f)$ where f is a nondegenerate quadratic form over K in $n \geq 5$ variables and conjectured that the result should still hold for $G = \mathrm{Spin}_3(f)$ (the case $n = 4$ easily reduces to the case $n = 3$) if the form f is isotropic over all nonarchimedean completions of K (i.e. $T = \emptyset$ in our notations). Kneser's conjecture was generalized to arbitrary simple simply connected groups by V.P. Platonov in his ICM-74 talk in the form of a local-global principle: *the group $G(K)$ is projectively simple if and only if the local groups $G(K_v)$ are projectively simple for all nonarchimedean v.* We observe that the latter condition is precisely equivalent to $T = \emptyset$, and that it is satisfied automatically if G is not of type A_n. To include groups with anisotropic nonarchimedean completions, Platonov's conjecture was generalized by G.A. Margulis [9] who gave the above formulation of (MP).

For a K-isotropic group G, the conjecture (MP) simply claims that the group $G(K)$ should be projectively simple. This has already been proved for all groups over global fields except one rank one form of type E_6, so the main emphasis in (MP) is really made on anisotropic groups. However, for more than twenty years Kneser's theorem for the spinor groups remained the only result about simplicity which allowed anisotropic groups. A breakthrough occurred in the late 70-s and 80-s. First, it was shown in [12] that if D is quaternion algebra and $G = \mathbf{SL}_{1,D}$, then under the assumption that $T = \emptyset$, the group $G(K)$ is perfect, i.e. $G(K) = [G(K), G(K)]$ (we observe that this group is the same as in Kneser's original conjecture, viz. $G \simeq \mathrm{Spin}_3(f)$ for some quadratic form f). Using some techniques of [12], Margulis [10] proved (MP) for such G in full. Subsequently, the result of [10] played a role of "the base of induction" in the proof of (MP) for the groups of classical types B_n $(n \geq 2)$, C_n $(n \geq 2)$, D_n $(n \geq 4)$ and the special unitary groups $\mathbf{SU}_n(f)$ $(n \geq 3)$ of type A_{n-1} associated with a nondegenerate hermitian form f over a quadratic extension L/K (M. Borovoi, Platonov-Rapinchuk, G. Tomanov) as well as exceptional types E_7, E_8, F_4 and some forms of type E_6 (V. Chernousov); type G_2 can be treated along with either classical or exceptional groups. Returning to the groups of the form $G = \mathbf{SL}_{1,D}$ associated with a central division algebra D over K, in [13] the methods and results of [12] were extended from quaternion algebras to algebras of arbitrary degree to prove (MP) for the commutator subgroup $N = [G(K), G(K)]$. Raghunathan [17] elaborated on this result and showed that if a noncentral normal subgroup $N \subseteq G(K)$ satisfies (MP) (i.e. is T-adically open), then so does its commutator subgroup $[N, N]$. (It was shown in [14], Sect. 9.2, that this can also be derived from the result in [13] and computations of H^2 for the norm 1 group of a local division algebra carried out by Prasad-Raghunathan in [16]). Tomanov [32] extended Margulis's result [10] from quaternion algebras to algebras of degree 2^d, $d \geq 1$, and moreover reduced (MP) to algebras of odd degree.

All aforementioned results were obtained on the basis of arithmetic methods (local-global principle, etc.), however subsequent attempts to complete with their help the proof of (MP) for the groups of the form $\mathbf{SL}_{1,D}$ for an arbitrary division algebra D turned out to be unsuccessful which suggested that new methods were needed. Such methods were found by departing from purely arithmetic techniques and putting the problem in a more general algebraic context. First, in [18] Margulis's finiteness theorem and the results from [17] were used to reduce (MP) for $G = \mathbf{SL}_{1,D}$ to the assertion that the multiplicative group $D^\times$ does not have normal subgroups N such that $D^\times/N$ is a (nonabelian) finite simple group. As this statement (in contrast to (MP)) makes sense for division algebras over arbitrary fields, its truth was conjectured in [18] for all finite dimensional division algebras and verified for algebras of degree 2 and 3 over arbitrary fields. This conjecture was proved in full in the papers [27] and [31] which completed the proof of (MP) for the groups of the form $\mathbf{SL}_{1,D}$[2].

More precisely, in [27], Segev proved that a finite simple group cannot appear as a quotient of $D^\times$ if either its commuting graph has diameter ≥ 5, or is balanced, and then in [31] Segev and Seitz verified using the classification of finite simple groups that *any* nonabelian finite simple group satisfies one of these conditions. It should be pointed out that the success in [27] was achieved due to a partial return to arithmetic methods in the context of arbitrary fields as some constructions therein were equivalent to defining valuations in the general setting (cf. Appendix B for more details), though valuations did not appear in [27] explicitly. After reading [27], Rapinchuk pointed out in [19] the expediency of valuation theory in this context and subsequently proved in [20] Theorem 1. This result led Segev to extend some of the results of Sects. 2–4 in [27] to construct what we now call valuation-like maps under weaker assumptions than in [27], viz. assuming only that the diameter of $\Delta(D^\times/N)$ is ≥ 4 (Theorem 2). Combin-

[2] We note that this result in combination with the local results mentined in the beginning implies that for any noncentral normal subgroup $N \subseteq G(K)$ the quotient $G(K)/N$ is solvable. Since the quotient $D^\times/G(K)$ is abelian, one immediately obtains that for any noncentral normal subgroup $N \subseteq D^\times$, the quotient $D^\times/N$ is solvable; in particular Conjecture F.So.Q. holds over global fields.

ing Theorems 1 and 2, one obtains Theorem 3 which, generally speaking, gives the best possible answer to the question when a finite index normal subgroup $N \subseteq D^{\times}$ is open with respect to a nontrivial valuation of D (see, however, the congruence subgroup theorem obtained in [22]). How does this result relate to (MP)? The point is that for the group $G = \mathbf{SL}_{1,D}$ the set T introduced in the statement of (MP) coincides with the set of all nonarchimedean valuations v of K for which $D \otimes_K K_v$ remains a division algebra, and these are precisely the (nonarchimedean) valuations of K that extend to D. In effect, what (MP) claims is that any noncentral normal subgroup $N \subseteq G(K)$ (which is equivalent to N having a finite index) should be open with respect to the topology defined by the valuations of D. So, Theorem 3 is precisely this kind of a result for the finite index normal subgroups in $D^{\times}$ and a single valuation. The next natural step in this direction should be the investigation of the full analog of (MP), i.e. the determination of general conditions that ensure the openness of a finite index normal subgroup $N \subseteq D^{\times}$ with respect to the topology defined by a finite collection of valuations – we hope that the condition $\operatorname{diam}(\Delta(D^{\times}/N)) \geq 3$ will be sufficient. As we pointed out in the introduction, this will give an alternative proof (i.e., different from the one obtained in [22]) of Conjecture F.So.Q. that all finite quotients of $D^{\times}$ are solvable.

In the conclusion, we remark that the case of anisotropic outer forms of type A_n in (MP) still remains open, and its resolution will most probably require additional new methods.

Appendix B. The local ring R of [27] is a valuation ring

In this short appendix we show that the local ring R constructed in Sects. 7–10 of [27] is in fact a valuation ring of a valuation extending the canonical homomorphism

$$v \colon N \to \Gamma$$

from N onto a totally ordered group Γ given in Sect. 6 of [27]. This fact was neither noted, nor proved in [27]. We continue with the notation of Sect. 6 of this paper. We emphasize that (unlike in this paper), in Sects. 7–10 of [27] it is assumed that G^* *is a nonabelian simple group*. In addition it is assumed that the commuting graph $\Delta(G^*)$ of G^* either has diameter ≥ 5 or is balanced, this hypothesis was later proved to hold true for all nonabelian finite simple groups in [31]. Recall that balanced is defined by: there are nonidentity elements $x^*, y^* \in G^*$ such that all the distances $d(x^*, y^*)$, $d(x^*, x^*y^*)$, $d(x^*, (x^{-1})^*y^*)$, $d(y^*, x^*y^*)$ and $d(y^*, (x^{-1})^*y^*)$ are ≥ 4. As in [27], let U be the kernel of v. By 7.1.1 in [27], $G = \mathbb{O}N$, where in the notation of [27], $G = D^{\times}$ and $\mathbb{O} = SL_1(D)$ are the elements of reduced norm 1. Define a map

$$w : D^{\times} \to \Gamma$$

as follows. Let $d \in D^\times$ and write $d = xn$, with $x \in \mathbb{O}$ and $n \in N$. Define $w(d) = Un \in \Gamma$. By 7.1.2 in [27], $\mathbb{O}U \cap N = U$, so w is a well defined homomorphism. We claim that w is a valuation whose valuation ring is the ring R defined in 10.1.1 and 10.1.3 of [27] as follows:

(i) $R = \{d \in D^\times \mid d = xn, \text{ for some } x \in \mathbb{O} \text{ and } n \in U \cup \bar{\mathbb{N}}\} \cup \{0\},$

where $\bar{\mathbb{N}} = \{n \in N \mid v(n) > 0\}$; in particular, for $a \in D^\times$, $w(a) \geq 0$ iff $a \in R$. Notice that R is a ring by 10.2.3 of [27]. To show that w is a valuation, it suffices to show that if $-1 \neq a \in D^\times$, with $w(a) \geq 0$, then $w(a+1) \geq 0$. Since R is a ring, for $-1 \neq a \in R$, we have $a + 1 \in R$, so $w(a+1) \geq 0$, completing the proof that w is a valuation. It is immediate from the definitions that w extends the canonical map $v : N \to \Gamma$, and it is immediate from (i) that R is the valuation ring of w. Note that the ideal I defined in [27] Sect. 10 by

$$I = \{d \in D^\times \mid d = xn, \text{ for some } x \in \mathbb{O} \text{ and } n \in \bar{\mathbb{N}}\} \cup \{0\}$$

is the valuation ideal of w and so $\bar{D} := R/I$ is the residue division algebra of w.

Appendix C. A brief account of the notes [19]

As we mentioned in Sect. 5, for conjugation invariant valuation-like maps one can give an alternative proof of Theorem 5.1. This argument first appeared in [19], and we reproduce it here with minor changes. So, suppose $N \lhd D^\times$ and let $\varphi\colon N \to \Gamma$ be a conjugation invariant valuation-like map having a level α. It is a consequence of Wedderburn's factorization theorem (cf. [23], p. 253) that for $x \in D^\times$, the reduced norm $\mathrm{Nrd}_{D/K}(x)$ is the product of n conjugates of x, where n is the degree of D, implying that for $x \in N$, one has $\mathrm{Nrd}_{D/K}(x) \in N_K$, and

(i) $$\varphi(x^n) = \varphi(\mathrm{Nrd}_{D/K}(x)).$$

In the begining of the proof of Theorem 5.1 we have seen that there exists a height one valuation $v_0\colon K^\times \to \Gamma_0$ associated with the restriction $\varphi_K\colon N_K \to \Gamma_K = \varphi(N_K)$, so that $v_0 \mid_{N_K} = \theta \circ \varphi \mid_{N_K}$ for a certain homomorphism of ordered groups $\theta\colon \Gamma_K \to \Gamma_0$. Consider the following map:

(ii) $$v\colon D^\times \to \tilde{\Gamma} := \frac{1}{n}\Gamma_0, \quad v(x) = \frac{1}{n}v_0(\mathrm{Nrd}_{D/K}(x)).$$

(recall that Γ_0 is isomorphic to a subgroup of $(\mathbb{R}, +)$, so multiplication by $\frac{1}{n}$ makes sense). It follows from (i) that $n\varphi(N) \subseteq \Gamma_K$ implying that θ extends to a homomorphism of ordered groups $\tilde{\theta}\colon \varphi(N) \to \tilde{\Gamma}$, and then in view of (ii) we have $v \mid_N = \theta \circ \varphi$, i.e. v is associated with φ. So, it remains to prove that v is a valuation of D. Since v is a group homomorphism by construction, for the property $v(a+b) \geq \min(v(a), v(b))$ it suffices to verify the following implication:

$$\text{(iii)} \qquad v(a) \geq 0 \implies v(a+1) \geq 0$$

for any $a \in D^\times$, $a \neq -1$. Now, suppose we know that for any maximal subfield $P \subseteq D$ such that P/K is separable, $\tilde{P} := P \otimes_K K_{v_0}$ is a field, where K_{v_0} is the completion of K. We will derive (iii) from this assumption. Since $v((1+a)^{p^r}) = v(1+a^{p^r})$ if $p = \text{char } D > 0$, it suffices to prove (iii) only for elements which are separable over K. Any such element a is contained in a maximal subfield P which is separable over K (cf. [11], Sect. 13.5). Since $\tilde{P}$ is a field, v_0 admits a unique extension u to P given by:

$$\text{(iv)} \qquad u(x) = \frac{1}{n} v_0(\mathrm{N}_{P/K}(x)),$$

where $\mathrm{N}_{P/K}$ is the usual norm in the field extension (cf. [8]). Using the fact that $\mathrm{Nrd}_{D/K}(x) = \mathrm{N}_{P/K}(x)$ for all $x \in P$ (cf. [4], p. 28), we conclude from (ii) and (iv) that $u = v\,|_P$. In other words, the restriction $v\,|_P$ is a valuation, and (iii) follows.

It remains to show that condition (VL) in the definition of a valuation-like map, implies that for every maximal separable subfield $P \subseteq D$, $\tilde{P} = P \otimes_K K_{v_0}$ is a field. Suppose otherwise. Then $\tilde{P} \simeq \prod_{i=1}^r P_i$, where P_i/K_{v_0} is a finite extension of degree n_i, $n_1 + \cdots + n_r = n$, and $r > 1$. (We recall that P_i's are precisely the completions of P with respect to different extensions of v_0 to P (cf. [5], Sect. 8, Cor. 2), and $\tilde{P}$ gets endowed with the direct product of the topologies defined by those extensions; this topology coincides with the unique topology on $\tilde{P}$ as a finite dimensional vector space over K_{v_0}.) We will assume (as we may) that $n_1 \leq n_i$ for all i (in particular, $n_1 \leq n/2$). Pick $a \in N_{>\alpha}$. According to (v) in the proof of Lemma 5.7, there exists $b \in N_K$ such that $\varphi(b) \geq \varphi(a)$. Since v_0 is nontrivial, there exists $x \in K$ such that $v_0(x) > n v_0(b) =: \beta$. We let

$$y = (x^{-1} + 1, x^{2m} + 1, \ldots, x^{2m} + 1) \in \tilde{P},$$

where $m = |D^\times : N|$, and using weak approximation pick a sequence $z_1, \ldots, z_d, \ldots \in P$ converging to y. Observe that the function $f \colon \tilde{P} \to K_{v_0}$ defined by:

$$f(t) = \prod_{i=1}^{r} \mathrm{N}_{P_i/K_{v_0}}(t_i) \quad \text{for} \quad t = (t_1, \ldots t_r) \in \tilde{P},$$

is continuous and coincides with the norm $\mathrm{N}_{P/K}(t)$ on P. It follows that

$$f(z_d) \to f(y) = x^{-n_1}(x+1)^{n_1}(x^{2m}+1)^{n-n_1},$$

implying that $v_0(f(y)) < -n_1\beta \leq -\beta$, and hence that

$$v_0(f(z_d)) = v_0(\mathrm{N}_{P/K}(z_d)) = v_0(\mathrm{Nrd}_{D/K}(z_d)) < -\beta$$

for all sufficiently large d. We have: $z_d^m \in N$ and

$$n\theta\big(\varphi\big(z_d^m\big)\big) = v_0\big(\mathrm{Nrd}_{D/K}\big(z_d^m\big)\big) < -\beta,$$

and therefore $\varphi(z_d^m) < -\alpha$, i.e. $z_d^m \in N_{<-\alpha}$. On the other hand,

$$f\big(-z_d^m + 1\big) \to f(-y^m + 1) = (-(x^{-1}+1)^m + 1)^{n_1}(-(x^{2m}+1)^m + 1)^{n-n_1}.$$

We have:

$$-(x^{-1}+1)^m + 1 = x^{-m}a, \quad -(x^{2m}+1)^m + 1 = x^{2m}b$$

where $a = -(1+x)^m + x^m$ and $b = -(1 + (x^{2m}+1) + \cdots + (x^{2m}+1)^{m-1})$, and therefore

$$f(-y^m + 1) = x^{m(2n-3n_1)} \cdot a^{n_1} \cdot b^{n-n_1}.$$

So, we conclude from $v_0(a) \geq 0$ and $v_0(b) \geq 0$ that $v_0(f(-y^m + 1)) > 0$, implying that $v_0(f(-z_d^m + 1)) = v_0(\mathrm{Nrd}_{D/K}(-z_d^m + 1)) > 0$ for all sufficiently large d. This shows that for those d's the inclusion $-z_d^m + 1 \in N_{<-\alpha}$ is impossible as it would imply that

$$v_0\big(\mathrm{Nrd}_{D/K}\big(-z_d^m + 1\big)\big) = n\theta\big(\varphi\big(-z_d^m + 1\big)\big) \leq 0.$$

Thus, we obtain a contradiction to condition (VL) in the definition of a valuation-like map.

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