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Tenth order mock theta functions in Ramanujan's Lost Notebook

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1 Introduction

In S. Ramanujan's last letter to G.H. Hardy, Ramanujan proclaimed, "I discovered very interesting functions recently which I call 'Mock' ϑ-functions." He then provided a long list of 'third order,' 'fifth order,' and 'seventh order' mock theta functions together with identities satisfied by them. The introduction to this letter has evidently been lost; a portion of it can be found in Ramanujan's *Collected Papers* [R1, p. xxxi]. However, the 'mathematical' portion of the letter has been completely preserved. Extracts from it can be found in the *Collected Papers* [R1, pp. xxxi–xxxii, 354–355]. The complete mathematical portion is given in G.N. Watson's paper [W1], with the publication of Ramanujan's lost notebook [R2, pp. 127–131] (a photocopy of the original letter), in G.E. Andrews' survey paper [A3], and in B.C. Berndt and R.A. Rankin's book [B4, pp. 220–223]. All of the results on third and fifth order mock theta functions were proved in two long papers by Watson in 1936 and 1937 [W1], [W2]. Three seventh order mock theta functions were recorded, but no identities for them were given.

In 1976, Andrews rediscovered Ramanujan's lost notebook in the library of Trinity College, Cambridge. This lost notebook contains many further results on mock theta functions. In particular, several further results on fifth order mock theta functions were established by the combined efforts of Andrews [A1], [A2], Andrews and F.G. Garvan [A4], and Hickerson [H1]. Ramanujan's results on seventh order mock theta functions, absent in his letter to Hardy, can be found in the lost notebook, and these were proved by Andrews [A1] and Hickerson [H2]. Eleven identities for sixth order mock theta functions are found in the lost notebook; these were established by Andrews and Hickerson [A5]. Lastly, the lost notebook contains eight identities for tenth order mock theta functions which heretofore have not been proved. It is the purpose of this paper to prove the first two of Ramanujan's tenth order mock theta function identities. Further identities will be proved in subsequent papers.

In [A5, p. 63], we can find a definition of a mock theta function. A mock theta function is a function $f(q)$ defined by a q-series which converges for $|q| < 1$ and which satisfies the following two conditions:

(0) For every root of unity ζ, there is a theta function $\theta_\zeta(q)$ such that the difference $f(q) - \theta_\zeta(q)$ is bounded as $q \to \zeta$ radially.

(1) There is no single theta function which works for all ζ: i.e., for every theta function $\theta(q)$ there is some root of unity ζ for which $f(q) - \theta(q)$ is unbounded as $q \to \zeta$ radially.

But, no one has proved that mock theta functions given by Ramanujan satisfy the definition of a mock theta function.

In [R2, p. 9], Ramanujan gave a list of eight identities involving the following four mock theta functions:

$$\phi(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}} \quad , \tag{1.1}$$

$$\psi(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}} \quad , \tag{1.2}$$

$$X(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}} \quad , \tag{1.3}$$

and

$$\chi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}} \quad , \tag{1.4}$$

where

$$(a;q)_n := \prod_{m=0}^{n-1} (1 - aq^m) \quad .$$

Among the eight identities, the first two identities are related to $\phi(q)$ and $\psi(q)$. The main purpose of this paper is to prove the following two identities related to the mock theta functions $\phi(q)$ and $\psi(q)$:

$$q^{2/3}\phi(q^3) - \frac{\psi(\omega q^{1/3}) - \psi(\omega^2 q^{1/3})}{\omega - \omega^2}$$

$$= -q^{1/3} \frac{1 - 2q^{1/3} + 2q^{4/3} - 2q^{9/3} + \cdots}{1 - 2q + 2q^4 - 2q^9 + \cdots} \frac{1 - q - q^4 + q^7 + \cdots}{(1-q)(1-q^3)(1-q^5)\cdots}$$

and

$$q^{-2/3}\psi(q^3) + \frac{\omega\phi(\omega q^{1/3}) - \omega^2\phi(\omega^2 q^{1/3})}{\omega - \omega^2}$$
$$= \frac{1 - 2q^{1/3} + 2q^{4/3} - 2q^{9/3} + \cdots}{1 - 2q + 2q^4 - 2q^9 + \cdots} \frac{1 - q^2 - q^3 + q^7 + \cdots}{(1-q)(1-q^3)(1-q^5)\cdots} \ ,$$

where ω is a primitive cube root of unity, and $|q| < 1$. Since Ramanujan's identities involve theta functions, we reformulate them by replacing q by q^3, and using the Jacobi triple product identity. Therefore, we will prove the two identities above in the forms:

$$q^2\phi(q^9) - \frac{\psi(\omega q) - \psi(\omega^2 q)}{\omega - \omega^2}$$
$$= -q\frac{(q;q)_\infty (q;q^2)_\infty}{(q^3;q^3)_\infty (q^3;q^6)_\infty} \frac{(q^3;q^{15})_\infty (q^{12};q^{15})_\infty (q^{15};q^{15})_\infty}{(q^3;q^6)_\infty} \tag{1.5}$$

and

$$q^{-2}\psi(q^9) + \frac{\omega\phi(\omega q) - \omega^2\phi(\omega^2 q)}{\omega - \omega^2}$$
$$= \frac{(q;q)_\infty (q;q^2)_\infty}{(q^3;q^3)_\infty (q^3;q^6)_\infty} \frac{(q^6;q^{15})_\infty (q^9;q^{15})_\infty (q^{15};q^{15})_\infty}{(q^3;q^6)_\infty} \ . \tag{1.6}$$

We use the following notations:

Definition 1.1 *For a complex number q with $|q| < 1$, $|bc| < 1$, and an integer n,*

$$(a;q)_\infty := \prod_{m=0}^{\infty}(1 - aq^m) \ ,$$
$$(a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty} \ ,$$

and

$$f(b,c) := \sum_{j=-\infty}^{\infty} b^{j(j+1)/2} c^{j(j-1)/2} \ .$$

Let $R(F;z)$ denote the residue of F at a pole z.

We can easily verify the following identity

$$(a^2;q)_\infty = (a^2;q^2)_\infty (a^2q;q^2)_\infty = (-a;q)_\infty (a;q)_\infty (a^2q;q^2)_\infty \ . \tag{1.7}$$

Note that

$$\sum_{j=-\infty}^{\infty} b^{j(j+1)/2} c^{j(j-1)/2} = (-b;bc)_\infty (-c;bc)_\infty (bc;bc)_\infty \tag{1.8}$$

is called *the Jacobi triple product identity*, and

$$f(-q,-q^2)=(q;q)_\infty \tag{1.9}$$

is called *Euler's pentagonal number theorem*. In this paper, we will use the Jacobi triple product identity and Euler's pentagonal number theorem many times.

In Section 2, we derive two Hecke type identities for $\phi(q)$ and $\psi(q)$, and define a theta function $D(q,z)$ whose coefficients of z and z^2 are $\psi(q)$ and $\phi(q)$, respectively. And then we develop two identities relating $D(q,z)$, $\phi(q)$, $\psi(q)$, and Lambert series. By similar methods, we define a function $h(x,q)$, a theta function $A(z,x,q)$, and generalized Lambert series, and derive two identities relating $D(q,z)$, these functions and Lambert series. From the four identities that we obtain, we can derive two identities which represent $\phi(q)$ and $\psi(q)$ by theta functions and Lambert series. Then we prove two theta function identities by using properties of modular form. In Section 3, we develop several theta function identities and apply the previous two identities for $\phi(q)$ and $\psi(q)$ to Ramanujan's first identity. Then we prove Ramanujan's first identity. In Section 4, we prove Ramanujan's second identity by methods similar to those in Section 3.

The author would like to thank Dean Hickerson for his many helpful suggestions.

2 Preliminary results

2.1 Proof of a Lemma

In this section, we will prove that

$$\begin{aligned}&\frac{q^2(q^{10};q^{10})_\infty(q^{20};q^{20})_\infty f(-q^4,-q^{16})}{f(-q^2,-q^8)f(-q^8,-q^{12})}qf(-q^2,-q^3)\\&\quad+\frac{(q^{10};q^{10})_\infty(q^{20};q^{20})_\infty f(-q^8,-q^{12})}{f(-q^4,-q^6)f(-q^4,-q^{16})}q^2f(-q,-q^4)\\&\quad=\frac{q^2(q^{10};q^{10})^3_\infty f(-q^5,-q^{15})f(-q^{10},-q^{30})}{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty f(q^5,q^5)}\quad.\end{aligned}$$

In this paper, we will use the next lemma many times.

Lemma 2.1

(i) $f(a,b)=f(b,a)$,
(ii) $f(1,a)=2f(a,a^3)$,
(iii) $f(-1,a)=0$,
and, if n is an integer,
(iv) $f(a,b)=a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n,b(ab)^{-n})$.

Proof. See Entry 18 in [B2, p. 34].

Theorem 2.1 *If* $ab = cd$, *then*

(i) $f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc)$

and

(ii) $f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c},\frac{c}{b}abcd\right) \times f\left(\frac{b}{d},\frac{d}{b}abcd\right).$

Proof. See Entry 29 in [B2, p. 45].

Theorem 2.2 *For* $0 < |q| < 1,\ x \neq 0,$ *and* $y \neq 0$,

$$\begin{aligned} &f(x,x^{-1}q)f(-y,-y^{-1}q) - f(-x,-x^{-1}q)f(y,y^{-1}q) \\ &= 2xf(-x^{-1}y,-xy^{-1}q^2)f(-xyq,-x^{-1}y^{-1}q)\ . \end{aligned}$$

Proof. Replace a, b, c, and d by x, $x^{-1}q$, $-y$, and $-y^{-1}q$, respectively, in Theorem 2.1 (ii).

Theorem 2.3 *For* $0 < |q| < 1$, $x \neq 0$, *and* $y \neq 0$,

$$\begin{aligned} f(-x,-x^{-1}q)f(-y,-y^{-1}q) &= f(xy,(xy)^{-1}q^2)f(x^{-1}yq,xy^{-1}q) \\ &\quad - xf(xyq,(xy)^{-1}q)f(x^{-1}y,xy^{-1}q^2)\ . \end{aligned}$$

Proof. See Theorem 1.1 in [H1, p. 643].

Theorem 2.4 *For* $0 < |q| < 1,\ x \neq 0,$ *and* $y \neq 0$,

$$\begin{aligned} &f(x,x^{-1}q)f(-y,-y^{-1}q) + f(-x,-x^{-1}q)f(y,y^{-1}q) \\ &= 2f(-xy,-x^{-1}y^{-1}q^2)f(-x^{-1}yq,-xy^{-1}q)\ . \end{aligned}$$

Proof. Replace a, b, c, and d by x, $x^{-1}q$, $-y$, and $-y^{-1}q$, respectively, in Theorem 2.1 (i).

Lemma 2.2 *Assume that*

$$\begin{aligned} &-a_1(q^2)qf(-q^2,-q^3) + a_2(q^2)q^2f(-q,-q^4) \\ &= \frac{q^2(q^{10};q^{10})^3_\infty f(-q^5,-q^{15})f(-q^{10},-q^{30})}{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty f(q^5,q^5)}\ , \end{aligned} \tag{2.1}$$

where $a_1(q)$ *and* $a_2(q)$ *are functions of* q. *Then,*

$$a_1(q) = -\frac{q(q^5;q^5)_\infty(q^{10};q^{10})_\infty f(-q^2,-q^8)}{f(-q,-q^4)f(-q^4,-q^6)}$$

and

$$a_2(q) = \frac{(q^5;q^5)_\infty (q^{10};q^{10})_\infty f(-q^4,-q^6)}{f(-q^2,-q^3) f(-q^2,-q^8)} \ .$$

Proof. Assume that (2.1) is given, and replace q by $-q$ in (2.1). Then,

$$\begin{aligned} &a_1(q^2) q f(-q^2,q^3) + a_2(q^2) q^2 f(q,-q^4) \\ &\quad = \frac{q^2 (q^{10};q^{10})_\infty^3 f(q^5,q^{15}) f(-q^{10},-q^{30})}{(q^{20};q^{20})_\infty (q^{40};q^{40})_\infty f(-q^5,-q^5)} \ . \end{aligned} \tag{2.2}$$

Now, multiplying (2.1) by $f(q,-q^4)$ and (2.2) by $f(-q,-q^4)$ and then subtracting the resulting two equalities, we find that

$$\begin{aligned} &a_1(q^2) q (f(-q,-q^4) f(-q^2,q^3) + f(-q^2,-q^3) f(q,-q^4)) \\ &= \frac{f(-q,-q^4) q^2 (q^{10};q^{10})_\infty^3 f(q^5,q^{15}) f(-q^{10},-q^{30})}{(q^{20};q^{20})_\infty (q^{40};q^{40})_\infty f(-q^5,-q^5)} \\ &\quad - \frac{f(q,-q^4) q^2 (q^{10};q^{10})_\infty^3 f(-q^5,-q^{15}) f(-q^{10},-q^{30})}{(q^{20};q^{20})_\infty (q^{40};q^{40})_\infty f(q^5,q^5)} \\ &= \frac{q^2 (q^{10};q^{10})_\infty^3 f(-q^{10},-q^{30})}{(q^{20};q^{20})_\infty (q^{40};q^{40})_\infty} \\ &\quad \times \left(\frac{f(-q,-q^4) f(q^5,q^{15})}{f(-q^5,-q^5)} - \frac{f(q,-q^4) f(-q^5,-q^{15})}{f(q^5,q^5)} \right) \ . \end{aligned} \tag{2.3}$$

Let $L_1(q)$ be the left hand side of (2.3) and $R_1(q)$ be the right hand side of (2.3). By (1.8),

$$\left.\begin{aligned} f(-q,-q^4) &= f(-q,-q^9) f(-q^4,-q^6) (q^5;q^{10})_\infty / (q^{10};q^{10})_\infty \\ f(-q^2,-q^3) &= f(-q^2,-q^8) f(-q^3,-q^7) (q^5;q^{10})_\infty / (q^{10};q^{10})_\infty \\ f(-q^2,q^3) &= f(-q^2,-q^8) f(q^3,q^7) (-q^5;q^{10})_\infty / (q^{10};q^{10})_\infty \\ f(q,-q^4) &= f(-q^4,-q^6) f(q,q^9) (-q^5;q^{10})_\infty / (q^{10};q^{10})_\infty \ . \end{aligned}\right\} \tag{2.4}$$

So, we find that

$$\begin{aligned} L_1(q) &= a_1(q^2) q \big(f(-q,-q^4) f(-q^2,q^3) + f(-q^2,-q^3) f(q,-q^4) \big) \\ &= \frac{a_1(q^2) q (q^{10};q^{20})_\infty f(-q^4,-q^6) f(-q^2,-q^8)}{(q^{10};q^{10})_\infty^2} \\ &\quad \times \big(f(-q,-q^9) f(q^3,q^7) + f(q,q^9) f(-q^3,-q^7) \big) \ . \end{aligned} \tag{2.5}$$

Now, we will consider $R_1(q)$. By (1.8),

$$\begin{aligned}
&\frac{f(-q,-q^4)f(q^5,q^{15})}{f(-q^5,-q^5)} \\
&\quad= \frac{f(-q,-q^9)f(-q^4,-q^6)(q^5;q^{10})_\infty}{(q^{10};q^{10})_\infty} \\
&\qquad\times \frac{(-q^5;q^{20})_\infty(-q^{15};q^{20})_\infty(q^{20};q^{20})_\infty}{(q^5;q^{10})^2_\infty(q^{10};q^{10})_\infty} \\
&\quad= \frac{f(-q,-q^9)f(-q^4,-q^6)}{(q^{10};q^{10})_\infty}\frac{(-q^5;q^{10})_\infty(q^{20};q^{20})_\infty}{(q^5;q^{10})_\infty(q^{10};q^{10})_\infty} \\
&\quad= \frac{f(-q,-q^9)f(-q^4,-q^6)}{(q^{10};q^{10})_\infty} \\
&\qquad\times \frac{(-q^5;q^{10})_\infty(-q^5;q^{10})_\infty(q^5;q^{10})_\infty(q^{10};q^{10})_\infty}{(q^5;q^{10})_\infty(q^{10};q^{10})_\infty} \\
&\quad= \frac{f(-q,-q^9)f(-q^4,-q^6)f(q^5,q^5)(-q^{10};q^{10})^2_\infty}{(q^{10};q^{10})^2_\infty} \quad ,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{f(q,-q^4)f(-q^5,-q^{15})}{f(q^5,q^5)} \\
&\quad= \frac{f(-q^4,-q^6)f(q,q^9)(-q^5;q^{10})_\infty}{(q^{10};q^{10})_\infty}\frac{(q^5;q^{20})_\infty(q^{15};q^{20})_\infty(q^{20};q^{20})_\infty}{f(q^5,q^5)} \\
&\quad= \frac{f(-q^4,-q^6)f(q,q^9)}{(q^{10};q^{10})_\infty}\frac{(q^{10};q^{20})_\infty(q^{20};q^{20})_\infty}{f(q^5,q^5)}\frac{(q^5;q^{10})^2_\infty}{(q^5;q^{10})^2_\infty} \\
&\quad= \frac{f(-q^4,-q^6)f(q,q^9)}{(q^{10};q^{10})_\infty}\frac{f(-q^5,-q^5)}{(q^{10};q^{20})^2_\infty(q^{10};q^{10})_\infty} \\
&\quad= \frac{f(-q^4,-q^6)f(q,q^9)f(-q^5,-q^5)(-q^{10};q^{10})^2_\infty}{(q^{10};q^{10})^2_\infty} \quad .
\end{aligned}$$

Therefore, we find that

$$\begin{aligned}
R_1(q) &= \frac{q^2(q^{10};q^{10})^3_\infty f(-q^{10},-q^{30})}{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty}\frac{f(-q^4,-q^6)(-q^{10};q^{10})^2_\infty}{(q^{10};q^{10})^2_\infty} \\
&\qquad\times\left(f(-q,-q^9)f(q^5,q^5)-f(q,q^9)f(-q^5,-q^5)\right) \\
&= \frac{q^2(q^{10};q^{10})_\infty f(-q^{10},-q^{30})f(-q^4,-q^6)(-q^{10};q^{10})^2_\infty}{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty} \\
&\qquad\times\left(f(-q,-q^9)f(q^5,q^5)-f(q,q^9)f(-q^5,-q^5)\right) \quad . \qquad (2.6)
\end{aligned}$$

From (2.5) and (2.6),

$$a_1(q^2) = \frac{q(q^{10};q^{10})_\infty^3 f(-q^{10},-q^{30}) f(-q^4,-q^6)(-q^{10};q^{10})_\infty^2}{(q^{20};q^{20})_\infty (q^{40};q^{40})_\infty (q^{10};q^{20})_\infty f(-q^4,-q^6) f(-q^2,-q^8)} \times \left(\frac{f(-q,-q^9)f(q^5,q^5) - f(q,q^9)f(-q^5,-q^5)}{f(-q,-q^9)f(q^3,q^7) + f(q,q^9)f(-q^3,-q^7)}\right) .$$

By (1.8) and $f(-q^{10},-q^{30}) = (q^{10};q^{40})_\infty (q^{30};q^{40})_\infty (q^{40};q^{40})_\infty = (q^{10};q^{20})_\infty (q^{40};q^{40})_\infty$,

$$a_1(q^2) = \frac{q(q^{10};q^{10})_\infty^3 (-q^{10};q^{10})_\infty^2}{(q^{20};q^{20})_\infty f(-q^2,-q^8)} \times \left(\frac{f(-q,-q^9)f(q^5,q^5) - f(q,q^9)f(-q^5,-q^5)}{f(-q,-q^9)f(q^3,q^7) + f(q,q^9)f(-q^3,-q^7)}\right) .$$

Replacing q, x and y by q^{10}, q^5 and q, respectively, in Theorem 2.2, we find that

$$f(-q,-q^9)f(q^5,q^5) - f(q,q^9)f(-q^5,-q^5) = 2q^5 f(-q^{-4},-q^{24}) f(-q^4,-q^{16}) , \tag{2.7}$$

and replacing q, x and y by q^{10}, q^3 and q, respectively, in Theorem 2.4, we find that

$$f(-q,-q^9)f(q^3,q^7) + f(q,q^9)f(-q^3,-q^7) = 2f(-q^4,-q^{16}) f(-q^8,-q^{12}) . \tag{2.8}$$

Using (2.7) and (2.8), and applying Lemma 2.1(iv), we find that

$$\begin{aligned} a_1(q^2) &= \frac{q(q^{10};q^{10})_\infty^3 (-q^{10};q^{10})_\infty^2}{(q^{20};q^{20})_\infty f(-q^2,-q^8)} \frac{2q^5 f(-q^{-4},-q^{24}) f(-q^4,-q^{16})}{2f(-q^4,-q^{16}) f(-q^8,-q^{12})} \\ &= -\frac{q^2(q^{10};q^{10})_\infty (q^{20};q^{20})_\infty^2 f(-q^4,-q^{16})}{(q^{20};q^{20})_\infty f(-q^2,-q^8) f(-q^8,-q^{12})} \\ &= -\frac{q^2(q^{10};q^{10})_\infty (q^{20};q^{20})_\infty f(-q^4,-q^{16})}{f(-q^2,-q^8) f(-q^8,-q^{12})} . \end{aligned}$$

Replacing q^2 by q, we finally deduce that

$$a_1(q) = -\frac{q(q^5;q^5)_\infty (q^{10};q^{10})_\infty f(-q^2,-q^8)}{f(-q,-q^4) f(-q^4,-q^6)} .$$

We next consider $a_2(q)$. Multiplying (2.1) by $f(-q^2,q^3)$ and (2.2) by $f(-q^2,-q^3)$ and then adding the resulting two equations, we find that

$$\begin{aligned}
&a_2(q^2)q^2\big(f(-q,-q^4)f(-q^2,q^3)+f(-q^2,-q^3)f(q,-q^4)\big)\\
&\quad=\frac{q^2(q^{10};q^{10})_\infty^3 f(-q^{10},-q^{30})}{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty}\\
&\qquad\times\left(\frac{f(-q^2,q^3)f(-q^5,-q^{15})}{f(q^5,q^5)}+\frac{f(-q^2,-q^3)f(q^5,q^{15})}{f(-q^5,-q^5)}\right)\ . \qquad (2.9)
\end{aligned}$$

Let $L_2(q)$ be the left hand side of (2.9) and $R_2(q)$ be the right hand side of (2.9). Applying (2.4), we find that

$$\begin{aligned}
L_2(q)&=\frac{a_2(q^2)q^2(q^{10};q^{20})_\infty f(-q^4,-q^6)f(-q^2,-q^8)}{(q^{10};q^{10})_\infty^2}\\
&\qquad\times\big(f(-q,-q^9)f(q^3,q^7)+f(q,q^9)f(-q^3,-q^7)\big)\ . \qquad (2.10)
\end{aligned}$$

We consider $R_2(q)$. By (1.8),

$$\begin{aligned}
&\frac{f(-q^2,q^3)f(-q^5,-q^{15})}{f(q^5,q^5)}\\
&\quad=\frac{f(-q^2,-q^8)f(q^3,q^7)(-q^5;q^{10})_\infty}{(q^{10};q^{10})_\infty}\frac{(q^5;q^{20})_\infty(q^{15};q^{20})_\infty(q^{20};q^{20})_\infty}{f(q^5,q^5)}\\
&\quad=\frac{f(-q^2,-q^8)f(q^3,q^7)}{(q^{10};q^{10})_\infty}\frac{(q^{10};q^{20})_\infty(q^{20};q^{20})_\infty}{f(q^5,q^5)}\frac{(q^5;q^{10})_\infty^2}{(q^5;q^{10})_\infty^2}\\
&\quad=\frac{f(-q^2,-q^8)f(q^3,q^7)}{(q^{10};q^{10})_\infty}\frac{f(-q^5,-q^5)}{(q^{10};q^{20})_\infty^2(q^{10};q^{10})_\infty}\\
&\quad=\frac{f(-q^2,-q^8)f(q^3,q^7)f(-q^5,-q^5)(-q^{10};q^{10})_\infty^2}{(q^{10};q^{10})_\infty^2}\ ,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{f(-q^2,-q^3)f(q^5,q^{15})}{f(-q^5,-q^5)}\\
&\quad=\frac{f(-q^2,-q^8)f(-q^3,-q^7)(q^5;q^{10})_\infty}{(q^{10};q^{10})_\infty}\\
&\qquad\times\frac{(-q^5;q^{20})_\infty(-q^{15};q^{20})_\infty(q^{20};q^{20})_\infty}{(q^5;q^{10})_\infty^2(q^{10};q^{10})_\infty}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{f(-q^2,-q^8)f(-q^3,-q^7)}{(q^{10};q^{10})_\infty} \frac{(-q^5;q^{10})_\infty (q^{20};q^{20})_\infty}{(q^5;q^{10})_\infty (q^{10};q^{10})_\infty} \\
&= \frac{f(-q^2,-q^8)f(-q^3,-q^7)}{(q^{10};q^{10})_\infty} \\
&\quad \times \frac{(-q^5;q^{10})_\infty (-q^5;q^{10})_\infty (q^5;q^{10})_\infty (q^{10};q^{10})_\infty (-q^{10};q^{10})^2_\infty}{(q^5;q^{10})_\infty (q^{10};q^{10})_\infty} \\
&= \frac{f(-q^2,-q^8)f(-q^3,-q^7)f(q^5,q^5)(-q^{10};q^{10})^2_\infty}{(q^{10};q^{10})^2_\infty} \ .
\end{aligned}
$$

We find that

$$
\begin{aligned}
R_2(q) = &\frac{q^2(q^{10};q^{10})_\infty f(-q^{10},-q^{30})f(-q^2,-q^8)(-q^{10};q^{10})^2_\infty}{(q^{20};q^{20})_\infty (q^{40};q^{40})_\infty} \\
&\times \left(f(q^3,q^7)f(-q^5,-q^5) + f(-q^3,-q^7)f(q^5,q^5) \right) \ . \qquad (2.11)
\end{aligned}
$$

From (2.10), (2.11), and $f(-q^{10},-q^{30}) = (q^{10};q^{20})_\infty (q^{40};q^{40})_\infty$, we find that

$$
\begin{aligned}
a_2(q^2) = &\frac{(q^{10};q^{10})^3_\infty (-q^{10};q^{10})^2_\infty}{(q^{20};q^{20})_\infty f(-q^4,-q^6)} \\
&\times \left(\frac{f(q^3,q^7)f(-q^5,-q^5) + f(-q^3,-q^7)f(q^5,q^5)}{f(-q,-q^9)f(q^3,q^7) + f(q,q^9)f(-q^3,-q^7)} \right) \ .
\end{aligned}
$$

We need two identities to complete the proof. In Theorem 2.4, replacing q, x and y by q^{10}, q^5 and q^3, respectively, we find that

$$
f(q^3,q^7)f(-q^5,-q^5) + f(-q^3,-q^7)f(q^5,q^5) = 2f(-q^8,-q^{12})^2 \ ,
$$

and in Theorem 2.4, replacing q, x and y by q^{10}, q^3 and q, respectively, we also have that

$$
\begin{aligned}
&f(-q,-q^9)f(q^3,q^7) + f(q,q^9)f(-q^3,-q^7) \\
&\quad = 2f(-q^4,-q^{16})f(-q^8,-q^{12}) \ .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
a_2(q^2) &= \frac{(q^{10};q^{10})^3_\infty (-q^{10};q^{10})^2_\infty}{(q^{20};q^{20})_\infty f(-q^4,-q^6)} \frac{2f(-q^8,-q^{12})^2}{2f(-q^4,-q^{16})f(-q^8,-q^{12})} \\
&= \frac{(q^{10};q^{10})_\infty (q^{20};q^{20})_\infty f(-q^8,-q^{12})}{f(-q^4,-q^6)f(-q^4,-q^{16})} \ .
\end{aligned}
$$

Replacing q^2 by q, we conclude that

$$a_2(q) = \frac{(q^5;q^5)_\infty (q^{10};q^{10})_\infty f(-q^4,-q^6)}{f(-q^2,-q^3) f(-q^2,-q^8)} \quad .$$

This completes the proof.

2.2 Hecke type identities for $\phi(q)$ and $\psi(q)$

In this section, we will develop two Hecke type identities for $\phi(q)$ and $\psi(q)$. This idea was developed by G. Andrews [A1]. To develop Hecke type identities, we need Bailey's Lemma [B1] and two more lemmas [A1].

Definition 2.1 *For an integer s,*

$$sg(s) := \begin{cases} 1, & \text{if } s \geq 0, \\ -1, & \text{if } s < 0 \end{cases} \quad .$$

Lemma 2.3 (Bailey's Lemma) *If for $n \geq 0$ the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are related by*

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \quad ,$$

then for $n \geq 0$,

$$\beta'_n = \sum_{r=0}^{n} \frac{\alpha'_r}{(q;q)_{n-r}(aq;q)_{n+r}} \quad , \tag{2.12}$$

where

$$\beta'_n = \frac{1}{(aq/\rho_1;q)_n (aq/\rho_2;q)_n} \times \sum_{j=0}^{n} \frac{(\rho_1;q)_j (\rho_2;q)_j (aq/\rho_1\rho_2;q)_{n-j} (aq/\rho_1\rho_2)^j \beta_j}{(q;q)_{n-j}} \quad ,$$

and

$$\alpha'_r = \frac{(\rho_1;q)_r (\rho_2;q)_r (aq/\rho_1\rho_2)^r \alpha_r}{(aq/\rho_1;q)_r (aq/\rho_2;q)_r}$$

for any given number ρ_1 and ρ_2.

Lemma 2.4 *Let*

$$\beta_n = \frac{1}{(q^{n+1};q)_{n+1}} \ ,$$
$$\alpha_{2n} = \frac{1}{1-q}\left(q^{3n^2+n}\sum_{|j|\le n} q^{-j^2} + 2q^{3n^2+2n}\sum_{j=0}^{n-1} q^{-j^2-j}\right) \ ,$$

and

$$\alpha_{2n+1} = -\frac{1}{1-q}\left(2q^{3n^2+4n+1}\sum_{j=0}^{n} q^{-j^2-j} + q^{3n^2+5n+2}\sum_{|j|\le n} q^{-j^2}\right) \ .$$

Then $(\alpha_n,\ \beta_n)$ *form a Bailey pair.*

Proof. See Lemma 12 in [A1, p. 131].

Lemma 2.5 *Let*

$$\beta_n = \frac{1}{(q^n;q)_n}, \qquad if\ n > 0 \ ,$$
$$\beta_0 = 0 \ ,$$

and

$$\alpha_{2n} = -2q^{3n^2-2n}(1-q^{4n})\sum_{j=0}^{n-1} q^{-j^2-j} \ ,$$
$$\alpha_{2n+1} = q^{3n^2+n}(1-q^{4n+2})\sum_{|j|\le n} q^{-j^2} \ .$$

Then $(\alpha_n,\ \beta_n)$ *form a Bailey pair.*

Proof. See Lemma 12 in [A1, p. 131].

We are ready to prove the next theorem.

Theorem 2.5 *With* $sg(s)$ *defined as above*,

$$\text{(i)}\quad \phi(q) = \frac{1}{f(-q,-q)}\sum_{\substack{r,s=-\infty\\ sg(r)=sg(s)}}^{\infty} sg(r)(-1)^{r+s}q^{(r+s)^2+rs+r+s} \ ,$$

and

$$\text{(ii)}\quad \psi(q) = \frac{1}{f(-q,-q)}\sum_{\substack{r,s=-\infty\\ sg(r)=sg(s)}}^{\infty} sg(r)(-1)^{r+s+1}q^{(r+s)^2+rs+3(r+s)+2} \ .$$

Let L_b be the left hand side of (2.12) and R_b be the right hand side of (2.12).

Proof of (i). By (1.1),

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+1)/2}}{(q^{n+1};q)_{n+1}} \quad .$$

In Bailey's Lemma, relpace ρ_1, ρ_2, and a by $-\rho_1$, $-q$, and q, respectively, and let n and ρ_1 tend to ∞. Now, applying Bailey's Lemma to $\phi(q)$, we find that

$$L_b = \frac{1}{(-q;q)_\infty (q;q)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2} (-q;q)_n \beta_n \quad , \tag{2.13}$$

and

$$R_b = \frac{1}{(q;q)_\infty (q^2;q)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2} \alpha_n \quad . \tag{2.14}$$

Applying (2.13), (2.14) and Lemma 2.4, we find that

$$\begin{aligned}
\phi(q) &= \sum_{n=0}^{\infty} q^{n(n+1)/2} (-q;q)_n \frac{1}{(q^{n+1};q)_{n+1}} = \frac{(-q;q)_\infty}{(q^2;q)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2} \alpha_n \\
&= \frac{1}{(q^2;q)_\infty (q;q^2)_\infty} \left(\sum_{n=0}^{\infty} q^{n(2n+1)} \alpha_{2n} + \sum_{n=0}^{\infty} q^{(2n+1)(n+1)} \alpha_{2n+1} \right) \\
&= \frac{1-q}{f(-q,-q)} \sum_{n=0}^{\infty} q^{n(2n+1)} \frac{1}{1-q} \\
&\quad \times \left(q^{3n^2+n} \sum_{|j| \le n} q^{-j^2} + 2q^{3n^2+2n} \sum_{j=0}^{n-1} q^{-j^2-j} \right) \\
&\quad + \frac{1-q}{f(-q,-q)} \sum_{n=0}^{\infty} q^{(2n+1)(n+1)} \frac{-1}{1-q} \\
&\quad \times \left(2q^{3n^2+4n+1} \sum_{j=0}^{n} q^{-j^2-j} + q^{3n^2+5n+2} \sum_{|j| \le n} q^{-j^2} \right) \quad .
\end{aligned}$$

Replacing n by $n+1$, we find that

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n-1} q^{5n^2+3n-j^2-j} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} q^{5n^2+13n+8-j^2-j} \quad .$$

Thus,

$$\phi(q) = \frac{1}{f(-q,-q)} \sum_{n=0}^{\infty} \sum_{|j| \le n} q^{5n^2+2n-j^2}\left(1-q^{6n+3}\right) - \frac{1}{f(-q,-q)} 2 \sum_{n=0}^{\infty} \sum_{j=0}^{n} q^{5n^2+7n+2-j^2-j}\left(1-q^{6n+6}\right) . \quad (2.15)$$

We need to show that

$$\phi(q) = \frac{1}{f(-q,-q)} \sum_{\substack{r,s=-\infty \\ sg(r)=sg(s)}}^{\infty} sg(r)(-1)^{r+s} q^{(r+s)^2+rs+r+s} .$$

Let $R(q)$ denote the right hand side of (2.15). First, assume that N and J are even. Replacing n and j by $N/2$ and $J/2$, respectively, in the first sum of $R(q)$, we find that this sum equals

$$\sum_{\substack{N=0 \\ N\equiv 0 (\mathrm{mod}\ 2)}}^{\infty} \sum_{\substack{|J| \le N \\ J \equiv 0 (\mathrm{mod}\ 2)}} q^{(5N^2+4N-J^2)/4}\left(1-q^{3N+3}\right) .$$

Second, assume that N and J are odd. Replacing n and j by $(N-1)/2$ and $(J-1)/2$, respectively, in the second sum of $R(q)$, we find that the second sum equals

$$\sum_{\substack{N=0 \\ N\equiv 1 (\mathrm{mod}\ 2)}}^{\infty} \sum_{\substack{|J| \le N \\ J \equiv 1 (\mathrm{mod}\ 2)}} q^{5\left(\frac{N-1}{2}\right)^2+7\left(\frac{N-1}{2}\right)+2-\left(\frac{J-1}{2}\right)^2-\frac{J-1}{2}}\left(1-q^{3N+3}\right)$$
$$= \sum_{\substack{N=0 \\ N\equiv 1 (\mathrm{mod}\ 2)}}^{\infty} \sum_{\substack{|J| \le N \\ J \equiv 1 (\mathrm{mod}\ 2)}} q^{(5N^2+4N-J^2)/4}\left(1-q^{3N+3}\right) ,$$

since $2\sum_{j=0}^{n} q^{-j^2-j} = \sum_{j=-n-1}^{n} q^{-j^2-j}$. Therefore, we find that

$$\phi(q) = \frac{1}{f(-q,-q)} \sum_{\substack{N=0 \\ N\equiv 0 (\mathrm{mod}\ 2)}}^{\infty} \sum_{\substack{|J| \le N \\ J \equiv 0 (\mathrm{mod}\ 2)}} q^{(5N^2+4N-J^2)/4}\left(1-q^{3N+3}\right)$$
$$- \frac{1}{f(-q,-q)} \sum_{\substack{N=0 \\ N\equiv 1 (\mathrm{mod}\ 2)}}^{\infty} \sum_{\substack{|J| \le N \\ J \equiv 1 (\mathrm{mod}\ 2)}} q^{(5N^2+4N-J^2)/4}\left(1-q^{3N+3}\right)$$
$$= \frac{1}{f(-q,-q)} \sum_{N=0}^{\infty} \sum_{\substack{|J| \le N \\ J \equiv N (\mathrm{mod}\ 2)}} (-1)^N q^{(5N^2+4N-J^2)/4}\left(1-q^{3N+3}\right)$$

$$= \frac{1}{f(-q,-q)} \sum_{N=0}^{\infty} \sum_{\substack{|J| \le N \\ J \equiv N \pmod 2}} (-1)^N q^{(5N^2+4N-J^2)/4}$$

$$- \frac{1}{f(-q,-q)} \sum_{N=0}^{\infty} \sum_{\substack{|J| \le N \\ J \equiv N \pmod 2}} (-1)^N q^{(5N^2+16N+12-J^2)/4} .$$

Replacing N by $r+s$ and J by $r-s$ in the first sum, and replacing N by $-r-s-2$ and J by $r-s$ in the second sum, we find that

$$\phi(q) = \frac{1}{f(-q,-q)} \left(\sum_{r \ge 0,\, s \ge 0} (-1)^{r+s} q^{(r+s)^2+rs+r+s} - \sum_{r \le -1,\, s \le -1} (-1)^{r+s} q^{(r+s)^2+rs+r+s} \right)$$

$$= \frac{1}{f(-q,-q)} \sum_{\substack{r,s=-\infty \\ sg(r)=sg(s)}}^{\infty} sg(r)(-1)^{r+s} q^{(r+s)^2+rs+r+s} .$$

Proof of (ii). The proof of (ii) is similar to the proof of (i). By (1.2),

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_n} = \sum_{n=1}^{\infty} \frac{(-q;q)_{n-1} q^{n(n+1)/2}}{(q^n;q)_n} .$$

In Bailey's Lemma, replace ρ_1, ρ_2, and a by -1, $-\rho_2$, and 1, respectively, and let n and ρ_2 tend to ∞. Now, applying Bailey's Lemma to $\psi(q)$, we find that

$$L_b = \frac{2}{(-q;q)_\infty (q;q)_\infty} \sum_{j=0}^{\infty} (-q;q)_{j-1} q^{j(j+1)/2} \beta_j , \tag{2.16}$$

and

$$R_b = \frac{2}{(q;q)_\infty^2} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} \alpha_n}{1+q^n} . \tag{2.17}$$

Using (2.16), (2.17) and Lemma 2.5, we find that

$$\psi(q) = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} \alpha_n}{1+q^n}$$

$$= \frac{1}{f(-q,-q)} \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{1+q^{2n}} \left(-2q^{3n^2-2n}(1-q^{4n}) \sum_{j=0}^{n-1} q^{-j^2-j} \right)$$

$$+ \frac{1}{f(-q,-q)} \sum_{n=0}^{\infty} \frac{q^{(2n+1)(n+1)}}{1+q^{2n+1}} \left(q^{3n^2+n}(1-q^{4n+2}) \sum_{|j| \le n} q^{-j^2} \right) .$$

Replacing n by $n+1$, we find that

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n-1}(1-q^{2n})q^{5n^2-n-j^2-j}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}(1-q^{2n+2})q^{5n^2+9n+4-j^2-j} \ .$$

So,

$$\psi(q)=\frac{1}{f(-q,-q)}\sum_{n=0}^{\infty}\sum_{|j|\le n}q^{5n^2+4n+1-j^2}\left(1-q^{2n+1}\right)$$
$$-\frac{1}{f(-q,-q)}2\sum_{n=0}^{\infty}\sum_{j=0}^{n}q^{5n^2+9n+4-j^2-j}\left(1-q^{2n+2}\right) \ . \qquad (2.18)$$

We need to show that

$$\psi(q)=\frac{1}{f(-q,-q)}\sum_{\substack{r,s=-\infty\\ sg(r)=sg(s)}}^{\infty} sg(r)(-1)^{r+s+1}q^{(r+s)^2+rs+3(r+s)+2} \ .$$

Let $R(q)$ denote the right hand side of (2.18). First, let us assume that N and J are even. Replacing n and j by $N/2$ and $J/2$, respectively, in the first sum of $R(q)$, we find that the first sum of $R(q)$ equals

$$\sum_{\substack{N=0\\ N\equiv 0 (\mathrm{mod}\ 2)}}^{\infty}\ \sum_{\substack{|J|\le N\\ J\equiv 0 (\mathrm{mod}\ 2)}} q^{(5N^2+8N+4-J^2)/4}(1-q^{N+1}) \ .$$

Second, assume that N and J are odd. Replacing n and j by $(N-1)/2$ and $(J-1)/2$, respectively, in the second sum of $R(q)$, we find that the second sum of $R(q)$ equals

$$\sum_{\substack{N=0\\ N\equiv 1 (\mathrm{mod}\ 2)}}^{\infty}\ \sum_{\substack{|J|\le N\\ J\equiv 1 (\mathrm{mod}\ 2)}} q^{5\left(\frac{N-1}{2}\right)^2+9\left(\frac{N-1}{2}\right)+4-\left(\frac{J-1}{2}\right)^2-\frac{J-1}{2}}(1-q^{N+1})$$
$$=\sum_{\substack{N=0\\ N\equiv 1 (\mathrm{mod}\ 2)}}^{\infty}\ \sum_{\substack{|J|\le N\\ J\equiv 1 (\mathrm{mod}\ 2)}} q^{(5N^2+8N+4-J^2)/4}(1-q^{N+1}) \ ,$$

since $2\sum_{j=0}^{n}q^{-j^2-j}=\sum_{j=-n-1}^{n}q^{-j^2-j}$. Hence, from (2.18),

$$\psi(q)=\frac{1}{f(-q,-q)}\sum_{\substack{N=0\\ N\equiv 0 (\mathrm{mod}\ 2)}}^{\infty}\ \sum_{\substack{|J|\le N\\ J\equiv 0 (\mathrm{mod}\ 2)}} q^{(5N^2+8N+4-J^2)/4}(1-q^{N+1})$$
$$-\frac{1}{f(-q,-q)}\sum_{\substack{N=0\\ N\equiv 1 (\mathrm{mod}\ 2)}}^{\infty}\ \sum_{\substack{|J|\le N\\ J\equiv 1 (\mathrm{mod}\ 2)}} q^{(5N^2+8N+4-J^2)/4}(1-q^{N+1})$$

$$= \frac{1}{f(-q,-q)} \sum_{N=0}^{\infty} \sum_{\substack{|J| \le N \\ J \equiv N \pmod 2}} (-1)^N q^{(5N^2+8N+4-J^2)/4} (1 - q^{N+1})$$

$$= \frac{1}{f(-q,-q)} \sum_{N=0}^{\infty} \sum_{\substack{|J| \le N \\ J \equiv N \pmod 2}} (-1)^N q^{(5N^2+8N+4-J^2)/4}$$

$$- \frac{1}{f(-q,-q)} \sum_{N=0}^{\infty} \sum_{\substack{|J| \le N \\ J \equiv N \pmod 2}} (-1)^N q^{(5N^2+12N+8-J^2)/4} .$$

Replacing N by $-r-s-2$ and J by $r-s$ in the first sum, and replacing N by $r+s$, and J by $r-s$ in the second sum, we find that

$$\psi(q) = \frac{1}{f(-q,-q)} \sum_{r \le -1,\, s \le -1} (-1)^{r+s} q^{(r+s)^2+rs+3(r+s)+2}$$

$$- \frac{1}{f(-q,-q)} \sum_{r \ge 0,\, s \ge 0} (-1)^{r+s} q^{(r+s)^2+rs+3(r+s)+2}$$

$$= \frac{1}{f(-q,-q)} \sum_{\substack{r,s=-\infty \\ sg(r)=sg(s)}}^{\infty} sg(r)(-1)^{r+s+1} q^{(r+s)^2+rs+3(r+s)+2} .$$

2.3 Mock theta functions $\phi(q)$ and $\psi(q)$ as coefficients

In this section, we will define a theta function $D(q,z)$, and develop its relation with $\phi(q)$ and $\psi(q)$.

Definition 2.2 *For $|q| < 1$ and z neither zero nor an integral power of q, let*

$$D(q,z) := D(z) := \frac{z^2 (q;q)_\infty^2 f(-z^2, -z^{-2}q) f(-z, -z^{-1}q^2)}{(q;q^2)_\infty f(-z, -z^{-1}q)^2} . \qquad (2.19)$$

By (1.8) and (1.7), we find that

$D(q,z)$

$$= \frac{z^2 (q;q)_\infty^2 f(-z^2, -z^{-2}q) f(-z, -z^{-1}q^2)}{(q;q^2)_\infty f(-z, -z^{-1}q)^2}$$

$$= \frac{z^2 (q;q)_\infty (z^2;q)_\infty (z^{-2}q;q)_\infty (z;q^2)_\infty (z^{-1}q^2;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (z;q)_\infty^2 (z^{-1}q;q)_\infty^2}$$

$$= \frac{z^2(q;q)_\infty(q^2;q^2)^2_\infty(-z;q)_\infty(-z^{-1}q;q)_\infty(z^2q;q^2)_\infty(z^{-2}q;q^2)_\infty}{(q;q^2)_\infty(zq;q^2)_\infty(z^{-1}q;q^2)_\infty(q^2;q^2)_\infty}$$

$$= \frac{z^2(q^2;q^2)_\infty f(z,z^{-1}q)f(-z^2q,-z^{-2}q)}{(q;q^2)_\infty f(-zq,-z^{-1}q)} \quad , \tag{2.20}$$

and clearly $D(z)$ is meromorphic for $z \neq 0$ with simple poles at $z = q^{2k+1}$ for each integer k.

Lemma 2.6 $D(z)$ *satisfies the following functional equations*

(i) $D(z^{-1}) = z^{-5}D(z)$,

and

(ii) $D(zq^2) = -z^{-5}D(z)$.

Proof (i). Applying the three identities

$$\begin{aligned} f(-z^{-2},-z^2q) &= (z^{-2};q)_\infty(z^2q;q)_\infty(q;q)_\infty \\ &= (1-z^{-2})(z^{-2}q;q)_\infty \frac{1}{1-z^2}(z^2;q)_\infty(q;q)_\infty \\ &= -z^{-2}f(-z^2,-z^{-2}q) \quad , \end{aligned} \tag{2.21}$$

$$\begin{aligned} f(-z^{-1},-zq^2) &= (z^{-1};q^2)_\infty(zq^2;q^2)_\infty(q^2;q^2)_\infty \\ &= (1-z^{-1})(z^{-1}q^2;q^2)_\infty \frac{1}{1-z}(z;q^2)_\infty(q^2;q^2)_\infty \\ &= -z^{-1}f(-z,-z^{-1}q^2) \quad , \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} f(-z^{-1},-zq) &= (z^{-1};q)_\infty(zq;q)_\infty(q;q)_\infty \\ &= (1-z^{-1})(z^{-1}q;q)_\infty \frac{1}{1-z}(z;q)_\infty(q;q)_\infty \\ &= -z^{-1}f(-z,-z^{-1}q) \quad , \end{aligned} \tag{2.23}$$

we find that

$$\begin{aligned} D(z^{-1}) &= \frac{z^{-2}(q;q)^2_\infty f(-z^{-2},-z^2q)f(-z^{-1},-zq^2)}{(q;q^2)_\infty f(-z^{-1},-zq)^2} \\ &= \frac{z^{-2}(q;q)^2_\infty(-z^{-2})f(-z^2,-z^{-2}q)(-z^{-1})f(-z,-z^{-1}q^2)}{(q;q^2)_\infty z^{-2}f(-z,-z^{-1}q)^2} \\ &= z^{-5}D(z) \quad . \end{aligned}$$

(ii) Applying the three identities

$$\begin{aligned}
f(-z^2q^4,-z^{-2}q^{-3}) &= (z^2q^4;q)_\infty(z^{-2}q^{-3};q)_\infty(q;q)_\infty \\
&= \frac{1}{(1-z^2)(1-z^2q)(1-z^2q^2)(1-z^2q^3)}(z^2;q)_\infty \\
&\quad \times(1-z^{-2}q^{-3})(1-z^{-2}q^{-2})(1-z^{-2}q^{-1})(1-z^{-2})(z^{-2}q;q)_\infty(q;q)_\infty \\
&= q^{-6}z^{-8}f(-z^2,-z^{-2}q)\ ,
\end{aligned} \tag{2.24}$$

$$f(-zq^2,-z^{-1}) = -\,z^{-1}f(-z,-z^{-1}q^2)\ , \tag{2.25}$$

and

$$\begin{aligned}
f(-zq^2,-z^{-1}q^{-1}) &= (zq^2;q)_\infty(z^{-1}q^{-1};q)_\infty(q;q)_\infty \\
&= \frac{1}{(1-z)(1-zq)}(z;q)_\infty \\
&\quad \times(1-z^{-1}q^{-1})(1-z^{-1})(z^{-1}q;q)_\infty(q;q)_\infty \\
&= q^{-1}z^{-2}f(-z,-z^{-1}q)\ ,
\end{aligned} \tag{2.26}$$

we find that

$$\begin{aligned}
D(zq^2) &= \frac{q^4z^2(q;q)_\infty^2 f(-z^2q^4,-z^{-2}q^{-3})f(-zq^2,-z^{-1})}{(q;q^2)_\infty f(-zq^2,-z^{-1}q^{-1})^2} \\
&= \frac{q^4z^2(q;q)_\infty^2 q^{-6}z^{-8}f(-z^2,-z^{-2}q)(-z^{-1})f(-z,-z^{-1}q^2)}{(q;q^2)_\infty q^{-2}z^{-4}f(-z,-z^{-1}q)^2} \\
&= -z^{-5}D(z)\ .
\end{aligned}$$

Lemma 2.7 *For* $|q|<|x|<1$ *and* $|q|<|y|<1$,

$$\sum_{\substack{r,s=-\infty \\ sg(r)=sg(s)}}^{\infty} sg(r)q^{rs}x^ry^s = \frac{(q;q)_\infty^3 f(-xy,-x^{-1}y^{-1}q)}{f(-x,-x^{-1}q)f(-y,-y^{-1}q)}\ .$$

Proof. This Lemma is a special case of Entry 17 in [B3, p. 152]. Replace ab, n and a by q, $-1/x$ and z, respectively, in Entry. Also see Theorem 1.5 in [H1, p. 646].

Theorem 2.6 *In the annulus* $|q|<|z|<1$,

(i) $\phi(q)$ *is the coefficient of* z^2 *in* $D(z)$,

and

(ii) $\psi(q)$ *is the coefficient of* z *in* $D(z)$.

Proof. From the definition of $f(b,c)$,

$$f(-z,-z^{-1}q^2)=\sum_{t=-\infty}^{\infty}(-1)^t q^{t(t-1)}z^t \ ,$$

and from Lemma 2.7 with x and y replaced by z

$$\frac{(q;q)_\infty^3 f(-z^2,-z^{-2}q)}{f(-z,-z^{-1}q)^2}=\sum_{\substack{r,s=-\infty\\ sg(r)=sg(s)}}^{\infty} sg(r)q^{rs}z^{r+s} \ .$$

By (1.8), the two previous equalities, and (2.19), we easily find that

$$\begin{aligned} f(-q,-q)D(z) &= (q;q)_\infty (q;q^2)_\infty D(z) \\ &= \frac{z^2(q;q)_\infty^3 f(-z^2,-z^{-2}q)f(-z,-z^{-1}q^2)}{f(-z,-z^{-1}q)^2} \\ &= z^2\sum_{\substack{r,s=-\infty\\ sg(r)=sg(s)}}^{\infty} sg(r)q^{rs}z^{r+s}\sum_{t=-\infty}^{\infty}(-1)^t q^{t(t-1)}z^t \ . \end{aligned} \tag{2.27}$$

Using Theorem 2.5(i), we see that $\phi(q)$ is the coefficient of z^2 in the Laurent expansion of $D(z)$. This proves (i). Similarly, using Theorem 2.5(ii), we find that the coefficient of z in this Laurent expansion is $\psi(q)$, and so the proof of (ii) is complete.

2.4 The Lambert series $L(z)$ and $M(z)$

In this section, we will define Lambert series $L(z)$ and $M(z)$, and develop the connections of $D(z)$ with $\phi(q)$, $\psi(q)$, $L(z)$, and $M(z)$.

Definition 2.3 *Let q be fixed. Define*

$$L(z):=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n(n+1)+1}z^{5n+5}}{1-q^{2n+1}z} \ , \tag{2.28}$$

$$M(z):=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n(n+1)+1}z^{-5n}}{1-q^{2n+1}z^{-1}} \ , \tag{2.29}$$

and

$$V(z):=D(z)+L(z)+M(z) \ . \tag{2.30}$$

Clearly, L and M are meromorphic for $z\neq 0$, and L and M have simple poles at $z=q^{2k+1}$ for each integer k.

Theorem 2.7 *Let q be fixed, $0 < |q| < 1$. Let a, b, and m be fixed integers with $b \neq 0$ and $m \geq 1$. Define*

$$F(z) = \frac{1}{f(-z^b q^a, -z^{-b} q^{m-a})} \quad .$$

Then F is meromorphic for $z \neq 0$, with simple poles at all points z_0 such that $z_0^b = q^{km-a}$ for some integer k. The residue of $F(z)$ at such a point z_0 is

$$\frac{(-1)^{k+1} q^{mk(k-1)/2} z_0}{b(q^m; q^m)_\infty^3} \quad .$$

Proof. See Theorem 1.3 in [H1, p. 644].

Theorem 2.8 *Suppose that*

$$F(z) = \sum_{r=-\infty}^{\infty} F_r z^r$$

for all $z \neq 0$ and that $F(z)$ satisfies $F(zq) = Cz^{-n}F(z)$, where $0 < |q| < 1$ and $C \neq 0$.

(a) *Then*

$$F(z) = \sum_{r=0}^{n-1} F_r z^r f(C^{-1} z^n q^r, Cz^{-n} q^{n-r}) \quad .$$

(b) *If, in addition, n is odd, $C = \pm 1$, and $F(z)$ satisfies*

$$F(z^{-1}) = -\, Cz^{-n} F(z) \quad ,$$

then

$$F(z) = \sum_{r=1}^{\frac{n-1}{2}} F_r [z^r f(Cz^n q^r, C^{-1} z^{-n} q^{n-r}) - Cz^{n-r} f(Cz^n q^{n-r}, C^{-1} z^{-n} q^r)] \quad .$$

Proof. See Theorem 1.8 in [H1, p. 647].

Lemma 2.8 *Let q be fixed. Then,*

(i) $R(D(z); q) = -2q^2$,
(ii) $R(L(z) + M(z); q) = 2q^2$,

and

(iii) $D(z) + L(z) + M(z)$ *is analytic at* $z = q$.

Proof of (i). By (2.20), Theorem 2.7 and (1.8),

$$R(D(z);q) = \frac{q^2(q^2;q^2)_\infty f(q,1)f(-q^3,-q^{-1})}{(q;q^2)_\infty}\frac{(-1)^2q}{(q^2;q^2)^3_\infty}$$
$$= \frac{q^3 f(q,1)f(-q^3,-q^{-1})}{(q;q^2)_\infty(q^2;q^2)^2_\infty}$$
$$= -2q^2 \ .$$

Proof of (ii). The residue of $L(z)$ at $z=q$ is q^2 and the residue of $M(z)$ at $z=q$ is q^2 since $(-1)q/(1-q^{-1}z) = q^2/(z-q)$ and $q/(1-qz^{-1}) = q + q^2/(z-q)$. So, $R(L(z)+M(z);q) = 2q^2$.

Proof of (iii). Since $D(z)$ and $-(L(z)+M(z))$ have a simple pole at $z=q$, and have same residue at $z=q$, $D(z)+L(z)+M(z)$ is analytic at $z=q$.

Lemma 2.9 *The function $V(z)$ is analytic at $z=q$ and satisfies the following two functional equations*

$$V(z^{-1}) = z^{-5}V(z), \qquad \textit{and} \qquad V(zq^2) = -z^{-5}V(z) \ .$$

Proof. The function $V(z)$ is analytic at $z=q$ by Lemma 2.8 (iii). Replacing z by z^{-1} in $L(z)$ and $M(z)$, we find that

$$L(z^{-1}) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)+1}z^{-5n-5}}{1-q^{2n+1}z^{-1}} = z^{-5}M(z) \ ,$$

and

$$M(z^{-1}) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)+1}z^{5n}}{1-q^{2n+1}z} = z^{-5}L(z) \ .$$

So, we find that $L(z^{-1}) + M(z^{-1}) = z^{-5}(L(z)+M(z))$. Next, replacing z by zq^2 and then replacing n by $n-1$ in $L(z)$, we find that

$$L(zq^2) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)+1}q^{10n+10}z^{5n+5}}{1-q^{2n+3}z}$$
$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{5n(n-1)+1}q^{10n}z^{5n}}{1-q^{2n+1}z}$$
$$= -z^{-5}L(z) \ .$$

Replacing z by q^2z and then n by $n+1$ in $M(z)$, we find that

$$\begin{aligned} M(zq^2) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)+1} q^{-10n} z^{-5n}}{1 - q^{2n-1} z^{-1}} \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n-1)+1} z^{-5n}}{1 - q^{2n-1} z^{-1}} \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{5n(n+1)+1} z^{-5n-5}}{1 - q^{2n+1} z^{-1}} \\ &= -z^{-5} M(z) \ . \end{aligned}$$

Since we already know that $D(z)$ satisfies the given functional equations in Lemma 2.6, $V(z)$ satisfies the same functional equations.

By Lemma 2.8(iii) and Lemma 2.9, $V(z)$ is analytic at $z = q^{2k+1}$ for each integer k. So $V(z)$ is analytic for $z \neq 0$. Thus $V(z)$ has a Laurent expansion which is valid for all $z \neq 0$.

Lemma 2.10 *Suppose that* $V(z) = \sum_{r=-\infty}^{\infty} V_r z^r$ *for all* $z \neq 0$. *Then*

$$\begin{aligned} V(z) = {} & V_1[zf(-z^5q^2, -z^{-5}q^8) + z^4 f(-z^5q^8, -z^{-5}q^2)] \\ & + V_2[z^2 f(-z^5q^4, -z^{-5}q^6) + z^3 f(-z^5q^6, -z^{-5}q^4)]. \end{aligned}$$

Proof. In Theorem 2.8 (b), replace q, C, and n by q^2, -1, and 5. We complete the proof of Lemma 2.10.

Lemma 2.11

$$V_1 = \psi(q), \qquad \mathit{and} \qquad V_2 = \phi(q) \ .$$

Proof. Assume $|q| < |z| < 1$. Then $|q^{2n+1}z| < 1$ if and only if $n \geq 0$, and by (2.28),

$$\begin{aligned} L(z) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)+1} z^{5n+5}}{1 - q^{2n+1} z} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(n+1)+1} z^{5n+5} sg(n) \sum_{\substack{m=-\infty \\ sg(m)=sg(n)}}^{\infty} (q^{2n+1}z)^m \\ &= \sum_{\substack{n,m=-\infty \\ sg(n)=sg(m)}}^{\infty} sg(n)(-1)^n q^{5n(n+1)+1+(2n+1)m} z^{5n+m+5} \ . \end{aligned}$$

If $sg(n) = sg(m)$, then $5n + m + 5 \geq 5$, or ≤ -1. Thus, the coefficients of z and z^2 in $L(z)$ are 0. Similarly, the coefficients of z and z^2 in $M(z)$ are 0. In

Theorem 2.6, we proved that $\psi(q)$ is the coefficient of z in $D(z)$, and $\phi(q)$ is the coefficient of z^2 in $D(z)$. And $f(-z^5q^2, -z^{-5}q^8)$ and $f(-z^5q^4, -z^{-5}q^6)$ have only powers of 5 in z variable, and the constant term in $f(-z^5q^2, -z^{-5}q^8)$ and $f(-z^5q^4, -z^{-5}q^6)$ is 1. Then by (2.30) and Lemma 2.10, Lemma 2.11 is proved.

By (2.30), Lemma 2.10, and Lemma 2.11, we can easily verify the next theorem.

Theorem 2.9 *If* $0 < |q| < 1$ *and* z *is neither zero nor an integral power of* q, *then*

$$\begin{aligned} D(z) = {} & \psi(q)[zf(-z^5q^2, -z^{-5}q^8) + z^4 f(-z^5q^8, -z^{-5}q^2)] \\ & + \phi(q)[z^2 f(-z^5q^4, -z^{-5}q^6) + z^3 f(-z^5q^6, -z^{-5}q^4)] \\ & - L(z) - M(z) \ . \end{aligned}$$

2.5 Another formulation of D(z)

We easily verify that

$$\frac{1}{1 - q^{2n+1}z} = \frac{1}{1 - q^{10n+5}z^5} \sum_{i=0}^{4} z^i q^{(2n+1)i}$$

and

$$\frac{1}{1 - q^{2n+1}z^{-1}} = \frac{z^{-5}}{1 - q^{10n+5}z^{-5}} \sum_{i=1}^{5} z^i q^{(2n+1)(5-i)} \ .$$

By (2.28), (2.29), and Theorem 2.9, we find that

$$\begin{aligned} D(z) = {} & z\psi(q)f(-z^5q^2, -z^{-5}q^8) - z\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(5n+7)+2} z^{5n+5}}{1 - q^{10n+5}z^5} \right. \\ & \left. + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(5n+13)+5} z^{-5n-5}}{1 - q^{10n+5}z^{-5}} \right) \\ & + z^2\phi(q)f(-z^5q^4, -z^{-5}q^6) - z^2\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(5n+9)+3} z^{5n+5}}{1 - q^{10n+5}z^5} \right. \\ & \left. + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(5n+11)+4} z^{-5n-5}}{1 - q^{10n+5}z^{-5}} \right) \end{aligned}$$

$$+z^3\phi(q)f(-z^5q^6,-z^{-5}q^4)-z^3\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(5n+11)+4}z^{5n+5}}{1-q^{10n+5}z^5}\right.$$

$$\left.+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(5n+9)+3}z^{-5n-5}}{1-q^{10n+5}z^{-5}}\right)$$

$$+z^4\psi(q)f(-z^5q^8,-z^{-5}q^2)-z^4\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(5n+13)+5}z^{5n+5}}{1-q^{10n+5}z^5}\right.$$

$$\left.+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(5n+7)+2}z^{-5n-5}}{1-q^{10n+5}z^{-5}}\right)$$

$$-\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n(n+1)+1}z^{5n+5}}{1-q^{10n+5}z^5}+z^5\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n(n+1)+1}z^{-5n-5}}{1-q^{10n+5}z^{-5}}\right).$$

Define

$$D_1(z^5)=\psi(q)f(-z^5q^2,-z^{-5}q^8)-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+7n+2}z^{5n+5}}{1-q^{10n+5}z^5}$$

$$-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+13n+5}z^{-5n-5}}{1-q^{10n+5}z^{-5}}, \tag{2.31}$$

$$D_2(z^5)=\phi(q)f(-z^5q^4,-z^{-5}q^6)-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+9n+3}z^{5n+5}}{1-q^{10n+5}z^5}$$

$$-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+11n+4}z^{-5n-5}}{1-q^{10n+5}z^{-5}}, \tag{2.32}$$

$$D_3(z^5)=\phi(q)f(-z^5q^6,-z^{-5}q^4)-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+11n+4}z^{5n+5}}{1-q^{10n+5}z^5}$$

$$-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+9n+3}z^{-5n-5}}{1-q^{10n+5}z^{-5}},$$

$$D_4(z^5)=\psi(q)f(-z^5q^8,-z^{-5}q^2)-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+13n+5}z^{5n+5}}{1-q^{10n+5}z^5}$$

$$-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+7n+2}z^{-5n-5}}{1-q^{10n+5}z^{-5}},$$

and

$$D_0(z^5) = -\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)+1} z^{5n+5}}{1 - q^{10n+5} z^5}$$
$$-z^5 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)+1} z^{-5n-5}}{1 - q^{10n+5} z^{-5}} \ .$$

By the previous six equalities, we rewrite

$$D(z) = \sum_{i=0}^{4} z^i D_i(z^5) \ .$$

Now, we will define $A(z,x,q)$, $h(x,q)$, and $Q(z)$, and show that

$$A(z,x,q) = f(-x^2 z, -x^{-2} z^{-1} q^2) h(x,q) + Q(z) \ .$$

Definition 2.4 *Let* $|q| < 1$, *and let* x *be neither zero nor an integral power of* q. *Define*

$$A(z,x,q) := \frac{(q;q)_\infty^3 f(-zx, -z^{-1}x^{-1}q) f(-z, -z^{-1}q^2)}{f(-q,-q) f(-x, -x^{-1}q) f(-z, -z^{-1}q)} \ , \qquad (2.33)$$

$$h(x,q) := \frac{1}{f(-q,-q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{1 - q^n x} \ , \qquad (2.34)$$

and

$$Q(z,x,q) := Q(z) :$$
$$= -\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)} x^{2n+1} z^{n+1}}{1 - q^{2n+1} z}$$
$$-\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+3)+1} x^{-2n-1} z^{-n-1}}{1 - q^{2n+1} z^{-1}} \ . \qquad (2.35)$$

By (1.8) and (1.7),

$$A(z,x,q) = \frac{(q;q)_\infty^2 (q^2;q^2)_\infty^2 f(-zx, -z^{-1}x^{-1}q)}{f(-q,-q) f(-x, -x^{-1}q) f(-zq, -z^{-1}q)} \ . \qquad (2.36)$$

the functions $A(z,x,q)$ and $Q(z)$ are clearly meromorphic for $z \neq 0$ with simple poles at $z = q^{2k+1}$ for each integer k.

Lemma 2.12 *Let* q *and* x *be fixed where* $0 < |q| < 1$, *and suppose that* x *is neither zero nor an integral power of* q. *Then* $h(x,q)$ *is the coefficient of* z^0 *in the Laurent series expansion of* $A(z,x,q)$ *in the annulus* $|q| < |z| < 1$.

Proof. By (2.33), (2.34) and Lemma 2.7,

$$\begin{aligned}h(x,q) &= \text{the coefficient of } z^0 \text{ in}\\ &\quad \frac{1}{f(-q,-q)}\sum_{n=-\infty}^{\infty}\frac{z^n}{1-q^n x}\sum_{s=-\infty}^{\infty}(-1)^s q^{s(s+1)}z^{-s}\\ &= \text{the coefficient of } z^0 \text{ in}\\ &\quad \frac{(q;q)_\infty^3 f(-zx,-z^{-1}x^{-1}q)}{f(-q,-q)f(-x,-x^{-1}q)f(-z,-z^{-1}q)}f(-z,-z^{-1}q^2)\\ &= \text{the coefficient of } z^0 \text{ in } A(z,x,q)\ .\end{aligned}$$

Lemma 2.13 *The function $Q(z)$ satisfies the functional equation*

$$Q(zq^2) = -x^{-2}z^{-1}Q(z)\ ,$$

and $R(Q(z);q) = -x^{-1}q$.

Proof. By (2.35), we find that

$$\begin{aligned}Q(zq^2) &= -\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)+2(n+1)}x^{2n+1}z^{n+1}}{1-q^{2n+3}z}\\ &\quad -\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+3)+1-2(n+1)}x^{-2n-1}z^{-n-1}}{1-q^{2n-1}z^{-1}}\ .\end{aligned}$$

Replacing n by $n-1$ in the first sum, and replacing n by $n+1$ in the second sum, we find that

$$\begin{aligned}Q(zq^2) &= -\frac{1}{2}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^{n-1}q^{n^2+n}x^{2n-1}z^n}{1-q^{2n+1}z}\right.\\ &\qquad\left.+\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}q^{n^2+3n+1}x^{-2n-3}z^{-n-2}}{1-q^{2n+1}z^{-1}}\right)\\ &= \frac{x^{-2}z^{-1}}{2}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)}x^{2n+1}z^{n+1}}{1-q^{2n+1}z}\right.\\ &\qquad\left.+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+3)+1}x^{-2n-1}z^{-n-1}}{1-q^{2n+1}z^{-1}}\right)\\ &= -x^{-2}z^{-1}Q(z)\ .\end{aligned}$$

And $R(Q(z);q) = -x^{-1}q$, by (2.35) and the equalities

$$-\frac{1}{2}\left(\frac{(-1)^{-1}x^{-1}}{1-q^{-1}z}+\frac{qx^{-1}z^{-1}}{1-qz^{-1}}\right)=-\frac{1}{2}\left(\frac{x^{-1}q}{z-q}+\frac{x^{-1}q}{z-q}\right)=-\frac{x^{-1}q}{z-q}\ .$$

Lemma 2.14 *The function $A(z,x,q)$ satisfies the functional equation*

$$A(zq^2,x,q)=-x^{-2}z^{-1}A(z,x,q)\ ,$$

and $R(A(z,x,q);q)=-x^{-1}q$.

Proof. By (2.22), (2.26), (2.33), and (1.8),

$$\begin{aligned}A(zq^2,x,\ q)&=\frac{(q;q)_\infty^3 f(-zxq^2,-z^{-1}x^{-1}q^{-1})f(-zq^2,-z^{-1})}{f(-q,-q)f(-x,-x^{-1}q)f(-zq^2,-z^{-1}q^{-1})}\\&=\frac{(q;q)_\infty^3 q^{-1}z^{-2}x^{-2}f(-zx,-z^{-1}x^{-1}q)(-z^{-1})f(-z,-z^{-1}q^2)}{f(-q,-q)f(-x,-x^{-1}q)q^{-1}z^{-2}f(-z,-z^{-1}q)}\\&=-z^{-1}x^{-2}A(z,x,q)\ .\end{aligned}$$

By Theorem 2.7,

$$\begin{aligned}R(A(z);q)&=\frac{(q;q)_\infty^3 f(-xq,-x^{-1})f(-q,-q)}{f(-q,-q)f(-x,-x^{-1}q)}\frac{q}{(q;q)_\infty^3}\\&=\frac{qf(-xq,-x^{-1})}{f(-x,-x^{-1}q)}\\&=-x^{-1}q\ .\end{aligned}$$

Theorem 2.10 *Let $|q|<1$, and suppose that x is neither zero nor an integral power of q. Then*

$$A(z,x,q)=f(-x^2z,-x^{-2}z^{-1}q^2)h(x,q)+Q(z)\ . \tag{2.37}$$

Proof. Let q and x be fixed. Define

$$E(z):=A(z,x,q)-Q(z)\ . \tag{2.38}$$

Note that $A(z,x,q)$ and $Q(z)$ have a simple for at $z=q^{2k+1}$ for each integer k. By Lemma 2.13 and Lemma 2.14, $E(z)$ satisfies the functional equation

$$E(q^2z)=-x^{-2}z^{-1}E(z)\ ,$$

and $E(z)$ is analytic at $z=q$. By the functional equation, $E(z)$ is analytic at $z\neq 0$. Thus we can say that $E(z)$ must have a Laurent expansion valid for all $z \neq 0$. Let $E(z) = \sum_{n=-\infty}^{\infty} E_n z^n$ be a Laurent expansion valid for all $z \neq 0$. Applying Theorem 2.8(a) with $n = 1$, $C = -x^{-2}$, and q replaced by q^2, we find that

$$E(z) = E_0 f(-x^2 z, -x^{-2} z^{-1} q^2) \ . \tag{2.39}$$

For $|q| < |z| < 1$, the coefficient of z^0 in $A(z, x, q)$ is $h(x, q)$ by Lemma 2.12. Also,

$$Q(z) = -\frac{1}{2} \sum_{\substack{n,m=-\infty \\ sg(n)=sg(m)}}^{\infty} sg(n)(-1)^n q^{n(n+1)+(2n+1)m} x^{2n+1} z^{n+m+1}$$
$$-\frac{1}{2} \sum_{\substack{n,m=-\infty \\ sg(n)=sg(m)}}^{\infty} sg(n)(-1)^n q^{n(n+3)+1+(2n+1)m} x^{-2n-1} z^{-n-m-1} \ .$$

If $sg(n) = sg(m)$, then $n + m + 1$ is either ≤ -1 or ≥ 1. So the coefficient of z^0 in $Q(z)$ is 0. Thus, equating coefficients of z^0 in the definition (2.38) of $E(z)$ and using (2.39), we deduce that $E_0 = h(x, q) + 0$. Hence, (2.37) follows.

We will define $G_m(z, q)$ and $H_m(z, q)$, and develop some properties of $G_m(z, q)$ and $H_m(z, q)$.

Definition 2.5 *Let* $|q| < 1$ *and* $1 \leq m \leq 4$. *Define*

$$G_m(z, q) := G_m(z) := a_m(q) z^m f(-z^5 q^{2m}, -z^{-5} q^{10-2m})$$

where

$$a_1(q) = a_4(q) = -\frac{q(q^5; q^5)_\infty (q^{10}; q^{10})_\infty f(-q^2, -q^8)}{f(-q, -q^4) f(-q^4, -q^6)}$$

and

$$a_2(q) = a_3(q) = \frac{(q^5; q^5)_\infty (q^{10}; q^{10})_\infty f(-q^4, -q^6)}{f(-q^2, -q^3) f(-q^2, -q^8)} \ . \tag{2.40}$$

Furthermore, define

$$H_m(z, q) := H_m(z) := 2 q z^m A(z^5, q^m, q^5) \tag{2.41}$$

and

$$H_0(z, q) := H_0(z) := -\frac{q(q^5; q^5)_\infty^3 f(z^5 q^5, z^{-5} q^5) f(-z^{10}, -z^{-10} q^{20})}{(q^{10}; q^{10})_\infty (q^{20}; q^{20})_\infty f(-z^5, -z^{-5} q^5)} \ . \tag{2.42}$$

By (2.33),

$$H_m(z) = 2qz^m \frac{(q^5;q^5)_\infty^3 f(-z^5q^m, -z^{-5}q^{5-m}) f(-z^5, -z^{-5}q^{10})}{f(-q^5,-q^5) f(-q^m,-q^{5-m}) f(-z^5,-z^{-5}q^5)} \ .$$

Lemma 2.15 *Let* $|q| < 1$. *Then* $G_m(z)$, $1 \le m \le 4$, *satisfies the functional equations*

$$G_m(zq^2) = -z^{-5}G_m(z), \quad \textit{and} \quad G_m(z^{-1}) = z^{-5}G_{5-m}(z) \ .$$

Proof. By (1.8),

$$\begin{aligned}
G_m(zq^2) &= a_m(q)z^m q^{2m} f(-z^5q^{2m+10}, -z^{-5}q^{-2m}) \\
&= a_m(q)z^m q^{2m}(z^5q^{2m+10};q^{10})_\infty (z^{-5}q^{-2m};q^{10})_\infty (q^{10};q^{10})_\infty \\
&= a_m(q)z^m q^{2m} \frac{(z^5q^{2m};q^{10})_\infty}{1-z^5q^{2m}} (1-z^{-5}q^{-2m})(z^{-5}q^{10-2m};q^{10})_\infty \\
&\quad \times (q^{10};q^{10})_\infty \\
&= -a_m(q)z^m z^{-5} f(-z^5q^{2m}, -z^{-5}q^{10-2m}) \\
&= -z^{-5}G_m(z)
\end{aligned}$$

and

$$\begin{aligned}
G_m(z^{-1}) &= a_m(q)z^{-m} f(-z^{-5}q^{2m}, -z^5q^{10-2m}) \\
&= a_m(q)z^{-m} f(-z^5q^{2(5-m)}, -z^{-5}q^{10-2(5-m)}) \\
&= z^{-5}G_{5-m}(z) \ .
\end{aligned}$$

Lemma 2.16 *Let* $|q| < 1$. *Then* $H_m(z)$, $0 \le m \le 4$, *satisfies the functional equations*

$$H_m(zq^2) = -z^{-5}H_m(z), \quad \textit{and} \quad H_m(z^{-1}) = z^{-5}H_{5-m}(z) \ .$$

Proof. By (2.23), Lemma 2.14, and (2.41), for $1 \le m \le 4$,

$$\begin{aligned}
H_m(zq^2) &= 2z^m q^{2m+1} A(z^5q^{10}, q^m, q^5) \\
&= -2qz^{m-5} A(z^5, q^m, q^5) \\
&= -z^{-5}H_m(z) \ ,
\end{aligned}$$

and

$$
\begin{aligned}
&H_m(z^{-1})\\
&\quad = \frac{2qz^{-m}(q^5;q^5)_\infty^3 f(-z^{-5}q^m,-z^5q^{5-m})f(-z^{-5},-z^5q^{10})}{f(-q^5,-q^5)f(-q^m,-q^{5-m})f(-z^{-5},-z^5q^5)}\\
&\quad = z^{-5}\frac{2qz^{5-m}(q^5;q^5)_\infty^3 f(-z^5q^{5-m},-z^{-5}q^m)(-z^{-5})f(-z^5,-z^{-5}q^{10})}{f(-q^5,-q^5)f(-q^m,-q^{5-m})(-z^{-5})f(-z^5,-z^{-5}q^5)}\\
&\quad = z^{-5}H_{5-m}(z) \ .
\end{aligned}
$$

Replacing q by q^5 and z by $-z^5q^5$ in (2.26), we find that

$$
f(z^5q^{15},z^{-5}q^{-5}) = q^{-5}z^{-5}f(z^5q^5,z^{-5}q^5) \ ,
$$

and replacing q by q^5 and z by z^5 in (2.26), we find that

$$
f(-z^5q^{10},-z^{-5}q^{-5}) = q^{-5}z^{-10}f(-z^5,z^{-5}q^5) \ .
$$

For $m = 0$,

$$
\begin{aligned}
H_0(zq^2) &= -\frac{q(q^5;q^5)_\infty^3 f(z^5q^{15},z^{-5}q^{-5})f(-z^{10}q^{20},-z^{-10})}{(q^{10};q^{10})_\infty(q^{20};q^{20})_\infty f(-z^5q^{10},-z^{-5}q^{-5})}\\
&= -\frac{q(q^5;q^5)_\infty^3(q^{-5}z^{-5})f(z^5q^5,z^{-5}q^5)(-z^{-10})f(-z^{10},-z^{-10}q^{20})}{(q^{10};q^{10})_\infty(q^{20};q^{20})_\infty(q^{-5}z^{-10})f(-z^5,-z^{-5}q^5)}\\
&= -z^{-5}H_0(z) \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
H_0(z^{-1}) &= -\frac{q(q^5;q^5)_\infty^3 f(z^{-5}q^5,z^5q^5)f(-z^{-10},-z^{10}q^{20})}{(q^{10};q^{10})_\infty(q^{20};q^{20})_\infty f(-z^{-5},-z^5q^5)}\\
&= -\frac{q(q^5;q^5)_\infty^3 f(z^5q^5,z^{-5}q^5)(-z^{-10})f(-z^{10},-z^{-10}q^{20})}{(q^{10};q^{10})_\infty(q^{20};q^{20})_\infty(-z^{-5})f(-z^5,-z^{-5}q^5)}\\
&= z^{-5}H_0(z) \ .
\end{aligned}
$$

Lemma 2.17 *For* $0 \le m \le 4$,

$$
R(H_m(z);\zeta q) = -\frac{2\zeta^{m+1}q^2}{5} \ ,
$$

where $\zeta^5 = 1$ *and* $\zeta \neq 1$.

Proof. For $1 \le m \le 4$, by Theorem 2.7, (2.41), and (1.8),

$$R(H_m(z);\zeta q) = 2q(\zeta q)^m (q^5;q^5)_\infty^3$$
$$\times \frac{f(-(\zeta q)^5 q^m, -(\zeta q)^{-5} q^{5-m}) f(-(\zeta q)^5, -(\zeta q)^{-5} q^{10})}{f(-q^5,-q^5) f(-q^m,-q^{5-m})} \frac{\zeta q}{5(q^5;q^5)_\infty^3}$$
$$= \frac{2\zeta^{m+1} q^{m+2} f(-q^{5+m}, -q^{-m})}{5 f(-q^m, -q^{5-m})}$$
$$= -\frac{2\zeta^{m+1} q^2}{5} \quad ,$$

and for $m = 0$, by (2.42) and (1.8),

$$R(H_0(z);\zeta q) = -\frac{q(q^5;q^5)_\infty^3 f((\zeta q)^5 q^5, (\zeta q)^{-5} q^5) f(-(\zeta q)^{10}, -(\zeta q)^{-10} q^{20})}{(q^{10};q^{10})_\infty (q^{20};q^{20})_\infty}$$
$$\times \frac{\zeta q}{5(q^5;q^5)_\infty^3}$$
$$= -\frac{2\zeta q^2}{5} \quad .$$

Lemma 2.18 *If* $|q| < 1$ *and* k *is an arbitrary integer*, *then*

$$\sum_{m=1}^{4} H_m(-q^k) = 0 \quad . \tag{2.43}$$

Proof. For $1 \le m \le 4$, by (2.41),

$$\frac{H_{5-m}(z)}{H_m(z)} = \frac{\dfrac{2qz^{5-m}(q^5;\ q^5)_\infty^3 f(-z^5 q^{5-m}, -z^{-5} q^m) f(-z^5, -z^{-5} q^{10})}{f(-q^5,-q^5) f(-q^{5-m}, -q^m) f(-z^5, -z^{-5} q^5)}}{\dfrac{2qz^m (q^5;\ q^5)_\infty^3 f(-z^5 q^m, -z^{-5} q^{5-m}) f(-z^5, -z^{-5} q^{10})}{f(-q^5,-q^5) f(-q^m,-q^{5-m}) f(-z^5,-z^{-5} q^5)}}$$
$$= \frac{z^{5-m} f(-z^5 q^{5-m}, -z^{-5} q^m)}{z^m f(-z^5 q^m, -z^{-5} q^{5-m})} \quad . \tag{2.44}$$

Replacing a by $-z^{-5}q^m$, b by $-z^5 q^{5-m}$, and n by $2k$ in Lemma 2.1(iv), we deduce that

$$f(-z^{-5}q^m, -z^5 q^{5-m})$$
$$= (-1)^{2k} q^{10k^2-5k+2mk} z^{-10k} f(-z^{-5} q^{10k+m}, -z^5 q^{5-m-10k}) \quad . \tag{2.45}$$

For $z = -q^k$, by (2.45),

$$\begin{aligned} f(-z^5q^m, -z^{-5}q^{5-m}) &\\ &= f(-z^{-5}q^{10k+m}, -z^5q^{5-10k-m}) \\ &= (-1)^{2k}(q^5)^{-2k(2k-1)/2}(q^mz^{-5})^{-2k}f(-z^{-5}q^m, -z^5q^{5-m}) \\ &= q^{-10k^2+5k-2km+10k^2}f(-z^{-5}q^m, -z^5q^{5-m}) \\ &= q^{5k-2km}f(-z^{-5}q^m, -z^5q^{5-m}) \\ &= -z^{5-2m}f(-z^{-5}q^m, -z^5q^{5-m}) \ . \end{aligned} \tag{2.46}$$

By (2.44) and (2.46),

$$\frac{H_{5-m}(z)}{H_m(z)} = -1 \quad \text{if } z = -q^k \ .$$

The equality (2.43) now easily follows.

Now, we will show that

$$D(z) = \sum_{m=1}^{4} G_m(z,q) + \sum_{m=0}^{4} H_m(z,q) \ .$$

Recall that the definition of $D(z)$ is (2.19).

Definition 2.6 *Let* $|q| < 1$. *Define*

$$W(z,q) := W(z) := D(z) - \sum_{m=1}^{4} G_m(z,q) - \sum_{m=0}^{4} H_m(z,q) \tag{2.47}$$

for $z \neq 0$ *and* $z \neq \zeta q^k$ *where* k *is an integer and* ζ *is a fifth root of unity.*

Theorem 2.11 *Let* q *and* C *be complex numbers with* $0 < |q| < 1$ *and* $C \neq 0$ *and let* n *be a nonnegative integer. Suppose that* $F(z)$ *is analytic for* $z \neq 0$ *and satisfies a functional equation* $F(qz) = Cz^{-n}F(z)$. *Then either* $F(z)$ *has exactly* n *zeros in the annulus* $|q| < |z| \leq 1$ *or* $F(z) \equiv 0$.

Proof. See Theorem 1.7 in [H1, p. 647].

Theorem 2.12 *If* n *is a positive integer*, $0 < |q| < 1$, *and* $x \neq 0$, *then*,

$$\begin{aligned} f(-x, -x^{-1}q) = &\sum_{k=0}^{n-1}(-1)^k q^{k(k-1)/2}x^k \\ &\times f((-1)^n x^n q^{n(n-1)/2+kn}, (-1)^n x^{-n} q^{n(n+1)/2-kn}) \ . \end{aligned}$$

Proof. See Theorem 1.2 in [A5, p. 67].

In Lemma 2.6, Lemma 2.15 and Lemma 2.16, we showed that $D(z)$, $\sum_{m=1}^{4} G_m(z)$, and $\sum_{m=0}^{4} H_m(z)$ satisfy the two same functional equations. So, we clearly see that $W(z)$ satisfies these functional equations

$$W(zq^2) = -z^{-5}W(z), \quad \text{and} \quad W(z^{-1}) = z^{-5}W(z) \ . \tag{2.48}$$

Also, in Lemma 2.17, we showed that the residues of $H_m(z)$, $0 \le m \le 4$, at $z = \zeta q$, where $\zeta^5 = 1$ and $\zeta \ne 1$, are $-2\zeta^{m+1}q^2/5$. Since

$$\sum_{m=0}^{4} \frac{2\omega^{m+1}q^2}{5} = 0 \ ,$$

it follows that $W(z)$ is analytic at $z = \zeta q^{2k+1}$ for each integer k, since $D(z)$ and $G_m(z)$, $1 \le m \le 4$, are analytic at $z = \zeta q^{2k+1}$ for each integer k. We will prove that $W(z)$ has 6 zeros in the annulus $|q^2| < |z| \le 1$. This implies that $W(z) \equiv 0$ by Theorem 2.11. We will show that $W(z) = 0$ at $z = -1, \ -q, \ \pm q^{1/2}, \ \pm q^{3/2}$.

Lemma 2.19

$$W(-1) = 0 \ .$$

Proof. By (2.19), (2.42), and Lemma 2.1(iii),

$$D(-1) = 0 \quad \text{and} \quad H_0(-1) = 0 \ . \tag{2.49}$$

By Lemma 2.18 with $k = 0$,

$$\sum_{m=1}^{4} H_m(-1) = 0 \ . \tag{2.50}$$

By (2.40),

$$\begin{aligned}\sum_{m=1}^{4} G_m(-1) &= \sum_{m=1}^{4} (-1)^m a_m(q) f(q^{2m}, q^{10-2m}) \\ &= (-a_1(q) + a_4(q)) f(q^2, q^8) + (a_2(q) - a_3(q)) f(q^4, q^6) \\ &= 0 \ . \end{aligned} \tag{2.51}$$

By (2.47), (2.49), (2.50), and (2.51), $W(-1) = 0$.

Lemma 2.20

$$W(-q) = 0 \ .$$

Proof. By (2.19), (2.42), Lemma 2.1(iii) and (iv),

$$D(-q) = 0 \quad \text{and} \quad H_0(-q) = 0 \ . \tag{2.52}$$

By Lemma 2.18 with $k = 1$,

$$\sum_{m=1}^{4} H_m(-q) = 0 \ . \tag{2.53}$$

By (2.40) and Lemma 2.1(iv),

$$\begin{aligned}\sum_{m=1}^{4} G_m(-q) &= \sum_{m=1}^{4} (-q)^m a_m(q) f(q^{2m+5}, q^{5-2m}) \\ &= -qa_1(q)f(q^3, q^7) + q^2 a_2(q) f(q, q^9) \\ &\quad -q^3 a_3(q) f(q^{-1}, q^{11}) + q^4 a_4(q) f(q^{-3}, q^{13}) \\ &= (-a_1(q) + a_4(q))qf(q^3, q^7) + (a_2(q) - a_3(q))q^2 f(q, q^9) \\ &= 0 \ . \end{aligned} \tag{2.54}$$

By (2.47), (2.52), (2.53), and (2.54), $W(-q) = 0$.

By (2.48), it follows that $W(\pm q^{3/2}) = -W(\pm q^{1/2})$. Thus, we only need to show that $W(\pm q^{1/2}) = 0$.

Lemma 2.21

$$W(\pm q^{1/2}) = 0 \ .$$

Proof. We will show that $W(-q, q^2) = 0$ instead of $W(-q^{1/2}, q) = 0$.

By (2.19) and Lemma 2.1(iii),

$$D(-q, q^2) = 0 \ . \tag{2.55}$$

By (2.41),

$$\frac{H_{5-m}(z)}{H_m(z)} = \frac{z^{5-m} f(-z^5 q^{5-m}, -z^{-5} q^m)}{z^m f(-z^5 q^m, -z^{-5} q^{5-m})} \ .$$

Replacing z and q by $-q$ and q^2, respectively, we find that by Lemma 2.1(iv)

$$\frac{H_{5-m}(-q, q^2)}{H_m(-q, q^2)} = \frac{-q^{5-2m} f(q^{15-2m}, q^{2m-5})}{f(q^{2m+5}, q^{5-2m})} = -1 \ .$$

It follows that

$$\sum_{m=1}^{4} H_m(-q, q^2) = 0 \ . \tag{2.56}$$

By (2.47), (2.55), and (2.56),

$$W(-q, q^2) = -\sum_{m=1}^{4} G_m(-q, q^2) - H_0(-q, q^2) \ . \tag{2.57}$$

We need Theorem 2.12 to finish the proof of Lemma 2.21.

By (2.40),

$$\begin{aligned}\sum_{m=1}^{4} G_m(-q, q^2) &= -qa_1(q^2)f(q^9, q^{11}) + q^2a_2(q^2)f(q^7, q^{13}) \\ &\quad -q^3a_3(q^2)f(q^3, q^{17}) + q^4a_4(q^2)f(q^{-1}, q^{21}) \\ &= -qa_1(q^2)\big(f(q^9, q^{11}) - q^3f(q^{-1}, q^{21})\big) \\ &\quad +q^2a_2(q^2)\big(f(q^7, q^{13}) - qf(q^3, q^{17})\big) \ .\end{aligned}$$

For the first term, replace n, q, and x by 2, q^5, and q^2, respectively, in Theorem 2.12, and for the second term, replace n, q, and x by 2, q^5, and q^4, respectively, in Theorem 2.12. Applying Lemma 2.1(iv), we have

$$\sum_{m=1}^{4} G_m(-q, q^2) = -qa_1(q^2)f(-q^2, -q^3) + q^2a_2(q^2)f(-q, -q^4) \ .$$

By Lemma 2.2, the equality above, (2.42) and (1.8),

$$\begin{aligned}\sum_{m=1}^{4} G_m(-q, q^2) &= \frac{q^2(q^{10}; q^{10})_\infty^3 f(-q^5, -q^{15})f(-q^{10}, -q^{30})}{(q^{20}; q^{20})_\infty (q^{40}; q^{40})_\infty f(q^5, q^5)} \\ &= -H_0(-q, q^2) \ .\end{aligned}$$

It follows from (2.57) that

$$W(-q, q^2) = 0 \ .$$

Replacing q by $q^{1/2}$, we find that

$$W(-q^{1/2}, q) = 0 \ .$$

Also, replacing q by $-q^{1/2}$, we find that

$$W(q^{1/2}, q) = 0 \ .$$

In conclusion, we have shown that $W(z)$ has zeros -1, $-q$, $\pm q^{1/2}$, and $\pm q^{3/2}$ in the annulus $|q^2| < |z| \leq 1$. We are ready to prove the next theorem.

Theorem 2.13 *Let $z \neq 0$ and $z \neq \zeta q^k$ where k is an integer and ζ is a fifth root of unity. Then,*

$$D(z) = \sum_{m=1}^{4} G_m(z,q) + \sum_{m=0}^{4} H_m(z,q)$$

Proof. We showed that $W(z)$ has 6 zeros in the annulus $|q^2| < |z| \leq 1$. Thus, by Theorem 2.11, $W(z) \equiv 0$. Hence, by (2.47), the proof is complete.

By (2.31) and (2.35),

$$D_1(z^5) = \psi(q)f(-z^5q^2, -z^{-5}q^8) + 2qQ(z^5, q, q^5) \ . \tag{2.58}$$

By Theorem 2.13, (2.40), and (2.41),

$$\begin{aligned} D_1(z^5) &= z^{-1}(G_1(z) + H_1(z)) \\ &= a_1(q)f(-z^5q^2, -z^{-5}q^8) + 2qA(z^5, q, q^5) \ . \end{aligned} \tag{2.59}$$

Thus, by (2.58) and (2.59),

$$\begin{aligned} \psi(q)f(-z^5q^2, -z^{-5}q^8) = a_1(q)f(-z^5q^2, -z^{-5}q^8) \\ + 2qA(z^5, q, q^5) - 2qQ(z^5, q, q^5) \end{aligned} \tag{2.60}$$

and by Theorem 2.10,

$$\psi(q) = a_1(q) + 2qh(q, q^5) \ .$$

By (2.32) and (2.35),

$$D_2(z^5) = \phi(q)f(-z^5q^4, -z^{-5}q^6) + 2qQ(z^5, q^2, q^5). \tag{2.61}$$

By Theorem 2.13, (2.40), and (2.41),

$$\begin{aligned} D_2(z^5) &= z^{-2}(G_2(z) + H_2(z)) \\ &= a_2(q)f(-z^5q^4, -z^{-5}q^6) + 2qA(z^5, q^2, q^5) \ . \end{aligned} \tag{2.62}$$

Thus, by (2.61) and (2.62),

$$\begin{aligned} \phi(q)f(-z^5q^4, -z^{-5}q^6) = a_2(q)f(-z^5q^4, -z^{-5}q^6) \\ + 2qA(z^5, q^2, q^5) - 2qQ(z^5, q^2, q^5) \end{aligned} \tag{2.63}$$

and by Theorem 2.10,

$$\phi(q) = a_2(q) + 2qh(q^2, q^5) \ .$$

In [A5], [H1], and [H2], it was shown that the 5th, 6th, and 7th order mock theta functions can be expressed in terms of the function $g(x, q)$, where

$$g(x, q) = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1 - q^n x} \ .$$

We have shown that the 10th order mock theta functions $\psi(q)$ and $\phi(q)$ can be expressed in terms of $h(x, q)$, which is an analogue of $g(x, q)$.

2.6 Proof of two theta function identities

In this section, we will prove two eta-function identities. Then we will derive two theta function identities from these 2 eta-function identities.

We need the definitions in [KM].

Definition (the Dedekind eta-function) *Let* $\mathbf{H} = \{z\colon Im\ z > 0\}$. *For* $z \in \mathbf{H}$ *and* $q = e^{2\pi i z}$, *define*

$$\eta(z) := e^{\frac{\pi i z}{12}} \prod_{m=1}^{\infty} (1 - e^{2\pi i m z}) = q^{\frac{1}{24}} (q;q)_\infty \ .$$

By (1.9),

$$\eta(z) = q^{\frac{1}{24}} f(-q, -q^2) \ .$$

Definition 2.8 (the modular group) *The modular group is the set of linear fractional transformations* T, $T(z) = \frac{az+b}{cz+d}$, *where* $a,\ b,\ c$, *and* d *are rational integers such that* $ad - bc = 1$. *The modular group is denoted by* $\Gamma(1)$. *Let* $\Gamma_0(N)$, *where* N *is a positive integer, be the set of linear factional transformations* U, $U(z) = \frac{az+b}{cz+d}$, *where* $a,\ b,\ c$, *and* d *are rational integers such that* $ad - bc = 1$ *and* $c \equiv 0 (\mathrm{mod}\, N)$.

Clearly, $\Gamma_0(N)$ is a subgroup of $\Gamma(1)$.

Definition 2.9 (a fundamental region) *Let* Γ *be a subgroup of* $\Gamma(1)$. *A fundamental region for* Γ *is an open subset* R *of* $\mathbf{H}$ *such that* (a) *for any distinct points* z_1, z_2 *in* R, *there is no* $T \in \Gamma$ *such that* $T(z_1) = z_2$, *and* (b) *for any point* z *in* H, *there is some point* z_3 *in the closure of* R *such that for some* $T' \in \Gamma$, $T'(z) = z_3$.

Definition 2.10 (a standard fundamental region) *Let* Γ *be a subgroup of* $\Gamma(1)$ *with cosets* $A_1, A_2, \ldots, A_\mu$ *in the sense that* $\Gamma(1) = \cup_{i=1}^{\mu} \Gamma A_i$. *Then R is called a standard fundamental region for* Γ.

Definition 2.11 (a cusp) *Let R be a fundamental region of* Γ. *A parabolic cusp of* Γ *in R is any real point q, or* $q = \infty$, *such that* $q \in \bar{R}$, *the closure of R in the topology of the Riemann sphere.*

Definition 2.12 (a modular form of weight *r*) *Let r be a real number. A function* $F(z)$, *defined and meromorphic in* $\mathbf{H}$, *is said to be a modular form of weight r with respect to* Γ , *with multiplier system v, if*

(a) $F(z)$ *satisfies*

$$F(Mz) = v(M)(cz + d)^r F(z)$$

for any $z \in \mathbf{H}$ *and* $M \in \Gamma$,

(b) *there exists a standard fundamental region R such that* $F(z)$ *has at most finitely many poles in* $\bar{R} \cap \mathbf{H}$,

and

(c) $F(z)$ *is meromorphic at* q_j, *for each cusp* q_j *in* $\bar{R}$.

The multiplier system $v = v(M)$ *for the group* Γ, *is a complex-valued function of absolute value* 1, *satisfying the equation*

$$v(M_1M_2)(c_3z + d_3)^r = v(M_1)v(M_2)(c_1M_2z + d_1)^r(c_2z + d_2)^r$$

for $M_1,\ M_2 \in \Gamma$ *where* $M_1(z) = \frac{a_1z+b_1}{c_1z+d_1}$, $M_2(z) = \frac{a_2z+b_2}{c_2z+d_2}$, *and* $M_3(z) = (M_1M_2)(z) = \frac{a_3z+b_3}{c_3z+d_3}$.

Let $\{\Gamma, r, v\}$ denote the space of modular forms of weight r and multiplier system v on Γ, where Γ is a subgroup of $\Gamma(1)$ of finite index.

Let $\operatorname{ord}(f; z)$ denote the invariant order of a modular form f at z. Let $\operatorname{Ord}_\Gamma(f; z)$ denote the order of f with respect to Γ, defined by $\operatorname{Ord}_\Gamma(f; z) := \frac{1}{\ell}\operatorname{ord}(f; z)$, where ℓ is the order of f at z as a fixed point of Γ.

Theorem 2.14 *The Dedekind eta-function* $\eta(z)$ *is a modular form of weight* 1/2 *on the full modular group* $\Gamma(1)$.

Proof. See Theorem 10 in [KM, p. 43].

Theorem 2.15 *The multiplier system* v_η *of the modular form* $\eta(z)$ *is given by the following formula*: *for each* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$,

$$\upsilon_\eta(M) = \begin{cases} \left(\frac{d}{|c|}\right)\zeta_{24}^{bd(1-c^2)+c(a+d)-3c}, & \text{if } c \text{ is odd },\\ \left(\frac{c}{|d|}\right)\zeta_{24}^{ac(1-d^2)+d(b-c)+3(d-1)}, & \text{if } d \text{ is odd and either } c \geq 0 \\ & \quad \text{or } d \geq 0,\\ -\left(\frac{c}{|d|}\right)\zeta_{24}^{ac(1-d^2)+d(b-c)+3(d-1)}, & \text{if } d \text{ is odd } c < 0,\ d < 0 \ , \end{cases}$$

where ζ_{24} *is a 24th root of unity.*

Proof. See Theorem 2 in [KM, p. 51].

Theorem 2.16 *If* $f \in \{\Gamma, r, v\}$ *and* $f \neq 0$, *then*

$$\sum_{z \in R} \mathrm{Ord}_\Gamma(f; z) = \mu r \ ,$$

where R *is any fundamental region for* Γ, *and* $\mu := \frac{1}{12}[\Gamma(1) : \Gamma]$.

This theorem is called the valence formula.

Proof. See Theorem 4.1.4 in [R3].

Theorem 2.17 *If* σ_∞ *denotes the number of inequivalent cusps of* $\Gamma_0(N)$, *then*

$$\sigma_\infty = \sum_{d|N} \varphi((d, N/d)) \ ,$$

where φ *denotes Euler's* φ*-function and* (a, b) *denotes the greatest common divisor of* a *and* b.

Proof. See page 102 in [SB].

Theorem 2.22 *If* $m_1, m_2, \ldots, m_8$ *are positive integers*, $(m_i, 4) = 2$, $1 \leq i \leq 4$, $(m_i, 4) = 1,\ 5 \leq i \leq 8$, *and the least common multiple of* $m_1, m_2, \ldots, m_8$ *is* 90, *then, for* $z \in \mathbf{H}$,

$$\eta(m_1 z)\eta(m_2 z)\cdots\eta(m_8 z) \in \{\Gamma_0(90), 4, v\} \ ,$$

where for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(90)$ *and a 24th root of unity* ζ_{24} ,

$$v(M) = \prod_{i=1}^{8} \left(\frac{c/m_i}{|d|}\right) \zeta_{24}^{ac(1-d^2)/m_i + d(m_i b - c/m_i) + 3(d-1)} \ .$$

Proof. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(90) \ .$$

Then d is odd since $ad - bc = 1$ and c is even. Then for $z \in \mathbf{H}$,

$$\eta(Az) = v_\eta(A)(cz+d)^{1/2}\eta(z) \ ,$$

and for $m \mid 90$,

$$\eta(mAz) = \eta\left(\frac{a(mz)+mb}{\frac{c}{m}(mz)+d}\right) = v_\eta\begin{pmatrix} a & mb \\ c/m & d \end{pmatrix}(cz+d)^{1/2}\eta(mz) \ .$$

Let $\delta(z) = \eta(m_1 z)\eta(m_2 z)\cdots\eta(m_8 z)$. Then

$$\delta(Az) = v(A)(cz+d)^4\delta(z) \ ,$$

where

$$A = v_\eta\begin{pmatrix} a & m_1 b \\ c/m_1 & d \end{pmatrix} v_\eta\begin{pmatrix} a & m_2 b \\ c/m_2 & d \end{pmatrix}\cdots v_\eta\begin{pmatrix} a & m_8 b \\ c/m_8 & d \end{pmatrix} \ .$$

For $1 \le i \le 8$, by Theorem 2.15,

$$v_\eta\begin{pmatrix} a & m_i b \\ c/m_i & d \end{pmatrix} = \pm\left(\frac{c/m_i}{|d|}\right)\zeta_{24}^{ac(1-d^2)/m_i + d(m_i b - c/m_i) + 3(d-1)} \ .$$

Hence,

$$v(A) = \prod_{i=1}^{8}\left(\frac{c/m_i}{|d|}\right)\zeta_{24}^{ac(1-d^2)/m_i + d(m_i b - c/m_i) + 3(d-1)} \ .$$

Theorem 2.18 *For* $z \in \mathbf{H}$,

$$\begin{aligned} &\eta(2z)\eta(3z)^2\eta(5z)\eta(18z)\eta(30z)\eta(45z)\eta(90z) \\ &\quad = \eta(z)\eta(6z)^2\eta(18z)\eta(30z)\eta(45z)^3 \\ &\qquad + \eta(z)\eta(6z)\eta(9z)^2\eta(30z)^2\eta(45z)\eta(90z) \\ &\qquad + \eta(z)\eta(6z)\eta(9z)\eta(15z)^2\eta(18z)\eta(90z)^2 \end{aligned} \tag{2.64}$$

and

$$\begin{aligned} &\eta(z)^2\eta(6z)\eta(9z)\eta(18z)\eta(30z)^2\eta(45z) \\ &\quad = \eta(2z)\eta(6z)\eta(9z)^3\eta(30z)^2\eta(45z) \\ &\qquad - \eta(2z)\eta(3z)^2\eta(18z)^2\eta(30z)\eta(45z)^2 \\ &\qquad - \eta(2z)\eta(6z)\eta(9z)^2\eta(15z)^2\eta(18z)\eta(90z) \ . \end{aligned} \tag{2.65}$$

Proof. Each of the 8 products in (2.64) and (2.65) has 8 eta-functions. From Lemma 2.22, by a straightforward calculation, each term of (2.64) is a modular form of weight 4 on $\Gamma_0(90)$ with the multiplier system v_1, where for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(90)$,

$$v_1(A) = \left(\frac{3}{|d|}\right) \zeta_{24}^{\frac{14}{90}c(a-ad^2-d)+4bd+3(d-1)} .$$

And from Lemma 2.22, by a straightforward calculation, each term of (2.65) is a modular form of weight 4 on $\Gamma_0(90)$ with the multiplier system v_2 where for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(90)$,

$$v_2(A) = \left(\frac{6}{|d|}\right) \zeta_{24}^{\frac{2}{90}c(a-ad^2-d)+20bd+3(d-1)} .$$

From [R3, p. 26], $[\Gamma(1) : \Gamma_0(n)] = n \prod_{p|n}(1+\frac{1}{p})$, where the product is over all primes p dividing n. It follows that $[\Gamma(1) : \Gamma_0(90)] = 216$. Let $F_1(q)$ denote the difference of the left and right sides of (2.64). Similarly, define $F_2(q)$. By Theorem 2.17, the number of inequivalent cusps of $\Gamma_0(90)$ is 16, and by [NM, p. 337], a complete set of inequivalent cusps are 0, ∞, and $\Delta/90$ where $|\Delta| \leq 90/2 = 45$ and $(\Delta, 90) > 1$. From [B5, p. 282], if $r, s \in Z$, $(r, s) = 1$, then for any pair of positive integers m, n

$$\operatorname{ord}\left(\eta(mnz); \frac{r}{s}\right) = \frac{(mn, s)^2}{24mn} .$$

So, we can say that

$$\operatorname{ord}\left(\eta(mnz); \frac{r}{s}\right) = \frac{(mn, s)^2}{24mn} \geq \frac{1}{24mn}$$

for $r, s \in Z$ and $(r, s) = 1$. Applying Theorem 2.16 (the valence formula), for a fundamental region R for $\Gamma_0(90)$ with a set of inequivalent cusps $c_1, \ldots, c_{15}, \infty$, we deduce that for $F = F_1$ or F_2,

$$\begin{aligned}
\sum_{z \in R} \operatorname{Ord}_{\Gamma_0(90)}(F; z) = \frac{4 \cdot 216}{12} = 72 & \\
& \geq \operatorname{ord}(F; \infty) + \sum_{i=1}^{15} \operatorname{ord}(F; c_i) \\
& \geq \operatorname{ord}(F; \infty) + 15 \frac{1}{24 \cdot 90} \\
& = \operatorname{ord}(F; \infty) + \frac{1}{144} . \qquad (2.66)
\end{aligned}$$

Using *Mathematica*, we calculated the Taylor series of F_1 and F_2 about $q=0$ (or about the cusp $x=\infty$) and found that $F_1 = O(q^{73}) = F_2$. Unless each of F_1 and F_2 is a constant, we have contradictions to (2.66). We have thus completed the proof of Theorem 2.18.

Now, we will derive two theta function identities from (2.64) and (2.65).

Theorem 2.19 *If* $|q|<1$, *then*

$$\frac{(q^2;q^2)_\infty (q^5;q^5)_\infty}{(q;q)_\infty} = \frac{(q^6;q^6)_\infty^2 f(-q^{45},-q^{45})}{(q^3;q^3)_\infty^2} + \frac{q(q^9;q^9)_\infty (q^{18};q^{18})_\infty (q^{30};q^{30})_\infty}{(q^3;q^3)_\infty f(-q^3,-q^{15})} + \frac{q^2 (q^{15};q^{15})_\infty (q^{18};q^{18})_\infty^2 f(-q^{15},-q^{75})}{(q^3;q^3)_\infty (q^{90};q^{90})_\infty f(-q^3,-q^{15})} .$$

Proof. Dividing both sides of (2.64) by $\eta(z)\eta(3z)^2\eta(18z)\eta(30z)\eta(45z)\eta(90z)$, we have

$$\frac{\eta(2z)\eta(5z)}{\eta(z)} = \frac{\eta(6z)^2\eta(45z)^2}{\eta(3z)^2\eta(90z)} + \frac{\eta(6z)\eta(9z)^2\eta(30z)}{\eta(3z)^2\eta(18z)} + \frac{\eta(6z)\eta(9z)\eta(15z)^2\eta(90z)}{\eta(3z)^2\eta(30z)\eta(45z)} .$$

Replacing $\eta(nz)$ by $q^{\frac{n}{24}}(q^n;q^n)_\infty$ and multiplying by $q^{-\frac{1}{4}}$, we find that

$$\frac{(q^2;q^2)_\infty (q^5;q^5)_\infty}{(q;q)_\infty} = \frac{(q^6;q^6)_\infty^2 (q^{45};q^{45})_\infty^2}{(q^3;q^3)_\infty^2 (q^{90};q^{90})_\infty} + q\frac{(q^6;q^6)_\infty (q^9;q^9)_\infty^2 (q^{30};q^{30})_\infty}{(q^3;q^3)_\infty^2 (q^{18};q^{18})_\infty} + q^2 \frac{(q^6;q^6)_\infty (q^9;q^9)_\infty (q^{15};q^{15})_\infty^2 (q^{90};q^{90})_\infty}{(q^3;q^3)_\infty^2 (q^{30};q^{30})_\infty (q^{45};q^{45})_\infty} .$$

Since

$$f(-q^n,-q^n) = \frac{(q^n;q^n)_\infty^2}{(q^{2n};q^{2n})_\infty} \tag{2.67}$$

and

$$f(-q^n,-q^{5n})=\frac{(q^n;q^n)_\infty(q^{6n};q^{6n})_\infty^2}{(q^{2n};q^{2n})_\infty(q^{3n};q^{3n})_\infty} \tag{2.68}$$

for each positive integer n, we finally have

$$\begin{aligned}\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty}{(q;q)_\infty}=&\frac{(q^6;q^6)_\infty^2\,f(-q^{45},-q^{45})}{(q^3;q^3)_\infty^2}\\&+\frac{q(q^9;q^9)_\infty(q^{18};q^{18})_\infty(q^{30};q^{30})_\infty}{(q^3;q^3)_\infty\,f(-q^3,-q^{15})}\\&+\frac{q^2(q^{15};q^{15})_\infty(q^{18};q^{18})_\infty^2\,f(-q^{15},-q^{75})}{(q^3;q^3)_\infty(q^{90};q^{90})_\infty\,f(-q^3,-q^{15})}\ ,\end{aligned}$$

which complete the proof.

Theorem 2.20 *If* $|q|<1$, *then*

$$\begin{aligned}&-\frac{q(q^6;q^6)_\infty\,f(-q,-q)f(-q^3,-q^{12})}{(q^3;q^3)_\infty\,f(-q^3,-q^3)}\\&\quad=\frac{q^2(q^{18};q^{18})_\infty\,f(-q^{12},-q^{18})}{f(-q^9,-q^{21})}\frac{(q^{45};q^{45})_\infty}{(q^9;q^9)_\infty}\\&\qquad-\frac{q^2(q^{18};q^{18})_\infty\,f(-q^{12},-q^{18})}{f(-q^9,-q^{21})}\frac{(q^9;q^9)_\infty(q^{30};q^{30})_\infty}{q(q^3;q^3)_\infty\,f(-q^3,-q^{15})}\\&\qquad+\frac{q^2(q^{18};q^{18})_\infty\,f(-q^{12},-q^{18})}{f(-q^9,-q^{21})}\frac{(q^{15};q^{15})_\infty(q^{18};q^{18})_\infty\,f(-q^{15},-q^{75})}{(q^3;q^3)_\infty(q^{90};q^{90})_\infty\,f(-q^3,-q^{15})}\ .\end{aligned}$$

Proof. This proof is similar to the proof of Theorem 2.19. Dividing both sides of (2.65) by $\eta(2z)\eta(3z)^2\eta(9z)\eta(18z)^2\eta(30z)\eta(45z)$, we find that

$$\begin{aligned}\frac{\eta(z)^2\eta(6z)\eta(30z)}{\eta(2z)\eta(3z)^2\eta(18z)}=&\frac{\eta(6z)\eta(9z)^2\eta(30z)}{\eta(3z)^2\eta(18z)^2}-\frac{\eta(45z)}{\eta(9z)}\\&-\frac{\eta(6z)\eta(9z)\eta(15z)^2\eta(90z)}{\eta(3z)^2\eta(18z)\eta(30z)\eta(45z)}\ .\end{aligned}$$

Replacing $\eta(nz)$ by $q^{\frac{n}{24}}(q^n;q^n)_\infty$ and multiplying both sides by $-q^{-\frac{1}{2}}$, we find that

$$\begin{aligned}&-\frac{(q;q)_\infty^2(q^6;q^6)_\infty(q^{30};q^{30})_\infty}{(q^2;q^2)_\infty(q^3;q^3)_\infty^2(q^{18};q^{18})_\infty}\\&\quad=-\frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^2(q^{30};q^{30})_\infty}{(q^3;q^3)_\infty^2(q^{18};q^{18})_\infty^2}+q\frac{(q^{45};q^{45})_\infty}{(q^9;q^9)_\infty}\\&\qquad+q\frac{(q^6;q^6)_\infty(q^9;q^9)_\infty(q^{15};q^{15})_\infty^2(q^{90};q^{90})_\infty}{(q^3;q^3)_\infty^2(q^{18};q^{18})_\infty(q^{30};q^{30})_\infty(q^{45};q^{45})_\infty}\ .\end{aligned}$$

By (1.8),

$$\frac{f(-q^{12},-q^{18})}{f(-q^3,-q^{12})f(-q^9,-q^{21})}=\frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty(q^{30};q^{30})_\infty}\quad . \tag{2.69}$$

Multiplying both sides by $q(q^{18};q^{18})_\infty f(-q^{12},-q^{18})/f(-q^9,-q^{21})$, and using (2.67), (2.68), and (2.69), we finally find that

$$\begin{aligned}
&-\frac{q(q^6;q^6)_\infty f(-q,-q)f(-q^3,-q^{12})}{(q^3;q^3)_\infty f(-q^3,-q^3)}\\
&=\frac{q(q^{18};q^{18})_\infty f(-q^{12},-q^{18})}{f(-q^9,-q^{21})}\frac{q(q^{45};q^{45})_\infty}{(q^9;q^9)_\infty}\\
&\quad-\frac{q(q^{18};q^{18})_\infty f(-q^{12},-q^{18})}{f(-q^9,-q^{21})}\frac{(q^9;q^9)_\infty(q^{30};q^{30})_\infty}{(q^3;q^3)_\infty f(-q^3,-q^{15})}\\
&\quad+\frac{q(q^{18};q^{18})_\infty f(-q^{12},-q^{18})}{f(-q^9,-q^{21})}\frac{q(q^{15};q^{15})_\infty(q^{18};q^{18})_\infty f(-q^{15},-q^{75})}{(q^3;q^3)_\infty(q^{90};q^{90})_\infty f(-q^3,-q^{15})}\quad ,
\end{aligned}$$

which completes the proof.

3 Proof of the first identity

We will use (2.60) and (2.63) and need several further theorems to prove (1.5).

Theorem 3.1 *If* $0<|q|<1$ *and* x *is neither zero nor an integral power of* q, *then*

$$f(-x^3q,-x^{-3}q^2)+xf(-x^3q^2,-x^{-3}q)=\frac{(q;q)_\infty f(-x^2,-x^{-2}q)}{f(-x,-x^{-1}q)}\quad .$$

Proof. See Theorem 1.0 in [H1, p. 643].

Theorem 3.2 *For* $0<|q|<1,\ x\neq 0,\ y\neq 0,$ *and* $z\neq 0$,

$$\begin{aligned}
&f(-x,-x^{-1}q)^2f(-yz,-(yz)^{-1}q)f(-yz^{-1},-y^{-1}zq)\\
&\quad=f(-y,-y^{-1}q)^2f(-xz,-(xz)^{-1}q)f(-xz^{-1},-x^{-1}zq)\\
&\qquad-yz^{-1}f(-z,-z^{-1}q)^2f(-xy,-(xy)^{-1}q)f(-xy^{-1},-x^{-1}yq)\quad .
\end{aligned}$$

Proof. See Theorem 1.0 in [H2, p. 663].

Theorem 3.3 *If* $|q| < 1$, *then*

$$\frac{(q^2;q^2)_\infty f(-z,-z^{-1}q^2)f(z,z^{-1}q^3)}{(q;q)_\infty} =$$
$$f(-zq,-z^{-1}q^5)f(-z^2q,-z^{-2}q^5) - z^2 f(-zq^5,-z^{-1}q)f(-z^2q^5,-z^{-2}q) \ .$$

Proof. This can be proved using a functional equation. Define

$$\begin{aligned} F(z) := {} & \frac{(q^2;q^2)_\infty f(-z,-z^{-1}q^2)f(z,z^{-1}q^3)}{(q;q)_\infty} \\ & - f(-zq,-z^{-1}q^5)f(-z^2q,-z^{-2}q^5) \\ & + z^2 f(-zq^5,-z^{-1}q)f(-z^2q^5,-z^{-2}q) \ . \end{aligned} \tag{3.1}$$

By (1.8),

$$\begin{aligned} & f(-zq^6,-z^{-1}q^{-4})f(zq^6,z^{-1}q^{-3}) \\ & = (zq^6;q^2)_\infty (z^{-1}q^{-4};q^2)_\infty (q^2;q^2)_\infty (-zq^6;q^3)_\infty \\ & \quad \times (-z^{-1}q^{-3};q^3)_\infty (q^3;q^3)_\infty \\ & = \frac{(z;q^2)_\infty}{(1-z)(1-zq^2)(1-zq^4)} \\ & \quad \times (1-z^{-1}q^{-4})(1-z^{-1}q^{-2})(1-z^{-1})(z^{-1}q^2;q^2)_\infty (q^2;q^2)_\infty \\ & \quad \times \frac{(-z;q^3)_\infty}{(1+z)(1+zq^3)} (1+z^{-1}q^{-3})(1+z^{-1})(-z^{-1}q^{-3};q^3)_\infty \\ & \quad \times (q^3;q^3)_\infty \\ & = -z^{-5}q^{-9} f(-z,-z^{-1}q^2)f(z,z^{-1}q^3) \ , \end{aligned} \tag{3.2}$$

$$\begin{aligned} & f(-zq^7,-z^{-1}q^{-1})f(-z^2q^{13},-z^{-2}q^{-7}) \\ & = (zq^7;q^6)_\infty (z^{-1}q^{-1};q^6)_\infty (q^6;q^6)_\infty (z^2q^{13};q^6)_\infty \\ & \quad \times (z^{-2}q^{-7};q^6)_\infty (q^6;q^6)_\infty \\ & = \frac{(zq;q^6)_\infty}{(1-zq)} (1-z^{-1}q^{-1})(z^{-1}q^5;q^6)_\infty (q^6;q^6)_\infty \\ & \quad \times \frac{(z^2q;q^6)_\infty}{(1-z^2q)(1-z^2q^7)} (1-z^{-2}q^{-7})(1-z^{-2}q^{-1})(z^{-2}q^5;q^6)_\infty \\ & \quad \times (q^6;q^6)_\infty \\ & = -z^{-5}q^{-9} f(-zq,-z^{-1}q^5)f(-z^2q,-z^{-2}q^5) \ , \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&f(-zq^{11},-z^{-1}q^{-5})f(-z^2q^{17},-z^{-2}q^{-11})\\
&\quad=(zq^{11};q^6)_\infty(z^{-1}q^{-5};q^6)_\infty(q^6;q^6)_\infty(z^2q^{17};q^6)_\infty\\
&\qquad\times(z^{-2}q^{-11};q^6)_\infty(q^6;q^6)_\infty\\
&\quad=\frac{(zq^5;q^6)_\infty}{(1-zq^5)}(1-z^{-1}q^{-5})(z^{-1}q;q^6)_\infty(q^6;q^6)_\infty\\
&\qquad\times\frac{(z^2q^5;q^6)_\infty}{(1-z^2q^5)(1-z^2q^{11})}(1-z^{-2}q^{-11})(1-z^{-2}q^{-5})(z^{-2}q;q^6)_\infty\\
&\qquad\times(q^6;q^6)_\infty\\
&\quad=-z^{-5}q^{-21}f(-zq^5,-z^{-1}q)f(-z^2q^5,-z^{-2}q)\ .
\end{aligned}\tag{3.4}$$

From (3.1), we see that $F(z)$ is analytic for all $z\neq 0$, and from (3.2), (3.3) and (3.4), we conclude that $F(z)$ satisfies the functional equation

$$F(zq^6)=-z^{-5}q^{-9}F(z)\ .$$

By Theorem 2.11, either $F(z)$ has exactly 5 zeros in the annulus $|q^6|<|z|\leq 1$ or $F(z)\equiv 0$. We shall prove that in this annulus $F(z)$ has zeros at 1, -1, q, q^2, q^4, and q^5. It's clear from (3.1) and Lemma 2.1(iii) that 1 and -1 are zeros of $F(z)$. By (1.8), (1.9), Lemma 2.1(iii) and (iv),

$$\begin{aligned}
F(q)&=\frac{(q^2;q^2)_\infty f(-q,-q)f(q,q^2)}{(q;q)_\infty}-f(-q^2,-q^4)f(-q^3,-q^3)\\
&=(q;q)_\infty f(q,q^2)-(q^2;q^2)_\infty f(-q^3,-q^3)\\
&=(q;q)_\infty(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty\frac{(-q^3;q^3)_\infty}{(-q^3;q^3)_\infty}\\
&\quad-(q^2;q^2)_\infty(q^3;q^6)_\infty^2(q^6;q^6)_\infty\\
&=0\ ,\\
F(q^2)&=-f(-q^3,-q^3)f(-q,-q^5)+q^4f(-q^{-1},-q^7)f(-q^{-3},-q^9)\\
&=-f(-q^3,-q^3)f(-q,-q^5)\\
&\quad+q^4(-q^{-1})f(-q,-q^5)(-q^{-3})f(-q^3,-q^3)\\
&=0\ ,\\
F(q^4)&=-f(-q,-q^5)f(-q^{-3},-q^9)+q^8f(-q^{-3},-q^9)f(-q^{-7},-q^{13})\\
&=-f(-q,-q^5)f(-q^{-3},-q^9)+q^8f(-q^{-3},-q^9)q^{-8}f(-q,-q^5)\\
&=0\ ,
\end{aligned}$$

and

$$F(q^5) = \frac{(q^2;q^2)_\infty f(-q^{-3},-q^5)f(q^{-2},q^5)}{(q;q)_\infty} + q^{10}f(-q^{-4},-q^{10})f(-q^{-9},-q^{15})$$
$$= q^{-6}\frac{(q^2;q^2)_\infty f(-q,-q)f(q,q^2)}{(q;q)_\infty} - q^{-6}f(-q^2,-q^4)f(-q^3,-q^3)$$
$$= 0 \ .$$

Thus, by Theorem 2.11, $F(z) \equiv 0$, which completes the proof.

Theorem 3.4 *If* $|q| < 1$, *then*

$$\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty(q^{30};q^{30})_\infty f(-q^{12},-q^{18})}{(q;q)_\infty f(-q^9,-q^{21})}$$
$$= -\frac{q(q^5;q^5)_\infty(q^{10};q^{10})_\infty(q^{30};q^{30})_\infty f(-q^2,-q^8)}{f(-q,-q^4)f(-q^4,-q^6)}$$
$$+ \frac{(q^{30};q^{30})_\infty f(-q^2,-q^8)f(-q^3,-q^7)^2 f(q^7,q^8)}{(q;q)_\infty(q^{10};q^{10})_\infty f(-q^9,-q^{21})}$$
$$+ \frac{q(q^{30};q^{30})_\infty f(-q^2,-q^8)f(-q^3,-q^7)^2 f(q^2,q^{13})}{(q;q)_\infty(q^{10};q^{10})_\infty f(-q^9,-q^{21})} \ .$$

Proof. By Theorem 3.1 with q replaced by q^{10} and x by q^4, and Lemma 2.1(iv),

$$-\frac{q(q^5;q^5)_\infty(q^{30};q^{30})_\infty}{f(-q,-q^4)}\frac{(q^{10};q^{10})_\infty f(-q^2,-q^8)}{f(-q^4,-q^6)}$$
$$= -\frac{q(q^5;q^5)_\infty(q^{30};q^{30})_\infty}{f(-q,-q^4)}\left(f(-q^8,-q^{22}) - q^2 f(-q^2,-q^{28})\right) \ . \tag{3.5}$$

By Theorem 3.3 with q replaced by q^5 and z by q^7, (1.8), and Lemma 2.1(iv),

$$\frac{(q^{30};q^{30})_\infty f(-q^2,-q^8)f(-q^3,-q^7)^2 f(q^7,q^8)}{(q;q)_\infty(q^{10};q^{10})_\infty f(-q^9,-q^{21})}$$
$$+ \frac{q^3(q^5;q^5)_\infty(q^{30};q^{30})_\infty f(-q^2,-q^{28})}{f(-q,-q^4)}$$

$$= \frac{(q^5;q^5)_\infty (q^{30};q^{30})_\infty}{f(-q,-q^4)f(-q^9,-q^{21})} \left(\frac{(q^{10};q^{10})_\infty f(-q^3,-q^7) f(q^7,q^8)}{(q^5;q^5)_\infty} + q^3 f(-q^2,-q^{28}) f(-q^9,-q^{21}) \right)$$

$$= \frac{(q^5;q^5)_\infty (q^{30};q^{30})_\infty}{f(-q,-q^4)f(-q^9,-q^{21})} f(-q^{11},-q^{19}) f(-q^{12},-q^{18}) \ . \tag{3.6}$$

By Theorem 3.3 with q replaced by q^5 and z by q^{13}, and Lemma 2.1(iv),

$$\frac{q(q^{30};q^{30})_\infty f(-q^2,-q^8) f(-q^3,-q^7)^2 f(q^2,q^{13})}{(q;q)_\infty (q^{10};q^{10})_\infty f(-q^9,-q^{21})} - \frac{q(q^5;q^5)_\infty (q^{30};q^{30})_\infty f(-q^8,-q^{22})}{f(-q,-q^4)}$$

$$= \frac{q(q^5;q^5)_\infty (q^{30};q^{30})_\infty}{f(-q,-q^4)f(-q^9,-q^{21})} \left(\frac{(q^{10};q^{10})_\infty f(-q^3,-q^7) f(q^2,q^{13})}{(q^5;q^5)_\infty} - f(-q^8,-q^{22}) f(-q^9,-q^{21}) \right)$$

$$= \frac{q(q^5;q^5)_\infty (q^{30};q^{30})_\infty}{f(-q,-q^4)f(-q^9,-q^{21})} q^2 f(-q,-q^{29}) f(-q^{12},-q^{18}) \ . \tag{3.7}$$

Adding (3.5), (3.6), (3.7), and applying Theorem 3.1 again with q replaced by q^{10} and x by q^3, and using Lemma 2.1(iv) and (1.8), we find that

$$- \frac{q(q^5;q^5)_\infty (q^{10};q^{10})_\infty (q^{30};q^{30})_\infty f(-q^2,-q^8)}{f(-q,-q^4) f(-q^4,-q^6)} + \frac{(q^{30};q^{30})_\infty f(-q^2,-q^8) f(-q^3,-q^7)^2 f(q^7,q^8)}{(q;q)_\infty (q^{10};q^{10})_\infty f(-q^9,-q^{21})} + \frac{q(q^{30};q^{30})_\infty f(-q^2,-q^8) f(-q^3,-q^7)^2 f(q^2,q^{13})}{(q;q)_\infty (q^{10};q^{10})_\infty f(-q^9,-q^{21})}$$

$$= \frac{(q^5;q^5)_\infty (q^{30};q^{30})_\infty f(-q^{12},-q^{18})}{f(-q,-q^4) f(-q^9,-q^{21})} \times \left(f(-q^{11},-q^{19}) + q^3 f(-q,-q^{29}) \right)$$

$$= \frac{(q^5;q^5)_\infty (q^{30};q^{30})_\infty f(-q^{12},-q^{18})}{f(-q,-q^4) f(-q^9,-q^{21})} \frac{(q^{10};q^{10})_\infty f(-q^4,-q^6)}{f(-q^3,-q^7)}$$

$$= \frac{(q^2;q^2)_\infty (q^5;q^5)_\infty (q^{30};q^{30})_\infty f(-q^{12},-q^{18})}{(q;q)_\infty f(-q^9,-q^{21})} \ .$$

This completes the proof of Theorem 3.4.

Theorem 3.5 *If* $0 < |q| < 1$ *and* a, b, c, *and* d *are nonzero complex numbers, then*

$$
\begin{aligned}
0 = {} & af(-ab, -(ab)^{-1}q)f\left(-\frac{b}{a}, -\frac{a}{b}q\right) \\
& \times f(-cd, -(cd)^{-1}q)f\left(-\frac{c}{d}, -\frac{d}{c}q\right) \\
& + bf(-bc, -(bc)^{-1}q)f\left(-\frac{c}{b}, -\frac{b}{c}q\right) \\
& \times f(-ad, -(ad)^{-1}q)f\left(-\frac{a}{d}, -\frac{d}{a}q\right) \\
& + cf(-ac, -(ac)^{-1}q)f\left(-\frac{a}{c}, -\frac{c}{a}q\right) \\
& \times f(-bd, -(bd)^{-1}q)f\left(-\frac{b}{d}, -\frac{d}{b}q\right). \qquad (3.8)
\end{aligned}
$$

Proof. First assume that a, b, c, and q are fixed complex numbers with $|q| < |a| \leq 1$, $|q| < |b| \leq 1$, and $|q| < |c| \leq 1$, and that a, b, and c are distinct. Let $g(d)$ be the right side of (3.8). Then it is easy to verify that $g(d)$ is analytic for $d \neq 0$ and satisfies $g(dq) = q^{-1}d^{-2}g(d)$ by Lemma 2.1(iv). By Theorem 2.11, either $g(d)$ has exactly 2 zeros in the annulus $|q| < |d| \leq 1$ or $g(d) \equiv 0$ for all $d \neq 0$. Applying Lemma 2.1(iii) three times, we can show that $g(d)$ has 3 zeros at $d = a,\ b$, and c. Hence (3.8) is true for all nonzero d in this case. Now we can apply the same argument to each one of $a,\ b$, and c. So by analytic continuation, (3.8) holds for all nonzero values of $a,\ b,\ c$, and d.

Theorem 3.6 *If* $|q| < 1$, *then*

$$
\begin{aligned}
& \frac{(q^{18};q^{18})_\infty(q^{30};q^{30})_\infty(q^{45};q^{45})_\infty f(-q^{12},-q^{18})}{(q^9;q^9)_\infty f(-q^9,-q^{21})} \\
& = \frac{(q^{30};q^{30})_\infty(q^{45};q^{45})_\infty(q^{90};q^{90})_\infty f(-q^{36},-q^{54})}{f(-q^{18},-q^{27})f(-q^{18},-q^{72})} \\
& \quad + \frac{q^9(q^{45};q^{45})_\infty^2(q^{90};q^{90})_\infty^2}{f(-q^{45},-q^{45})f(-q^{18},-q^{27})} \\
& \quad \times \left(\frac{f(-q^{12},-q^{33})}{f(-q^{39},-q^{51})} + \frac{f(-q^3,-q^{42})}{f(-q^{21},-q^{69})}\right). \qquad (3.9)
\end{aligned}
$$

Proof. In Theorem 3.5, replacing q by q^{90}, a by q^{33}, b by q^3, c by q^{24}, and d by q^{15} and applying Lemma 2.1(iv), we find that

$$
\begin{aligned}
0 = {} & q^{33}f(-q^{36},-q^{54})f(-q^{-30},-q^{120})f(-q^{39},-q^{51})f(-q^9,-q^{81}) \\
& + q^3f(-q^{27},-q^{63})f(-q^{21},-q^{69})f(-q^{48},-q^{42})f(-q^{18},-q^{72}) \\
& + q^{24}f(-q^{33},-q^{57})f(-q^9,-q^{81})f(-q^{18},-q^{72})f(-q^{-12},-q^{102})
\end{aligned}
$$

$$= -q^3 f(-q^{36},-q^{54})f(-q^{30},-q^{60})f(-q^{39},-q^{51})f(-q^9,-q^{81})$$
$$+ q^3 f(-q^{27},-q^{63})f(-q^{21},-q^{69})f(-q^{48},-q^{42})f(-q^{18},-q^{72})$$
$$-q^{12} f(-q^{33},-q^{57})f(-q^9,-q^{81})f(-q^{18},-q^{72})f(-q^{12},-q^{78}) \ .$$

This implies that

$$\begin{aligned} &f(-q^{27},-q^{63})f(-q^{21},-q^{69})f(-q^{48},-q^{42})f(-q^{18},-q^{72}) \\ &\quad = f(-q^{36},-q^{54})f(-q^{30},-q^{60})f(-q^{39},-q^{51})f(-q^9,-q^{81}) \\ &\qquad + q^9 f(-q^{33},-q^{57})f(-q^9,-q^{81})f(-q^{18},-q^{72})f(-q^{12},-q^{78}) \ . \end{aligned} \tag{3.10}$$

In Theorem 3.2, replacing q by q^{90}, x by q^{18}, y by q^{21}, and z by q^9, and applying Lemma 2.1(iv), we find that

$$\begin{aligned} &f(-q^{18},-q^{72})^2 f(-q^{30},-q^{60})f(-q^{12},-q^{78}) \\ &\quad = f(-q^{21},-q^{69})^2 f(-q^{27},-q^{63})f(-q^9,-q^{81}) \\ &\qquad + q^9 f(-q^9,-q^{81})^2 f(-q^{39},-q^{51})f(-q^3,-q^{87}) \ . \end{aligned} \tag{3.11}$$

Then, by (3.10), (3.11), and (1.8), the right hand side of (3.9) equals

$$\begin{aligned} &\frac{(q^{30};q^{30})_\infty (q^{45};q^{45})_\infty (q^{90};q^{90})_\infty f(-q^{36},-q^{54})}{f(-q^{18},-q^{27})f(-q^{18},-q^{72})} \\ &\quad + \frac{q^9 (q^{45};q^{45})_\infty^2 (q^{90};q^{90})_\infty^2 f(-q^{12},-q^{33})}{f(-q^{45},-q^{45})f(-q^{18},-q^{27})f(-q^{39},-q^{51})} \\ &\quad + \frac{q^9 (q^{45};q^{45})_\infty^2 (q^{90};q^{90})_\infty^2 f(-q^3,-q^{42})}{f(-q^{45},-q^{45})f(-q^{18},-q^{27})f(-q^{21},-q^{69})} \\ &= \frac{(q^{30};q^{30})_\infty (q^{90};q^{90})_\infty^3 f(-q^{36},-q^{54})}{f(-q^{27},-q^{63})f(-q^{18},-q^{72})^2} \\ &\quad + \frac{q^9 (q^{90};q^{90})_\infty^3 f(-q^{12},-q^{78})f(-q^{33},-q^{57})}{f(-q^{18},-q^{72})f(-q^{27},-q^{63})f(-q^{39},-q^{51})} \\ &\quad + \frac{q^9 (q^{45};q^{45})_\infty^2 (q^{90};q^{90})_\infty^2 f(-q^3,-q^{42})}{f(-q^{45},-q^{45})f(-q^{18},-q^{27})f(-q^{21},-q^{69})} \\ &= \frac{(q^{90};q^{90})_\infty^3}{f(-q^9,-q^{81})f(-q^{18},-q^{72})^2 f(-q^{27},-q^{63})f(-q^{39},-q^{51})} \\ &\quad \times (f(-q^{36},-q^{54})f(-q^{30},-q^{60})f(-q^{39},-q^{51})f(-q^9,-q^{81}) \\ &\quad + q^9 f(-q^{33},-q^{57})f(-q^9,-q^{81})f(-q^{18},-q^{72})f(-q^{12},-q^{78})) \\ &\quad + \frac{q^9 (q^{45};q^{45})_\infty^2 (q^{90};q^{90})_\infty^2 f(-q^3,-q^{42})}{f(-q^{45},-q^{45})f(-q^{18},-q^{27})f(-q^{21},-q^{69})} \end{aligned}$$

$$= \frac{(q^{90};q^{90})_\infty^3}{f(-q^9,-q^{81})f(-q^{18},-q^{72})^2f(-q^{27},-q^{63})f(-q^{39},-q^{51})}$$

$$\times f(-q^{27},-q^{63})f(-q^{21},-q^{69})f(-q^{48},-q^{42})f(-q^{18},-q^{72})$$

$$+ \frac{q^9(q^{90};q^{90})_\infty^3 f(-q^3,-q^{87})f(-q^{42},-q^{48})}{f(-q^{18},-q^{72})f(-q^{27},-q^{63})f(-q^{21},-q^{69})}$$

$$= \frac{(q^{90};q^{90})_\infty^3}{f(-q^9,-q^{81})^2f(-q^{18},-q^{72})f(-q^{27},-q^{63})f(-q^{39},-q^{51})}$$

$$\times \frac{f(-q^{42},-q^{48})}{f(-q^{21},-q^{69})}(f(-q^{21},-q^{69})^2f(-q^{27},-q^{63})f(-q^9,-q^{81})$$

$$+ q^9f(-q^9,-q^{81})^2f(-q^{39},-q^{51})f(-q^3,-q^{87}))$$

$$= \frac{(q^{30};q^{30})_\infty f(-q^{42},-q^{48})}{f(-q^9,-q^{21})f(-q^9,-q^{81})f(-q^{18},-q^{72})f(-q^{27},-q^{63})}$$

$$\times f(-q^{18},-q^{72})^2f(-q^{30},-q^{60})f(-q^{12},-q^{78})$$

$$= \frac{(q^{18};q^{18})_\infty(q^{30};q^{30})_\infty(q^{45};q^{45})_\infty f(-q^{12},-q^{18})}{(q^9;q^9)_\infty f(-q^9,-q^{21})} \quad .$$

We are now ready to prove the main identity.

Theorem 3.7 (First 10th Order Mock Theta Function Identity) *For* $|q| < 1$,

$$q^2\phi(q^9) - \frac{\psi(\omega q) - \psi(\omega^2 q)}{\omega - \omega^2} = -\frac{q(q^6;q^6)_\infty f(-q,-q)f(-q^3,-q^{12})}{(q^3;q^3)_\infty f(-q^3,-q^3)} \quad .$$

Proof. First we will examine $\phi(q)$.

Observe that

$$F_n := \frac{(-1)^n q^{5n^2+15n+7}z^{-n-1}}{1-q^{10n+9}z^{-1}}\frac{q^{-10n-9}z}{q^{-10n-9}z} = \frac{(-1)^{n+1}q^{5n^2+5n-2}z^{-n}}{1-q^{-10n-9}z} \quad .$$

Then replacing n by $-n-1$, we find that

$$F_{-n-1} = \frac{(-1)^n q^{5n^2+5n-2}z^{n+1}}{1-q^{10n+1}z} \quad .$$

So, by (2.35),

$$\begin{aligned}
&-2Q(zq^{-4},q^2,q^5)\\
&\quad=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n(n+1)}q^{2(2n+1)}(zq^{-4})^{n+1}}{1-q^{5(2n+1)}zq^{-4}}\\
&\qquad+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n(n+3)+5}q^{2(-2n-1)}(zq^{-4})^{-n-1}}{1-q^{5(2n+1)}(zq^{-4})^{-1}}\\
&\quad=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n^2+5n-2}z^{n+1}}{1-q^{10n+1}z}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n^2+15n+7}z^{-n-1}}{1-q^{10n+9}z^{-1}}\\
&\quad=2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n^2+5n-2}z^{n+1}}{1-q^{10n+1}z}\quad. \qquad (3.12)
\end{aligned}$$

Replacing z^5q^4 by z in (2.63) and using (3.12), we find that

$$\begin{aligned}
\phi(q)f(-z,-z^{-1}q^{10})&=2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n^2+5n-1}z^{n+1}}{1-q^{10n+1}z}\\
&\quad+a_2(q)f(-z,-z^{-1}q^{10})+2qA(zq^{-4},q^2,q^5). \qquad (3.13)
\end{aligned}$$

Replacing q and z by q^9 and q^{30}, respectively, in (3.13), multiplying both sides of (3.13) by q^2, and using (1.9), we find that

$$\begin{aligned}
q^2(q^{30};q^{30})_\infty\phi(q^9)&=2q^2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+45n-9}q^{30n+30}}{1-q^{90n+39}}\\
&\quad+q^2a_2(q^9)(q^{30};q^{30})_\infty+2q^{11}A(q^{-6},q^{18},q^{45}). \qquad (3.14)
\end{aligned}$$

Replacing q and z by q^9 and q^{60}, respectively, in (3.13), multiplying both sides of (3.13) by q^2, and using (1.9), we find that

$$\begin{aligned}
q^2(q^{30};q^{30})_\infty\phi(q^9)&=2q^2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+45n-9}q^{60n+60}}{1-q^{90n+69}}\\
&\quad+q^2a_2(q^9)(q^{30};q^{30})_\infty+2q^{11}A(q^{24},q^{18},q^{45})\quad. \qquad (3.15)
\end{aligned}$$

Adding (3.14) and (3.15), dividing by 2, and applying (2.36), (2.40), and Theorem 3.6, we find that

$$\begin{aligned}
&q^2(q^{30};q^{30})_\infty\phi(q^9)\\
&\quad=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+75n+23}}{1-q^{90n+39}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+105n+53}}{1-q^{90n+69}}\\
&\qquad+q^2a_2(q^9)(q^{30};q^{30})_\infty+q^{11}\left(A(q^{-6},q^{18},q^{45})+A(q^{24},q^{18},q^{45})\right)
\end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{45n^2+75n+23}}{1-q^{90n+39}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{45n^2+105n+53}}{1-q^{90n+69}}$$

$$+ \frac{q^2 (q^{30};q^{30})_\infty (q^{45};q^{45})_\infty (q^{90};q^{90})_\infty f(-q^{36},-q^{54})}{f(-q^{18},-q^{27}) f(-q^{18},-q^{72})}$$

$$+ \frac{q^{11} (q^{45};q^{45})_\infty^2 (q^{90};q^{90})_\infty^2}{f(-q^{45},-q^{45}) f(-q^{18},-q^{27})} \left(\frac{f(-q^{12},-q^{33})}{f(-q^{39},-q^{51})} + \frac{f(-q^{3},-q^{42})}{f(-q^{21},-q^{69})} \right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{45n^2+75n+23}}{1-q^{90n+39}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{45n^2+105n+53}}{1-q^{90n+69}}$$

$$+ q^2 \frac{(q^{18};q^{18})_\infty (q^{30};q^{30})_\infty (q^{45};q^{45})_\infty f(-q^{12},-q^{18})}{(q^9;q^9)_\infty f(-q^9,-q^{21})} \quad . \tag{3.16}$$

Now, consider $\psi(q)$.
First, observe that

$$G_n := \frac{(-1)^n q^{5n^2+15n+6} z^{-n-1}}{1-q^{10n+7} z^{-1}} \frac{q^{-10n-7} z}{q^{-10n-7} z} = \frac{(-1)^{n+1} q^{5n^2+5n-1} z^{-n}}{1-q^{-10n-7} z}$$

and replacing n by $-n-1$, we find that

$$G_{-n-1} = \frac{(-1)^n q^{5n^2+5n-1} z^{n+1}}{1-q^{10n+3} z} \quad .$$

Thus, by (2.35), we find that

$$-2Q(zq^{-2},q,q^5) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)+2n+1} (zq^{-2})^{n+1}}{1-q^{5(2n+1)} zq^{-2}}$$

$$+ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+3)+5-2n-1} (zq^{-2})^{-n-1}}{1-q^{5(2n+1)} (zq^{-2})^{-1}}$$

$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n^2+5n-1} z^{n+1}}{1-q^{10n+3} z} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n^2+15n+6} z^{-n-1}}{1-q^{10n+7} z^{-1}}$$

$$= 2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n^2+5n-1} z^{n+1}}{1-q^{10n+3} z} \quad . \tag{3.17}$$

Replacing z^5q^2 by z in (2.60) and using (3.17), we find that

$$\psi(q)f(-z,-z^{-1}q^{10}) = 2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n^2+5n}z^{n+1}}{1-q^{10n+3}z} + a_1(q)f(-z,-z^{-1}q^{10}) + 2qA(zq^{-2},q,q^5)\ . \tag{3.18}$$

If ω is a primitive cube root of unity, then

$$\begin{aligned} f(-\omega,-\omega^2q^{10}) &= (\omega;q^{10})_\infty(\omega^2q^{10};q^{10})_\infty(q^{10};q^{10})_\infty \\ &= (1-\omega)(q^{30};q^{30})_\infty\ . \end{aligned}$$

So, replacing z by ω in (3.18), and applying (2.36) and (2.40), we find that

$$\begin{aligned} &(q^{30};q^{30})_\infty\psi(q) \\ &\quad= \frac{2}{1-\omega}\sum_{n=-\infty}^{\infty}\frac{(-1)^n\omega^{n+1}q^{5n^2+5n}}{1-\omega q^{10n+3}} \\ &\qquad + a_1(q)(q^{30};q^{30})_\infty + \frac{2q}{1-\omega}A(\omega q^{-2},q,q^5) \\ &\quad= \frac{2}{1-\omega}\sum_{n=-\infty}^{\infty}\frac{(-1)^n\omega^{n+1}q^{5n^2+5n}}{1-\omega q^{10n+3}} \\ &\qquad - \frac{q(q^5;q^5)_\infty(q^{10};q^{10})_\infty(q^{30};q^{30})_\infty f(-q^2,-q^8)}{f(-q,-q^4)f(-q^4,-q^6)} \\ &\qquad - \frac{2(\omega-1)}{3}\frac{(q^5;q^5)^2_\infty(q^{10};q^{10})^2_\infty f(-\omega^2q,-\omega q^4)}{f(-q^5,-q^5)f(-q,-q^4)f(-\omega q^3,-\omega^2q^7)}\ . \end{aligned} \tag{3.19}$$

In Theorem 3.1, replacing q by q^5 and x by $-\omega^2 q$, we find that

$$f(q^7,q^8) - \omega^2qf(q^2,q^{13}) = \frac{(q^5;q^5)_\infty f(-\omega q^2,-\omega^2q^3)}{f(\omega^2q,\omega q^4)}\ . \tag{3.20}$$

We examine the last expression in (3.19). By (3.19), (3.20), and (1.8),

$$\begin{aligned} &\frac{2(\omega-1)}{3}\frac{(q^5;q^5)^2_\infty(q^{10};q^{10})^2_\infty f(-\omega^2q,-\omega q^4)}{f(-q^5,-q^5)f(-q,-q^4)f(-\omega q^3,-\omega^2q^7)} \\ &\quad= \frac{2(\omega-1)}{3}\frac{(q^{10};q^{10})^3_\infty f(-\omega^2q,-\omega q^4)}{f(-q,-q^4)f(-\omega q^3,-\omega^2q^7)}\frac{f(-\omega^2q^3,-\omega q^7)}{f(-\omega^2q^3,-\omega q^7)} \\ &\quad= \frac{2(\omega-1)}{3}\frac{(q^{30};q^{30})_\infty f(-q^3,-q^7)f(-\omega^2q,-\omega q^4)f(-\omega^2q^3,-\omega q^7)}{f(-q,-q^4)f(-q^9,-q^{21})} \end{aligned}$$

$$= \frac{2(\omega-1)}{3} \frac{(q^{30};q^{30})_\infty f(-q^3,-q^7) f(-\omega^2 q,-\omega q^4) f(-\omega^2 q^3,-\omega q^7)}{f(-q,-q^4) f(-q^9,-q^{21})}$$
$$\times \frac{f(\omega^2 q,\omega q^4)}{f(\omega^2 q,\omega q^4)}$$
$$= \frac{2(\omega-1)}{3} \frac{(q^{30};q^{30})_\infty f(-q^2,-q^8) f(-q^3,-q^7)^2}{(q;q)_\infty (q^{10};q^{10})_\infty f(-q^9,-q^{21})}$$
$$\times \frac{(q^5;q^5)_\infty f(-\omega q^2,-\omega^2 q^3)}{f(\omega^2 q,\omega q^4)}$$
$$= \frac{2(\omega-1)}{3} \frac{(q^{30};q^{30})_\infty f(-q^2,-q^8) f(-q^3,-q^7)^2}{(q;q)_\infty (q^{10};q^{10})_\infty f(-q^9,-q^{21})}$$
$$\times \left(f(q^7,q^8) - \omega^2 q f(q^2,q^{13})\right)$$
$$= \frac{(q^{30};q^{30})_\infty f(-q^2,-q^8) f(-q^3,-q^7)^2}{3(q;q)_\infty (q^{10};q^{10})_\infty f(-q^9,-q^{21})}$$
$$\times \left(2(\omega-1) f(q^7,q^8) - (4+2\omega) q f(q^2,q^{13})\right) . \tag{3.21}$$

By (3.19) and (3.21),

$$(q^{30};q^{30})_\infty \psi(q)$$
$$= \frac{2}{1-\omega} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^{n+1} q^{5n^2+5n}}{1-\omega q^{10n+3}}$$
$$- \frac{q(q^5;q^5)_\infty (q^{10};q^{10})_\infty (q^{30};q^{30})_\infty f(-q^2,-q^8)}{f(-q,-q^4) f(-q^4,-q^6)}$$
$$- \frac{(q^{30};q^{30})_\infty f(-q^2,-q^8) f(-q^3,-q^7)^2}{3(q;q)_\infty (q^{10};q^{10})_\infty f(-q^9,-q^{21})}$$
$$\times \left(2(\omega-1) f(q^7,q^8) - (4+2\omega) q f(q^2,q^{13})\right) . \tag{3.22}$$

If we repalce ω by ω^2 in (3.22), we also find that

$$(q^{30};q^{30})_\infty \psi(q)$$
$$= \frac{2}{1-\omega^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^{2(n+1)} q^{5n^2+5n}}{1-\omega^2 q^{10n+3}}$$
$$- \frac{q(q^5;q^5)_\infty (q^{10};q^{10})_\infty (q^{30};q^{30})_\infty f(-q^2,-q^8)}{f(-q,-q^4) f(-q^4,-q^6)}$$
$$- \frac{(q^{30};q^{30})_\infty f(-q^2,-q^8) f(-q^3,-q^7)^2}{3(q;q)_\infty (q^{10};q^{10})_\infty f(-q^9,-q^{21})}$$
$$\times \left(2(\omega^2-1) f(q^7,q^8) - (4+2\omega^2) q f(q^2,q^{13})\right) . \tag{3.23}$$

Separating terms in (3.22) and (3.23) according to the residue classes $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$, and $n \equiv 2 \pmod 3$, we find that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^{n+1} q^{5n^2+5n}}{1-\omega q^{10n+3}} = &\sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega q^{45n^2+15n}}{1-\omega q^{30n+3}} \\ &- \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^2 q^{45n^2+45n+10}}{1-\omega q^{30n+13}} \\ &+ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{45n^2+75n+30}}{1-\omega q^{30n+23}}\end{aligned}$$

and

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^{2n+2} q^{5n^2+5n}}{1-\omega^2 q^{10n+3}} = &\sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^2 q^{45n^2+15n}}{1-\omega^2 q^{30n+3}} \\ &- \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega q^{45n^2+45n+10}}{1-\omega^2 q^{30n+13}} \\ &+ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{45n^2+75n+30}}{1-\omega^2 q^{30n+23}} \quad .\end{aligned}$$

Next,

$$\begin{aligned}&\frac{1}{1-\omega} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega q^{45n^2+15n}}{1-\omega q^{30n+3}} + \frac{1}{1-\omega^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^2 q^{45n^2+15n}}{1-\omega^2 q^{30n+3}} \\ &= \frac{1}{1-\omega^2} \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+15n} \left(\frac{-1}{1-\omega q^{30n+3}} + \frac{\omega^2}{1-\omega^2 q^{30n+3}} \right) \\ &= \frac{1}{1-\omega^2} \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+15n} \frac{(-1+\omega^2)(1+q^{30n+3})}{(1-\omega q^{30n+3})(1-\omega^2 q^{30n+3})} \\ &= -\sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+15n} \frac{1-q^{60n+6}}{1-q^{90n+9}} \quad ,\end{aligned}$$

$$\begin{aligned}&\frac{1}{1-\omega} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^2 q^{45n^2+45n+10}}{1-\omega q^{30n+13}} + \frac{1}{1-\omega^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega q^{45n^2+45n+10}}{1-\omega^2 q^{30n+13}} \\ &= \frac{1}{1-\omega^2} \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+45n+10} \left(\frac{-\omega}{1-\omega q^{30n+13}} + \frac{\omega}{1-\omega^2 q^{30n+13}} \right) \\ &= \frac{1}{1-\omega^2} \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+45n+10} \frac{(1-\omega^2) q^{30n+13}}{(1-\omega q^{30n+13})(1-\omega^2 q^{30n+13})} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+75n+23} \frac{1-q^{30n+13}}{1-q^{90n+39}} \quad ,\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{1-\omega}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+75n+30}}{1-\omega q^{30n+23}}+\frac{1}{1-\omega^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+75n+30}}{1-\omega^2 q^{30n+23}}\\
&\quad=\frac{1}{1-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+30}\left(\frac{1+\omega}{1-\omega q^{30n+23}}+\frac{1}{1-\omega^2 q^{30n+23}}\right)\\
&\quad=\frac{1}{1-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+30}\frac{1-\omega^2}{(1-\omega q^{30n+23})(1-\omega^2 q^{30n+23})}\\
&\quad=\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+30}\frac{1-q^{30n+23}}{1-q^{90n+69}}\ .
\end{aligned}$$

Adding (3.22) and (3.23), dividing by 2, and using the three identities above, we can easily verify that

$$\begin{aligned}
(q^{30};q^{30})_\infty\psi(q)&=-\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+15n}\frac{1-q^{60n+6}}{1-q^{90n+9}}\\
&\quad-\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+23}\frac{1-q^{30n+13}}{1-q^{90n+39}}\\
&\quad+\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+30}\frac{1-q^{30n+23}}{1-q^{90n+69}}\\
&\quad-\frac{q(q^5;q^5)_\infty(q^{10};q^{10})_\infty(q^{30};q^{30})_\infty f(-q^2,-q^8)}{f(-q,-q^4)f(-q^4,-q^6)}\\
&\quad-\frac{(q^{30};q^{30})_\infty f(-q^2,-q^8)f(-q^3,-q^7)^2}{3(q;q)_\infty(q^{10};q^{10})_\infty f(-q^9,-q^{21})}\\
&\quad\times\big((\omega-1)f(q^7,q^8)-(2+\omega)qf(q^2,q^{13})\big)\\
&\quad-\frac{(q^{30};q^{30})_\infty f(-q^2,-q^8)f(-q^3,-q^7)^2}{3(q;q)_\infty(q^{10};q^{10})_\infty f(-q^9,-q^{21})}\\
&\quad\times\big((\omega^2-1)f(q^7,q^8)-(2+\omega^2)qf(q^2,q^{13})\big)\\
&=-\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+15n}\frac{1-q^{60n+6}}{1-q^{90n+9}}\\
&\quad-\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+23}\frac{1-q^{30n+13}}{1-q^{90n+39}}\\
&\quad+\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+30}\frac{1-q^{30n+23}}{1-q^{90n+69}}\\
&\quad-\frac{q(q^5;q^5)_\infty(q^{10};q^{10})_\infty(q^{30};q^{30})_\infty f(-q^2,-q^8)}{f(-q,-q^4)f(-q^4,-q^6)}
\end{aligned}$$

$$+\frac{(q^{30};q^{30})_\infty f(-q^2,-q^8)f(-q^3,-q^7)^2 f(q^7,q^8)}{(q;q)_\infty (q^{10};q^{10})_\infty f(-q^9,-q^{21})}$$

$$+\frac{q(q^{30};q^{30})_\infty f(-q^2,-q^8)f(-q^3,-q^7)^2 f(q^2,q^{13})}{(q;q)_\infty (q^{10};q^{10})_\infty f(-q^9,-q^{21})} \quad . \qquad (3.24)$$

Applying Theorem 2.19 and Theorem 3.4 to (3.24), we find that

$$\begin{aligned}
&(q^{30};q^{30})_\infty \psi(q) \\
&= -\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+15n}\frac{1-q^{60n+6}}{1-q^{90n+9}} \\
&\quad -\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+23}\frac{1-q^{30n+13}}{1-q^{90n+39}} \\
&\quad +\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+30}\frac{1-q^{30n+23}}{1-q^{90n+69}} \\
&\quad +\frac{(q^2;q^2)_\infty (q^5;q^5)_\infty (q^{30};q^{30})_\infty f(-q^{12},-q^{18})}{(q;q)_\infty f(-q^9,-q^{21})}
\end{aligned}$$

$$\begin{aligned}
&= -\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+15n}\frac{1-q^{60n+6}}{1-q^{90n+9}} \\
&\quad -\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+23}\frac{1-q^{30n+13}}{1-q^{90n+39}} \\
&\quad +\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+30}\frac{1-q^{30n+23}}{1-q^{90n+69}} \\
&\quad +\frac{(q^{30};q^{30})_\infty f(-q^{12},-q^{18})}{f(-q^9,-q^{21})}\frac{(q^6;q^6)_\infty^2 f(-q^{45},-q^{45})}{(q^3;q^3)_\infty^2} \\
&\quad +\frac{(q^{30};q^{30})_\infty f(-q^{12},-q^{18})}{f(-q^9,-q^{21})}\frac{q(q^9;q^9)_\infty (q^{18};q^{18})_\infty (q^{30};q^{30})_\infty}{(q^3;q^3)_\infty f(-q^3,-q^{15})} \\
&\quad +\frac{(q^{30};q^{30})_\infty f(-q^{12},-q^{18})}{f(-q^9,-q^{21})} \\
&\quad \times\frac{q^2(q^{15};q^{15})_\infty (q^{18};q^{18})_\infty^2 f(-q^{15},-q^{75})}{(q^3;q^3)_\infty (q^{90};q^{90})_\infty f(-q^3,-q^{15})} \quad . \qquad (3.25)
\end{aligned}$$

Using the identity (3.25) twice, we find that

$$\begin{aligned}
&(q^{30};q^{30})_\infty \frac{\psi(\omega q)-\psi(\omega^2 q)}{\omega-\omega^2}\\
&\quad= -\frac{1}{\omega-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+23}\frac{\omega^2-q^{30n+13}}{1-q^{90n+39}}\\
&\qquad+\frac{1}{\omega-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+23}\frac{\omega-q^{30n+13}}{1-q^{90n+39}}\\
&\qquad+\frac{1}{\omega-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+30}\frac{1-\omega^2 q^{30n+23}}{1-q^{90n+69}}\\
&\qquad-\frac{1}{\omega-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+30}\frac{1-\omega q^{30n+23}}{1-q^{90n+69}}\\
&\qquad+\frac{(q^{30};q^{30})_\infty f(-q^{12},-q^{18})}{f(-q^9,-q^{21})}\\
&\qquad\times\left(\frac{q(q^9;q^9)_\infty(q^{18};q^{18})_\infty(q^{30};q^{30})_\infty}{(q^3;q^3)_\infty f(-q^3,-q^{15})}\right.\\
&\qquad\qquad\left.-\frac{q^2(q^{15};q^{15})_\infty(q^{18};q^{18})^2_\infty f(-q^{15},-q^{75})}{(q^3;q^3)_\infty(q^{90};q^{90})_\infty f(-q^3,-q^{15})}\right)\\
&=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+75n+23}}{1-q^{90n+39}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+105n+53}}{1-q^{90n+69}}\\
&\quad+\frac{q(q^{18};q^{18})_\infty(q^{30};q^{30})_\infty f(-q^{12},-q^{18})}{(q^3;q^3)_\infty f(-q^3,-q^{15})f(-q^9,-q^{21})}\\
&\quad\times\left((q^9;q^9)_\infty(q^{30};q^{30})_\infty-\frac{q(q^{15};q^{15})_\infty(q^{18};q^{18})_\infty f(-q^{15},-q^{75})}{(q^{90};q^{90})_\infty}\right).
\end{aligned} \tag{3.26}$$

By (3.16), (3.26) and Theorem 2.20, we find that

$$\begin{aligned}
&(q^{30};q^{30})_\infty\left(q^2\phi(q^9)-\frac{\psi(\omega q)-\psi(\omega^2 q)}{\omega-\omega^2}\right)\\
&\quad= q^2\frac{(q^{18};q^{18})_\infty(q^{30};q^{30})_\infty(q^{45};q^{45})_\infty f(-q^{12},-q^{18})}{(q^9;q^9)_\infty f(-q^9,-q^{21})}\\
&\qquad-\frac{q(q^{18};q^{18})_\infty(q^{30};q^{30})_\infty f(-q^{12},-q^{18})}{(q^3;q^3)_\infty f(-q^3,-q^{15})f(-q^9,-q^{21})}\\
&\qquad\times\left((q^9;q^9)_\infty(q^{30};q^{30})_\infty-\frac{q(q^{15};q^{15})_\infty(q^{18};q^{18})_\infty f(-q^{15},-q^{75})}{(q^{90};q^{90})_\infty}\right)\\
&\quad=-\frac{q(q^6;q^6)_\infty(q^{30};q^{30})_\infty f(-q,-q)f(-q^3,-q^{12})}{(q^3;q^3)_\infty f(-q^3,-q^3)}.
\end{aligned} \tag{3.27}$$

Dividing both sides of (3.27) by $(q^{30};q^{30})_\infty$, we complete the proof of Theorem 3.7.

4 Proof of the second Identity

We will use (2.60) and (2.63) and need several further theorems to prove (1.6).

Theorem 4.1 *If* $|q|<1$, *then*

$$\begin{aligned}&\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty(q^{30};q^{30})_\infty f(-q^6,-q^{24})}{q(q;q)_\infty f(-q^3,-q^{27})}\\&\quad=\frac{(q^5;q^5)_\infty(q^{10};q^{10})_\infty(q^{30};q^{30})_\infty f(-q^4,-q^6)}{f(-q^2,-q^3)f(-q^2,-q^8)}\\&\qquad+\frac{(q^{30};q^{30})_\infty f(-q^4,-q^6)f(-q,-q^9)^2 f(q^4,q^{11})}{q(q;q)_\infty(q^{10};q^{10})_\infty f(-q^3,-q^{27})}\\&\qquad+\frac{(q^{30};q^{30})_\infty f(-q^4,-q^6)f(-q,-q^9)^2 f(q,q^{14})}{(q;q)_\infty(q^{10};q^{10})_\infty f(-q^3,-q^{27})}\ .\end{aligned}$$

Proof. By Theorem 3.1 with q replaced by q^{10} and x by q^2,

$$\begin{aligned}&\frac{(q^5;q^5)_\infty(q^{30};q^{30})_\infty}{f(-q^2,-q^3)}\frac{(q^{10};q^{10})_\infty f(-q^4,-q^6)}{f(-q^2,-q^8)}\\&\quad=\frac{(q^5;q^5)_\infty(q^{30};q^{30})_\infty}{f(-q^2,-q^3)}\left(f(-q^{14},-q^{16})+q^2f(-q^4,-q^{26})\right)\ .\end{aligned}\tag{4.1}$$

By Lemma 2.1(iv), (1.8), and Theorem 3.3 with q replaced by q^5 and z by q^{11}, we find that

$$\begin{aligned}&\frac{(q^{30};q^{30})_\infty f(-q^4,-q^6)f(-q,-q^9)^2 f(q^4,q^{11})}{q(q;q)_\infty(q^{10};q^{10})_\infty f(-q^3,-q^{27})}\\&\quad+\frac{(q^5;q^5)_\infty(q^{30};q^{30})_\infty f(-q^{14},-q^{16})}{f(-q^2,-q^3)}\\&=\frac{(q^5;q^5)_\infty(q^{30};q^{30})_\infty}{qf(-q^2,-q^3)f(-q^3,-q^{27})}\left(\frac{(q^{10};q^{10})_\infty f(-q,-q^9)f(q^4,q^{11})}{(q^5;q^5)_\infty}\right.\\&\qquad\left.+qf(-q^{14},-q^{16})f(-q^3,-q^{27})\right)\\&=\frac{(q^5;q^5)_\infty(q^{30};q^{30})_\infty}{qf(-q^2,-q^3)f(-q^3,-q^{27})}f(-q^6,-q^{24})f(-q^{13},-q^{17})\ .\end{aligned}\tag{4.2}$$

By Lemma 2.1(iv), (1.8), and Theorem 3.3 with q replaced by q^5 and z by q,

$$\begin{aligned}
&\frac{(q^{30};q^{30})_\infty f(-q^4,-q^6)f(-q,-q^9)^2 f(q,q^{14})}{(q;q)_\infty (q^{10};q^{10})_\infty f(-q^3,-q^{27})} \\
&\quad + \frac{q^2(q^5;q^5)_\infty (q^{30};q^{30})_\infty f(-q^4,-q^{26})}{f(-q^2,-q^3)} \\
&= \frac{(q^5;q^5)_\infty (q^{30};q^{30})_\infty}{f(-q^2,-q^3)f(-q^3,-q^{27})} \left(\frac{(q^{10};q^{10})_\infty f(-q,-q^9)f(q,q^{14})}{(q^5;q^5)_\infty} \right. \\
&\qquad \left. + q^2 f(-q^4,-q^{26})f(-q^3,-q^{27}) \right) \\
&= \frac{(q^5;q^5)_\infty (q^{30};q^{30})_\infty}{f(-q^2,-q^3)f(-q^3,-q^{27})} f(-q^6,-q^{24})f(-q^7,-q^{23}) \ . \qquad (4.3)
\end{aligned}$$

Adding (4.1), (4.2) and (4.3), and using (1.8) and Theorem 3.1 again with q replaced by q^{10} and x by q, we find that

$$\begin{aligned}
&\frac{(q^5;q^5)_\infty (q^{10};q^{10})_\infty (q^{30};q^{30})_\infty f(-q^4,-q^6)}{f(-q^2,-q^3)f(-q^2,-q^8)} \\
&\quad + \frac{(q^{30};q^{30})_\infty f(-q^4,-q^6)f(-q,-q^9)^2 f(q^4,q^{11})}{q(q;q)_\infty (q^{10};q^{10})_\infty f(-q^3,-q^{27})} \\
&\quad + \frac{(q^{30};q^{30})_\infty f(-q^4,-q^6)f(-q,-q^9)^2 f(q,q^{14})}{(q;q)_\infty (q^{10};q^{10})_\infty f(-q^3,-q^{27})} \\
&= \frac{(q^5;q^5)_\infty (q^{30};q^{30})_\infty f(-q^6,-q^{24})}{q f(-q^2,-q^3)f(-q^3,-q^{27})} \\
&\quad \times \left(f(-q^{13},-q^{17}) + q f(-q^7,-q^{23}) \right) \\
&= \frac{(q^5;q^5)_\infty (q^{30};q^{30})_\infty f(-q^6,-q^{24})}{q f(-q^2,-q^3)f(-q^3,-q^{27})} \frac{(q^{10};q^{10})_\infty f(-q^2,-q^8)}{f(-q,-q^9)} \\
&= \frac{(q^2;q^2)_\infty (q^5;q^5)_\infty (q^{30};q^{30})_\infty f(-q^6,-q^{24})}{q(q;q)_\infty f(-q^3,-q^{27})} \ .
\end{aligned}$$

We thus have completed the proof of Theorem 4.1.

Theorem 4.2 *If* $|q| < 1$, *then*

$$\begin{aligned}
&\frac{q(q^{18};q^{18})_\infty (q^{30};q^{30})_\infty (q^{45};q^{45})_\infty f(-q^6,-q^{24})}{(q^9;q^9)_\infty f(-q^3,-q^{27})} \\
&\quad = \frac{q^7(q^{30};q^{30})_\infty (q^{45};q^{45})_\infty (q^{90};q^{90})_\infty f(-q^{18},-q^{72})}{f(-q^9,-q^{36})f(-q^{36},-q^{54})}
\end{aligned}$$

$$- \frac{q^7 (q^{45}; q^{45})_\infty^2 (q^{90}; q^{90})_\infty^2}{f(-q^{45}, -q^{45}) f(-q^9, -q^{36})}$$
$$\times \left(\frac{f(-q^{21}, -q^{24})}{f(-q^{33}, -q^{57})} - \frac{f(-q^6, -q^{39})}{q^6 f(-q^3, -q^{87})} \right) . \tag{4.4}$$

Proof. In Theorem 3.5, replacing q by q^{90}, a by q^{51}, b by q^{21}, c by q^{48}, and d by q^{15}, we find that

$$\begin{aligned} 0 &= q^{51} f(-q^{18}, -q^{72}) f(-q^{-30}, -q^{120}) f(-q^{27}, -q^{63}) f(-q^{33}, -q^{57}) \\ &\quad + q^{21} f(-q^{21}, -q^{69}) f(-q^{27}, -q^{63}) f(-q^{24}, -q^{66}) f(-q^{36}, -q^{54}) \\ &\quad + q^{48} f(-q^{-9}, -q^{99}) f(-q^3, -q^{87}) f(-q^{36}, -q^{54}) f(-q^6, -q^{84}) \\ &= -q^{21} f(-q^{18}, -q^{72}) f(-q^{30}, -q^{60}) f(-q^{27}, -q^{63}) f(-q^{33}, -q^{57}) \\ &\quad + q^{21} f(-q^{21}, -q^{69}) f(-q^{27}, -q^{63}) f(-q^{24}, -q^{66}) f(-q^{36}, -q^{54}) \\ &\quad - q^{39} f(-q^9, -q^{81}) f(-q^3, -q^{87}) f(-q^{36}, -q^{54}) f(-q^6, -q^{84}) . \end{aligned}$$

This implies that

$$\begin{aligned} &q^{25} f(-q^9, -q^{81}) f(-q^3, -q^{87}) f(-q^{36}, -q^{54}) f(-q^6, -q^{84}) \\ &\quad = -q^7 f(-q^{18}, -q^{72}) f(-q^{30}, -q^{60}) f(-q^{27}, -q^{63}) f(-q^{33}, -q^{57}) \\ &\qquad + q^7 f(-q^{21}, -q^{69}) f(-q^{27}, -q^{63}) f(-q^{24}, -q^{66}) f(-q^{36}, -q^{54}) . \end{aligned} \tag{4.5}$$

Replacing q by q^{90}, x by q^{36}, y by q^3, and z by q^{27} in Theorem 3.2, and applying Lemma 2.1(iv), we find that

$$\begin{aligned} &-q f(-q^{36}, -q^{54})^2 f(-q^{30}, -q^{60}) f(-q^{24}, -q^{66}) \\ &\quad = q^{25} f(-q^3, -q^{87})^2 f(-q^{27}, -q^{63}) f(-q^9, -q^{81}) \\ &\qquad - q f(-q^{27}, -q^{63})^2 f(-q^{39}, -q^{51}) f(-q^{33}, -q^{57}) . \end{aligned} \tag{4.6}$$

Let $R(q)$ denote the right side of (4.4). Then, by (1.8), (1.9), (4.5) and (4.6),

$$\begin{aligned} R(q) &= \frac{q^7 (q^{30}; q^{30})_\infty (q^{45}; q^{45})_\infty (q^{90}; q^{90})_\infty f(-q^{18}, -q^{72})}{f(-q^9, -q^{36}) f(-q^{36}, -q^{54})} \\ &\quad - \frac{q^7 (q^{45}; q^{45})_\infty^2 (q^{90}; q^{90})_\infty^2 f(-q^{21}, -q^{24})}{f(-q^{45}, -q^{45}) f(-q^9, -q^{36}) f(-q^{33}, -q^{57})} \\ &\quad + \frac{q (q^{45}; q^{45})_\infty^2 (q^{90}; q^{90})_\infty^2 f(-q^6, -q^{39})}{f(-q^{45}, -q^{45}) f(-q^9, -q^{36}) f(-q^3, -q^{87})} \end{aligned}$$

$$= \frac{q^7(q^{30};q^{30})_\infty(q^{90};q^{90})_\infty^3 f(-q^{18},-q^{72})}{f(-q^9,-q^{81})f(-q^{36},-q^{54})^2}$$
$$- \frac{q^7(q^{90};q^{90})_\infty^3 f(-q^{21},-q^{69})f(-q^{24},-q^{66})}{f(-q^9,-q^{81})f(-q^{36},-q^{54})f(-q^{33},-q^{57})}$$
$$+ \frac{q(q^{45};q^{45})_\infty^2(q^{90};q^{90})_\infty^2 f(-q^6,-q^{39})}{f(-q^{45},-q^{45})f(-q^9,-q^{36})f(-q^3,-q^{87})}$$

$$= \frac{(q^{90};q^{90})_\infty^3}{f(-q^9,-q^{81})f(-q^{36},-q^{54})^2 f(-q^{33},-q^{57})f(-q^{27},-q^{63})}$$
$$\times (q^7 f(-q^{18},-q^{72})f(-q^{30},-q^{60})f(-q^{27},-q^{63})f(-q^{33},-q^{57})$$
$$- q^7 f(-q^{21},-q^{69})f(-q^{27},-q^{63})f(-q^{24},-q^{66})f(-q^{36},-q^{54}))$$
$$+ \frac{q(q^{45};q^{45})_\infty^2(q^{90};q^{90})_\infty^2 f(-q^6,-q^{39})}{f(-q^{45},-q^{45})f(-q^9,-q^{36})f(-q^3,-q^{87})}$$

$$= - \frac{(q^{90};q^{90})_\infty^3}{f(-q^9,-q^{81})f(-q^{36},-q^{54})^2 f(-q^{33},-q^{57})f(-q^{27},-q^{63})}$$
$$\times q^{25} f(-q^9,-q^{81})f(-q^3,-q^{87})f(-q^{36},-q^{54})f(-q^6,-q^{84})$$
$$+ \frac{q(q^{90};q^{90})_\infty^3 f(-q^6,-q^{84})f(-q^{39},-q^{51})}{f(-q^9,-q^{81})f(-q^{36},-q^{54})f(-q^3,-q^{87})}$$

$$= - \frac{(q^{90};q^{90})_\infty^3}{f(-q^9,-q^{81})f(-q^{36},-q^{54})f(-q^{33},-q^{57})f(-q^{27},-q^{63})^2}$$
$$\times \frac{f(-q^6,-q^{84})}{f(-q^3,-q^{87})}(q^{25} f(-q^3,-q^{87})^2 f(-q^{27},-q^{63})f(-q^9,-q^{81})$$
$$- q f(-q^{27},-q^{63})^2 f(-q^{39},-q^{51})f(-q^{33},-q^{57}))$$

$$= \frac{(q^{30};q^{30})_\infty f(-q^6,-q^{84})}{f(-q^9,-q^{21})f(-q^9,-q^{81})f(-q^{36},-q^{54})f(-q^{27},-q^{63})}$$
$$\times (q f(-q^{36},-q^{54})^2 f(-q^{30},-q^{60})f(-q^{24},-q^{66}))$$

$$= \frac{q(q^{18};q^{18})_\infty(q^{30};q^{30})_\infty(q^{45};q^{45})_\infty f(-q^6,-q^{24})}{(q^9;q^9)_\infty f(-q^3,-q^{27})} \quad .$$

We are ready to prove the main theorem.

Theorem 4.3 (Second 10th Order Mock Theta Function Identity) *For* $|q| < 1$,

$$q^{-2}\psi(q^9)+\frac{\omega\phi(\omega q)-\omega^2\phi(\omega^2 q)}{\omega-\omega^2}=\frac{f(-q,-q)f(-q^6,-q^9)(q^3;q^3)_\infty}{f(-q^3,-q^3)^2} \ .$$

Proof. The proof is similar to that of Theorem 3.7 in the previous chapter. First we will examine $\psi(q)$. Replace q by q^9 in (3.18) and multiply both sides of (3.18) by q^{-2}. If $z=q^{30}$, then, by (1.9),

$$\begin{aligned} q^{-2}(q^{30};q^{30})_\infty\psi(q^9) &= 2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+75n+28}}{1-q^{90n+57}} \\ &\quad + q^{-2}a_1(q^9)(q^{30};q^{30})_\infty + 2q^7A(q^{12},q^9,q^{45}) \ . \end{aligned} \tag{4.7}$$

Replacing q by q^9 and z by q^{60} in (3.18), multiplying both sides by q^{-2}, and using (1.9), we find that

$$\begin{aligned} q^{-2}(q^{30};q^{30})_\infty\psi(q^9) &= 2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+105n+58}}{1-q^{90n+87}} \\ &\quad + q^{-2}a_1(q^9)(q^{30};q^{30})_\infty + 2q^7A(q^{42},q^9,q^{45}) \ . \end{aligned} \tag{4.8}$$

Adding (4.7) and (4.8), dividing by 2, using (2.36), (2.40), and Lemma 2.1(iv), and applying Theorem 4.2, we find that

$$\begin{aligned} &q^{-2}(q^{30};q^{30})_\infty\psi(q^9) \\ &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+75n+28}}{1-q^{90n+57}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+105n+58}}{1-q^{90n+87}} \\ &\quad + q^{-2}a_1(q^9)(q^{30};q^{30})_\infty + q^7\big(A(q^{12},q^9,q^{45})+A(q^{42},q^9,q^{45})\big) \\ &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+75n+28}}{1-q^{90n+57}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+105n+58}}{1-q^{90n+87}} \\ &\quad - \frac{q^7(q^{30};q^{30})_\infty(q^{45};q^{45})_\infty(q^{90};q^{90})_\infty f(-q^{18},-q^{72})}{f(-q^9,-q^{36})f(-q^{36},-q^{54})} \\ &\quad + \frac{q^7(q^{45};q^{45})_\infty^2(q^{90};q^{90})_\infty^2}{f(-q^{45},-q^{45})f(-q^9,-q^{36})} \\ &\quad \times\left(\frac{f(-q^{21},-q^{24})}{f(-q^{33},-q^{57})}-\frac{q^{-6}f(-q^6,-q^{39})}{f(-q^3,-q^{87})}\right) \\ &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+75n+28}}{1-q^{90n+57}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+105n+58}}{1-q^{90n+87}} \\ &\quad - \frac{q(q^{18};q^{18})_\infty(q^{30};q^{30})_\infty(q^{45};q^{45})_\infty f(-q^6,-q^{24})}{(q^9;q^9)_\infty f(-q^3,-q^{27})} \ . \end{aligned} \tag{4.9}$$

We now consider $\phi(q)$. Replacing z by ω in (3.13) and using (2.36), (2.40), and Lemma 2.1(iv), we find that

$$\begin{aligned}
&(q^{30};q^{30})_\infty\phi(q)\\
&\quad=\frac{2}{1-\omega}\sum_{n=-\infty}^{\infty}\frac{(-1)^n\omega^{n+1}q^{5n^2+5n-1}}{1-\omega q^{10n+1}}+a_2(q)(q^{30};q^{30})_\infty\\
&\qquad+\frac{2q}{1-\omega}A(\omega q^{-4},q^2,q^5)\\
&\quad=\frac{2}{1-\omega}\sum_{n=-\infty}^{\infty}\frac{(-1)^n\omega^{n+1}q^{5n^2+5n-1}}{1-\omega q^{10n+1}}\\
&+\frac{(q^5;q^5)_\infty(q^{10};q^{10})_\infty(q^{30};q^{30})_\infty f(-q^4,-q^6)}{f(-q^2,-q^3)f(-q^2,-q^8)}\\
&-\frac{2(\omega-1)}{3q}\frac{(q^5;q^5)^2_\infty(q^{10};q^{10})^2_\infty f(-\omega^2q^2,-\omega q^3)}{f(-q^5,-q^5)f(-q^2,-q^3)f(-\omega q,-\omega^2q^9)}\ .
\end{aligned}\tag{4.10}$$

In Theorem 3.1, replacing q by q^5 and x by $-\omega^2q^2$, and using Lemma 2.1(iv), we have

$$f(q^4,q^{11})-\omega^2qf(q,q^{14})=\frac{(q^5;q^5)_\infty f(-\omega^2q,-\omega q^4)}{f(\omega^2q^2,\omega q^3)}\ .$$

Let $F(q)$ denote the last expression on the right side of (4.10), then using (1.8) several times and the formula above, we find that

$$\begin{aligned}
F(q)&=-\frac{2(\omega-1)}{3q}\frac{(q^{10};q^{10})^3_\infty}{f(-q^2,-q^3)}\\
&\qquad\times\frac{(q^{10};q^{10})_\infty(q^5;q^5)_\infty f(-\omega^2q,-\omega q^4)}{f(\omega^2q^2,\omega q^3)f(-\omega^2q,-\omega q^9)f(-\omega q,-\omega^2q^9)}\\
&=-\frac{2(\omega-1)}{3q}\frac{(q^{30};q^{30})_\infty f(-q^4,-q^6)f(-q,-q^9)^2}{(q;q)_\infty(q^{10};q^{10})_\infty f(-q^3,-q^{27})}\\
&\qquad\times\frac{(q^5;q^5)_\infty f(-\omega^2q,-\omega q^4)}{f(\omega^2q^2,\omega q^3)}\\
&=-\frac{2(\omega-1)}{3q}\frac{(q^{30};q^{30})_\infty f(-q^4,-q^6)f(-q,-q^9)^2}{(q;q)_\infty(q^{10};q^{10})_\infty f(-q^3,-q^{27})}\\
&\qquad\times\big(f(q^4,q^{11})-\omega^2qf(q,q^{14})\big)\\
&=-\frac{(q^{30};q^{30})_\infty f(-q^4,-q^6)f(-q,-q^9)^2}{3q(q;q)_\infty(q^{10};q^{10})_\infty f(-q^3,-q^{27})}\\
&\qquad\times\big(2(\omega-1)f(q^4,q^{11})-(4+2\omega)qf(q,q^{14})\big)\ .
\end{aligned}\tag{4.11}$$

Putting (4.11) in (4.10), we deduce that

$$
\begin{aligned}
(q^{30};q^{30})_\infty \phi(q) = & \frac{2}{1-\omega} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^{n+1} q^{5n^2+5n-1}}{1-\omega q^{10n+1}} \\
& + \frac{(q^5;q^5)_\infty (q^{10};q^{10})_\infty (q^{30};q^{30})_\infty f(-q^4,-q^6)}{f(-q^2,-q^3) f(-q^2,-q^8)} \\
& - \frac{(q^{30};q^{30})_\infty f(-q^4,-q^6) f(-q,-q^9)^2}{3q(q;q)_\infty (q^{10};q^{10})_\infty f(-q^3,-q^{27})} \\
& \times \left(2(\omega-1) f(q^4,q^{11}) - (4+2\omega) q f(q,q^{14})\right) . \quad (4.12)
\end{aligned}
$$

If we repalce ω by ω^2 in (4.12), we find that

$$
\begin{aligned}
(q^{30};q^{30})_\infty \phi(q) = & \frac{2}{1-\omega^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^{2(n+1)} q^{5n^2+5n-1}}{1-\omega^2 q^{10n+1}} \\
& + \frac{(q^5;q^5)_\infty (q^{10};q^{10})_\infty (q^{30};q^{30})_\infty f(-q^4,-q^6)}{f(-q^2,-q^3) f(-q^2,-q^8)} \\
& - \frac{(q^{30};q^{30})_\infty f(-q^4,-q^6) f(-q,-q^9)^2}{3q(q;q)_\infty (q^{10};q^{10})_\infty f(-q^3,-q^{27})} \\
& \times \left(2(\omega^2-1) f(q^4,q^{11}) - (4+2\omega^2) q f(q,q^{14})\right) . \quad (4.13)
\end{aligned}
$$

Splitting each sum in (4.12) and (4.13) into three parts according as $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$, and $n \equiv 2 \pmod 3$, we deduce that

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^{n+1} q^{5n^2+5n-1}}{1-\omega q^{10n+1}} = & \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega q^{45n^2+15n-1}}{1-\omega q^{30n+1}} \\
& - \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^2 q^{45n^2+45n+9}}{1-\omega q^{30n+11}} \\
& + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{45n^2+75n+29}}{1-\omega q^{30n+21}} , \quad (4.14)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^{2n+2} q^{5n^2+5n-1}}{1-\omega^2 q^{10n+1}} = & \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega^2 q^{45n^2+15n-1}}{1-\omega^2 q^{30n+1}} \\
& - \sum_{n=-\infty}^{\infty} \frac{(-1)^n \omega q^{45n^2+45n+9}}{1-\omega^2 q^{30n+11}} \\
& + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{45n^2+75n+29}}{1-\omega^2 q^{30n+21}} . \quad (4.15)
\end{aligned}
$$

Next,

$$\frac{1}{1-\omega}\sum_{n=-\infty}^{\infty}\frac{(-1)^n\omega q^{45n^2+15n-1}}{1-\omega q^{30n+1}}+\frac{1}{1-\omega^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n\omega^2 q^{45n^2+15n-1}}{1-\omega^2 q^{30n+1}}$$

$$=\frac{1}{1-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+15n-1}\left(\frac{-1}{1-\omega q^{30n+1}}+\frac{\omega^2}{1-\omega^2 q^{30n+1}}\right)$$

$$=\frac{1}{1-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+15n-1}\frac{(-1+\omega^2)(1+q^{30n+1})}{(1-\omega q^{30n+1})(1-\omega^2 q^{30n+1})}$$

$$=-\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+15n-1}\frac{1-q^{60n+2}}{1-q^{90n+3}} \quad ,$$

$$\frac{1}{1-\omega}\sum_{n=-\infty}^{\infty}\frac{(-1)^n\omega^2 q^{45n^2+45n+9}}{1-q^{30n+11}\omega}+\frac{1}{1-\omega^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n\omega q^{45n^2+45n+9}}{1-\omega^2 q^{30n+11}}$$

$$=\frac{1}{1-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+45n+9}\left(\frac{-\omega}{1-\omega q^{30n+11}}+\frac{\omega}{1-\omega^2 q^{30n+11}}\right)$$

$$=\frac{1}{1-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+45n+9}\frac{(1-\omega^2)q^{30n+11}}{(1-\omega q^{30n+11})(1-\omega^2 q^{30n+11})}$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+20}\frac{1-q^{30n+11}}{1-q^{90n+33}} \quad ,$$

and

$$\frac{1}{1-\omega}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+75n+29}}{1-\omega q^{30n+21}}+\frac{1}{1-\omega^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{45n^2+75n+29}}{1-\omega^2 q^{30n+21}}$$

$$=\frac{1}{1-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+29}\left(\frac{1+\omega}{1-\omega q^{30n+21}}+\frac{1}{1-\omega^2 q^{30n+21}}\right)$$

$$=\frac{1}{1-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+29}\frac{1-\omega^2}{(1-\omega q^{30n+21})(1-\omega^2 q^{30n+21})}$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+29}\frac{1-q^{30n+21}}{1-q^{90n+63}} \quad .$$

Adding (4.12) and (4.13), dividing by 2, and using (4.14), (4.15), and the above three identities, we can easily verify that

$$
\begin{aligned}
(q^{30};q^{30})_\infty \phi(q) = &- \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+15n-1} \frac{1-q^{60n+2}}{1-q^{90n+3}} \\
&- \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+75n+20} \frac{1-q^{30n+11}}{1-q^{90n+33}} \\
&+ \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+75n+29} \frac{1-q^{30n+21}}{1-q^{90n+63}} \\
&\quad + \frac{(q^5;q^5)_\infty (q^{10};q^{10})_\infty (q^{30};q^{30})_\infty f(-q^4,-q^6)}{f(-q^2,-q^3) f(-q^2,-q^8)} \\
&\quad - \frac{(q^{30};q^{30})_\infty f(-q^4,-q^6) f(-q,-q^9)^2}{3q(q;q)_\infty (q^{10};q^{10})_\infty f(-q^3,-q^{27})} \\
&\quad \times \big((\omega-1) f(q^4,q^{11}) - (2+\omega) q f(q,q^{14})\big) \\
&\quad - \frac{(q^{30};q^{30})_\infty f(-q^4,-q^6) f(-q,-q^9)^2}{3q(q;q)_\infty (q^{10};q^{10})_\infty f(-q^3,-q^{27})} \\
&\quad \times \big((\omega^2-1) f(q^4,q^{11}) - (2+\omega^2) q f(q,q^{14})\big) \\
= &- \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+15n-1} \frac{1-q^{60n+2}}{1-q^{90n+3}} \\
&- \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+75n+20} \frac{1-q^{30n+11}}{1-q^{90n+33}} \\
&+ \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+75n+29} \frac{1-q^{30n+21}}{1-q^{90n+63}} \\
&+ \frac{(q^5;q^5)_\infty (q^{10};q^{10})_\infty (q^{30};q^{30})_\infty f(-q^4,-q^6)}{f(-q^2,-q^3) f(-q^2,-q^8)} \\
&+ \frac{(q^{30};q^{30})_\infty f(-q^4,-q^6) f(-q,-q^9)^2 f(q^4,q^{11})}{q(q;q)_\infty (q^{10};q^{10})_\infty f(-q^3,-q^{27})} \\
&+ \frac{(q^{30};q^{30})_\infty f(-q^4,-q^6) f(-q,-q^9)^2 f(q,q^{14})}{(q;q)_\infty (q^{10};q^{10})_\infty f(-q^3,-q^{27})} \quad . \qquad (4.16)
\end{aligned}
$$

Applying Theorem 2.19 and Theorem 4.1 to (4.16), we find that

$$
\begin{aligned}
(q^{30};q^{30})_\infty \phi(q) = &- \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+15n-1} \frac{1-q^{60n+2}}{1-q^{90n+3}} \\
&- \sum_{n=-\infty}^{\infty} (-1)^n q^{45n^2+75n+20} \frac{1-q^{30n+11}}{1-q^{90n+33}}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+29}\frac{1-q^{30n+21}}{1-q^{90n+63}}\\
&+\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty(q^{30};q^{30})_\infty f(-q^6,-q^{24})}{q(q;q)_\infty f(-q^3,-q^{27})}\\
=&-\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+15n-1}\frac{1-q^{60n+2}}{1-q^{90n+3}}\\
&-\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+20}\frac{1-q^{30n+11}}{1-q^{90n+33}}\\
&+\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+29}\frac{1-q^{30n+21}}{1-q^{90n+63}}\\
&+\frac{(q^{30};q^{30})_\infty f(-q^6,-q^{24})}{qf(-q^3,-q^{27})}\frac{(q^6;q^6)_\infty^2 f(-q^{45},-q^{45})}{(q^3;q^3)_\infty^2}\\
&+\frac{(q^{30};q^{30})_\infty f(-q^6,-q^{24})}{qf(-q^3,-q^{27})}\\
&\times\left(\frac{q(q^9;q^9)_\infty(q^{18};q^{18})_\infty(q^{30};q^{30})_\infty}{(q^3;q^3)_\infty f(-q^3,-q^{15})}\right.\\
&\quad\left.+\frac{q^2(q^{15};q^{15})_\infty(q^{18};q^{18})_\infty^2 f(-q^{15},-q^{75})}{(q^3;q^3)_\infty(q^{90};q^{90})_\infty f(-q^3,-q^{15})}\right)\ . \qquad (4.17)
\end{aligned}
$$

Using the identity (4.17), we find that

$$
\begin{aligned}
&(q^{30};q^{30})_\infty\frac{\omega\phi(\omega q)-\omega^2\phi(\omega^2 q)}{\omega-\omega^2}\\
&=-\frac{1}{\omega-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+15n-1}\frac{1-\omega^2 q^{60n+2}}{1-q^{90n+3}}\\
&+\frac{1}{\omega-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+15n-1}\frac{1-\omega q^{60n+2}}{1-q^{90n+3}}\\
&-\frac{1}{\omega-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+20}\frac{1-\omega^2 q^{30n+11}}{1-q^{90n+33}}\\
&+\frac{1}{\omega-\omega^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{45n^2+75n+20}\frac{1-\omega q^{30n+11}}{1-q^{90n+33}}\\
&+\frac{(q^{30};q^{30})_\infty f(-q^6,-q^{24})}{f(-q^3,-q^{27})}\\
&\times\left(\frac{(q^9;q^9)_\infty(q^{18};q^{18})_\infty(q^{30};q^{30})_\infty}{(q^3;q^3)_\infty f(-q^3,-q^{15})}\right.\\
&\quad\left.-\frac{q(q^{15};q^{15})_\infty(q^{18};q^{18})_\infty^2 f(-q^{15},-q^{75})}{(q^3;q^3)_\infty(q^{90};q^{90})_\infty f(-q^3,-q^{15})}\right)
\end{aligned}
$$

$$= -\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{45n^2+75n+1}}{1-q^{90n+3}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{45n^2+105n+31}}{1-q^{90n+33}}$$
$$+ \frac{(q^{18};q^{18})_\infty (q^{30};q^{30})_\infty f(-q^6,-q^{24})}{(q^3;q^3)_\infty f(-q^3,-q^{15}) f(-q^3,-q^{27})}$$
$$\times \Bigg((q^9;q^9)_\infty (q^{30};q^{30})_\infty - \frac{q(q^{15};q^{15})_\infty (q^{18};q^{18})_\infty f(-q^{15},-q^{75})}{(q^{90};q^{90})_\infty} \Bigg) .$$

In the first sum on the far right side above, multiply the numerator and denominator by q^{-90n-3}, and replace n by $-(n+1)$. In the second summation, multiply the numerator and denominator by $q^{-90n-33}$, and replace n by $-(n+1)$. Then we find that

$$(q^{30};q^{30})_\infty \frac{\omega\phi(\omega q) - \omega^2\phi(\omega^2 q)}{\omega - \omega^2}$$
$$= -\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{45n^2+105n+58}}{1-q^{90n+87}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{45n^2+75n+28}}{1-q^{90n+57}}$$
$$+ \frac{(q^{18};q^{18})_\infty (q^{30};q^{30})_\infty f(-q^6,-q^{24})}{(q^3;q^3)_\infty f(-q^3,-q^{15}) f(-q^3,-q^{27})}$$
$$\times \Bigg((q^9;q^9)_\infty (q^{30};q^{30})_\infty - \frac{q(q^{15};q^{15})_\infty (q^{18};q^{18})_\infty f(-q^{15},-q^{75})}{(q^{90};q^{90})_\infty} \Bigg) . \tag{4.18}$$

By (4.9), (4.18), and Theorem 2.20,

$$(q^{30}q^{30})_\infty \left(q^{-2}\psi(q^9) + \frac{\omega\phi(\omega q) - \omega^2\phi(\omega^2 q)}{\omega - \omega^2} \right)$$
$$= -\frac{q(q^{18};q^{18})_\infty (q^{30};q^{30})_\infty (q^{45};q^{45})_\infty f(-q^6,-q^{24})}{(q^9;q^9)_\infty f(-q^3,-q^{27})}$$
$$+ \frac{(q^{18};q^{18})_\infty (q^{30};q^{30})_\infty f(-q^6,-q^{24})}{(q^3;q^3)_\infty f(-q^3,-q^{15}) f(-q^3,-q^{27})}$$
$$\times \left((q^9;q^9)_\infty (q^{30};q^{30})_\infty - \frac{q(q^{15};q^{15})_\infty (q^{18};q^{18})_\infty f(-q^{15},-q^{75})}{(q^{90};q^{90})_\infty} \right)$$

$$= -\frac{(q^{18};q^{18})_\infty (q^{30};q^{30})_\infty f(-q^6,-q^{24})}{f(-q^3,-q^{27})} \frac{q(q^{45};q^{45})_\infty}{(q^9;q^9)_\infty}$$
$$+\frac{(q^{18};q^{18})_\infty (q^{30};q^{30})_\infty f(-q^6,-q^{24})}{f(-q^3,-q^{27})} \frac{(q^9;q^9)_\infty (q^{30};q^{30})_\infty}{(q^3;q^3)_\infty f(-q^3,-q^{15})}$$
$$-\frac{(q^{18};q^{18})_\infty (q^{30};q^{30})_\infty f(-q^6,-q^{24})}{f(-q^3,-q^{27})}$$
$$\times \frac{q(q^{15};q^{15})_\infty (q^{18};q^{18})_\infty f(-q^{15},-q^{75})}{(q^{90};q^{90})_\infty (q^3;q^3)_\infty f(-q^3,-q^{15})}$$
$$= (q^{30};q^{30})_\infty \frac{f(-q,-q)f(-q^6,-q^9)(q^3;q^3)_\infty}{f(-q^3,-q^3)^2}, \tag{4.19}$$

after several applications of (1.8). Dividing both sides of (4.19) by $(q^{30};q^{30})_\infty$, we complete the proof of Theorem 4.3.

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